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"JÚLIO DE MESQUITA FILHO"
Câmpus de São José do Rio Preto

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Some Persistent Cohomology Invariants and an Axiomatic Version of Persistent Homology

São José do Rio Preto
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Tese apresentada como parte dos requisitos para obtenção do título de Doutor em Matemática, junto ao Programa de Pós-Graduação em Matemática, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista "Júlio de Mesquita Filho", Câmpus de São José do Rio Preto.

Orientadora: Profa. Dra. Alice Kimie Miwa Libardi
Coorientador: Prof. Dr. Roberto Facundo Mémoli Techera

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To God, my family and
everyone who supported me.

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Sometimes we feel that what we are doing
is nothing but a drop of water in the sea.
But the sea would be smaller if it lacked a drop.

Por vezes sentimos que aquilo que fazemos
não é senão uma gota de água no mar.
Mas o mar seria menor se lhe faltasse uma gota.
Madre Tereza de Calcutá ([Ter10], p.213)

RESUMO

Neste trabalho encontramos dois grandes capítulos que têm como foco duas das mais importantes ferramentas da Análise Topológica de Dados (TDA): homologia de persistência e cohomologia de persistência. As abordagens dadas a essas duas ferramentas são de natureza e objetivos muitos distintos.

Com inúmeras aplicações nas mais variadas áreas, a homologia de persistência já se mostrou uma ferramenta muito poderosa, porém pouco se estudou a respeito de uma abordagem axiomática sobre a mesma. Definimos adaptações persistentes dos axiomas de Eilenberg-Steenrod, com os quais podemos desenvolver e construir as propriedades da mesma. Para concluir, provamos um teorema de unicidade, mostrando a total caracterização de nossa teoria por meio desses axiomas.

Considerando a ferramenta dual da anterior, temos a cohomologia de persistência. Muito estudada em artigos recentes, a cohomologia vem como uma forma alternativa, mais rápida e de mesma eficiência que a homologia de persistência, já que devido às dualidades temos construções semelhantes. Porém, pouquíssima abordada nesses trabalhos, a estrutura de anel que se ganha ao trabalhar com cohomologia não teve desenvolvimento relevante em TDA. Nesse trabalho, definiremos dois invariantes totalmente relacionados a essa estrutura de anel, que surge através dos produtos cup. Calcularemos vários exemplos desses invariantes, mostrando situações em que eles são capazes de nos dar informações mais completas que as antigas ferramentas.

Palavras-chave: Análise Topológica de Dados, Homologia de Persistência , Axiomatização, Cohomologia de Persistência, Invariantes Persistentes.

ABSTRACT

In this work we find two chapters focussing on two of the most important tools for topological data analysis: persistent homology and persistent cohomology. The approaches given to these two tools are very different in nature and objectives.
With numerous applications in the most varied areas, the persistent homology has already proved to be a very powerful tool, but there are no study about an axiomatic approach of it. We define persistent versions of the Eilenberg-Sttenrod axioms, with which we can develop and construct its properties. To conclude, we prove a uniqueness theorem, showing the full characterization of our theory through these axioms.
Considering the dual tool of the previous one, we have the persistence cohomology. Much studied in recent articles, cohomology comes as an alternative form, faster and with the same efficiency than persistence homology, since due to the dualities, we have similar constructions. However, very little addressed in these works, the ring structure that is gained by working with cohomology did not have relevant development in TDA. In this work, we will define two invariants totally related to this ring structure, which arises through the cup products. We will calculate several examples of these invariants, showing situations in which they are able to give us more information than the old tools.

Keywords: Topological Data Analysis, Persistent Homology, Axiomatization, Persistent Cohomology, Persistent Invariants.

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1 Introduction

To live in a digital era with daily advanced in technology is something that is incredible for all of us. With just a click we can discover our exact coordinates on Earth, we can join a call with a friend in another country, or you can discover the nationality of your grand-grandfather if you want. All these advanced come to help the people to live more and better, but in this meantime all the information about everything we can think are storage as Data. The preference of a population for an app, the increasing of a healthy style of life in a region, the difference in a lifestyle of a person that uses a product or not... these bunch of information are all storage as datasets and to be able to retrieves important information about them a growing need of understand, process and working with data has demand innovation in many areas. Essentially most of this cloud of data has discreet nature and applying the standard topological methods to study patterns, shapes or features do not return relevant information about our set.

Following this necessity, Topological Data Analysis (TDA) emerges being a powerful tool combining the classical definitions of topology with data analysis, which is able to discover, extract and understand the structure of different set of data and points. The development of this new field increases since it was proposed by Frosini, in [Fro90], using persistent to obtain and define a distance between certain topological spaces and by Robins, in [Rob99], combining persistent ideas with the study of fractal sets. It didn't take long until one of the most powerful TDA tools started to grow, develop and have interesting applications in the most diverse areas of knowledge: The Persistent Homology.

Persistent homology is an algebraic method to capture and measure topological features of functions and shapes, that is basically the study of the evolution of homology across a filtration of spaces [Fro92, ZC05, CSEH07]. It is able to extract the time when a topological feature is 'born' and the time when it 'dies' in the filtration, and collect all information in pairs, called barcode or persistence diagram of the filtration. We list several applications of this method.

◦ **Biogenetics**

- In [SVV19], the authors apply persistent homology to observe properties of gene expression in post-mortem brain tissue (cerebral cortex) of individuals with autism spectrum disorders (ASD) and matched controls. They also use it to recover a difference in the shape of these controls when compare autism and healthy controls, observed using the Euler characteristic and death times of zero-dimensional components. (See Fig. 1.1) The study provides a novel framework for persistent homology analysis of gene expression data for genetically complex disorders, since it was the first study to apply persistent homology for brain co-expression networks. By assessing topological invariants of the inter-sample distance matrices, the authors demonstrate that gene expression data from ASD individuals are more heterogeneous than gene expression from controls, since both SDT and the Euler characteristic were significantly higher for the ASD group in all datasets.

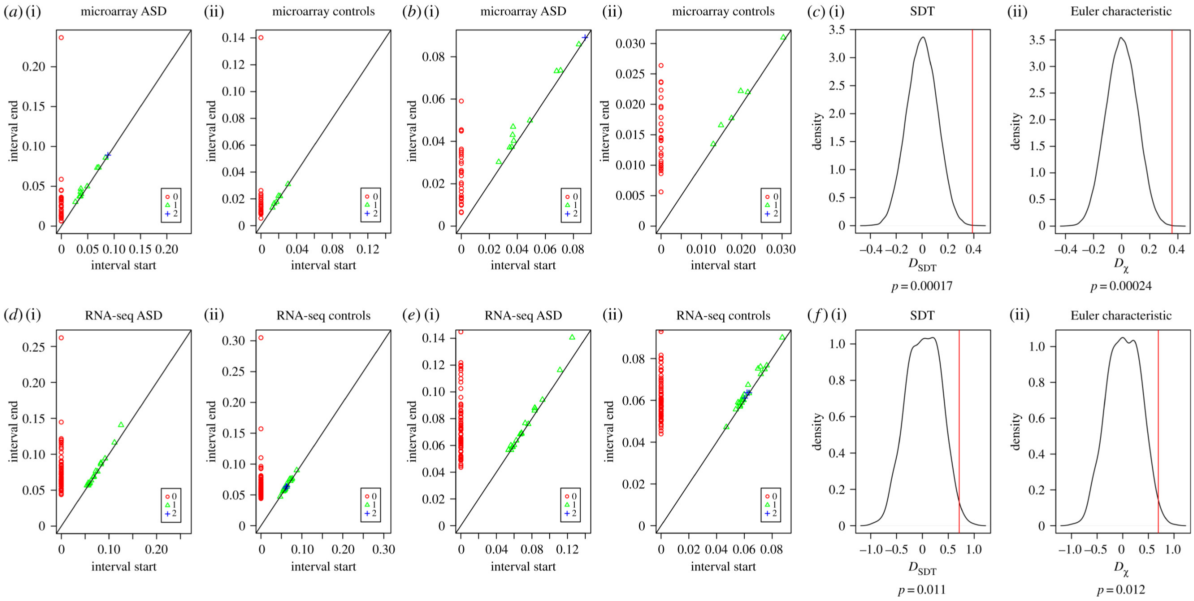


Figure 1.1: Persistent homology analysis of sample-level point clouds. Persistence diagrams of ASD, SDT and control groups, density plot of SDT difference between ASD and controls (DSDT) and of Euler characteristic difference between ASD and controls. Source of the image: [SVV19]

- In [KNBNH16], the authors apply topological techniques to study a protein, the maltose-binding protein (MBP), represented by 370 points in $\mathbb{R}^3$. This set is constantly changed by biological functions, turning it into a not static set of points. They use topology to construct summary statistics of these changes to extract information about the points.

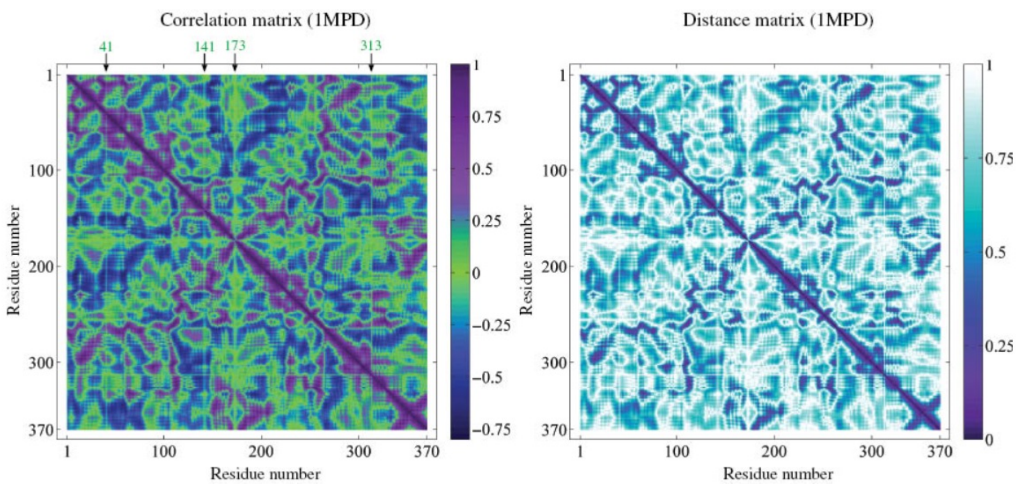


Figure 1.2: Visualization of the cross-correlation matrix and the dynamical distance matrix of the 1MPD structure. (Left) Cross-correlation matrix for the 1MPD structure. (Right) Dynamical distance matrix for the 1MPD structure. Source of the image: [KNBNH16].

Using the ideas of [EH10], where one studies computational ways of predicting protein interactions, the authors used topological methods to describe the function of the MBP using the spatial coordinates of some residues, but this approach was not efficient to distinguishing closed and the open MBP conformation. Therefore, they tried to calculate fourteen cross-correlation matrices (C) (See Fig. 1.2) of size 370×370 using the ideas founded in [EYB06] and define a metric space in which highly correlated residues lie close to each other.

Barcodes, persistent diagrams, snapshots and landscapes were developed and obtained following these ideas and statistically significant difference between the closed and the open conformation could be determined using filtered Vietoris-Rips complex and topological methods, specially persistent homology.

- In [DP18], the authors compute the persistent homology of weighted networks constructed from microarray data sets to distinguishing the stress factors. In [CCR13], an identification of a evolution of viruses is possible using the cycles of evolutionaries trees via persistent homology. In [MAL+20], the authors recall weighted persistent homology (WPH) models and their applications in biomolecular data analysis, foccusing the application in the study of DNA structures.

◦ **Orthodontia**

- Combining persistent homology and dimensionally reduction methods, the authors of [GH10] use a three-dimensional landmark-based data set of an orthodontic study to generates persistent diagram and compute de distances between them.

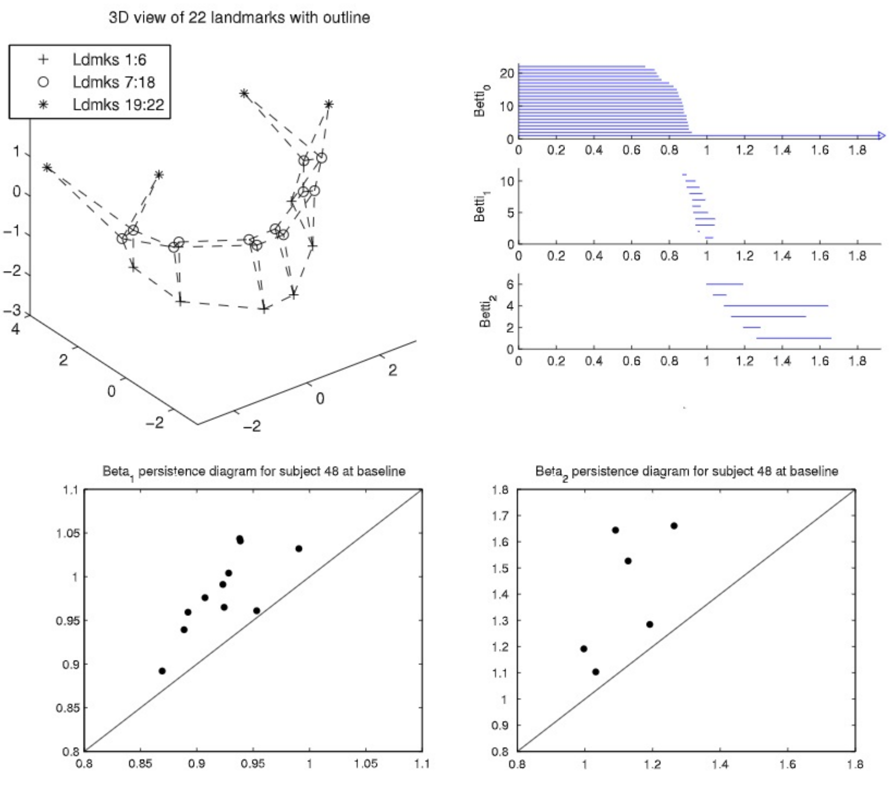


Figure 1.3: Three-dimensional plot of the 22 landmarks. The barcode and persistent diagrams of dimension one and two associated to the points. Source of the image: [GH10]

The results are able to distinguish effects of clinical treatments. Each subject of the work is represented by a set of 22 three-dimensional landmarks that were previously chosen. With this representation, the authors computed the barcodes and persistent diagrams of dimension one and two (see Fig. 1.3) before and after the treatments and also a computation of a distance between the barcodes and persistent diagrams.

The analysis of the distances between these objects, when the authors focus on persistent diagrams of dimension one, were effective to distinguish the effects of the maxilla treatments in the patients. They also realize that if more landmarks or another kind of filtration were used, the distance between the persistent diagrams of dimension two would be more effective to the main idea of the study.

This work suggested another interesting feature about the persistent homology ideas: There is a better chosen of a homology degree and a special filtration for each case and problem. The richest thing about this persistence idea is that there is no right answer, but for sure there is one that gives us more and richer information.

- **Machine Learning**

  One of the most recent application for persistent homology is in machine learning architecture optimization and selection of model. Different from the previous applications, where it was a tool used to find, recover and extract features of topological features in a set, persistent cohomology is being used to improve the yield of models. Several recent paper bring us the richness of this new way to use the TDA tool, [CNBW19, GC19, RVM19].

- **Persistent Cohomology**

  Recent years have seen a very rapid advance on the development of techniques based on applied algebraic topology for the analysis of data [Car09]. There are recent indications that ideas related to cohomology, often lead to algorithms which exhibit higher memory efficiency and lower overall computational complexity in practical applications.

  Persistent cohomology has been studied in [Bau21, DSMVJ11a, DSMVJ11b], but only in [Yar10] this study explore the ring structure in cohomology. From classical topology, the main difference between homology and cohomology is the fact that we can define a structure with a multiplicative operation for cohomology, obtaining not a group, but a ring. Therefore, an motivation of the last chapter of this thesis emerges when we realize that few of the authors took this structure into account.

  See Chapter 3 for a deeper understanding of the subject.

2 Persistent Homology: A Homology Theory

2.1 The Eilenberg-Steenrod axioms and axiomatic approach

Before [ES45, ES15], the usual approach to homology groups and homology theory was based completely over a complicated idea of complex. All the definitions and results were based in heavy and not so natural constructions. In this new approach proposed in 1945 by the authors, they proposed 7 axioms that full characterize the *homology theories* and provide a construction of the statements in a way that makes concepts and constructions with an extended point of view and logical simplicity.

Following this idea of an 'new' approach, In [CM+10] the authors extended their previous work on hierarchical clustering of dissimilarity networks under an axiomatic point of view. The results and the different framework generates an analogous existence and uniqueness theorem to a non-existence theorem stated before and, with this new theorem of existence, questions as stability could be explored. In [MP20], the authors generalize the hierarchical clustering of dissimilarity networks ideas and work with more general pairs, called extended network, where is studied an axiomatic for clustering ideas.

With all this successful axiomatic frameworks, we also develop in this chapter an axiomatic approach to Persistent Homology, defining a persistent version of the Eilenberg-Steenrod axioms with a simplification of them. After this, we construct an environment where it will be possible to prove a uniqueness theorem for our theory.

2.2 Persistent Homology

In this work we will consider real numbers $\varepsilon \leq \varepsilon' \leq M$, where $M \gg 0$ is a sufficient large fixed real positive number.

Definition 2.1. Let X be a finite set. A **filtration** over X is a monotone function $F_X \colon \mathrm{pow}(X) \longrightarrow \mathbb{R}$ which gives rise to a family of subspaces $K^\varepsilon = F_X^{-1}((-\infty, \varepsilon])$ and inclusions $\{K^\varepsilon \xrightarrow{k_{\varepsilon,\varepsilon'}} K^{\varepsilon'}\}$, where $\varepsilon \leq \varepsilon' \in \mathbb{R}$. By a monotone function F_X, we mean that if $\sigma, \tau \in \mathrm{pow}(X)$ and $\sigma \subseteq \tau$ then $F_X(\sigma) \leq F_X(\tau)$.

A **filtered pair** will be any pair $\mathbb{X} = (X, F_X)$, where X is a finite set and F_X is a filtration over X. $\mathcal{F}$ denotes the set of all finite filtered pairs and $\mathcal{F}(X)$ is the set of all possible filtrations $F_X \colon \mathrm{pow}(X) \longrightarrow \mathbb{R}$ on X.

Given any $\mathbb{X}$ a filtered pair, each K^ε of the filtration is a simplicial complex, considering for $\sigma \in \mathrm{pow}(X)$ that σ is an n-simplex of K^ε if, and only if, $\#\sigma = n+1$ and $F_X(\sigma) \leq \varepsilon$. The family $K(\mathbb{X}) = \{K^\varepsilon\}_{\varepsilon \in \mathbb{R}}$ is a family of simplicial complexes.

Definition 2.2. Let $\mathbb{X}$ and $\mathbb{Y}$ be filtered pairs and let $\{K^{\varepsilon}\}_{\varepsilon\in\mathbb{R}}$ and $\{L^{\varepsilon}\}_{\varepsilon\in\mathbb{R}}$ be the respective associated filtration over X and Y. A map $f\colon \mathbb{X} \longrightarrow \mathbb{Y}$ is a **filtration preserving map** if $F_X = F_Y \circ f$.This map induces a family $\{f^{\varepsilon}\}_{\varepsilon\in\mathbb{R}}$, where $f^{\varepsilon}\colon K^{\varepsilon} \longrightarrow L^{\varepsilon}$ is such that for $\varepsilon \leq \varepsilon'$ the following diagram commutes:

$$\begin{tikzcd} K^{\varepsilon} \arrow[r, hook, "k^{\varepsilon,\varepsilon'}"] \arrow[d, "f^{\varepsilon}"] & K^{\varepsilon'} \arrow[d, "f^{\varepsilon'}"] \\ L^{\varepsilon} \arrow[r, hook, "l^{\varepsilon,\varepsilon'}"] & L^{\varepsilon'} \end{tikzcd}$$

For $n \geq 0$ and $\varepsilon \leq \varepsilon'$, we define $f^{\varepsilon,\varepsilon'} := f^{\varepsilon'} k^{\varepsilon,\varepsilon'} = l^{\varepsilon,\varepsilon'} f^{\varepsilon}$ and we denote by $k_n^{\varepsilon,\varepsilon'}$ the induced n-chain map of $k^{\varepsilon,\varepsilon'}$.

For a fixed $\varepsilon \in \mathbb{R}$, to construct the chain complex associated to K^{ε}, we need to have an orientation in the simplicial complex of K^{ε}. We begin by giving an orientation to the simplices of K^{ε}. Let $\sigma_n = \{x_0, x_1, \ldots, x_n\}$ be an n-simplex of K^{ε}. We say that two orderings of the elements of σ_n are equivalent whenever they differ by an even permutation; an orientation of σ_n, $n > 0$, is an equivalence class of orderings of the vertices of σ_n. By this definition, an n-simplex $\sigma_n = \{x_0, x_1, \ldots, x_n\}$ has two orientations. A 0-simplex has only one orientation which is given by ± 1.

We remark that if $\sigma = \{x_0, x_1, \ldots, x_n\}$ is oriented by the natural ordering of the indices of its vertices, its $(n-1)$-faces

$$\sigma_{n-1,i} = \{x_0, x_1, \ldots, \widehat{x_i}, \ldots, x_n\} = \{x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$$

have an orientation given by $(-1)^i \sigma_{n-1,i}$.

Now, we give an order to the simplicial complex K^{ε} by taking a partial ordering of the set K^{ε} in such a way that the set of vertices of each simplex σ_n is ordered. Then we obtain an ordering class for each simplex. A simplicial complex whose simplices are all oriented is said to be oriented.

Definition 2.3. Let K^{ε} be an ordered simplicial complex. For every $n \in \mathbb{Z}$, with $n \geq 0$, the free abelian group $C_n(K^{\varepsilon})$ consists of all linear combinations with coefficients in $\mathbb{Z}$ of oriented n-simplices of K^{ε}. If $n < 0$, we define $C_n(K^{\varepsilon}) = 0$. $C_n(K^{\varepsilon})$ is called the **group of n-chains of K^{ε}**.

For every $n \in \mathbb{Z}$, $n > 0$, we first define $\partial_n^{\varepsilon}\colon C_n(K^{\varepsilon}) \longrightarrow C_{n-1}(K^{\varepsilon})$ over an oriented n-simplex $\{x_0, x_1, \ldots, x_n\}$ as

$$\partial_n^{\varepsilon}(\{x_0, x_1, \ldots, x_n\}) = \sum_{i=0}^{n} (-1)^i \{x_0, \ldots, \widehat{x_i}, \ldots, x_n\};$$

and extend by linearity over an oriented n-chain. For $n \leq 0$, ∂_n^{ε} is the null homomorphism. This homomorphism is called **boundary homomorphism**.

This construction is possible for each K^{ε} of our filtration and then we obtain ∂_n^{ε}, for all K^{ε} and $n \in \mathbb{Z}$ such that $\partial_{n-1}^{\varepsilon} \circ \partial_n^{\varepsilon} = 0$. Therefore,

$$C\colon \quad \cdots \longrightarrow C_n(K^{\varepsilon}) \xrightarrow{\partial_n^{\varepsilon}} C_{n-1}(K^{\varepsilon}) \xrightarrow{\partial_{n-1}^{\varepsilon}} \cdots$$

is a **chain complex** and the elements of $C_n(K^{\varepsilon})$ are called n-chains.

Definition 2.4. Let us denote by $Z_n^\varepsilon = ker(\partial_n^\varepsilon)$, the set of all n-**cycles** and by $B_n^\varepsilon = \mathrm{Im}(\partial_{n+1}^\varepsilon)$ the set of all n-**boundaries**.

Consider now the poset $Int(\mathbb{R}) = \{[\varepsilon, \varepsilon'], \varepsilon \leq \varepsilon'\}$. For $I, J \in Int(\mathbb{R})$, one has $I \leq J \leftrightarrow I \subseteq J$. Then we define the **persistent homology group** of X as the functor from $Int(\mathbb{R})$ into the category of Abelian groups, for $\varepsilon \leq \varepsilon'$ as:

$$H_n^{\varepsilon,\varepsilon'}(X) := \frac{k_n^{\varepsilon,\varepsilon'}(Z_n^\varepsilon)}{B_n^{\varepsilon'} \cap k_n^{\varepsilon,\varepsilon'}(Z_n^\varepsilon)}.$$

For each pair $\varepsilon \leq \varepsilon'$ we are able to compute the Persistent Homology of X associated to this pair, but for us will be interested if we have a way to visualize the homology more globally in our filtration, in other words, be able to see the evolution of classes in the entire filtration. For this reason, we have the idea of barcode. We will see that the algorithms that compute the persistent homology, output for us intervals or bars, that represents cycles that persisted in this time.

2.2.1 Barcodes

Let $\mathbb{X}$ be a filtered pair. A critical point of F_X is a real value r such that $K^r = F_X^{-1}((-\infty, r]) \neq K^{r-\delta} = F_X^{-1}((-\infty, r-\delta])$, for any $\delta < 0$.

Since $F_X\colon \mathrm{pow}(X) \longrightarrow \mathbb{R}$ and X is finite, there exists a finite number m of critical point of F_X. Let us denote these critical points by $c_1, \ldots, c_m$.

Then, our filtration of X can be seen as:

$$\mathbb{X}\colon \quad K^{c_1} \subset K^{c_2} \subset \cdots \subset K^{c_n} = X.$$

Applying the homology functor over a field $\mathbb{F}$ to this filtration, we obtain a **persistent module**, which is a diagram of finite dimensional vector spaces and linear maps.

$$H(\mathbb{X})\colon \quad H_*(K^{c_1}) \longrightarrow H_*(K^{c_2}) \longrightarrow \cdots \longrightarrow H_*(K^{c_n}).$$

This kind of persistent module decomposes as a direct sum of interval modules [ZC05]. Each interval $[c_i, c_j]$ in the decomposition represents a lifespan of a feature, or a class of a cycle, which means that the cycle associated to this interval was born at the time c_i and persisted until the time c_j, and died at time c_{j+1}.

The **barcode** is a multiset of pairs $[c_i, c_j]$ in this decomposition. We often associate the interval $[c_i, c_j]$ to the interval $[i, j]$, simply because it is easy to represent in a matrix, since the columns and rows are non-negative integers. If we have a feature that survive until the last step of the filtration, we put the interval associate to this feature with the end point being infinite.

$$\mathcal{B}(H_*(\mathbb{X})) := \mathrm{Barc}(H_*(\mathbb{X})) = \{[i_0, j_0], \ldots, [i_m, j_m]\} = \{[c_{i_0}, c_{j_0}], \ldots, [c_{i_m}, c_{j_m}]\}.$$

2.2.2 Algorithms

In [CSEM06] the authors explain how to compute the persistent as a decomposition of a matrix D, the boundary matrix. Using the operation low, we find a factorize D as the product of two other matrix R and U, where R is a **reduced** matrix and U a invertible upper-triangular one. We will remember just the essential definitions and concepts.

For any matrix M, we define $low_M(j)$ to be the index of the lowest non-zero entry in the j-th column of M, and it is not defined if the j-th column is a zero column. The matrix M is said to be reduced if low_M is injective over its domain.

Since the matrix U obtained is invertible, it will be more convenient for us to use the inverse matrix $V = U^{-1}$ and use the decomposition $R = DV$ instead $D = RU$. This is clarified in the paper, since using this decomposition we are able to associate columns and rows of theses matrix of the decomposition with representatives for our classes of cycles and boundaries. To find more details, see [CSEM06] and [DSMVJ11a].

In [ZC05], the original persistence algorithm finds the pairs of the barcode by processing the boundary matrix D using column operations, to obtain the reduced matrix R.

Algorithm 1: Persistent Homology by column

Input : The Boundary matrix D.
Output : The decomposition $R = DV$.

```
R ← D
V ← I
for i ← 1 to n do
    while ∃ j < i such that low_R(j) = low_R(i) do
        c ← R[i][low_R(j)] / R[j][low_R(i)]
        R[i] ← R[i] − cR[j]
        V[i] ← V[i] − cV[j]
```

The previous algorithm is Gaussian elimination using column operations. Another way to compute the decomposition is to use row operations instead of columns operation.

Algorithm 2: Persistent Homology by row

Input : The Boundary matrix D.
Output : The decomposition $R = DV$.

```
R ← D
V ← I
for i ← n downto 1 do
    indices ← [j | low_R(j) = i]
    p ← indices[0]
    for j ∈ indices[1 ...] do
        c ← R[j][low_R(j)] / R[p][low_R(p)]
        R[j] ← R[j] − cR[p]
        V[j] ← V[j] − cV[p]
```

We can see that, due to the operations that the algorithms do with the matrix R, when the algorithm ends, what we have is a reduced matrix R. Another thing that is proved in [DSMVJ11a] is that the two decompositions are the same, e.g. the two algorithm produce the same matrices R e V.

2.3 The Persistent version of Eilenberg-Steenrod axioms

We recall some definitions and results that will be fundamental to define the persistent version of the Eilenberg-Steenrod axiom, inspired by [ES45], on which our full theory will be constructed.

Definition 2.5. Let $\mathbb{X}$ and $\mathbb{Y}$ be filtered pairs and let $\{K^{\varepsilon}\}_{\varepsilon\in\mathbb{R}}$ and $\{L^{\varepsilon}\}_{\varepsilon\in\mathbb{R}}$ be the associated filtrations, respectively. Let $f\colon \mathbb{X} \longrightarrow \mathbb{Y}$ be a filtration preserving map. If $\{v_0, \ldots, v_n\}$ is a simplex of K^{ε}, then $f^{\varepsilon}(v_0), \ldots, f^{\varepsilon}(v_n)$ span a simplex of L^{ε}, $n > 0$.

The homomorphism $(f^\varepsilon)_\#\colon C_n(K^\varepsilon) \longrightarrow C_n(L^\varepsilon)$ is defined on oriented simplices as follows:

$$(f^\varepsilon)_\#(\{v_0,\dots,v_n\}) := \begin{cases} \{f^\varepsilon(v_0),\dots,f^\varepsilon(v_n)\}, & \text{if } f^\varepsilon(v_0),\dots,f^\varepsilon(v_n) \text{ are distinct} \\ 0, & \text{otherwise} \end{cases}$$

For $\varepsilon \le \varepsilon'$, we define $f_\#^{\varepsilon,\varepsilon'} := l_n^{\varepsilon,\varepsilon'} f_\#^\varepsilon = f_\#^{\varepsilon'} k_n^{\varepsilon,\varepsilon'}$

Lemma 2.6. *For $\varepsilon \in \mathbb{R}$, the homomorphism $f_\#^\varepsilon$ commutes with ∂_n^ε, the usual boundary homomorphism. For $\varepsilon \le \varepsilon'$, the homomorphism $f_\#^{\varepsilon,\varepsilon'}$ commutes with $\partial_n^{\varepsilon,\varepsilon'}$, where $\partial_n^{\varepsilon,\varepsilon'} = k_{n-1}^{\varepsilon,\varepsilon'} \circ \partial_n^\varepsilon = \partial_n^{\varepsilon'} \circ k_n^{\varepsilon,\varepsilon'}$.*

The family of homomorphism $\{f_\#^{\varepsilon,\varepsilon'}\}$, for $\varepsilon \le \varepsilon'$, is called the **chain map induced by the filtration preserving map** f.

Given a filtration preserving map $f\colon \mathbb{X} \longrightarrow \mathbb{Y}$, the chain map induced by f satisfies: $f_\#^\varepsilon(Z_{n,X}^\varepsilon) \subset Z_{n,Y}^\varepsilon$ and $f_\#^\varepsilon(B_{n,X}^\varepsilon) \subset B_{n,Y}^\varepsilon$, for each $\varepsilon > 0$.

It follows that for $\varepsilon \le \varepsilon'$, we have $f_\#^{\varepsilon'}(k_n^{\varepsilon,\varepsilon'}(Z_{n,X}^\varepsilon)) \subseteq k_n^{\varepsilon,\varepsilon'}(Z_{n,Y}^\varepsilon)$ and $f_\#^{\varepsilon'}(B_{n,X}^{\varepsilon'}) \subseteq B_{n,Y}^{\varepsilon'}$. So we have the inclusion

$$f_\#^{\varepsilon'}(B_{n,X}^{\varepsilon'} \cap k_n^{\varepsilon,\varepsilon'}(Z_{n,X}^\varepsilon)) \subseteq f_\#^{\varepsilon'}(B_{n,X}^\varepsilon) \cap f_\#^{\varepsilon'}(k_n^{\varepsilon,\varepsilon'}(Z_{n,X}^\varepsilon)) \subseteq B_{n,Y}^{\varepsilon'} \cap k_n^{\varepsilon,\varepsilon'}(Z_{n,Y}^{\varepsilon'}).$$

Therefore, $f_\#^{\varepsilon,\varepsilon'}$ induces a well defined **homomorphism in persistent homology**

$$f_*^{\varepsilon,\varepsilon'}\colon H_n^{\varepsilon,\varepsilon'}(X) \longrightarrow H_n^{\varepsilon,\varepsilon'}(Y)$$

given by $f_*^{\varepsilon,\varepsilon'}[d] := [f_\#^{\varepsilon'}(d)]$, where $[d] \in H_n^{\varepsilon,\varepsilon'}(X)$.

Definition 2.7. A **relative filtered pair** $(\mathbb{X},\mathbb{A}) := ((X,A),(F_X,F_A))$, is a pair (X,A) of finite sets, where $A \subset X$, equipped with a pair of maps (F_X,F_A) defined in the following way: The maps $F_X\colon \mathrm{pow}(X) \longrightarrow \mathbb{R}$ and $F_A\colon \mathrm{pow}(A) \longrightarrow \mathbb{R}$ are filtrations and satisfy $F_A(\sigma) \ge F_X(\sigma)$ for all $\sigma \in \mathrm{pow}(A)$. By a **filtration preserving map of relative filtered pairs** $f\colon (\mathbb{X},\mathbb{A}) \longrightarrow (\mathbb{Y},\mathbb{B})$ we mean a filtration preserving map $f\colon \mathbb{X} \longrightarrow \mathbb{Y}$ such that $f(A) \subset B$ and the restriction map $f|_A\colon \mathbb{A} \longrightarrow \mathbb{B}$ is also a filtration preserving map.

Remark 2.8. It is important to note that even using the word filtration in the previous definition, it does not mean that there is a filtration on the set X. The name is related to the fact that from the finite set X with the map F_X the following filtration of simplicial complex is derived

$$K^\varepsilon = F_X^{-1}((-\infty,\varepsilon])$$

and

$$A^\varepsilon = F_A^{-1}((-\infty,\varepsilon]).$$

Definition 2.9. The **persistent homology boundary homomorphism**

$$\Delta_A\colon H_n^{\varepsilon,\varepsilon'}(X,A) \longrightarrow H_{n-1}^{\varepsilon,\varepsilon'}(A)$$

is the homomorphism induced by the boundary map from the group of n-chains of $(K^{\varepsilon'},A^{\varepsilon'})$ to the group of $(n-1)$-chains of $A^{\varepsilon'}$.

Remark 2.10. Let $\mathbb{X}$ and $\mathbb{Y}$ be filtered pairs and fix $\alpha \in \mathbb{R}$. Two maps $f,g\colon F_X^{-1}((-\infty,\alpha]) \longrightarrow F_Y^{-1}((-\infty,\alpha])$ are **contiguous** if for every $\sigma \in F_X^{-1}((-\infty,\alpha])$, there exists $\tau \in F_Y^{-1}((-\infty,\alpha])$ such that $f(\sigma) \subset \tau$ and $g(\sigma) \subset \tau$.

Definition 2.11. Let $(\mathbb{X},\mathbb{A})$ and $(\mathbb{Y},\mathbb{B})$ be relative filtered pairs and let $f, g\colon (\mathbb{X},\mathbb{A}) \longrightarrow (\mathbb{Y},\mathbb{B})$ be two filtration preserving maps. We say that f and g are **contiguous maps** in $[\varepsilon,\varepsilon']$ if for all $\delta \in [\varepsilon,\varepsilon']$, $f_\delta, g_\delta\colon (K^\delta, L^\delta) \longrightarrow (M^\delta, N^\delta)$ are contiguous maps, where $K^\delta = F_X^{-1}((-\infty,\delta])$, $L^\delta = F_A^{-1}((-\infty,\delta])$, $M^\delta = F_Y^{-1}((-\infty,\delta])$, $N^\delta = F_B^{-1}((-\infty,\delta])$.

The contiguous maps f and g are **sequentially contiguous** maps if there exists a finite sequence $f_1,\ldots,f_m\colon (\mathbb{X},\mathbb{A}) \longrightarrow (\mathbb{Y},\mathbb{B})$ such that $f_1 = f$, $f_m = g$ and f_i and f_{i+1} are contiguous maps for $i = 1,\ldots,m-1$.

Definition 2.12. Let $(\mathbb{X},\mathbb{A})$ and $(\mathbb{Y},\mathbb{B})$ be relative filtered pairs and let $f\colon (\mathbb{X},\mathbb{A}) \longrightarrow (\mathbb{Y},\mathbb{B})$ and $g\colon (\mathbb{Y},\mathbb{B}) \longrightarrow (\mathbb{X},\mathbb{A})$ be filtration preserving maps. f and g are **contiguously equivalent maps** if $g\circ f$ and $\mathrm{Id}_{(\mathbb{X},\mathbb{A})}$ are contiguous maps and $f\circ g$ and $\mathrm{Id}_{(\mathbb{Y},\mathbb{B})}$ are contiguous maps.

If there exist contiguously equivalent filtration preserving maps $f\colon (\mathbb{X},\mathbb{A}) \longrightarrow (\mathbb{Y},\mathbb{B})$ and $g\colon (\mathbb{Y},\mathbb{B}) \longrightarrow (\mathbb{X},\mathbb{A})$, we will say that $(\mathbb{X},\mathbb{A})$ and $(\mathbb{Y},\mathbb{B})$ are **equi-contiguous sets** and the maps f and g are said to be **contiguity maps**.

We now formulate a persistent version of the Eilenberg-Steenrod axioms and also prove that the **Persistent Homology** satisfies these axioms. For the following axioms we assume $\varepsilon \le \varepsilon' < M$, where M is a sufficiently large fixed real positive number.

Persistent Axiom 1. Let $f\colon \mathbb{X} \longrightarrow \mathbb{X}$ be a filtration preserving map. If f is the identity map of X then the induced map in persistent homology $f_*^{\varepsilon,\varepsilon'}\colon H_*^{\varepsilon,\varepsilon'}(X) \longrightarrow H_*^{\varepsilon,\varepsilon'}(X)$ is the identity homomorphism.

Proof of Persistent Axiom 1. Let $\varepsilon > 0$ fixed and $n \ge 0$. A generic element $c \in C_n(K^\varepsilon)$ can be described as a finite sum $c = \sum m_i\sigma_n^i$, where each $m_i \in \mathbb{Z}$ and each σ_n^i is an n-simplex. Since $f_\#^\varepsilon$ was defined by linearity, it is enough to show that $f_\#^\varepsilon(\sigma_n^i) = \sigma_n^i$ for all n-simplex σ_n^i in K^ε.

If $\sigma_n^i = \{v_0, v_1, \ldots, v_n\}$, then by definition $f_\#^\varepsilon(\sigma_n^i) = \{f^\varepsilon(v_0), f^\varepsilon(v_1), \ldots, f^\varepsilon(v_n)\} = \{v_0, v_1, \ldots, v_n\} = \sigma_n^i$, because f is the identity map.

Thus, $f_\#^\varepsilon$ is the identity homomorphism and for $\varepsilon \le \varepsilon'$ and $[c] \in H_*^{\varepsilon,\varepsilon'}(X)$,

$$f_*^{\varepsilon,\varepsilon'}([c]) = [f_\#^{\varepsilon'}(c)] = [c]. \qquad \square$$

Persistent Axiom 2. Let $f\colon \mathbb{X} \longrightarrow \mathbb{Y}$ and $g\colon \mathbb{Z} \longrightarrow \mathbb{X}$ be filtration preserving maps. Then for $\varepsilon \le \varepsilon'$, $(f\circ g)_*^{\varepsilon,\varepsilon'} = f_*^{\varepsilon,\varepsilon'} \circ g_*^{\varepsilon,\varepsilon'}$. In other words, the following diagram commutes:

$$\begin{tikzcd} H_*^{\varepsilon,\varepsilon'}(Z) \arrow[rr, "(f\circ g)_*^{\varepsilon,\varepsilon'}"] \arrow[dr, "g_*^{\varepsilon,\varepsilon'}"'] & & H_*^{\varepsilon,\varepsilon'}(Y) \\ & H_*^{\varepsilon,\varepsilon'}(X) \arrow[ur, "f_*^{\varepsilon,\varepsilon'}"'] & \end{tikzcd}$$

Proof of Persistent Axiom 2. For $\varepsilon > 0$ fixed and $n \ge 0$ one has that any $c \in C_n(K^\varepsilon)$ can be written as a finite sum $c = \sum m_i\sigma_n^i$, where $m_i \in \mathbb{Z}$ and σ_n^i is an n-simplex of K^ϵ.

Let us show that $(f\circ g)_\#^\varepsilon = f_\#^\varepsilon \circ g_\#^\varepsilon$ in these n-simplices. Given $\sigma_n^i = \{v_0, v_1, \ldots, v_n\}$, one has that $f_\#^\varepsilon \circ g_\#^\varepsilon(\sigma_n^i) = f_\#^\varepsilon(g_\#^\varepsilon(\{v_0, v_1, \ldots, v_n\}))$ and then we consider two cases:

(1) If $g^\varepsilon(v_i) = g^\varepsilon(v_j)$, for some $0 \le i < j \le n$, then $g_\#^\varepsilon(\{v_0, v_1, \ldots, v_n\}) = 0$ and $f_\#^\varepsilon(g_\#^\varepsilon(\{v_0, v_1, \ldots, v_n\})) = 0$. Observe that if $g^\varepsilon(v_i) = g^\varepsilon(v_j)$ then $(f\circ g)^\varepsilon(v_i) = (f\circ g)^\varepsilon(v_j)$, and therefore $(f\circ g)_\#^\varepsilon(\{v_0, v_1, \ldots, v_n\}) = 0$.

(2) On other hand, if $g^\varepsilon(v_i) \neq g^\varepsilon(v_j)$, for all $0 \le i < j \le n$, using the fact that $f^\varepsilon g^\varepsilon = (f \circ g)^\varepsilon$, we have

$$\begin{aligned} f^\varepsilon_\# \circ g^\varepsilon_\#(\sigma^i_n) &= f^\varepsilon_\#(g^\varepsilon_\#(\{v_0, v_1, \dots, v_n\})) \\ &= f^\varepsilon_\#(\{g^\varepsilon(v_0), g(v_1), \dots, g^\varepsilon(v_n)\}) \\ &= \{f^\varepsilon(g^\varepsilon(v_0)), f^\varepsilon(g^\varepsilon(v_1)), \dots, f^\varepsilon(g^\varepsilon(v_n)\} \\ &= \{f^\varepsilon \circ g^\varepsilon(v_0), f^\varepsilon \circ g^\varepsilon(v_1), \dots, f^\varepsilon \circ g^\varepsilon(v_n)\} \\ &= \{(f \circ g)^\varepsilon(v_0), (f \circ g)^\varepsilon(v_1), \dots, (f \circ g)^\varepsilon(v_n)\} \\ &= (f \circ g)^\varepsilon_\#(\{v_0, v_1, \dots, v_n\}). \end{aligned}$$

Hence, given $[d] \in \mathrm{H}^{\varepsilon,\varepsilon'}_*(Z)$,

$$(f \circ g)^{\varepsilon,\varepsilon'}_*([d]) = [(f \circ g)^{\varepsilon'}_\#(d)] = [f^{\varepsilon'}_\# \circ g^{\varepsilon'}_\#(d)] = f^{\varepsilon,\varepsilon'}_*[(g^{\varepsilon'}_\#(d))] = (f^{\varepsilon,\varepsilon'}_* \circ g^{\varepsilon,\varepsilon'}_*)[d]. \quad \square$$

Persistent Axiom 3. Let $(\mathbb{X}, \mathbb{A})$ and $(\mathbb{Y}, \mathbb{B})$ be relative filtered pairs and let $f\colon (\mathbb{X}, \mathbb{A}) \longrightarrow (\mathbb{Y}, \mathbb{B})$ be a filtration preserving map. Then, for $\varepsilon \le \varepsilon'$, the following diagram commutes:

$$\begin{tikzcd} \mathrm{H}^{\varepsilon,\varepsilon'}_n(X, A) \arrow[r, "f^{\varepsilon,\varepsilon'}_*"] \arrow[d, "\Delta_A"] & \mathrm{H}^{\varepsilon,\varepsilon'}_n(Y, B) \arrow[d, "\Delta_B"] \\ \mathrm{H}^{\varepsilon,\varepsilon'}_{n-1}(A) \arrow[r, "(f|_A)^{\varepsilon,\varepsilon'}_*"] & \mathrm{H}^{\varepsilon,\varepsilon'}_{n-1}(B) \end{tikzcd}$$

Proof of Persistent Axiom 3. Let c be a class in $\mathrm{H}^{\varepsilon,\varepsilon'}_n(X, A) = \dfrac{\overline{k}^{\varepsilon,\varepsilon'}_n(Z^\varepsilon_n(X, A))}{B^{\varepsilon'}_n(X, A) \cap \overline{k}^{\varepsilon,\varepsilon'}_n(Z^\varepsilon_n(X, A))}$, where $\overline{k}^{\varepsilon,\varepsilon'}_n : C_n(K^\varepsilon, A^\varepsilon) \longrightarrow C_n(K^{\varepsilon'}, A^{\varepsilon'})$ is the homomorphism induced by $k^{\varepsilon,\varepsilon'}_n : C_n(K^\varepsilon) \longrightarrow C_n(K^{\varepsilon'})$. Then we can write $c = [\overline{k}^{\varepsilon,\varepsilon'}_n(d)]$, where $d \in Z^\varepsilon_n(X, A)$.

$$\begin{aligned} (\Delta_B \circ f^{\varepsilon,\varepsilon'}_*)(c) &= \Delta_B \circ f^{\varepsilon,\varepsilon'}_*([\overline{k}^{\varepsilon,\varepsilon'}_n(d)]) = \Delta_B([f^{\varepsilon'}_\#(\overline{k}^{\varepsilon,\varepsilon'}_n(d))]) \\ &= \Delta_B([(\overline{k}^{\varepsilon,\varepsilon'}_n f^{\varepsilon'}_\#(d))]) = [\partial^{\varepsilon'}_{n,Y}(\overline{k}^{\varepsilon,\varepsilon'}_n f^{\varepsilon'}_\#(d))] \\ &= [(\overline{k}^{\varepsilon,\varepsilon'}_{n-1}(\partial^\varepsilon_{n,Y} f^\varepsilon_\#(d)))] = [(\overline{k}^{\varepsilon,\varepsilon'}_{n-1}(f^\varepsilon_\# \partial^\varepsilon_{n,X}(d)))] \\ &= [((f^{\varepsilon'}_\# \overline{k}^{\varepsilon,\varepsilon'}_{n-1}\partial^\varepsilon_{n,X}(d)))] = (f|_A)^{\varepsilon,\varepsilon'}_*([\overline{k}^{\varepsilon,\varepsilon'}_{n-1}\partial^\varepsilon_{n,X}(d)]) \\ &= (f|_A)^{\varepsilon,\varepsilon'}_*([\partial^\varepsilon_{n,X}(\overline{k}^{\varepsilon,\varepsilon'}_n d)]) = (f|_A)^{\varepsilon,\varepsilon'}_* \circ \Delta_A([(\overline{k}^{\varepsilon,\varepsilon'}_n d)]) \\ &= ((f|_A)^{\varepsilon,\varepsilon'}_* \circ \Delta_A)(c). \end{aligned} \quad \square$$

Persistent Axiom 4. Let $(\mathbb{X}, \mathbb{A})$ be a relative filtered pair and $i\colon A \longrightarrow X$ and $j\colon X \longrightarrow (X, A)$ the inclusion maps. The following sequence is exact:

$$\cdots \longrightarrow \mathrm{H}^{\varepsilon,\varepsilon'}_n(A) \xrightarrow{i^{\varepsilon,\varepsilon'}_*} \mathrm{H}^{\varepsilon,\varepsilon'}_n(X) \xrightarrow{j^{\varepsilon,\varepsilon'}_*} \mathrm{H}^{\varepsilon,\varepsilon'}_n(X, A) \xrightarrow{\Delta_A} \mathrm{H}^{\varepsilon,\varepsilon'}_{n-1}(A) \longrightarrow \cdots$$

Proof of Persistent Axiom 4. We just need to prove that $\ker(i^{\varepsilon,\varepsilon}_*) = \mathrm{Im}(\Delta_A)$ and $\ker(\Delta_A) = \mathrm{Im}(j^{\varepsilon,\varepsilon}_*)$.

1. $\mathrm{Im}(j_*^{\varepsilon,\varepsilon}) \subseteq \ker(\Delta_A)$

   Let $[c] \in \mathrm{Im}(j_*^{\varepsilon,\varepsilon})$. Then exists $[d] \in \mathrm{H}_n^{\varepsilon,\varepsilon'}(X)$, such that $j_*^{\varepsilon,\varepsilon'}([d]) = [c]$, with $d \in k_n^{\varepsilon,\varepsilon'}(Z_{n,X}^{\varepsilon})$.

   $\Delta_A([c]) = \Delta_A(j_*^{\varepsilon,\varepsilon'}([d])) = \Delta_A([j_\#^{\varepsilon'}(d)]) = [\partial_{n,X}^{\varepsilon'}(j_\#^{\varepsilon'}(d))] = [j_\#^{\varepsilon'}(\partial_{n,X}^{\varepsilon'}(d))]$.

   Since $d \in k_n^{\varepsilon,\varepsilon'}(Z_{n,X}^{\varepsilon})$, $\partial_{n,X}^{\varepsilon'}(d) = 0$.

   Therefore, $\Delta_A([c]) = [0]$, which means that $[c] \in \ker(\Delta_A)$.

2. $\ker(\Delta_A) \subseteq \mathrm{Im}(j_*^{\varepsilon,\varepsilon})$

   Let $[c] \in \ker(\Delta_A) \subseteq \mathrm{H}_n^{\varepsilon,\varepsilon'}(X, A)$. Then $\Delta_A([c]) = [\partial_{n,X}^{\varepsilon'}(c)] = [0]$, which means that $\partial_{n,X}^{\varepsilon'}(c) \in B_{n-1,A}^{\varepsilon'} \cap k_{n-1}^{\varepsilon,\varepsilon'}(Z_{n-1,A}^{\varepsilon})$.

   Since $B_{n-1,A}^{\varepsilon'} = \mathrm{Im}(\partial_{n,A}^{\varepsilon'})$, there is an element $m \in C_n(A^{\varepsilon'})$ such that $\partial_{n,A}^{\varepsilon'}(m) = \partial_{n,X}^{\varepsilon'}(c)$. Because $i_\#^{\varepsilon'} \circ \partial_{n,A}^{\varepsilon'} = \partial_{n,X}^{\varepsilon'} \circ i_\#^{\varepsilon'}$, we do have that $i_\#^{\varepsilon'}(m) \in C_n(K^{\varepsilon'})$ is such that $j_*^{\varepsilon,\varepsilon}([m]) = [c]$.

3. $\mathrm{Im}(\Delta_A) \subseteq \ker(i_*^{\varepsilon,\varepsilon})$

   Let $[c] \in \mathrm{Im}(\Delta_A)$. There exists a $[d] \in \mathrm{H}_p^{\epsilon,\epsilon'}(X, A)$ such that $\Delta_A([d]) = [\partial_{n,X}^{\varepsilon'}(d)] = [c]$.

   Since $j \colon X \longrightarrow (X, A)$ is the inclusion, we have that $j_\#^{\varepsilon'}$ is surjective. Because of this fact, there is an $m \in C_n(K^{\varepsilon'})$ such that $j_\#^{\varepsilon'}(m) = d$. Then

$$\begin{aligned}
i_*^{\varepsilon,\varepsilon'}([c]) &= i_*^{\varepsilon,\varepsilon'}([\partial_{n,X}^{\varepsilon'}(d)]) \\
&= [i_\#^{\varepsilon'} \circ \partial_{n,X}^{\varepsilon'}(d)] = [i_\#^{\varepsilon'} \circ \partial_{n,X}^{\varepsilon'}(j_\#^{\varepsilon'}(m))] \\
&= [i_\#^{\varepsilon'} \circ \partial_{n,X}^{\varepsilon'} \circ j_\#^{\varepsilon'}(m)] = [0].
\end{aligned}$$

4. $\ker(i_*^{\varepsilon,\varepsilon'}) \subseteq \mathrm{Im}(\Delta_A)$

   Let $[c] \in \ker(i_*^{\varepsilon,\varepsilon'})$. Then $i_*^{\varepsilon,\varepsilon'}[c] = [i_\#^{\varepsilon'}(c)] = [0]$. Therefore, $i_\#^{\varepsilon'}(c) \in B_{n-1}^{\varepsilon'} = \mathrm{Im}(\partial_{n,X}^{\varepsilon'})$.Because i is the inclusion, we have that $i_\#^{\varepsilon}$ is also an inclusion and $i_\#^{\varepsilon}(c) = c$.

   Since $c \in B_{n-1}^{\varepsilon'} = \mathrm{Im}(\partial_{n,X}^{\varepsilon'})$, there exist $m \in C_n(K^{\varepsilon'})$, such that $c = \partial_{n,X}^{\varepsilon'}(m)$. Hence,

$$\Delta_A([m]) = [\partial_{n,X}^{\varepsilon'}(m)] = [c]. \qquad \square$$

Persistent Axiom 5. Let $f, g \colon \mathbb{X} \longrightarrow \mathbb{Y}$ be filtration preserving maps. If f and g are contiguous maps in $[\varepsilon, \varepsilon']$, then $f_*^{\varepsilon,\varepsilon'} = g_*^{\varepsilon,\varepsilon'}$.

Proof of Persistent Axiom 5. Calling $C_n(K^{\varepsilon})$ the filtered chain complex of X and $D_n(L^{\varepsilon})$ the filtered chain complex of Y, ∂ the boundary homomorphism in X and α the boundary homomorphism in Y and $\Delta_n^{\varepsilon} = g_\#^{\varepsilon} - f_\#^{\varepsilon}$ we have the following diagram

$$\begin{tikzcd}
 & C_{n+1}(K^{\varepsilon}) \arrow[rr, "\partial_{n+1}^{\varepsilon}"] \arrow[dl, "i_{n+1}^{\varepsilon,\varepsilon'}"'] \arrow[dd, "\Delta_{n+1}^{\varepsilon}"] & & C_n(K^{\varepsilon}) \arrow[rr, "\partial_n^{\varepsilon}"] \arrow[dl, "i_n^{\varepsilon,\varepsilon'}"'] \arrow[dd, "\Delta_n^{\varepsilon}"] & & C_{n-1}(K^{\varepsilon}) \arrow[dl, "i_{n-1}^{\varepsilon,\varepsilon'}"'] \arrow[dd, "\Delta_{n-1}^{\varepsilon}"] \\
C_{n+1}(K^{\varepsilon'}) \arrow[rr, "\partial_{n+1}^{\varepsilon'}"] \arrow[dd, "\Delta_{n+1}^{\varepsilon'}"] & & C_n(K^{\varepsilon'}) \arrow[rr, "\partial_{n+1}^{\varepsilon'}"] \arrow[dd, "\Delta_n^{\varepsilon'}"] & & C_{n-1}(K^{\varepsilon'}) \arrow[dd, "\Delta_{n-1}^{\varepsilon'}"] & \\
 & D_{n+1}(L^{\varepsilon}) \arrow[rr, "\alpha_{n+1}^{\varepsilon}"] \arrow[dl, "j_{n+1}^{\varepsilon,\varepsilon'}"'] & & D_n(L^{\varepsilon}) \arrow[rr, "\alpha_n^{\varepsilon}"] \arrow[dl, "j_n^{\varepsilon,\varepsilon'}"'] & & D_{n-1}(L^{\varepsilon}) \arrow[dl, "j_{n-1}^{\varepsilon,\varepsilon'}"'] \\
D_{n+1}(L^{\varepsilon'}) \arrow[rr, "\alpha_{n+1}^{\varepsilon'}"] & & D_n(L^{\varepsilon'}) \arrow[rr, "\alpha_n^{\varepsilon'}"] & & D_{n-1}(L^{\varepsilon'}) &
\end{tikzcd}$$

From the contiguity map hypothesis we have that exists $\psi_n^\varepsilon\colon C_n(K^\varepsilon) \longrightarrow D_{n+1}(L^\varepsilon)$ such that $\Delta_n^\varepsilon = \psi_{n-1}^\varepsilon \circ \partial_n^\varepsilon + \alpha_{n+1}^\varepsilon \circ \psi_n^\varepsilon$.

$$\begin{tikzcd}
C_{n+1}(K^\varepsilon) \arrow[r, "\partial_{n+1}^\varepsilon"] \arrow[d, "\Delta_{n+1}^\varepsilon"] & C_n(K^\varepsilon) \arrow[r, "\partial_n^\varepsilon"] \arrow[d, "\Delta_n^\varepsilon"] \arrow[dl, "\psi_n^\varepsilon"] & C_{n-1}(K^\varepsilon) \arrow[d, "\Delta_{n-1}^\varepsilon"] \arrow[dl, "\psi_{n-1}^\varepsilon"] \\
D_{n+1}(L^\varepsilon) \arrow[r, "\alpha_{n+1}^\varepsilon"] & D_n(L^\varepsilon) \arrow[r, "\alpha_n^\varepsilon"] & D_{n-1}(L^\varepsilon)
\end{tikzcd}$$

Let n be a p-cycle of X. Then $g_\#^\varepsilon(z) - f_\#^\varepsilon(z) = \psi_{n-1}^\varepsilon \circ \partial_n^\varepsilon(z) + \alpha_{n+1}^\varepsilon \circ \psi_n^\varepsilon(z) = \alpha_{n+1}^\varepsilon(\psi_n^\varepsilon(z))$. In other words, they differ by a boundary, which implies that $g_*^{\varepsilon,\varepsilon'} = f_*^{\varepsilon,\varepsilon'}$. □

Persistent Axiom 6. (Dimension axiom) Let $\mathbb{X} = (X, F_X)$ be a filtered pair, where X a one-point space and $F_X(\sigma) = \alpha$, one has

$$\mathrm{H}_k^{\varepsilon,\varepsilon'}(\mathbb{X}) = \begin{cases} \mathbb{F}, & \text{if } k = 0 \text{ and } \varepsilon \geq \alpha; \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Proof of Persistent Axiom 6. Note that, if X is an one-point space, then $C_n^\varepsilon(X) = 0$, for $n > 0$ and $C_0^\varepsilon \approx \mathbb{F}$, if $K^\epsilon = X$ or $C_0^\varepsilon = 0$, if $K^\varepsilon = \emptyset$. This way, $Z_n^\varepsilon = \ker(\partial_n^\varepsilon) = 0$ if $n \neq 0$ and if $K^\varepsilon = \emptyset$.

Let us show that $\mathrm{H}_0^{\varepsilon,\varepsilon'}(X) = \mathbb{F}$ if $\varepsilon \geq \alpha$.

For $\varepsilon \leq \varepsilon'$, we have that $K^\varepsilon = K^{\varepsilon'} = X$ as well. We know that $k_n^{\varepsilon,\varepsilon'}(Z_n^\varepsilon) = \mathbb{F}$, since $k_n^{\varepsilon,\varepsilon'}$ is the inclusion map. Remember that $B_n^{\varepsilon'} = \mathrm{Im}(\partial_{n+1}^{\varepsilon'})$. But $C_1^{\varepsilon'}(X) = 0$, and since X is an one-point space, then $\mathrm{Im}(\partial_{n+1}^{\varepsilon'}) = 0$. Then

$$\mathrm{H}_0^{\varepsilon,\varepsilon'}(X) = \mathbb{F}.$$ □

Persistent Axiom 7. (Excision axiom) Let $(\mathbb{X}, \mathbb{A})$ and $(\mathbb{X}', \mathbb{A}')$ be a relative filtered pairs such that $X' \cup A = X$ and $X' \cap A = A'$ and the inclusion map $i\colon (X', A') \longrightarrow (X, A)$ is a filtration preserving map. Then the induced map:

$$i_*^{\varepsilon,\varepsilon'}\colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(X', A') \longrightarrow \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A)$$

is an isomorphism.

Proof of Persistent Axiom 7. For each ε let φ_ε be the composition of the morphism induced by the inclusion and the projection:

$$\begin{tikzcd}
C_p(X'^\varepsilon) \arrow[r] \arrow[rr, bend right, "\varphi_\varepsilon"'] & C_p(X'^\varepsilon \cup A^\varepsilon) = C_p(X^\varepsilon) \arrow[r] & C_p(X^\varepsilon)/C_p(A^\varepsilon) = C_p(X^\varepsilon, A^\varepsilon)
\end{tikzcd}$$

Since $X'^\varepsilon \cup A^\varepsilon = X^\varepsilon$, this map φ_ε is surjective. Moreover, the kernel of φ_ε is $C_p(A'^\varepsilon)$, because $X'^\varepsilon \cap A^\varepsilon = A'^\varepsilon$.

Therefore, φ_ε induces an isomorphism

$$C_p(X'^\varepsilon, A'^\varepsilon) = C_p(X'^\varepsilon)/C_p(A'^\varepsilon) \xrightarrow{\varphi_{\varepsilon\#}} C_p(X^\varepsilon)/C_p(A^\varepsilon).$$

Notice that $\varphi_{\varepsilon\#}$ preserves the boundary operator, that is, the following diagram commutes:

$$\begin{tikzcd}
C_p(X'^\varepsilon, A'^\varepsilon) \arrow[r, "\varphi_{\varepsilon\#}"] \arrow[d, "\partial"] & C_p(X^\varepsilon, A^\varepsilon) \arrow[d, "\partial"] \\
C_{p-1}(X'^\varepsilon, A'^\varepsilon) \arrow[r, "\varphi_{\varepsilon\#}"] & C_{p-1}(X^\varepsilon, A^\varepsilon)
\end{tikzcd}$$

Then, we have natural isomorphisms

$$Z_p^{\varepsilon}(X', A') \approx Z_p^{\varepsilon}(X, A)$$

and

$$B_p^{\varepsilon}(X', A') \approx B_p^{\varepsilon}(X, A),$$

so, in particular for a pair $\varepsilon \le \varepsilon'$, we have an isomorphism

$$\mathrm{H}_p^{\varepsilon,\varepsilon'}(X', A') \cong \mathrm{H}_p^{\varepsilon,\varepsilon'}(X, A). \qquad \square$$

As we proved, the Persistent Homology satisfies the previous persistent versions of the Eilenberg-Steenrod axioms. These axioms will be the blocks where we will build our new definitions and results of our axiomatic approach to it.

2.3.1 Simplification of the Eilenberg-Steenrod axioms

In this section we present a simplification of the Persistent Homology Axioms, inspired by Dawson in [Daw88] where it is given a simplification of the Eilenberg-Steenrod Axioms for the case of finite simplicial complexes.

Let X be a simplicial complex and let $F_1, F_2 \colon \mathrm{pow}(X) \longrightarrow \mathbb{R}$ be filtrations of X.

Let us consider the maps $\overline{F}, \underline{F} : \mathrm{pow}(X) \longrightarrow \mathbb{R}$ defined, for all $\sigma \in \mathrm{pow}(X)$, as

$$\overline{F}(\sigma) = \min\{F_1(\sigma), F_2(\sigma)\},$$

and

$$\underline{F}(\sigma) = \max\{F_1(\sigma), F_2(\sigma)\}.$$

The chosen of the notation of $\overline{F}$ and $\underline{F}$ seems not natural, since it is obvious that $\overline{F} = \min\{F_1(\sigma), F_2(\sigma)\} \le \max\{F_1(\sigma), F_2(\sigma)\} = \underline{F}$. But, looking to the simplicial complexes induced by the maps, we do have that $\underline{F}^{-1}(-\infty, t] \subseteq \overline{F}^{-1}(-\infty, t]$ and therefore our notation is according to the complexes.

Theorem 2.13. *There is a unique pair* (PH, ∂) *such that* PH *is a functor from the category ($\mathcal{F}ilt$) of filtered simplicial complexes to the category of functors $\mathcal{F}unct(Int(\mathbb{R}) \to \mathbf{Ab})$,* $\partial\colon \mathrm{PH} \longrightarrow \mathrm{PH}$ *is a natural transformation which reduces the grade by* 1 *and satisfies the following axioms:*

(SH1) *The inclusion map*

$$((X, X), (F_1, \underline{F})) \xhookrightarrow{I} ((X, X), (\overline{F}, F_2))$$

induces isomorphism in homology.

(SH2) *The triangle,*

$$\begin{tikzcd} \mathrm{PH}(X, F_X) \arrow[dr, "j_*"] & & \mathrm{PH}(A, F_A) \arrow[ll, "i_*"'] \\ & \mathrm{PH}((X, A), (F_X, F_A)) \arrow[ur, "\partial"] & \end{tikzcd}$$

is exact, where $i\colon A \hookrightarrow X$ *and* $j\colon (X, \emptyset) \hookrightarrow (X, A)$ *are inclusions maps.*

(SH3) *For X a simplicial complex and $F_X(\sigma) = \alpha$, $\forall\ \emptyset \neq \sigma \subset X$, one has*

$$\mathrm{PH}_k(X, F_X) = \begin{cases} \mathbf{1}_{[\alpha,\infty)}, & \text{if } k = 0 \\ \mathbf{0}, & \text{if } k > 0 \end{cases}$$

Proof. For $\varepsilon \leq \varepsilon'$, let us prove that the **Persistent Homology** $H_n^{\varepsilon,\varepsilon'}$ satisfies these three axioms.

(SH1) Let X be a simplicial complex and let $F_1, F_2\colon \mathrm{pow}(X) \longrightarrow \mathbb{R}$ be filtrations of X.

Denoting $K_i^\varepsilon = F_i^{-1}((\infty, \varepsilon]), i = 1, 2$, one has two increasing families of simplicial complexes $\{K_1^\varepsilon\}_{\varepsilon\in\mathbb{R}}$ and $\{K_2^\varepsilon\}_{\varepsilon\in\mathbb{R}}$.

Denote $\overline{K^\varepsilon} = \overline{F}^{-1}((\infty, \varepsilon])$ and $\underline{K^\varepsilon} = \underline{F}^{-1}((\infty, \varepsilon])$.

Therefore, one has

$$\begin{aligned} \overline{K^\varepsilon} &= \{\sigma \in \mathrm{pow}(X)/\overline{F}(\sigma) \leq \varepsilon\} \\ &= \{\sigma \in \mathrm{pow}(X)/\min(F_1(\sigma), F_2(\sigma)) \leq \varepsilon\} \\ &= \{\sigma \in \mathrm{pow}(X)/F_1(\sigma) \leq \varepsilon \text{ or } F_2(\sigma) \leq \varepsilon\} = K_1^\varepsilon \cup K_2^\varepsilon \end{aligned}$$

and

$$\begin{aligned} \underline{K^\varepsilon} &= \{\sigma \in \mathrm{pow}(X)/\underline{F}(\sigma) \leq \varepsilon\} \\ &= \{\sigma \in \mathrm{pow}(X)/\max(F_1(\sigma), F_2(\sigma)) \leq \varepsilon\} \\ &= \{\sigma \in \mathrm{pow}(X)/F_1(\sigma) \leq \varepsilon \text{ and } F_2(\sigma) \leq \varepsilon\} = K_1^\varepsilon \cap K_2^\varepsilon. \end{aligned}$$

Now, we need to show that the inclusion map

$$((X, X), (F_1, \underline{F})) \overset{I}{\hookrightarrow} ((X, X), (\overline{F}, F_2))$$

induces isomorphism in persistent homology.

For each $\varepsilon, \varepsilon' \in \mathbb{R}$, $\varepsilon \leq \varepsilon'$, and $n \in \mathbb{Z}^+$, we have the following diagram:

$$\begin{tikzcd}
 & C_{n+1}(M,N) \arrow[rr, "\overline{\partial}_{n+1}^{\varepsilon}"] \arrow[dl, "k_{n+1}^{\varepsilon,\varepsilon'}"'] \arrow[dd, "I_\#^\varepsilon"] & & C_n(M,N) \arrow[rr, "\overline{\partial}_{n}^{\varepsilon}"] \arrow[dl, "k_{n}^{\varepsilon,\varepsilon'}"'] \arrow[dd, "I_\#^\varepsilon"] & & C_{n-1}(M,N) \arrow[dl, "k_{n-1}^{\varepsilon,\varepsilon'}"'] \arrow[dd, "I_\#^\varepsilon"] \\
C_{n+1}(A,B) \arrow[rr, "\overline{\partial}_{n+1}^{\varepsilon'}"] \arrow[dd, "I_\#^{\varepsilon'}"] & & C_n(A,B) \arrow[rr, "\overline{\partial}_{n}^{\varepsilon'}"] \arrow[dd, "I_\#^{\varepsilon'}"] & & C_{n-1}(A,B) \arrow[dd, "I_\#^{\varepsilon'}"] & \\
 & C_{n+1}(E,F) \arrow[rr, "\overline{\partial}_{n+1}^{\varepsilon}"] \arrow[dl, "k_{n+1}^{\varepsilon,\varepsilon'}"'] & & C_n(E,F) \arrow[rr, "\overline{\partial}_{n}^{\varepsilon}"] \arrow[dl, "k_{n}^{\varepsilon,\varepsilon'}"'] & & C_{n-1}(E,F) \arrow[dl, "k_{n-1}^{\varepsilon,\varepsilon'}"'] \\
C_{n+1}(G,H) \arrow[rr, "\overline{\partial}_{n+1}^{\varepsilon'}"] & & C_n(G,H) \arrow[rr, "\overline{\partial}_{n}^{\varepsilon'}"] & & C_{n-1}(G,H) &
\end{tikzcd} \tag{2.1}$$

where $(M, N) = (K_1^\varepsilon, K_1^\varepsilon \cap K_2^\varepsilon)$, $(A, B) = (K_1^{\varepsilon'}, K_1^{\varepsilon'} \cap K_2^{\varepsilon'})$, $(E, F) = (K_1^\varepsilon \cup K_2^\varepsilon, K_2^\varepsilon)$ e $(G, H) = (K_1^{\varepsilon'} \cup K_2^{\varepsilon'}, K_2^{\varepsilon'})$.

Let us consider, for each ε fixed, the composition of the morphisms induced by the inclusion and the projection

$$C_p(K_1^\varepsilon) \overset{i}{\hookrightarrow} C_p(K_1^\varepsilon \cup K_2^\varepsilon) \overset{j}{\longrightarrow} \frac{C_p(K_1^\varepsilon \cup K_2^\varepsilon)}{C_p(K_2^\varepsilon)} = C_p(K_1^\varepsilon \cup K_2^\varepsilon, K_2^\varepsilon).$$

A basis for $C_p(K_1^\varepsilon \cup K_2^\varepsilon, K_2^\varepsilon)$ is the set of equivalent classes $[\alpha]$ such that α is a simplex in $K_1^\varepsilon \cup K_2^\varepsilon$ which doesn't belong to K_2^ε. Using this fact, we conclude that the composite above is surjective and the kernel is $C_p(K_1^\varepsilon \cap K_2^\varepsilon)$.

Then one has an isomorphism

$$\frac{C_p(K_1^\varepsilon)}{C_p(K_1^\varepsilon \cap K_2^\varepsilon)} \xrightarrow{I_\#^\varepsilon} \frac{C_p(K_1^\varepsilon \cup K_2^\varepsilon)}{C_p(K_2^\varepsilon)}.$$

We remark from the Diagram (2.1) that for ε fixed, the following diagram commutes:

$$\begin{tikzcd}
C_n(K_1^\varepsilon, K_1^\varepsilon \cap K_2^\varepsilon) \arrow[r, "I_\#^\varepsilon"] \arrow[d, "\overline{\partial_p^\varepsilon}"] & C_n(K_1^\varepsilon \cup K_2^\varepsilon, K_2^\varepsilon) \arrow[d, "\overline{\partial_p^\varepsilon}"] \\
C_{n-1}(K_1^\varepsilon, K_1^\varepsilon \cap K_2^\varepsilon) \arrow[r, "I_\#^\varepsilon"] & C_{n-1}(K_1^\varepsilon \cup K_2^\varepsilon, K_2^\varepsilon)
\end{tikzcd}$$

Then, we have natural isomorphism

$$Z_p^\varepsilon(K_1^\varepsilon, K_1^\varepsilon \cap K_2^\varepsilon) \cong Z_p^\varepsilon(K_1^\varepsilon \cup K_2^\varepsilon, K_2^\varepsilon)$$

and

$$B_p^{\varepsilon'}(K_1^{\varepsilon'}, K_1^{\varepsilon'} \cap K_2^{\varepsilon'}) \cong B_p^{\varepsilon'}(K_1^{\varepsilon'} \cup K_2^{\varepsilon'}, K_2^{\varepsilon'}).$$

From diagram (2.1), for $\varepsilon \leq \varepsilon'$, the following diagram commutes:

$$\begin{tikzcd}
C_n(K_1^\varepsilon, K_1^\varepsilon \cap K_2^\varepsilon) \arrow[r, "k_n^{\varepsilon,\varepsilon'}"] \arrow[d, "I_\#^\varepsilon"] & C_n(K_1^{\varepsilon'}, K_1^{\varepsilon'} \cap K_2^{\varepsilon'}) \arrow[d, "I_\#^{\varepsilon'}"] \\
C_n(K_1^\varepsilon \cup K_2^\varepsilon, K_2^\varepsilon) \arrow[r, "k_n^{\varepsilon,\varepsilon'}"] & C_n(K_1^{\varepsilon'} \cup K_2^{\varepsilon'}, K_2^{\varepsilon'})
\end{tikzcd}$$

Therefore, one has $k^{\varepsilon,\varepsilon'}(Z_n^\varepsilon(K_1^\varepsilon, K_1^\varepsilon \cap K_2^\varepsilon)) \cong k^{\varepsilon,\varepsilon'}(Z_n^\varepsilon(K_1^\varepsilon \cup K_2^\varepsilon, K_2^\varepsilon))$ and

$$H_n^{\varepsilon,\varepsilon'}((X,X),(F_1,\underline{F})) \xrightarrow{I_*^{\varepsilon,\varepsilon'}} H_n^{\varepsilon,\varepsilon'}((X,X),(\overline{F},F_2))$$

is an isomophism between the persistent homology groups, induced by the inclusion I.

(SH2) Note that this axiom states an exactness of a sequence that is the same sequence present in Axiom 4, where we proved the exactness of it. For this reason, the proof of SH2 is the same presented in Axiom 4.

(SH3) By hypothesis, we have that X is a simplicial complex and $F_X(\sigma) = \alpha, \forall \emptyset \neq \sigma \subseteq X$. One can see, there are just two possibilities for K^ε.

If $\varepsilon < \alpha$, then $K^\varepsilon = \emptyset$ and if $\varepsilon \geq \alpha$ then $K^\varepsilon = X$, since every face of our simplicial appears at the same time in the filtration.

By definition, for $\varepsilon \leq \varepsilon'$,

$$H_n^{\varepsilon,\varepsilon'}(X) = \frac{k_n^{\varepsilon,\varepsilon'}(Z_n^\varepsilon)}{B_n^{\varepsilon'} \cap k_n^{\varepsilon,\varepsilon'}(Z_n^\varepsilon)}.$$

Let us split the analysis into some cases:

1º) If $\varepsilon, \varepsilon' < \alpha$, we have that

$$C_n(K^\varepsilon) = 0 \Longrightarrow Z_n^\varepsilon = 0 \Longrightarrow k_n^{\varepsilon,\varepsilon'}(Z_n^\varepsilon) = 0 \text{ and } C_n(K^{\varepsilon'}) = 0 \Longrightarrow B_n^\varepsilon = 0.$$

Then, by the definition of persistent homology, $\mathrm{H}_n^{\varepsilon,\varepsilon'}(X) = 0$, which means that there is no generator of an element in PH before the moment α.

2º) If $\varepsilon < \alpha$ and $\varepsilon' \geq \alpha$, we have that
$C_n(K^\varepsilon) = 0 \Longrightarrow Z_n^\varepsilon = 0 \Longrightarrow k_n^{\varepsilon,\varepsilon'}(Z_n^\varepsilon) = 0$. Since, $B_n^{\varepsilon'} \subseteq k_n^{\varepsilon,\varepsilon'}(Z_n^\varepsilon)$, then $B_n^{\varepsilon'} = 0$. This way, the second case is equal to the first one.

3º) If $\varepsilon, \varepsilon' \geq \alpha$, we have that
$C_n(K^\varepsilon) = C_n(X) = C_n(K^{\varepsilon'})$. Then, $k_n^{\varepsilon,\varepsilon'}$ is the identity. Therefore,
$\mathrm{H}_n^{\varepsilon,\varepsilon'}(X) = \dfrac{k_n^{\varepsilon,\varepsilon'}(Z_n^\varepsilon)}{B_n^{\varepsilon'} \cap k_n^{\varepsilon,\varepsilon'}(Z_n^\varepsilon)} = \dfrac{Z_n^{\varepsilon'}}{B_n^{\varepsilon'}} = \mathrm{H}_n(X).$
We know, in classical simplicial homology, that the homolgoy group of a simplicial complex is 0 if $n \neq 0$ and G if $n = 0$, where G is the coefficients group. □

One of the future works will be to prove the equivalence of the first seven persistent versions of the axioms and the simplified ones. In the following sections, we assume that H is a Persistent Homology Theory satisfying the 7 axioms of §2.3.

2.4 Reduced Persistent Homology and Filtered Triads

In this section, we present results involving maps between persistent homology sequences and introduce the coefficient group of a persistent homology theory.

Theorem 2.14. *Given a filtration preserving map pair $f\colon (\mathbb{X}, \mathbb{A}) \longrightarrow (\mathbb{Y}, \mathbb{B})$ let us consider the restriction maps*

$$f_1\colon X \longrightarrow Y \text{ and } f\colon A \longrightarrow B.$$

The respective induced maps give rise to a homomorphism between the persistent homology sequences of $(\mathbb{X}, \mathbb{A})$ and of $(\mathbb{Y}, \mathbb{B})$.

Proof. Let us consider for $\varepsilon \leq \varepsilon'$ the following commutative diagram, where the rows are the exact sequence of the pairs (X, A) and (Y, B) and the vertical arrows are induced maps of f, f_1 and f_2.

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{H}_{n+1}^{\varepsilon,\varepsilon'}(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"] & \mathrm{H}_n^{\varepsilon,\varepsilon'}(A) \arrow[r, "i_*"] \arrow[d, "f_{2*}"] & \mathrm{H}_n^{\varepsilon,\varepsilon'}(X) \arrow[r, "j_*"] \arrow[d, "f_{1*}"] & \mathrm{H}_n^{\varepsilon,\varepsilon'}(X,A) \arrow[r] \arrow[d, "f_*"] & {} \\
{} \arrow[r] & \mathrm{H}_{n+1}^{\varepsilon,\varepsilon'}(Y,B) \arrow[r, "\partial'"] & \mathrm{H}_n^{\varepsilon,\varepsilon'}(B) \arrow[r, "i'_*"] & \mathrm{H}_n^{\varepsilon,\varepsilon'}(Y) \arrow[r, "j'_*"] & \mathrm{H}_n^{\varepsilon,\varepsilon'}(Y,B) \arrow[r] & {}
\end{tikzcd}$$

The commutativity of the diagram follows from the Axiom 2 and Axiom 3. □

Theorem 2.15. *Given a filtration preserving map pair $f\colon (\mathbb{X}, \mathbb{A}) \longrightarrow (\mathbb{Y}, \mathbb{B})$, let us suppose that the induced maps $f_{1*}\colon \mathrm{H}_n^{\varepsilon,\varepsilon'}(X) \longrightarrow \mathrm{H}_n^{\varepsilon,\varepsilon'}(Y)$, $f_{2*}\colon \mathrm{H}_n^{\varepsilon,\varepsilon'}(A) \longrightarrow \mathrm{H}_n^{\varepsilon,\varepsilon'}(B)$ are isomorphisms. Then $f_*\colon \mathrm{H}_n^{\varepsilon,\varepsilon'}(X,A) \longrightarrow \mathrm{H}_n^{\varepsilon,\varepsilon'}(Y,B)$ is an isomorphism.*

Proof. The result is a consequence of Five Lemma.

Let us consider the following diagram

$$
\begin{tikzcd}
{} \arrow[r] & \mathrm{H}_n^{\varepsilon,\varepsilon'}(X) \arrow[r, "j_*"] \arrow[d, "f_{1*}"] & \mathrm{H}_n^{\varepsilon,\varepsilon'}(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"] & \mathrm{H}_{n-1}^{\varepsilon,\varepsilon'}(A) \arrow[r, "i_*"] \arrow[d, "f_{2*}"] & \mathrm{H}_{n-1}^{\varepsilon,\varepsilon'}(X) \arrow[r] \arrow[d, "f_{1*}"] & {} \\
{} \arrow[r] & \mathrm{H}_n^{\varepsilon,\varepsilon'}(Y) \arrow[r, "j'_*"] & \mathrm{H}_n^{\varepsilon,\varepsilon'}(Y,B) \arrow[r, "\partial'"] & \mathrm{H}_{n-1}^{\varepsilon,\varepsilon'}(B) \arrow[r, "i'_*"] & \mathrm{H}_{n-1}^{\varepsilon,\varepsilon'}(Y) \arrow[r] & {}
\end{tikzcd}
$$

Since f_{1*} and f_{2*} are surjective maps and f_{1*} is an injective map it follows from the Five Lemma that f_* is surjective.

In the other hand, since f_{1*} and f_{2*} are injective maps and f_{2*} is a surjective map, the Five Lemma implies that f_* is injective.

Therefore $f_*\colon \mathrm{H}_n^{\varepsilon,\varepsilon'}(X,A) \longrightarrow \mathrm{H}_n^{\varepsilon,\varepsilon'}(Y,B)$ is an isomorphism. $\square$

Theorem 2.16. *A contiguity map f from $(\mathbb{X},\mathbb{A})$ onto $(\mathbb{Y},\mathbb{B})$ induces an isomorphism $f_*^{\varepsilon,\varepsilon'}\colon \mathrm{H}_n^{\varepsilon,\varepsilon'}(X,A) \longrightarrow \mathrm{H}_n^{\varepsilon,\varepsilon'}(Y,B)$.*

Proof. Since f is a contiguity map from $(\mathbb{X},\mathbb{A})$ onto $(\mathbb{Y},\mathbb{B})$ there exists a filtration preserving map $g\colon (\mathbb{Y},\mathbb{B}) \longrightarrow (\mathbb{X},\mathbb{A})$ such that f and g are contiguously equivalent. Then we have that $g\circ f$ and $\mathrm{Id}_{(\mathbb{X},\mathbb{A})}$ are contiguous maps and $f\circ g$ and $\mathrm{Id}_{(\mathbb{Y},\mathbb{B})}$ are contiguous maps.

It follows from the Axiom 5 that $(g\circ f)_*^{\varepsilon,\varepsilon'} = (\mathrm{Id}_{(\mathbb{X},\mathbb{A})})_*^{\varepsilon,\varepsilon'}$ and $(f\circ g)_*^{\varepsilon,\varepsilon'} = (\mathrm{Id}_{(\mathbb{Y},\mathbb{B})})_*^{\varepsilon,\varepsilon'}$ where $(\mathrm{Id}_{(\mathbb{X},\mathbb{A})})_*^{\varepsilon,\varepsilon'}$ and $(\mathrm{Id}_{(\mathbb{Y},\mathbb{B})})_*^{\varepsilon,\varepsilon'}$ are identity homomorphisms by Axiom 1.

Then, by Axiom 2, $(f\circ g)_*^{\varepsilon,\varepsilon'} = f_*^{\varepsilon,\varepsilon'} \circ g_*^{\varepsilon,\varepsilon'}$ and $(g\circ f)_*^{\varepsilon,\varepsilon'} = g_*^{\varepsilon,\varepsilon'} \circ f_*^{\varepsilon,\varepsilon'}$, which concludes the proof. $\square$

The next goal is to define the reduced persistent homology associated to a persistent homology theory. In order we will need some definitions and results.

Definition 2.17. Let P_0 be a fixed point. Given a persistent homology theory, **the persistent coefficient group relative to** $(\varepsilon,\varepsilon')$ is defined as the group $G^{\varepsilon,\varepsilon'} = \mathrm{H}_0^{\varepsilon,\varepsilon'}(P_0, F_{P_0})$, where $\varepsilon \le \varepsilon'$ and $F_{P_0}\colon P_0 \longrightarrow \mathbb{R}$ is such that $F_{P_0}(x) = 0$. We will denote (P_0, F_{P_0}) by $\mathbb{P}_0$. If $x \in X$ and $g \in G^{\varepsilon,\varepsilon'}$, then $(gx)_X$ denotes the image of g in $\mathrm{H}_0^{\varepsilon,\varepsilon'}(X)$ under the homomorphism induced by the map $f\colon \mathbb{P}_0 \longrightarrow \mathbb{X}$ defined by $f(P_0) = x$. The image of $G^{\varepsilon,\varepsilon'}$ in $\mathrm{H}_0^{\varepsilon,\varepsilon'}(X)$ under f_* is denoted by $(Gx)_X$.

Remark 2.18. An important remark here is that this map f is a filtration preserving map, since $F_{P_0} = F_X \circ f$.

Theorem 2.19. *If $f\colon \mathbb{X} \longrightarrow \mathbb{Y}$ is a filtration preserving map then for $x \in X$, $y = f(x) \in Y$ and $g \in G^{\varepsilon,\varepsilon'}$, one has $f_*^{\varepsilon,\varepsilon'}((gx)_X) = (gy)_Y$. Thus $f_*^{\varepsilon,\varepsilon'}$ maps $(Gx)_X$ onto $(Gy)_Y$.*

Proof. Consider the following commutative diagram:

$$
\begin{tikzcd}
P_0 \arrow[rr, "F_{X'}"] \arrow[dr, "F_{Y'}"'] & & X \arrow[dl, "f"] \\
& Y &
\end{tikzcd}
$$

where $F_{X'}\colon \mathbb{P}_0 \longrightarrow \mathbb{X}$ is defined by $F_{X'}(P_0) = x$ and $F_{Y'}\colon \mathbb{P}_0 \longrightarrow \mathbb{Y}$ is defined by $F_{Y'}(P_0) = y$. Then by Axiom 2 the diagram commutes

$$\begin{tikzcd} \mathrm{H}_*^{\varepsilon,\varepsilon'}(P_0) \arrow[rr, "F_{X'*}"] \arrow[dr, "F_{Y'*}"'] & & \mathrm{H}_*^{\varepsilon,\varepsilon'}(X) \arrow[dl, "f_*^{\varepsilon,\varepsilon'}"] \\ & \mathrm{H}_*^{\varepsilon,\varepsilon'}(Y) & \end{tikzcd}$$

For $x \in X$, $y \in Y$, $y = f(x)$ and $g \in G^{\varepsilon,\varepsilon'}$ we have

$$f_*^{\varepsilon,\varepsilon'} F_{X'*}(g) = F_{Y'*}(g) \Longrightarrow f_*^{\varepsilon,\varepsilon'}((gx)_X) = (gy)_Y.$$

Since $(Gx)_X$ is the image of $G^{\varepsilon,\varepsilon'}$ in $\mathrm{H}_0^{\varepsilon,\varepsilon'}(X)$ under $F_{X'*}$, $(Gy)_Y$ is the image of $G^{\varepsilon,\varepsilon'}$ in $\mathrm{H}_0^{\varepsilon,\varepsilon'}(Y)$ under $F_{Y'*}$ and from the commutative diagram we have that $f_*^{\varepsilon,\varepsilon'}$ maps $(Gx)_X$ onto $(Gy)_Y$. □

Definition 2.20. Let $f\colon \mathbb{X} \longrightarrow \mathbb{P}_0$ be a filtration preserving map. For any point P_0 we define the **reduced persistent homology group**

$$\tilde{\mathrm{H}}_n^{\varepsilon,\varepsilon'}(X) := \ker(f_*^{\varepsilon,\varepsilon'} : \mathrm{H}_n^{\varepsilon,\varepsilon'}(X) \longrightarrow \mathrm{H}_n^{\varepsilon,\varepsilon'}(P_0))$$

Remark 2.21. For $n > 0$ and $\varepsilon \le \varepsilon'$, $\mathrm{H}_n^{\varepsilon,\varepsilon'}(P_0) = 0$ because P_0 is a single point space. Then $\ker f_*^{\varepsilon,\varepsilon'} : \mathrm{H}_n^{\varepsilon,\varepsilon'}(X) \longrightarrow \mathrm{H}_n^{\varepsilon,\varepsilon'}(P_0)$ is $\mathrm{H}_n^{\varepsilon,\varepsilon'}(X)$.

Therefore, for $n > 0$ we also have $\tilde{\mathrm{H}}_n^{\varepsilon,\varepsilon'}(X) = \mathrm{H}_n^{\varepsilon,\varepsilon'}(X)$.

Theorem 2.22. *If P is a single point space, with the only possible filtration map F_0, then*

$$\tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(P) = 0 \textit{ and } \mathrm{H}_0^{\varepsilon,\varepsilon'}(P) = (Gx)P$$

Proof. Let $f\colon (P, F_0) \longrightarrow \mathbb{P}_0$ and $g\colon \mathbb{P}_0 \longrightarrow (P, F_0)$ be maps relating P and P_0. Since they are single point spaces, the maps are well defined.

Also $f \circ g$ and $g \circ f$ are contiguous to the respective identity maps, then f and g are contiguously equivalent, which means that f is a contiguity map.

Applying Theorem 2.16, we have that f induces an isomorphism

$$f_*^{\varepsilon,\varepsilon'} : \mathrm{H}_n^{\varepsilon,\varepsilon'}(P) \approx \mathrm{H}_n^{\varepsilon,\varepsilon'}(P_0), n \ge 0.$$

Since,

$$\tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(P) = \ker(f_*^{\varepsilon,\varepsilon'} : \mathrm{H}_0^{\varepsilon,\varepsilon'}(P) \longrightarrow \mathrm{H}_0^{\varepsilon,\varepsilon'}(P_0))$$

and $f_*^{\varepsilon,\varepsilon'}$ is an isomorphism, we have that $\ker(f_*^{\varepsilon,\varepsilon'}) = 0$ and, consequently, $\tilde{H}_0^{\varepsilon,\varepsilon'}(P) = 0$.

By definition, $(Gx)_P$ is the image of $G^{\varepsilon,\varepsilon'}$ in $\mathrm{H}_0^{\varepsilon,\varepsilon'}(P)$ under $g_*^{\varepsilon,\varepsilon'}$, which is also an isomorphism. Then

$$(Gx)P = g_*^{\varepsilon,\varepsilon'}(G^{\varepsilon,\varepsilon'}) = g_*^{\varepsilon,\varepsilon'}(\mathrm{H}_0^{\varepsilon,\varepsilon'}(P_0)) = \mathrm{H}_0^{\varepsilon,\varepsilon'}(P).$$ □

Theorem 2.23. *Let $\mathbb{X}$ be a filtered pair and $x \in X$. Then $\mathrm{H}_0^{\varepsilon,\varepsilon'}(X)$ decomposes into the direct sum*

$$\mathrm{H}_0^{\varepsilon,\varepsilon'}(X) = \tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(X) \oplus (Gx)_X$$

and the correspondence $g \longrightarrow (gx)_X$ maps $G^{\varepsilon,\varepsilon'}$ isomorphically onto $(Gx)_X$.

Proof. Let $f_1\colon \mathbb{P}_0 \longrightarrow \mathbb{X}$ be the filtration preserving map defined by $f_1(P_0) = x$ and let $f_2\colon \mathbb{X} \longrightarrow \mathbb{P}_0$ be the constant filtration preserving map.

Since $f_2 \circ f_1$ is the identity map of P_0, by Axioms 1 and 2, the composition of $f_{1*}^{\varepsilon,\varepsilon'}\colon G^{\varepsilon,\varepsilon'} \longrightarrow \mathrm{H}_0^{\varepsilon,\varepsilon'}(X)$ and $f_{2*}^{\varepsilon,\varepsilon'}\colon \mathrm{H}_0^{\varepsilon,\varepsilon'}(X) \longrightarrow G^{\varepsilon,\varepsilon'}$ is the identity map.

Then, this composition maps $G^{\varepsilon,\varepsilon'}$ isomorphically onto $(Gx)_X$ and maps $(Gx)_X$ isomorphically onto $G^{\varepsilon,\varepsilon'}$.

To prove that $\mathrm{H}_0^{\varepsilon,\varepsilon'}(X) = \tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(X) \oplus (Gx)_X$, first note that, $\tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(X) = \ker f_{2*}$ which implies that $(Gx)_X \cap \tilde{H}_0^{\varepsilon,\varepsilon'}(X) = 0$.

Let $m \in \mathrm{H}_0^{\varepsilon,\varepsilon'}(X)$. Define $m' := f_{1*} \circ f_{2*}(m)$ and $m'' = m - m'$.

Then $m' := f_{1*} \circ f_{2*}(m)$ implies $m' \in \mathrm{Im}(f_2*)$ and therefore $m' \in (Gx)_X$.

Furthermore $m'' = m - m'$ implies $f_{2*}(m'') = f_{2*}(m - m') = f_{2*}(m) - f_{2*}(m') = f_{2*}(m) - f_{2*}(f_{1*} \circ f_{2*}(m)) = f_{2*}(m) - (f_{2*} \circ f_{1*})(f_{2*}(m)) = f_{2*}(m) - f_{2*}(m) = 0$ and then $m'' \in \tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(X)$.

Then, $m' \in (Gx)_X$, $m'' \in \tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(X)$, $(Gx)_X \cap \tilde{H}_0^{\varepsilon,\varepsilon'}(X) = 0$ and $m = m' + m''$, which concludes the proof. □

Theorem 2.24. *If $f\colon \mathbb{X} \longrightarrow \mathbb{Y}$ is a filtration preserving map , $x \in X$, $y \in Y$ and $y = f(x)$, then $f_*^{\varepsilon,\varepsilon'}$ maps $\tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(X)$ into $\tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(Y)$ and maps $(Gx)_X$ isomorphic onto $(Gy)_Y$.*

Proof. Let $f_1\colon \mathbb{P}_0 \longrightarrow \mathbb{X}$ be the filtration preserving map defined by $f_1(P_0) = x$ and $f_2\colon \mathbb{Y} \longrightarrow \mathbb{P}_0$ be the constant map. then $f_2 \circ f \circ f_1$ is the identity map of P_0 and from Axioms 1 and 2, the composition

$$\mathrm{H}_0^{\varepsilon,\varepsilon'}(P_0) \xrightarrow{f_{1*}^{\varepsilon,\varepsilon'}} \mathrm{H}_0^{\varepsilon,\varepsilon'}(X) \xrightarrow{f_*^{\varepsilon,\varepsilon'}} \mathrm{H}_0^{\varepsilon,\varepsilon'}(Y) \xrightarrow{f_{2*}^{\varepsilon,\varepsilon'}} \mathrm{H}_0^{\varepsilon,\varepsilon'}(P_0)$$

and particularly

$$G^{\varepsilon,\varepsilon'} \xrightarrow{f_{1*}^{\varepsilon,\varepsilon'}} (Gx)_X \xrightarrow{f_*^{\varepsilon,\varepsilon'}} (Gy)_Y \xrightarrow{f_{2*}^{\varepsilon,\varepsilon'}} G^{\varepsilon,\varepsilon'}$$

are identity maps.

Since each homomorphism is injective and, by Theorem 2.19, each one is also surjective, then we have that each homomorphism is an isomorphism, which proves that $f_*^{\varepsilon,\varepsilon'}$ maps $(Gx)_X$ isomorphically onto $(Gy)_Y$.

Since $f_2 \circ f\colon X \longrightarrow P_0$ then by definition we have $\tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(X) = \ker(f_2 \circ f)_*^{\varepsilon,\varepsilon'}$. Then to prove that $f_*^{\varepsilon,\varepsilon'}$ maps $\tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(X)$ into $\tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(Y)$ we just note that

$$f_*^{\varepsilon,\varepsilon'}(\tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(X)) = f_*^{\varepsilon,\varepsilon'}(\ker(f_2 \circ f)_*^{\varepsilon,\varepsilon'}) = f_*^{\varepsilon,\varepsilon'}(\ker(f_{2*}^{\varepsilon,\varepsilon'} \circ f_*^{\varepsilon,\varepsilon'})) \subseteq \ker(f_{2*}^{\varepsilon,\varepsilon'}) = \tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(Y). \quad \square$$

Definition 2.25. Let $f\colon \mathbb{X} \longrightarrow \mathbb{Y}$ be a filtration preserving map. The map from $\tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(X)$ onto $\tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(Y)$ defined by $f_*^{\varepsilon,\varepsilon'}$ is denoted by $\tilde{f}_*^{\varepsilon,\varepsilon'}$.

Corollary 2.26. *The map previously defined has same kernel as the induced map, that is,* $\ker \tilde{f}_*^{\varepsilon,\varepsilon'} = \ker f_*^{\varepsilon,\varepsilon'}$

Proof. Since $\tilde{f}_*$ is the restriction of f_*, the $\ker(\tilde{f}_*) \subseteq \ker(f_*)$. The other inclusion follows from the definition of reduced persistent homology. □

Lemma 2.27. *For each filtered pair $\mathbb{X}$ and each integer q,*

$$\mathrm{H}_q^{\varepsilon,\varepsilon'}(X, X) = 0.$$

Proof. Let $i\colon X \longrightarrow X$ and $j\colon X \longrightarrow (X, X)$ be inclusion maps. Consider the following part of the persistent homology exact sequence of the filtered relative pair $(\mathbb{X}, \mathbb{X})$:

$$\mathrm{H}_q^{\varepsilon,\varepsilon'}(X) \xrightarrow{i_*^q} \mathrm{H}_q^{\varepsilon,\varepsilon'}(X) \xrightarrow{j_*^q} \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, X) \xrightarrow{\Delta_X} \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(X) \xrightarrow{i_*^{q-1}} \mathrm{H}_p^{\varepsilon,\varepsilon'}(X)$$

Since each i_* is an isomorphism, it follows by exactness that $0 = \ker(i_*^{q-1}) = \mathrm{Im}(\Delta_X)$. Therefore, $\mathrm{H}_q^{\varepsilon,\varepsilon'}(X, X) = \ker(\Delta_X)$. Similarly $\mathrm{H}_q^{\varepsilon,\varepsilon'}(X) = \mathrm{Im}(i_*^q) = \ker(j_*^q)$. Therefore $\mathrm{Im}(j_*) = 0$. Since by exactness $\mathrm{Im}(j_*^q) = \ker(\Delta_X)$, it follows that $\mathrm{H}_q^{\varepsilon,\varepsilon'}(X, X) = 0$. □

Theorem 2.28. *Δ_A maps $\mathrm{H}_1^{\varepsilon,\varepsilon'}(X, A)$ into $\tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(A)$ in the exact sequence of the relative filtered pair $(\mathbb{X}, \mathbb{A})$*

Proof. Let $f\colon (\mathbb{X}, \mathbb{A}) \longrightarrow (\mathbb{P}_0, \mathbb{P}_0)$ be the constant map. If $h \in \mathrm{H}_1^{\varepsilon,\varepsilon'}(X, A)$, then $f_*^{\varepsilon,\varepsilon'}(h) \in \mathrm{H}_1^{\varepsilon,\varepsilon'}(P_0, P_0) = 0$, by Lemma 2.27.

From the Axiom 3 the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}_1^{\varepsilon,\varepsilon'}(X, A) \arrow[r, "\Delta_A"] \arrow[d, "f_*"] & \mathrm{H}_0^{\varepsilon,\varepsilon'}(A) \arrow[d, "(f|_A)_*"] \\
\mathrm{H}_1^{\varepsilon,\varepsilon'}(P_0, P_0) \arrow[r, "\Delta_{P_0}"] & \mathrm{H}_0^{\varepsilon,\varepsilon'}(P_0)
\end{tikzcd}$$

Therefore $(f|_A)_* \circ \Delta_A = \Delta_B \circ f_*(h) = 0$ and, by definition, $\Delta_A(h) \in \tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(A)$. □

Definition 2.29. Let $f\colon (\mathbb{X}, \mathbb{A}) \longrightarrow (\mathbb{P}_0, \mathbb{P}_0)$ be a filtration preserving map. The reduced homology sequence of $(\mathbb{X}, \mathbb{A})$ is the sequence of kernels of the induced maps defined in Theorem 2.14.

Theorem 2.30. *The reduced homology sequence of $(\mathbb{X}, \mathbb{A})$ differs from the homology sequence of $(\mathbb{X}, \mathbb{A})$ only on the dimension where*

$$\mathrm{H}_1^{\varepsilon,\varepsilon'}(X, A) \xrightarrow{\Delta_A} \mathrm{H}_0^{\varepsilon,\varepsilon'}(A) \xrightarrow{i_*} \mathrm{H}_0^{\varepsilon,\varepsilon'}(X) \xrightarrow{j_*} \mathrm{H}_0^{\varepsilon,\varepsilon'}(X, A)$$

is replaced by

$$\mathrm{H}_1^{\varepsilon,\varepsilon'}(X, A) \xrightarrow{\tilde{\Delta}_A} \tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(A) \xrightarrow{\tilde{i}_*} \tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(X) \xrightarrow{\tilde{j}_*} \mathrm{H}_0^{\varepsilon,\varepsilon'}(X, A),$$

where, $\tilde{i}_*, \tilde{j}_*$ and $\tilde{\Delta}_A$ are the maps defined by i_*, j_* and Δ_A, respectively.

Proof. From Remark 2.21, it follows that for $n > 0$, one has $\mathrm{H}_0^{\varepsilon,\varepsilon'}(P_0) = 0$. Then $\ker(f_*^{\varepsilon,\varepsilon'}) = \mathrm{H}_n^{\varepsilon,\varepsilon'}(P)$.

For $n = 0$, there are changes in the sequence, but by definition $\tilde{\mathrm{H}}_n^{\varepsilon,\varepsilon'}(X) := \ker(f_*^{\varepsilon,\varepsilon'} : \mathrm{H}_n^{\varepsilon,\varepsilon'}(X) \longrightarrow \mathrm{H}_n^{\varepsilon,\varepsilon'}(P_0))$ and also for the pair (X, A) and for A, which concludes the proof. □

Corollary 2.31. *If $f\colon \mathbb{X} \longrightarrow \mathbb{Y}$ is a filtration preserving map, then f_* maps the reduced homology sequence of $(\mathbb{X}, \mathbb{A})$ into that of $(\mathbb{Y}, \mathbb{B})$.*

Definition 2.32. A filtered pair $\mathbb{X}$ is said to be **homologically trivial** if $\tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(X) = 0$, for all q. If $A \neq \emptyset$, $(\mathbb{X}, \mathbb{A})$ is said to be homologically trivial if $\tilde{\mathrm{H}}_q^{\varepsilon,\varepsilon'}(X, A) = 0$ for all q.

Theorem 2.33. *If* $(\mathbb{X}, \mathbb{A})$ *is homologically trivial and* $A \neq \emptyset$*, then the inclusion map* $i \colon A \longrightarrow X$ *induces isomorphism*

$$i_* \colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(A) \approx \mathrm{H}_q^{\varepsilon,\varepsilon'}(X), \forall q$$

and

$$\tilde{i}_* \colon \tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(A) \approx \tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(X).$$

Proof. Since $(\mathbb{X}, \mathbb{A})$ is homologically trivial, we have $\mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A) = 0$ for all q. Therefore the persistent homology sequence is

$$0 \longrightarrow \mathrm{H}_q^{\varepsilon,\varepsilon'}(A) \xrightarrow{i_*^q} \mathrm{H}_q^{\varepsilon,\varepsilon'}(X) \longrightarrow 0.$$

By exacteness of the persistent homology sequence we have that

$$i_* \colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(A) \approx \mathrm{H}_q^{\varepsilon,\varepsilon'}(X), \forall q.$$

Since $\tilde{\mathrm{H}}_n^{\varepsilon,\varepsilon'}(X) := \ker(f_*^{\varepsilon,\varepsilon'} : \mathrm{H}_n^{\varepsilon,\varepsilon'}(X) \longrightarrow \mathrm{H}_n^{\varepsilon,\varepsilon'}(P_0))$ and $\mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A) = 0$ for all q, we obtain that $\tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(X, A) = 0$, then we have the same behavior of the previous case, which is enough to obtain the isomorphism. □

Theorem 2.34. *If* $\mathbb{A}$ *is homologically trivial and* $\emptyset \neq A \subset X$*, then the inclusion map* $j \colon (\mathbb{X}, \emptyset) \longrightarrow (\mathbb{X}, \mathbb{A})$ *induces isomorphism*

$$j_* \colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(X) \approx \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A), \forall q$$

and

$$\tilde{j}_* \colon \tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(X) \approx \tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(X, A).$$

Proof. Since $\mathbb{A}$ is homologically trivial, we have $\mathrm{H}_q^{\varepsilon,\varepsilon'}(A) = 0$ for $q \neq 0$ and $\tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(X) = 0$. Therefore the persistent homology sequence is

$$0 \longrightarrow \mathrm{H}_q^{\varepsilon,\varepsilon'}(X) \xrightarrow{i_*^q} \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A) \longrightarrow 0.$$

By exacteness of the persistent homology sequence we have that

$$i_* \colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(X) \approx \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A), \forall q.$$

Now, for the second isomorphism, observe the following part of the reduced homological sequence:

$$\cdots \longrightarrow \tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(A) \xrightarrow{\tilde{i}_*} \tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(X) \xrightarrow{\tilde{j}_*} \mathrm{H}_0^{\varepsilon,\varepsilon'}(X, A) \longrightarrow 0.$$

Since $\mathrm{H}_0^{\varepsilon,\varepsilon'}(X, A) = \tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(X, A)$ and $\tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(A) \subseteq \mathrm{H}_0^{\varepsilon,\varepsilon'}(A) = 0$, by definition of homologically trivial, the sequence becomes

$$0 \longrightarrow \tilde{H}_0^{\varepsilon,\varepsilon'}(X) \xrightarrow{\tilde{j}_*} \tilde{H}_0^{\varepsilon,\varepsilon'}(X, A) \longrightarrow 0$$

and by exactness we obtain the required isomorphism. □

Theorem 2.35. *Let* $(\mathbb{X}, \mathbb{A})$ *be a relative filtered pair. If* $\mathbb{X}$ *is homologically trivial and* $A \neq \emptyset$*, then the boundary homomorphism of* $(\mathbb{X}, \mathbb{A})$ *is an isomorphism in each dimension*

$$\Delta_{A*} \colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A) \approx \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(A), \textit{ for } q \neq 1$$

and

$$\tilde{\Delta}_A \colon \mathrm{H}_1^{\varepsilon,\varepsilon'}(X, A) \approx \tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(A).$$

Proof. The proof of this theorem is analogous to the previous two theorems. We just need to verify that from the homological sequence we obtain the isomorphisms. □

Corollary 2.36. *If both $\mathbb{X}$ and $\mathbb{A}$ are homologically trivial, so also is* $(\mathbb{X}, \mathbb{A})$.

Proof. The proof follows from Theorems 2.33, 2.34 and 2.35. □

We now introduce triples, triads and properties of their persistent homology and its sequences.

Definition 2.37. Suppose that $B \subset A \subset X$ and that the inclusions

$$\bar{i} \colon (A, B) \subset (X, B) \text{ and } \bar{j} \colon (X, B) \subset (X, A)$$

are filtration preserving maps. Then $((X, A, B), (F_X, F_A, F_B))$, denoted by $(\mathbb{X}, \mathbb{A}, \mathbb{B})$, is called a **relative filtered triple**. The inclusion maps and boundary homomorphisms associated with the relative filtered pairs $(\mathbb{X}, \mathbb{A})$, $(\mathbb{X}, \mathbb{B})$ and $(\mathbb{A}, \mathbb{B})$ are, respectively, denoted by

$$\begin{array}{lll} i \colon A \longrightarrow X, & j \colon X \longrightarrow (X, A), & \Delta \colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A) \longrightarrow \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(A), \\ i' \colon B \longrightarrow X, & j' \colon X \longrightarrow (X, B), & \Delta' \colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, B) \longrightarrow \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(B), \\ i'' \colon B \longrightarrow A, & j'' \colon A \longrightarrow (A, B), & \Delta'' \colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(A, B) \longrightarrow \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(B). \end{array}$$

Definition 2.38. $\overline{\Delta} = j''_* \circ \Delta$ is called **the connecting operator** of the triple $(\mathbb{X}, \mathbb{A}, \mathbb{B})$:

$$\overline{\Delta} : \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A) \longrightarrow \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(A, B).$$

The sequence of groups

$$\cdots \longrightarrow \mathrm{H}_q^{\varepsilon,\varepsilon'}(A, B) \xrightarrow{\bar{i}_*} \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, B) \xrightarrow{\bar{j}_*} \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A) \xrightarrow{\overline{\Delta}} \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(A, B) \longrightarrow \cdots$$

is called **the persistent homology sequence of the triple** $(\mathbb{X}, \mathbb{A}, \mathbb{B})$.

Theorem 2.39. *The persistent homology sequence of a triple is exact.*

Proof. The proof of this Theorem is similar to the persistent homology sequence of the pair. □

Theorem 2.40. *A filtration preserving map* $g \colon (\mathbb{X}, \mathbb{A}, \mathbb{B}) \longrightarrow (\mathbb{X}', \mathbb{A}', \mathbb{B}')$ *induces a homomorphism of the homology sequence of* $(\mathbb{X}, \mathbb{A}, \mathbb{B})$ *into the homology sequence of* $(\mathbb{X}', \mathbb{A}', \mathbb{B}')$.

Proof. First note that a filtration preserving map $g \colon (\mathbb{X}, \mathbb{A}, \mathbb{B}) \longrightarrow (\mathbb{X}', \mathbb{A}', \mathbb{B}')$ defines three filtration preserving maps

$$g_1 \colon (\mathbb{A}, \mathbb{B}) \longrightarrow (\mathbb{A}', \mathbb{B}')\ ,\ g_2 \colon (\mathbb{X}, \mathbb{B}) \longrightarrow (\mathbb{X}', \mathbb{B}') \text{ and } g_3 \colon (\mathbb{X}, \mathbb{A}) \longrightarrow (\mathbb{X}', \mathbb{A}'),$$

obtained from g. To prove the theorem, consider the following diagram

$$\begin{tikzcd} {} \arrow[r] & \mathrm{H}_n^{\varepsilon,\varepsilon'}(A, B) \arrow[r, "\bar{i}_*"] \arrow[d, "g_{1*}"] & \mathrm{H}_n^{\varepsilon,\varepsilon'}(X, B) \arrow[r, "\bar{j}_*"] \arrow[d, "g_{2*}"] & \mathrm{H}_n^{\varepsilon,\varepsilon'}(X, A) \arrow[r, "\overline{\Delta}"] \arrow[d, "g_{3*}"] & \mathrm{H}_{n-1}^{\varepsilon,\varepsilon'}(A, B) \arrow[r] \arrow[d, "g_{1*}"] & {} \\ {} \arrow[r] & \mathrm{H}_n^{\varepsilon,\varepsilon'}(A', B') \arrow[r, "\overline{i'}_*"] & \mathrm{H}_n^{\varepsilon,\varepsilon'}(X', B') \arrow[r, "\overline{j'}_*"] & \mathrm{H}_{n-1}^{\varepsilon,\varepsilon'}(X', A') \arrow[r, "\overline{\Delta'}"] & \mathrm{H}_{n-1}^{\varepsilon,\varepsilon'}(A', B') \arrow[r] & {} \end{tikzcd}$$

where $\bar{i}, \bar{j}, \overline{i'}, \overline{j'}$ are the appropriate inclusions maps. To prove the commutativity we must verify the relations

$$g_{2*}\bar{i}_* = \overline{i'}_* g_{1*},\ g_{3*}\bar{j}_* = \overline{j'}_* g_{2*} \text{ and } g_{1*}\overline{\Delta} = \overline{\Delta'} g_{3*}.$$

Therefore, the proof is analogous to the proof of Theorem 2.14. □

Theorem 2.41. *If $B \neq \emptyset$ and any of the relative filtered pairs $(\mathbb{X}, \mathbb{A})$, $(\mathbb{X}, \mathbb{B})$ and $(\mathbb{A}, \mathbb{B})$ is homologically trivial, then the homology groups of the remaining two pairs are isomorphic under the maps of the homology sequence of $(\mathbb{X}, \mathbb{A}, \mathbb{B})$.*

1. *If $\mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A) = 0$ for each q, then $\bar{i}_*\colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(A, B) \approx \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, B)$ for each q.*
2. *If $\mathrm{H}_q^{\varepsilon,\varepsilon'}(X, B) = 0$ for each q, then $\overline{\Delta}\colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A) \approx \mathrm{H}_q^{\varepsilon,\varepsilon'}(A, B)$ for each q.*
3. *If $\mathrm{H}_q^{\varepsilon,\varepsilon'}(A, B) = 0$ for each q, then $\overline{j}_*\colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, B) \approx \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A)$ for each q.*

Conversely, any one of the three conclusions implies the corresponding hypothesis.

Proof. The proof of this theorem is analogous to the proof of Theorems 2.33,2.34 and 2.35 just changing the notation. Notice that we usedthe homological triviality to obtain that $\mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A) = 0$, $\mathrm{H}_q^{\varepsilon,\varepsilon'}(X, B) = 0$ and $\mathrm{H}_q^{\varepsilon,\varepsilon'}(A, B) = 0$. In this result we have it by hypothesis. □

Theorem 2.42. *Let $(\mathbb{X}, \mathbb{A}, \mathbb{B})$ a relative filtered triple. If the inclusion $\mathbb{B} \subset \mathbb{A}$ induces isomorphism $\mathrm{H}_q^{\varepsilon,\varepsilon'}(B) \approx \mathrm{H}_q^{\varepsilon,\varepsilon'}(A)$ for all values of q, then the inclusion map $(\mathbb{X}, \mathbb{B}) \subset (\mathbb{X}, \mathbb{A})$ also induces isomorphism $\mathrm{H}_q^{\varepsilon,\varepsilon'}(X, B) \approx \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A)$ for all q. Similarly, if $\mathbb{A} \subset \mathbb{X}$ induces $\mathrm{H}_q^{\varepsilon,\varepsilon'}(A) \approx \mathrm{H}_q^{\varepsilon,\varepsilon'}(X)$ for all q, then $(\mathbb{A}, \mathbb{B}) \subset (\mathbb{X}, \mathbb{B})$ induces $\mathrm{H}_q^{\varepsilon,\varepsilon'}(A, B) \approx \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, B)$ for all q.*

Proof. The proof of the second part is analogous to the first one. The first part of the theorem follows from the hypothesis $\mathrm{H}_q^{\varepsilon,\varepsilon'}(B) \approx \mathrm{H}_q^{\varepsilon,\varepsilon'}(A)$ for all values of q and the exactness of the homology sequence of a triple $(\mathbb{X}, \mathbb{A}, \mathbb{B})$.

Applying Theorem 2.15 to the inclusion $(\mathbb{X}, \mathbb{B}) \subset (\mathbb{X}, \mathbb{A})$, we obtain another proof of this assertion. □

To continue, it will be important to remember some definitions as contiguous and sequentially contiguous and Axiom 5.

Theorem 2.43. *A contiguity f from $(\mathbb{X}, \mathbb{A})$ to $(\mathbb{Y}, \mathbb{B})$ induces isomorphism $f_*\colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A) \approx \mathrm{H}_q^{\varepsilon,\varepsilon'}(Y, B)$ and $(f_*)^{-1} = g_*$, for all q.*

Proof. Remember that $f\colon (\mathbb{X}, \mathbb{A}) \longrightarrow (\mathbb{Y}, \mathbb{B})$ and $g\colon (\mathbb{Y}, \mathbb{B}) \longrightarrow (\mathbb{X}, \mathbb{A})$ are contiguously equivalent if $g \circ f$ and $\mathrm{Id}_{(\mathbb{X},\mathbb{A})}$ are contiguous and $f \circ g$ and $\mathrm{Id}_{(\mathbb{Y},\mathbb{B})}$ are contiguous.

Therefore $g \circ f$ and $\mathrm{Id}_{(\mathbb{X},\mathbb{A})}$ being contiguous and $f \circ g$ and $\mathrm{Id}_{(\mathbb{Y},\mathbb{B})}$ being contiguous, by Axiom 5, we have that $(f \circ g)_* = \mathrm{Id}_* = \mathrm{Id}$ and, since by Axiom 2 $(f \circ g)_* = f_* \circ g_*$ we do have $f_* \circ g_* = \mathrm{Id}$. For the same reason, $g_* \circ f_* = \mathrm{Id}$.

Then, $f_*\colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A) \approx \mathrm{H}_q^{\varepsilon,\varepsilon'}(Y, B)$ is an isomorphism with inverse $(f_*)^{-1} = g_*$. □

Corollary 2.44. *A contiguity map f from $(\mathbb{X}, \mathbb{A})$ to $(\mathbb{Y}, \mathbb{B})$ induces isomorphism of the ordinary and reduced persistent homology sequences of $(\mathbb{X}, \mathbb{A})$ with the corresponding sequences of $(\mathbb{Y}, \mathbb{B})$.*

Proof. Let f, g be contiguously equivalent from $(\mathbb{X}, \mathbb{A})$ to $(\mathbb{Y}, \mathbb{B})$.

Then there are defined contiguous equivalences f_1, g_1 from X into Y and f_2, g_2 from A into B.

On the one hand, Corollary 2.44 implies that f, f_1 and f_2 give rise to isomorphisms $f_*\colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A) \approx \mathrm{H}_q^{\varepsilon,\varepsilon'}(Y, B)$, $f_{1*}\colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(X) \approx \mathrm{H}_q^{\varepsilon,\varepsilon'}(Y)$ and $f_{2*}\colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(A) \approx \mathrm{H}_q^{\varepsilon,\varepsilon'}(B)$. In other hand, the same result implies that g, g_1 and g_2 give rise to inverse isomorphisms

$g_*\colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(Y,B) \approx \mathrm{H}_q^{\varepsilon,\varepsilon'}(X,A)$, $g_{1*}\colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(Y) \approx \mathrm{H}_q^{\varepsilon,\varepsilon'}(X)$ and $g_{2*}\colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(B) \approx \mathrm{H}_q^{\varepsilon,\varepsilon'}(A)$. Then, we have an isomorphism between the ordinary persistent homology sequence of $(\mathbb{X},\mathbb{A})$ with the corresponding sequence of $(\mathbb{Y},\mathbb{B})$.

The same argument holds for reduced persistent homology sequence, but instead of Theorem 2.43 we use the Theorem 2.24. □

Theorem 2.45. *Every filtered pair contiguously equivalent to a point is homologically trivial.*

Proof. A filtered pair $\mathbb{X}$ is said to be homologically trivial if $\mathrm{H}_q^{\varepsilon,\varepsilon'}(X) = 0$ for $q \neq 0$ and $\tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(X) = 0$.

Since the space is contiguously equivalent to a point P_0, using Theorem 2.43, there exists an isomorphism of the ordinary and reduced persistent homology sequences.

Also considering that $\mathrm{H}_q^{\varepsilon,\varepsilon'}(P_0) = 0$ for $q \neq 0$ and $\tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(P_0) = 0$, the result follows. □

Definition 2.46. Let $(\mathbb{X}',\mathbb{A}')$ and $(\mathbb{X},\mathbb{A})$ be relative filtered pairs with $(X',A') \subset (X,A)$ and $F_{X'} \leq F_X$. The pair $(\mathbb{X}',\mathbb{A}')$ is a **retract** of the pair $(\mathbb{X},\mathbb{A})$ if there exists a filtration preserving map $f\colon (\mathbb{X},\mathbb{A}) \longrightarrow (\mathbb{X}',\mathbb{A}')$, which we call **retraction**, such that $f(x) = x$ for each $x \in X'$. It is called a **deformation retract** if there is a retraction f and the composition of f with the inclusion map $(\mathbb{X}',\mathbb{A}') \subset (\mathbb{X},\mathbb{A})$ and the identity map of $(\mathbb{X},\mathbb{A})$ are contiguous maps.

Lemma 2.47. *If $(\mathbb{X}',\mathbb{A}')$ is a deformation retract of $(\mathbb{X},\mathbb{A})$, the inclusion map $i\colon (\mathbb{X}',\mathbb{A}') \subset (\mathbb{X},\mathbb{A})$ and the retraction f are contiguously equivalent maps.*

Proof. In order to prove the result, we must show that $i \circ f$ and $\mathrm{Id}_{(\mathbb{X},\mathbb{A})}$ are contiguous and $f \circ i$ and $\mathrm{Id}_{(\mathbb{X}',\mathbb{A}')}$ are contiguous.

Since f is the retraction, f is a filtration preserving map such that $f(x) = x$ for each $x \in X'$, then $f \circ i = \mathrm{Id}_{(\mathbb{X}',\mathbb{A}')}$ and, in particular, the maps are contiguous.

By definition, deformation retract means that the composition of f and i is contiguous to the identity map of $(\mathbb{X},\mathbb{A})$, which ends the proof. □

Theorem 2.48. *If $(\mathbb{X}',\mathbb{A}')$ is a deformation retract of $(\mathbb{X},\mathbb{A})$, the inclusion map $(\mathbb{X}',\mathbb{A}') \subset (\mathbb{X},\mathbb{A})$ induces an isomorphism of the persistent homology sequence of $(\mathbb{X}',\mathbb{A}')$ onto that of $(\mathbb{X},\mathbb{A})$.*

Proof. If $(\mathbb{X}',\mathbb{A}')$ is a deformation retract of $(\mathbb{X},\mathbb{A})$, by Lemma 2.47, the inclusion map i : $(\mathbb{X}',\mathbb{A}') \subset (\mathbb{X},\mathbb{A})$ and the retraction f forms a contiguous equivalence of $(\mathbb{X},\mathbb{A})$ and $(\mathbb{X}',\mathbb{A}')$.

Applying Corollary 2.44, we have that this contiguous equivalence induces isomorphism of the persistent homology sequences of $(\mathbb{X}',\mathbb{A}')$ with the corresponding sequence of $(\mathbb{X},\mathbb{A})$. □

Definition 2.49. Let $\mathbb{X}$ and $\mathbb{Y}$ be two filtered pairs and M a sufficiently large positive number such that $F_X(\sigma) \leq M$, for all $\sigma \in \mathrm{pow}(X)$ and $F_Y(\tau) \leq M$, for all $\tau \in \mathrm{pow}(Y)$. By the filtered pair $(X \cup Y, F_X \cup F_Y)$, simply denoted by $\mathbb{X} \cup \mathbb{Y}$, we understand a filtered pair where the finite set is $X \cup Y$ and the map $F_X \cup F_Y\colon \mathrm{pow}(X \cup Y) \longrightarrow \mathbb{R}$ is defined as follows:

$$(F_X \cup F_Y)(\sigma) = \begin{cases} F_X(\sigma), & \text{if } \sigma \subset X \text{ and } \sigma \not\subset X \cap Y;\\ F_Y(\sigma), & \text{if } \sigma \subset Y \text{ and } \sigma \not\subset X \cap Y;\\ \inf\{F_X(\sigma), F_Y(\sigma)\}, & \text{if } \sigma \subset X \cap Y;\\ M & \text{otherwise.} \end{cases}$$

The filtered pair $(X \cap Y, F_x \cap F_Y)$ is a filtered pair where the finite set is the intersection set $X \cap Y$ and the map $F_x \cap F_Y\colon \mathrm{pow}(X \cap Y) \longrightarrow \mathbb{R}$ is defined as follow:

$$(F_X \cap F_Y)(\sigma) = \sup\{F_X(\sigma), F_Y(\sigma)\}.$$

Definition 2.50. A **filtered triad** $((X; X_1, X_2), (F_X; F_{X_1}, F_{X_2}))$, denoted by $(\mathbb{X}; \mathbb{X}_1, \mathbb{X}_2)$, consists of filtered pairs $\mathbb{X}$, $\mathbb{X}_1$ and $\mathbb{X}_2$. The filtered triad $(\mathbb{X}; \mathbb{X}_1, \mathbb{X}_2)$ is called **proper** if the inclusion maps $k_1\colon (X_1, X_1 \cap X_2) \subset (X_1 \cup X_2, X_2)$ and $k_2\colon (X_2, X_1 \cap X_2) \subset (X_1 \cup X_2, X_1)$ induce isomorphism of the persistent homology groups in all dimensions.

A useful observation is that if $(\mathbb{X}; \mathbb{X}_1, \mathbb{X}_2)$ is a proper triad, then $(\mathbb{X}_1 \cup \mathbb{X}_2; \mathbb{X}_1, \mathbb{X}_2)$ is proper as well.

Theorem 2.51. *A filtered triad $(\mathbb{X}; \mathbb{X}_1, \mathbb{X}_2)$ is proper if, and only if, the inclusion map $i_j\colon (X_j, X_1 \cap X_2) \longrightarrow (X_1 \cup X_2, X_1 \cap X_2)$ yields, for each q, an injective representation of $\mathrm{H}_q^{\varepsilon,\varepsilon'}(X_1 \cup X_2, X_1 \cap X_2)$ as a direct sum, that is, every $u \in \mathrm{H}_q^{\varepsilon,\varepsilon'}(X_1 \cup X_2, X_1 \cap X_2)$ can be expressed uniquely as $u = i_{1*}u_1 + i_{2*}u_2$ for $u_j \in \mathrm{H}_q^{\varepsilon,\varepsilon'}(X_j, X_1 \cap X_2)$, $j = 1, 2$.*

Proof. Consider the diagram

$$\begin{tikzcd}
\mathrm{H}_n^{\varepsilon,\varepsilon'}(X_1 \cup X_2, X_1) & & \mathrm{H}_n^{\varepsilon,\varepsilon'}(X_1 \cup X_2, X_2) \\
& \mathrm{H}_n^{\varepsilon,\varepsilon'}(X_1 \cup X_2, X_1 \cap X_2) \arrow[ul, "j_{2*}"'] \arrow[ur, "j_{1*}"] & \\
\mathrm{H}_n^{\varepsilon,\varepsilon'}(X_2, X_1 \cap X_2) \arrow[uu, "k_{1*}"] \arrow[ur, "i_{2*}"] & & \mathrm{H}_{n-1}^{\varepsilon,\varepsilon'}(X_1, X_1 \cap X_2) \arrow[uu, "k_{2*}"'] \arrow[ul, "i_{1*}"']
\end{tikzcd}$$

From the exactness of the persistent homology sequence of the triple of $(\mathbb{X}_1 \cup \mathbb{X}_2, \mathbb{X}_1, \mathbb{X}_1 \cap \mathbb{X}_2)$ and $(\mathbb{X}_1 \cup \mathbb{X}_2, \mathbb{X}_2, \mathbb{X}_1 \cap \mathbb{X}_2)$ we have that $\ker(j_{1*}) = \mathrm{Im}(i_{1*})$ and $\ker(j_{2*}) = \mathrm{Im}(i_{2*})$. Supposing the triad proper, by definition, k_{1*} and k_{2*} are isomorphisms and 13.1 of [ES15] implies the decomposition as a direct sum of $\mathrm{H}_q^{\varepsilon,\varepsilon'}(X_1 \cup X_2, X_1 \cap X_2)$.

Assuming the direct sum condition, we have that $\ker(i_{1*}) = 0$, Then, studying the persistent homology sequence of the triple $(\mathbb{X}_1 \cup \mathbb{X}_2, \mathbb{X}_1, \mathbb{X}_1 \cap \mathbb{X}_2)$ we have $\overline{\Delta} = 0$. By exactness of this sequence, j_{1*} is surjective.

We must prove that k_{1*} is an isomorphism. Let $m \in \mathrm{H}_q^{\varepsilon,\varepsilon'}(X_1 \cup X_2, X_1)$, then $m = j_{1*}n$, for some $n \in \mathrm{H}_q^{\varepsilon,\varepsilon'}(X_1 \cup X_2, X_1 \cap X_2)$. By hypothesis, $n = i_{1*}n_1 + i_{2*}n_2$ and therefore $m = j_{1*}n = j_{1*}(i_{1*}n_1 + i_{2*}n_2) = j_{1*} \circ i_{1*}(n_1) + j_{1*} \circ i_{2*}(n_2) = k_{1*}n_2$ which implies that k_{1*} is onto.

Supposing $m \in \mathrm{H}_q^{\varepsilon,\varepsilon'}(X_2, X_1 \cap X_2)$ and $k_{1*}m = 0$, we have that $j_{1*} \circ i_{2*}(m) = 0$. By exactness, there exist an $n \in \mathrm{H}_q^{\varepsilon,\varepsilon'}(X_1, X_1 \cap X_2)$ with $i_{2*}m = i_{1*}n$. Then $i_{1*}(-n) + i_{2*}m = 0$ and, by the direct sum condition, $u = 0$, which implies that k_{1*} is a monomorphism.

Therefore k_{1*} is an isomorphism. The same arguments prove that k_{2*} is also an isomorphism. Thus, the triad $(\mathbb{X}; \mathbb{X}_1, \mathbb{X}_2)$ is proper. □

Definition 2.52. Let $(\mathbb{X}; \mathbb{X}_1, \mathbb{X}_2)$ be a proper triad. The composition of the homomorphisms

$$\mathrm{H}_q^{\varepsilon,\varepsilon'}(X, X_1 \cup X_2) \xrightarrow{\Delta} \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(X_1 \cup X_2) \xrightarrow{l_{2*}} \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(X_1 \cup X_2, X_2) \xrightarrow{k_{2*}^{-1}} \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(X_1, X_1 \cap X_2),$$

where k_2 and l_2 are inclusion maps, is called the **boundary operator** of the proper triad and is denoted, ambiguously, by ∂. The sequence

$$\mathrm{H}_q^{\varepsilon,\varepsilon'}(X_1, X_1 \cap X_2) \xrightarrow{i_*} \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, X_2) \xrightarrow{j_*} \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, X_1 \cup X_2) \xrightarrow{\Delta} \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(X_1, X_1 \cap X_2)$$

where $i\colon (X_1, X_1 \cap X_2) \hookrightarrow (X, X_2)$ and $j\colon (X, X_2) \hookrightarrow (X, X_1 \cup X_2)$ are inclusions, is called **persistent homology sequence** of the proper triad.

Theorem 2.53. *The persistent homology sequence of a proper triad is exact.*

Proof. Looking to the following diagram

$$\begin{tikzcd}
\mathrm{H}_n^{\varepsilon,\varepsilon'}(X,X_2) & \mathrm{H}_n^{\varepsilon,\varepsilon'}(X_1\cup X_2,X_2) \arrow[l, "\overline{i_*}"'] & \mathrm{H}_n^{\varepsilon,\varepsilon'}(X,X_1\cup X_2) \arrow[l, "\overline{\Delta}"'] \arrow[dl, "\Delta"] & \mathrm{H}_{n-1}^{\varepsilon,\varepsilon'}(X,X_2) \arrow[l, "\overline{j_*}"'] \\
& \mathrm{H}_n^{\varepsilon,\varepsilon'}(X_1,X_1\cap X_2) \arrow[ul, "i_*"] \arrow[u, "k_{2*}"] & &
\end{tikzcd}$$

we can note that the persistent homology sequence of the triad is obtained from the persistent homology sequence of the triple $(\mathbb{X},\mathbb{X}_1\cup\mathbb{X}_2,\mathbb{X}_2)$ by changing the group $\mathrm{H}_n^{\varepsilon,\varepsilon'}(X_1\cup X_2,X_2)$ by the group $\mathrm{H}_n^{\varepsilon,\varepsilon'}(X_1,X_1\cap X_2)$ under k_{2*} and defining Δ such that $k_{2*}\circ\Delta=\overline{\Delta}$.

Since the triad is proper, k_{2*} is an isomorphism and then the persistent homology sequences are isomorphic. From the fact that the persistent homology sequence of a triple is exact, follows that the persistent homology sequence of a proper triad is exact as desired. □

Note that, if $(\mathbb{X};\mathbb{X}_1,\mathbb{X}_2)$ is a triad with $X_2\subset X_1$, then $(X;X_1,X_2)$ is a proper triad and its persistent homology sequence reduces to the homology sequence of the triple $(\mathbb{X},\mathbb{X}_1,\mathbb{X}_2)$.

Theorem 2.54. *If* $(\mathbb{X};\mathbb{X}_1,\mathbb{X}_2)$ *and* $(\mathbb{Y};\mathbb{Y}_1,\mathbb{Y}_2)$ *are proper triads, and if* $f\colon(\mathbb{X};\mathbb{X}_1,\mathbb{X}_2)\longrightarrow(\mathbb{Y};\mathbb{Y}_1,\mathbb{Y}_2)$*, then* f *induces a homomorphism* f_* *from the persistent homology sequence of* $(\mathbb{X};\mathbb{X}_1,\mathbb{X}_2)$ *into that of* $(\mathbb{Y};\mathbb{Y}_1,\mathbb{Y}_2)$*. In particular, the boundary homomorphism for proper triads commutes with the respective induced homomorphisms.*

Proof. Let us consider the restriction maps $f_1\colon(X_1,X_1\cap X_2)\longrightarrow(Y_1,Y_1\cap Y_2)$, $f_2\colon(X,X_2)\longrightarrow(Y,Y_2)$ and $f_3\colon(X,X_1\cup X_2)\longrightarrow(X,X_1\cup X_2)$.

In the diagram

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{H}_q^{\varepsilon,\varepsilon'}(X_1,X_1\cap X_2) \arrow[r, "i_*"] \arrow[d, "f_{1*}"] & \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(X,X_2) \arrow[r, "j_*"] \arrow[d, "f_{2*}"] & \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(X,X_1\cup X_2) \arrow[r, "\Delta"] \arrow[d, "f_{3*}"] & \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(X_1,X_1\cap X_2) \arrow[r] \arrow[d, "f_{1*}"] & {} \\
{} \arrow[r] & \mathrm{H}_q^{\varepsilon,\varepsilon'}(Y_1,Y_1\cap Y_2) \arrow[r, "i'_*"] & \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(Y,Y_2) \arrow[r, "j'_*"] & \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(Y,Y_1\cup Y_2) \arrow[r, "\Delta'"] & \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(Y_1,Y_1\cap Y_2) \arrow[r] & {}
\end{tikzcd}$$

which induced maps are homomorphism maps. Since i_*, i'_*, j_* and j'_* are inclusion maps, then we have the commutativity of the first two squares. Let us prove the commutativity of the last one. We can extend the last square as

$$\begin{tikzcd}
\mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(X,X_1\cup X_2) \arrow[r, "\Delta"] \arrow[d, "f_{3*}"] & \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(X_1\cup X_2,X_2) \arrow[r, "k_*^{-1}"] \arrow[d, "f_{4*}"] & \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(X_1,X_1\cap X_2) \arrow[d, "f_{1*}"] \\
\mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(Y,Y_1\cup Y_2) \arrow[r, "\Delta'"] & \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(Y_1\cup Y_2,Y_2) \arrow[r, "k_*'^{-1}"] & \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(Y_1,Y_1\cap Y_2)
\end{tikzcd}$$

where $f_4\colon(X_1\cup X_2,X_2)\longrightarrow(Y_1\cup Y_2,Y_2)$ is a restriction of f. The commutative of the left square follows using Theorem 2.40 and, using that k and k' are inclusions and Theorem 2.51, we have the commutativity of the right one. From these commutativities we have the commutativity of the last square of first diagram, proving the result. □

Theorem 2.55. *Let $\mathbb{X}$ be a filtered pair with filtered pairs $(X_1, F_1), \dots, (X_r, F_r), (A, \overline{F})$ such that*

$$\mathbb{X} = (X_1 \cup \dots \cup X_r \cup A, F_1 \cup \dots \cup F_r \cup \overline{F})$$

and $X_i \cap X_j \subset A$ as a filtered set for $i \neq j$. Supposing $(A_i, \overline{F}_i) = (X_i \cap A, F_i \cap \overline{F})$ and $k_i \colon (X_i, A_i) \subset (X, A)$, the homomorphism $k_{i*} \colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(X_i, A_i) \longrightarrow \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A)(i = 1, \dots, r)$ form an injective representation of $\mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A)$ as a direct sum.

2.5 Persistent Homology Theory for Simplicial Complexes

Definition 2.56. By a q**-simplex** s^q we will understand a pair (S^q, F^q), where S^q is a (ordered) set of $q+1$ points and $F^q \colon \mathrm{pow}(S^q) \longrightarrow \mathbb{R}$ is defined by $F^q(\sigma) = \alpha_0$ for all σ.

The boundary of the simplex, denoted by $\dot{s}^q$ is the pair $(S^q, \dot{F}^q)$, where $\dot{F}^q(S^q) = M$, for a sufficiently large number $M \geq \alpha_0$, and $\dot{F}^q(\sigma) = \alpha_0$, otherwise.

Let s^{q-1} be a $(q-1)$-simplex satisfying $S^{q-1} \subset S^q$. Let A be the vertex in S^q not in S^{q-1}. We will denote by c^{q-1} the filtered complex (S^q, F^c), defined by

$$(F^c)(\sigma) = \begin{cases} M, & \text{if } \sigma = S^q \text{ or } S^{q-1}; \\ \alpha_0, & \text{otherwise.} \end{cases}$$

Lemma 2.57. *s^q, c^{q-1} and (s^q, c^{q-1}) are homologically trivial.*

Proof. We have that s^q and c^{q-1} are contiguously equivalent to a point and by Theorem 2.45 are homologically trivial. Since s^q and c^{q-1} are homologically trivial, the Corollary 2.36 implies that (s^q, c^{q-1}) is also homologically trivial. □

Theorem 2.58. *The group $\mathrm{H}_p^{\varepsilon,\varepsilon'}(s^q, \dot{s}^q)$ is isomorphic to the group $\mathrm{H}_{p-1}^{\varepsilon,\varepsilon'}(s^{q-1}, \dot{s}^{q-1})$.*

Proof. The homology sequence of the proper triad (s^q, s^{q-1}, c^{q-1}) is given by

$$\mathrm{H}_p^{\varepsilon,\varepsilon'}(s^q, c^{q-1}) \xrightarrow{j_*} \mathrm{H}_p^{\varepsilon,\varepsilon'}(s^q, s^{q-1} \cup c^{q-1}) \xrightarrow{\Delta} \mathrm{H}_{p-1}^{\varepsilon,\varepsilon'}(s^{q-1}, s^{q-1} \cap c^{q-1}) \xrightarrow{i_*} \mathrm{H}_{p-1}^{\varepsilon,\varepsilon'}(s^q, c^{q-1})$$

We have that $s^{q-1} \cap c^{q-1} = \dot{s}^{q-1}$, $s^{q-1} \cup c^{q-1} = \dot{s}^q$ and, by Lemma 2.57, (s^q, c^{q-1}) is homologically trivial. Then, the groups $\mathrm{H}_p^{\varepsilon,\varepsilon'}(s^q, c^{q-1})$ and $\mathrm{H}_{p-1}^{\varepsilon,\varepsilon'}(s^q, c^{q-1})$ are zero and the sequence becomes

$$0 \longrightarrow \mathrm{H}_p^{\varepsilon,\varepsilon'}(s^q, \dot{s}^q) \xrightarrow{\Delta} \mathrm{H}_{p-1}^{\varepsilon,\varepsilon'}(s^{q-1}, \dot{s}^{q-1}) \longrightarrow 0.$$

Therefore, the sequence consists of Δ and, by the exactness of the sequence, we have that Δ is an isomorphism. □

Theorem 2.59. *The persistent homology groups of $(s^q, \dot{s}^q)$ are as follow:*

$$\mathrm{H}_q^{\varepsilon,\varepsilon'}(s^q, \dot{s}^q) \approx G^{\varepsilon,\varepsilon'}$$

and

$$\mathrm{H}_p^{\varepsilon,\varepsilon'}(s^q, \dot{s}^q) = 0, \textit{ for } p \neq q.$$

Proof. By Theorem 2.58 we have $\Delta \colon \mathrm{H}_p^{\varepsilon,\varepsilon'}(s^q, \dot{s}^q) \approx \mathrm{H}_{p-1}^{\varepsilon,\varepsilon'}(s^{q-1}, \dot{s}^{q-1})$. Then, inductively we have $\mathrm{H}_p^{\varepsilon,\varepsilon'}(s^q, \dot{s}^q) \approx \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(s^{p-1}, \dot{s}^{q-1}) \approx \mathrm{H}_{p-q}^{\varepsilon,\varepsilon'}(s^0, \dot{s}^0) = \mathrm{H}_{p-q}^{\varepsilon,\varepsilon'}(s^0)$ and the result follows. □

Definition 2.60. The **incidence isomorphism**

$$[s^q : s^{q-1}] \colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(s^q, \dot{s}^q) \approx \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(s^{q-1}, \dot{s}^{q-1})$$

is defined to be the isomorphism Δ of the Theorem 2.58.

Corollary 2.61. *For every ordered q-simplex s^q, the correspondence $g \longrightarrow gs^q$ is an isomorphism between $G^{\varepsilon,\varepsilon'}$ and $\mathrm{H}_q^{\varepsilon,\varepsilon'}(s^q, \dot{s}^q)$.*

Proof. This result follows from Theorem 2.59, Theorem 2.24 and from the fact that incidence isomorphisms are isomorphisms. □

Lemma 2.62. *If $q > 0$ and $f \colon (s^q, \dot{s}^q) \longrightarrow (s_1^q, \dot{s}_1^q)$ is such that s^{q-1} and c^{q-1} are mapped into s_1^{q-1} and c_1^{q-1}, respectively, then the commutativity holds in the diagram*

$$\begin{array}{ccc}
\mathrm{H}_q^{\varepsilon,\varepsilon'}(s^q, \dot{s}^q) & \xrightarrow{[s^q:s^{q-1}]} & \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(s^{q-1}, \dot{s}^{q-1}) \\
\downarrow f_* & & \downarrow f_{1*} \\
\mathrm{H}_q^{\varepsilon,\varepsilon'}(s_1^q, \dot{s}_1^q) & \xrightarrow{[s_1^q:s_1^{q-1}]} & \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(s_1^{q-1}, \dot{s}_1^{q-1})
\end{array}$$

where f_1 is the map defined by f.

Proof. The incidence isomorphism coincides with the boundary operator of the proper triad (s^q, s^{q-1}, c^{q-1}), in the case $p = q$. Therefore, the commutativity with induced homomorphism is already proved in Theorem 2.54. □

Definition 2.63. If A is a 0-simplex and $g \in G^{\varepsilon,\varepsilon'}$, let gA denote the element $(gA)_A$ defined previously.

Let s^q be an ordered q-simplex with vertices $A^0 < A^1 < \cdots < A^q$. Define s^k, $k < q$, to be the ordered simplex with vertices $A^{q-k} < \cdots < A^q$. For each $g \in G^{\varepsilon,\varepsilon'}$, the element gs^q for $\mathrm{H}_q^{\varepsilon,\varepsilon'}(s^q, \dot{s}^q)$ is defined inductively by

$$gs^k = [s^k : s^{k-1}]^{-1} gs^{k-1}.$$

Thus

$$gs^q = [s^q : s^{q-1}]^{-1} \cdots [s^1 : A^q]^{-1} gA^q.$$

Using Lemma 2.62 we are able to prove the following results:

Theorem 2.64. *If s_1 and s_2 are ordered q-simplices and f is a preserving filtration map $(s_1, \dot{s}_1) \longrightarrow (s_2, \dot{s}_2)$ preserving the order, then $f_*^{\varepsilon,\varepsilon'}(gs_1) = gs_2$.*

Theorem 2.65. *Let A and B be the vertices of the 1-simplex s^1. Using a simplified notation $S^0 = \dot{s}^1$, every element $h \in \mathrm{H}_0^{\varepsilon,\varepsilon'}(S^0)$ can be written uniquely as*

$$h = (gA)_{S^0} + (g'B)_{S^0}, g, g' \in G^{\varepsilon,\varepsilon'}.$$

The element h is in $\tilde{\mathrm{H}}_0^{\varepsilon,\varepsilon'}(S^0)$ if, and only if,

$$h = (gA)_{S^0} - (gB)_{S^0}, g \in G^{\varepsilon,\varepsilon'}.$$

Theorem 2.66. *Let $f\colon (s,\dot s) \longrightarrow (s,\dot s)$ be a permutation of the vertices of s and $h \in \mathrm{H}_q^{\varepsilon,\varepsilon'}(s,\dot s)$. Then*

$$f_*^{\varepsilon,\varepsilon'}(h) = \begin{cases} h, & \text{if } f \text{ is even;} \\ -h, & \text{if } f \text{ is odd.} \end{cases}$$

Proof. For $q = 0$, the theorem is trivial. Let us consider the case $q = 1$.

Let $S^0 = \dot s^1$ consist of the points of A and B and let $f_0\colon S^0 \longrightarrow S^0$ be defined by f. If f is the identity, the result is trivial, thus we may assume $f(A) = B$ and $f(B) = A$. Then the commutativity holds in the diagram

$$\begin{tikzcd} \mathrm{H}_1^{\varepsilon,\varepsilon'}(s^1,S^0) \arrow[r, "f_*^{\varepsilon,\varepsilon'}"] \arrow[d, "\partial"] & \mathrm{H}_1^{\varepsilon,\varepsilon'}(s^1,S^0) \arrow[d, "\partial"] \\ \tilde H_0^{\varepsilon,\varepsilon'}(S^0) \arrow[r, "f_{0*}^{\varepsilon,\varepsilon'}"] & \tilde H_0^{\varepsilon,\varepsilon'}(S^0) \end{tikzcd}$$

Let $h \in \mathrm{H}_1^{\varepsilon,\varepsilon'}(s^1,S^0)$. Then by Theorem 2.14,

$$\partial h = (gA)_{S^0} - (gB)_{S^0}$$

for some $g \in G^{\varepsilon,\varepsilon'}$, and by Theorem 2.19,

$$\partial \circ f_*^{\varepsilon,\varepsilon'} h = f_{0*}^{\varepsilon,\varepsilon'} \circ \partial h = (g \circ f(A))_{S^0} - (g \circ f(B))_{S^0} = (gB)_{S^0} - (gA)_{S^0} = -\partial h.$$

Since $\mathrm{H}_1^{\varepsilon,\varepsilon'}(s^1) = 0$, the kernel of ∂ is zero. Hence $f_*^{\varepsilon,\varepsilon'} h = -h$.

Suppose, inductively, that the theorem is true for integers $q-1$, $q \geq 2$. Since every permutation is a product of simple permutations, it is sufficient to consider the case of a simple permutation f of the vertices of s. Since s has more than two vertices, there is a vertex A_0 such that $F(A_0) = A_0$. Let s' be the $(q-1)$-face of s opposite to the vertex A_0. Then f maps s' onto itself and defines a permutation

$$f\colon (s',\dot s') \longrightarrow (s',\dot s').$$

Moreover, by Lemma 2.62, the commutativity relation $[s:s']f_*^{\varepsilon,\varepsilon'} = f_*'^{\varepsilon,\varepsilon'}[s:s']$ holds.

Let $h \in \mathrm{H}_q^{\varepsilon,\varepsilon'}(s,\dot s)$. Since f' is a simple permutation of the vertices of s', and the theorem is assumed for the dimensions $q-1$, we have

$$f_*'^{\varepsilon,\varepsilon'}[s:s']h = -[s:s']h.$$

This implies that

$$[s:s']f_*^{\varepsilon,\varepsilon'} h = -[s:s'](-h)$$

and, since $[s:s']$ is an isomorphism, $f_*^{\varepsilon,\varepsilon'} h = -h$. □

Theorem 2.67. *Let s_1 and s_2 be two ordered simplices both carried by the same unordered q-simplex s. Then*

$$gs_1 = \pm gs_2$$

according as the order of s_2 differs by an even or odd permutation from the order of s_1.

Proof. Let us denote by f the filtration preserving map from $(s,\dot s)$ to $(s,\dot s)$ which maps s_2 onto s_1 preserving the order.

By Theorem 2.64 we have $f_*^{\varepsilon,\varepsilon'}(gs_2) = gs_1$. Furthermore because of Theorem 2.66, $f_*^{\varepsilon,\varepsilon'}(gs_2) = \pm gs_2$, and consequently $gs_1 = \pm gs_2$. □

Theorem 2.68. *Let s be an ordered simplex with vertices $A^0 < \cdots < A^q$ and let s_k be the face obtained by omitting the vertex A^k and not changing the order of the others. Then*

$$[s : s_k]gs = (-1)^k gs_k.$$

Proof. For the case $k = 0$, the formula is exactly the Definition 2.63. We will reduce the general case to complete the proof.

Let us obtain from s the new ordered simplex $\overline{s}$ by moving the vertex A^k in front of all the others. Then $s_k = \overline{s}_0$ and $g(s_k) = g(\overline{s}_0)$, while using Theorem 2.67 we have $gs = (-1)^k g\overline{s}$. To conclude,

$$[s : s_k]gs = [\overline{s} : \overline{s}_0](-1)^k g\overline{s} = (-1)^k g\overline{s}_0 = (-1)^k g\overline{s}_k. \qquad \square$$

Definition 2.69. Let $\mathbb{X}$ be a filtered pair and M a sufficiently large positive number such that $F_X(\sigma) \leq M$, for all $\sigma \in \text{pow}(X)$. The **q-dimensional skeleton** of $\mathbb{X}$, that will be denoted by $(X^{(q)}, F^{(q)})$, is the filtered set where the set $X^{(q)} = X$ and $F^{(q)}$ is defined by

$$F^{(q)}(\sigma) = \begin{cases} F(\sigma), & \text{if } \#\sigma \leq q; \\ M, & \text{otherwise.} \end{cases}$$

Then, given a filtered pair $((X, A), (F_X, F_A))$ we have

$$((X, X), (F^{(q)}, F^{(q-1)})) = ((X^{(q)} \cup A, X^{(q-1)} \cup A), ((F_X)^{(q)} \cup F_A, (F_X)^{(q-1)} \cup F_A)).$$

Lemma 2.70. *Let $((X, A), (F_X, F_A))$ be a filtered pair. Then the filtered pair $((X, X), (F^{(q)}, F^{(q-1)}))$ obtained from $((X, A), (F_X, F_A))$ has the property that if $p \neq q$,*

$$\mathrm{H}_p^{\varepsilon,\varepsilon'}((X, X), (F^{(q)}, F^{(q-1)})) = 0.$$

Proof. Let $S_1, \ldots, S_r$ be the collection of subsets of X such that $\#S_i = q$ and $S_i \not\subset A$ or $F_A(S_i) \geq M$. Consider the filtered sets (S_i, f_i), where $f_i = F_X|_{\text{pow}(S_i)}$. It allows us to define

$$(S_1 \cup \cdots \cup S_s \cup X^{(q-1)} \cup A, f_1 \cup \cdots \cup f_r \cup F_{(q-1)} \cup F_A) = (X^{(q)} \cup A, F_{(q)} \cup F_A).$$

$S_1, \ldots, S_r$ together with $X^{(q-1)} \cup A$ are in the conditions of Theorem 2.55.

Let $\dot{S}_i = S_i \cap (X^{(q-1)} \cup A)$. Then we need to show that

$$\mathrm{H}_p^{\varepsilon,\varepsilon'}(S_i, \dot{S}_i) = 0$$

for $p \neq q$ and the lemma follows.

Note that if S_i is a simplex, then this equality follows by Theorem 2.59. $\square$

Definition 2.71. The group of q-chains of (X, A), for $\varepsilon \leq \varepsilon'$, denoted by $C_q^{\varepsilon,\varepsilon'}(X, A)$ is defined by

$$C_q^{\varepsilon,\varepsilon'}(X, A) = \mathrm{H}_q^{\varepsilon,\varepsilon'}((X, X), (F^{(q)}, F^{(q-1)})).$$

If $f\colon (X, A) \longrightarrow (X', A')$ is a preserving filtered map, f induces homomorphism

$$f_q^{\varepsilon,\varepsilon'}\colon C_q^{\varepsilon,\varepsilon'}(X, A) \longrightarrow C_q^{\varepsilon,\varepsilon'}(X', A').$$

Theorem 2.72. $C_q^{\varepsilon,\varepsilon'}(X, A) = 0$ *for $q < 0$ and $q > \#X$.*

Proof. Even for the case $q < 0$ or $q > \#X$, the result is a consequence of Lemma 2.27 $\square$

Definition 2.73. Let $A^0, \ldots, A^q$ be a finite sequence of vertices of X. For each $g \in G^{\varepsilon,\varepsilon'}$, we define the element $gA^0 \cdots A^q \in C_q^{\varepsilon,\varepsilon'}(X, A)$ as follow:

Let s^q be an ordered simplex with vertices $B^0 < \cdots < B^q$. and $f\colon (s^q, \dot{s}^q) \longrightarrow (X, X)$ be the filtered map defined by $f(B^i) = A^i$. Then

$$gA^0 \cdots A^q = f_*^{\varepsilon,\varepsilon'}(qs^q).$$

Theorem 2.74. *The association of g to $gA^0 \cdots A^q$ defines a homomorphism $G^{\varepsilon,\varepsilon'} \longrightarrow C_q^{\varepsilon,\varepsilon'}(X, A)$, i.e.*

$$(g_1 + g_2)A^0 \cdots A^q = g_1A^0 \cdots A^q + g_2A^0 \cdots A^q.$$

If $i_0, \ldots, i_q$ is a permutation of the array $0, \ldots, q$, then

$$gA^{i_0} \cdots A^{i_q} = \pm gA^0 \cdots A^q$$

according if the permutation is even or odd. If some vertex occurs at least twice in $A^0, \ldots, A^q$, then $gA^0 \cdots A^q = 0$. If $A^0, \ldots, A^q$ are all in A, then $gA^0 \cdots A^q = 0$.

Proof. The first part of this theorem follows from Corollary 2.61 and the second part from Theorem 2.67. □

Theorem 2.75. *Let $S_1, \ldots, S_{\alpha_q}$ be the collection of subsets of X of order q such that $S_i \not\subset A$ or $F_A(S_i) \geq M$, where is a sufficient large fixed real positive number such that $F_X(\sigma) \leq M$, for all $\sigma \in \mathrm{pow}(X)$. Suppose that for each S_i an order of its vertices $A_i^0 < \cdots < A_i^q$ has been chosen. Then each q-chain c of (X, A) can be written uniquely in the form*

$$c = \sum_{i=1}^{\alpha_q} g_i A_i^0 \cdots A_i^q, g_i \in G^{\varepsilon,\varepsilon'}.$$

Proof. Consider the inclusion $i_m\colon (s_m, \dot{s}_m) \subset (K^q \cup L, K^{q-1} \cup L)$. By Corollary 2.55 and from the fact that $K^q \cup L = s_1 \cup s_{\alpha_q} \cup \cdots \cup K^{q-1} \cup L$ and $\dot{s}_m = s_m \cap (K^{q-1} \cup L)$ we have a unique representation

$$c = \sum_{i=1}^{\alpha_q} i_{m*}h_m.$$

For each h_m we do have a correspondence, by Theorem 2.61, and one can write it uniquely as $h_m = g_m s_m$. By Definition 2.73, $i_{m*}g_m s_m = g_m A_m^0 \cdots A_m^q$. Hence

$$c = \sum_{i=1}^{\alpha_q} i_{m*}h_m = \sum_{i=1}^{\alpha_q} i_{m*}g_m s_m = \sum_{i=1}^{\alpha_q} g_i A_i^0 \cdots A_i^q. \qquad \square$$

Theorem 2.76. *If $f\colon (X, A) \longrightarrow (X_1, A_1)$ is filtration preserving map, $g \in G^{\varepsilon,\varepsilon'}$ and $gA^0 \cdots A^q$ is an q-chain of (X, A), then*

$$f_q(gA^0 \cdots A^q) = gf(A^0) \cdots f(A^q).$$

The following results and definitions characterize properties of chains, boundary operator, cycles, boundaries and persistent homology group of filtered pairs. Most of them were already proved, but they are really important for us to be able to state the final lemmas and theorems that are necessary to prove the Uniqueness Theorem.

Theorem 2.77. *The sequence*

$$0 \longrightarrow C_q^{\varepsilon,\varepsilon'}(A) \xrightarrow{i_q} C_q^{\varepsilon,\varepsilon'}(X) \xrightarrow{j_q} C_q^{\varepsilon,\varepsilon'}(X, A) \longrightarrow 0$$

is exact and the image of i_q is a direct summand of $C_q^{\varepsilon,\varepsilon'}(X)$.

Proof. The proof of this Theorem is similar to the one we found previously, when we defined and proved properties of persistent homology of pairs. □

Definition 2.78. The boundary operator for chains

$$\partial_q : C_q^{\varepsilon,\varepsilon'}(X, A) \longrightarrow C_{q-1}^{\varepsilon,\varepsilon'}(X, A)$$

is defined to be the boundary operator of the triple $(X^{(q)} \cup A, X^{(q-1)} \cup A, X^{(q-2)} \cup A)$.

Explicitly, ∂_q is the composition of the homomorphisms

$$\mathrm{H}_q^{\varepsilon,\varepsilon'}(X^{(q)} \cup A, X^{(q-1)} \cup A) \longrightarrow \mathrm{H}_q^{\varepsilon,\varepsilon'}(X^{(q-1)} \cup A) \longrightarrow \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(X^{(q-1)} \cup A, X^{(q-2)} \cup A).$$

Theorem 2.79. *The boundary operator has the following properties:*

1. $\partial_q \circ \partial_{q+1} = 0$.
2. *If $f : (X, A) \longrightarrow (X', A')$ is a filtration preserving map, then $f_{q-1}^{\varepsilon,\varepsilon'} \partial_q = \partial_q f_q^{\varepsilon,\varepsilon'}$.*
3. *The boundary operator for q-chains ∂_q satisfies*

$$\partial_q(gA^0 \cdots A^q) = \sum_{k=0}^{q} (-1)^k gA^0 \cdots \hat{A}^k \cdots A^q.$$

Proof. The proof of this Theorem is also similar to the one we found previously, when we defined and proved properties of persistent homology of pairs. □

2.6 The Main Isomorphism

Recall that the group of q-chains of (X, A), for $\varepsilon \leq \varepsilon'$, denoted by $C_q^{\varepsilon,\varepsilon'}(X, A)$ is defined by

$$C_q^{\varepsilon,\varepsilon'}(X, A) = \mathrm{H}_q^{\varepsilon,\varepsilon'}((X, X), (F^{(q)}, F^{(q-1)})).$$

Definition 2.80. Let $(\mathbb{X}, \mathbb{A})$ be a filtered pair. The kernel of $\partial_q : C_q^{\varepsilon,\varepsilon'}(X, A) \longrightarrow C_{q-1}^{\varepsilon,\varepsilon'}(X, A)$ is called **the group of q-cycles of** (X, A) and it is denoted by $\mathcal{Z}_q^{\varepsilon,\varepsilon'}(X, A)$. The image of $\partial_{q+1} : C_{q+1}^{\varepsilon,\varepsilon'}(X, A) \longrightarrow C_q^{\varepsilon,\varepsilon'}(X, A)$ is called **the group of q-boundaries of** (X, A) and its denoted by $\mathcal{B}_q^{\varepsilon,\varepsilon'}(X, A)$. Since $\partial_q \circ \partial_{q+1} = 0$, $\mathcal{B}_q^{\varepsilon,\varepsilon'}(X, A)$ is a subgroup of $\mathcal{Z}_q^{\varepsilon,\varepsilon'}(X, A)$.

The factor group $\mathcal{H}_q^{\varepsilon,\varepsilon'}(X, A) := \dfrac{\mathcal{Z}_q^{\varepsilon,\varepsilon'}(X, A)}{\mathcal{B}_q^{\varepsilon,\varepsilon'}(X, A)}$ is called the **q-dimensional persistent homology of** (X, A) associated to $(\varepsilon, \varepsilon')$.

If $f : (X, A) \longrightarrow (X', A')$ is filtration preserving map, then $f_q^{\varepsilon,\varepsilon'}$ carries $\mathcal{Z}_q^{\varepsilon,\varepsilon'}(X, A)$ into $\mathcal{Z}_q^{\varepsilon,\varepsilon'}(X', A')$ and $\mathcal{B}_q^{\varepsilon,\varepsilon'}(X, A)$ into $\mathcal{B}_q^{\varepsilon,\varepsilon'}(X', A')$,thereby inducing homomorphisms

$$f_*^{\varepsilon,\varepsilon'} : \mathcal{H}_q^{\varepsilon,\varepsilon'}(X, A) \longrightarrow \mathcal{H}_q^{\varepsilon,\varepsilon'}(X', A').$$

Remark 2.81. Observe that the group $\mathcal{H}_q^{\varepsilon,\varepsilon'}(X,A)$ is not to be confused with the group $\mathrm{H}_q^{\varepsilon,\varepsilon}(X,A)$ given by the axioms. An isomorphism between these two groups will be established and it is the main step in the uniqueness proof.

Lemma 2.82. *Let $j_q\colon C_q^{\varepsilon,\varepsilon'}(X) \longrightarrow C_q^{\varepsilon,\varepsilon'}(X,A)$ be the induced by the inclusion $X \overset{j}{\hookrightarrow} (X,A)$.*

If we define $\overline{\mathcal{Z}}_q^{\varepsilon,\varepsilon'}(X,A) := j_q^{-1}[\mathcal{Z}_q^{\varepsilon,\varepsilon'}(X,A)]$,
$\overline{\mathcal{B}}_q^{\varepsilon,\varepsilon'}(X,A) := j_q^{-1}[\mathcal{B}_q^{\varepsilon,\varepsilon'}(X,A)]$ and $\overline{\mathcal{H}}_q^{\varepsilon,\varepsilon'}(X,A) := \dfrac{\overline{\mathcal{Z}}_q^{\varepsilon,\varepsilon'}(X,A)}{\overline{\mathcal{B}}_q^{\varepsilon,\varepsilon'}(X,A)}$, then

$$\overline{\mathcal{Z}}_q^{\varepsilon,\varepsilon'}(X,A) = \partial_q^{-1}[i_{q-1}C_q^{\varepsilon,\varepsilon'}(A)]$$

and

$$\overline{\mathcal{B}}_q^{\varepsilon,\varepsilon'}(X,A) = \mathcal{B}_q^{\varepsilon,\varepsilon'}(X,A) \cup i_q[C_q^{\varepsilon,\varepsilon'}(A)],$$

where $\cup$ is the smallest subgroup of $C_q^{\varepsilon,\varepsilon'}(X)$ containing the two groups. Further, the homomorphism j_q induces isomorphism

$$\overline{j}\colon \overline{\mathcal{H}}_q^{\varepsilon,\varepsilon'}(X,A) \longrightarrow \mathcal{H}_q^{\varepsilon,\varepsilon'}(X,A).$$

Proof. According to Theorem 2.77, the inclusion maps

$$A \xrightarrow{i} X \xrightarrow{j} (X,A)$$

induce an exact sequence

$$0 \longrightarrow C_q^{\varepsilon,\varepsilon'}(A) \xrightarrow{i_q} C_q^{\varepsilon,\varepsilon'}(X) \xrightarrow{j_q} C_q^{\varepsilon,\varepsilon'}(X,A) \longrightarrow 0.$$

Suppose $c \in \overline{\mathcal{B}}_q^{\varepsilon,\varepsilon'}(X,A)$. Then $j_q(c) \in \mathcal{B}_q^{\varepsilon,\varepsilon'}(X,A)$ and for some $b \in C_{q+1}^{\varepsilon,\varepsilon'}(X,A)$ we have that $j_q(c) = \partial_{q+1}(b)$. Since the sequence is exact, there is $m \in C_{q+1}^{\varepsilon,\varepsilon'}(X)$ such that $j_{q+1}(m) = b$. Therefore

$$j_q(c - \partial_{q+1}m) = j_q(c) - \partial_{q+1} \circ j_{q+1}(m) = j_q(c) - \partial_{q+1}(b) = 0.$$

Again, by exactness of the sequence, there is a $n \in C_q^{\varepsilon,\varepsilon'}(A)$ such that $i_q(n) = c - \partial_{q+1}d$, and follows that $c = \partial_{q+1}(m) + i_q(n) \in \mathcal{B}_q^{\varepsilon,\varepsilon'}(X,A) \cup i_q[C_q^{\varepsilon,\varepsilon'}(A)]$.

If $c = \partial_{q+1}(m) + i_q(n)$, then $j_q c = j_q(\partial_{q+1}(m) + i_q(n)) = j_q \circ \partial_{q+1}(m) + j_q(i_q(n))$. Since the sequence is exact, we have that $j_q(i_q(n)) = 0$ and, from the fact that $j_q \circ \partial_{q+1} = \partial_{q+1} \circ j_{q+1}$, follows that

$$j_q(c) = \partial_{q+1} \circ j_{q+1}(m) \text{ and } j_q(c) \in \overline{\mathcal{B}}_q^{\varepsilon,\varepsilon'}(X,A).$$

Supposing $e \in \overline{\mathcal{Z}}_q^{\varepsilon,\varepsilon'}(X,A)$ we have that $\partial_q \circ j_q(e) = 0$. Since $\partial_q \circ j_q = j_{q-1} \circ \partial_q$, $j_{q-1} \circ \partial_q(e) = 0$. From exactness of the sequence, $\ker(j_{q-1}) = \mathrm{Im}(i_{q-1})$.

Hence,

$$\partial_q(e) \in i_{q-1}[C_{q-1}^{\varepsilon,\varepsilon'}(A)] \text{ which means } e \in \partial_q^{-1}(i_{q-1}[C_{q-1}^{\varepsilon,\varepsilon'}(A)]). \qquad \square$$

Lemma 2.83. *The boundary homomorphism $\partial_q\colon C_q^{\varepsilon,\varepsilon'}(X,A) \longrightarrow C_{q-1}^{\varepsilon,\varepsilon'}(X,A)$ defines homomorphisms*

$$\overline{\mathcal{Z}}_q^{\varepsilon,\varepsilon'}(X,A) \longrightarrow i_{q-1}[\mathcal{Z}_q^{\varepsilon,\varepsilon'}(A)]$$

and

$$\overline{\mathcal{B}}_q^{\varepsilon,\varepsilon'}(X,A) \longrightarrow i_{q-1}[\mathcal{B}_q^{\varepsilon,\varepsilon'}(A)]$$

Since the kernel of i_{q-1} *is zero, the composition* $i_{q-1}^{-1}\partial_q$ *define homomorphisms*

$$\overline{\mathcal{Z}}_q^{\varepsilon,\varepsilon'}(X,A) \longrightarrow \mathcal{Z}_{q-1}^{\varepsilon,\varepsilon'}(A)$$

and

$$\overline{\mathcal{B}}_q^{\varepsilon,\varepsilon'}(X,A) \longrightarrow \mathcal{B}_{q-1}^{\varepsilon,\varepsilon'}(A)$$

and thereby induces a homomorphism

$$\Delta\colon \overline{\mathcal{H}}_q^{\varepsilon,\varepsilon'}(X,A) \longrightarrow \mathcal{H}_{q-1}^{\varepsilon,\varepsilon'}(X,A).$$

Proof. This lemma is a consequence of Lemma 2.82.

Supposing $c \in \overline{\mathcal{Z}}_q^{\varepsilon,\varepsilon'}(X,A)$, by Lemma 2.82 we have that $\partial_q(c) = i_{q-1}(d)$ for some $d \in C_{q-1}^{\varepsilon,\varepsilon'}(A)$. Using the commutativity of the boundary operator we have

$$i_{q-2} \circ \partial_{q-1}(d) = \partial_{q-1} \circ i_{q-1}(d) = \partial_{q-1} \circ \partial_q(c) = 0.$$

Since the kernel of i_{q-2} is zero, then $\partial_{q-1}(d) = 0$.

Therefore $d \in \mathcal{Z}_{q-1}^{\varepsilon,\varepsilon'}(A)$ and $\partial_q(c) \in i_{q-1}[\mathcal{Z}_{q-1}^{\varepsilon,\varepsilon'}(A)]$.

If we suppose $c \in \overline{\mathcal{B}}_q^{\varepsilon,\varepsilon'}(X,A)$, again by Lemma 2.82, $c = \partial_{q+1}(d) + i_q(e)$, for some $d \in C_{q+1}^{\varepsilon,\varepsilon'}(X)$ and $e \in C_q^{\varepsilon,\varepsilon'}(A)$. Then,

$$\partial_q(c) = \partial_q \circ \partial_{q+1}(d) + \partial_q \circ i_q(e) = i_{q-1} \circ \partial_q(e).$$

Hence, $\partial_q(c) \in i_{q-1}[\mathcal{B}_q^{\varepsilon,\varepsilon'}(A)]$. □

Definition 2.84. The homomorphism

$$\partial_*\colon \mathcal{H}_q^{\varepsilon,\varepsilon'}(X,A) \longrightarrow \mathcal{H}_{q-1}^{\varepsilon,\varepsilon'}(A)$$

is defined to be the composition $\partial_q \overline{j}^{-1}$.

Theorem 2.85 (Main Isomorphism)**.** *The group* $\mathrm{H}_q^{\varepsilon,\varepsilon'}(X,A)$ *is isomorphic to* $\mathcal{H}_q^{\varepsilon,\varepsilon'}(X,A)$.

Proof. Let us define an isomorphism between $\mathrm{H}_q^{\varepsilon,\varepsilon'}(X,A)$ and $\mathcal{H}_q^{\varepsilon,\varepsilon'}(X,A)$.

First, consider the following diagram,

$$\mathcal{H}_q^{\varepsilon,\varepsilon}(X,A) \xleftarrow{l_*} \mathcal{H}_q^{\varepsilon,\varepsilon}(X^{(q)} \cup A, A) \xrightarrow{j_*} \mathcal{H}_q^{\varepsilon,\varepsilon}(X^{(q)} \cup A, X^{(q-1)} \cup A)$$

where the inclusions $l\colon (X^{(q)} \cup A, A) \longrightarrow (X,A)$ and $j\colon (X^{(q)} \cup A, A) \longrightarrow (X^{(q)} \cup A, X^{(q-1)} \cup A)$ are filtration preserving maps. The following relations hold:

(a) j_* is mono and its image is $\mathcal{Z}_q^{\varepsilon,\varepsilon'}$;

(b) l_* is onto, whose kernel is $j_*^{-1}[\mathcal{B}_q^{\varepsilon,\varepsilon'}]$.

In fact, if $p \neq q$, then
$$\mathcal{H}_q^{\varepsilon,\varepsilon}(X^{(p)} \cup A, X^{(p-1)} \cup A) = 0. \tag{2.2}$$
We claim some properties of the homomorphism
$$i_* \colon \mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(p)} \cup A, A) \longrightarrow \mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(p+1)} \cup A, A)$$
induced by the inclusion $(X^{(p)} \cup A, A) \overset{i}{\hookrightarrow} (X^{(p-1)} \cup A, A)$.

Consider the homology sequence of the triple $(X^{(p+1)} \cup A, X^{(p)} \cup A, A)$ given by

$$\begin{tikzcd}
\mathcal{H}_{q+1}^{\varepsilon,\varepsilon'}(X^{(p+1)} \cup A, X^{(p)} \cup A) \arrow[r] & \mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(p)} \cup A, A) \arrow[d] & \\
& \mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(p+1)} \cup A, A) \arrow[r] & \mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(p+1)} \cup A, X^{(p)} \cup A)
\end{tikzcd}$$

Note that, if $q \neq p$, then (2.2) implies that $\mathcal{H}_{q+1}^{\varepsilon,\varepsilon'}(X^{(p+1)} \cup A, X^{(p)} \cup A) = 0$ and then by the exactness of the homology sequence of the triple we have that i_* has trivial kernel.

If $q \neq p+1$, then (2.2) implies that $\mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(p+1)} \cup A, X^{(p)} \cup A) = 0$ and then, again by the exactness of the homology sequence of the triple, we have that i_* is surjective.

Supposing $q \neq p, p+1$ we have that i_* is an isomorphism, since it is injective and surjective.

We also claim that
$$\mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(q-1)} \cup A, A) = 0. \tag{2.3}$$

Note that, using the previous properties about i_* one has that $\mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(q-1)} \cup A, A) \approx \mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(q-r)} \cup A, A)$, for $r = 2, \ldots, q+1$. Since $X^{-1} = \emptyset$, then $X^{-1} \cup A = A$ and therefore, $\mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(q-1)} \cup A, A) = \mathcal{H}_q^{\varepsilon,\varepsilon'}(A, A)$ that is zero by Lemma 2.27.

Let us now prove our claims. Consider the following part of the homology sequence of the triple $(X^{(q)} \cup A, X^{(q-1)} \cup A, A)$ given by

$$\mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(q-1)} \cup A, A) \xrightarrow{i_*} \mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(q)} \cup A, A) \xrightarrow{j_*} \mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(q)} \cup A, X^{(q-1)} \cup A)$$
$$\xrightarrow{\Delta} \mathcal{H}_{q-1}^{\varepsilon,\varepsilon'}(X^{(p-1)} \cup A, A)$$

Using (2.3) we have that $\mathrm{H}_q^{\varepsilon,\varepsilon'}(X^{(q-1)} \cup A, A) = 0$ and then, by exactness of the homology sequence of a triple, follow that the kernel of j_* is zero.

The second part of (a) follows by the following commutative diagram:

$$\begin{tikzcd}
\mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(q)} \cup A, A) \arrow[r, "j_*"] & \mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(q)} \cup A, X^{(q-1)} \cup A) \arrow[r, "\Delta"] \arrow[dr, "\partial_q"'] & \mathcal{H}_{q-1}^{\varepsilon,\varepsilon'}(X^{(q-1)} \cup A, A) \arrow[d, "j_*"] \\
& & \mathcal{H}_*^{\varepsilon,\varepsilon'}(X)
\end{tikzcd}$$

Since $\mathrm{Im} j_* = \ker \Delta$ and the kernel of j_* is trivial,
$$\ker \Delta = \ker(j_* \circ \Delta) = \ker \partial_q = \mathcal{Z}_q^{\varepsilon,\varepsilon'}.$$

In order to prove the statement (b), note that we can decompose the homomorphism $l_* \colon \mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(q)} \cup A, A) \longrightarrow \mathcal{H}_q^{\varepsilon,\varepsilon'}(X, A)$ into

$$\mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(q)} \cup A, A) \xrightarrow{l_{1*}} \mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(q+1)} \cup A, A) \longrightarrow \cdots \xrightarrow{l_{r*}} \mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(q+r)} \cup A, A) = \mathcal{H}_q^{\varepsilon,\varepsilon'}(X, A)$$

where $q + r = \dim(X)$ and $l_1, \dots, l_r$ are inclusions. Using the properties of $i_* \colon \mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(p)} \cup A, A) \longrightarrow \mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(p+1)} \cup A, A)$ previously proved, we have that l_{1*} is onto and $l_{2*}, \dots, l_{r*}$ are isomorphisms. Therefore, since $l_* = l_{r*} \circ \cdots \circ l_{1*}$, j_* is onto.

To conclude, let us study the following diagram:

$$\begin{tikzcd}
\mathcal{H}_{q+1}^{\varepsilon,\varepsilon'}(X^{(q+1)} \cup A, X^{(q)} \cup A) \arrow[d, "\Delta"] \arrow[dr, "\partial_{q+1}"] & \\
\mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(q)} \cup A, A) \arrow[r, "j_*"] \arrow[d, "l_{1*}"] & \mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(q)} \cup A, X^{(q-1)} \cup A) \\
\mathcal{H}_q^{\varepsilon,\varepsilon'}(X^{(q+1)} \cup A, A) &
\end{tikzcd}$$

Note that the vertical line is part of a homology sequence of a triple. Therefore, from the exactness and $\ker l_* = \ker l_{1*}$, one concludes

$$j_*[\ker l_*] = j_*[\ker l_{1*}] = j_*[\mathrm{Im}\Delta] = \mathrm{Im}(j_* \circ \Delta) = \mathrm{Im}\partial_{q+1} = \mathcal{B}_q^{\varepsilon,\varepsilon'}.$$

Since $\ker j_*$ is zero, then the kernel of l_* is $j_*^{-1}[B_q^{\varepsilon,\varepsilon'}]$, which conclude the statement (b).

To define the isomorphism, consider the diagram

$$\mathrm{H}_q^{\varepsilon,\varepsilon}(X, A) \xleftarrow{l_*} \mathrm{H}_q^{\varepsilon,\varepsilon}(X^{(q)} \cup A, A) \xrightarrow{j_*} C_q^{\varepsilon,\varepsilon'}(X, A) \xleftarrow{\eta} \mathcal{Z}_q^{\varepsilon,\varepsilon'}(X, A) \xrightarrow{\tau} \mathcal{H}_q^{\varepsilon,\varepsilon}(X, A)$$

where j_* and l_* are given before, η is an inclusion and τ the natural map.

The isomorphism is given by $\Theta = \tau \circ \eta^{-1} \circ j_* \circ l_*^{-1}$. □

Theorem 2.86. $\mathcal{H}_q^{\varepsilon,\varepsilon'}(X, A) = 0$, *for* $q < 0$.

Proof. By Theorem 2.72, we have that $C_q^{\varepsilon,\varepsilon'}(X, A) = 0$ for $q < 0$. This implies that $\mathcal{Z}_q^{\varepsilon,\varepsilon'}(X, A) = 0$ and thus $\mathcal{H}_q^{\varepsilon,\varepsilon'}(X, A) = 0$. □

2.7 Uniqueness

Theorem 2.87. *Given H and $\overline{H}$ two persistent homology theories and a homomorphism $h_0^{\varepsilon,\varepsilon'} \colon \mathrm{H}_0^{\varepsilon,\varepsilon'}(P_0, F_\alpha) \longrightarrow \overline{H}_0^{\varepsilon,\varepsilon'}(P_0, F_\alpha)$ (where $\alpha \le \varepsilon \le \varepsilon'$) between their coefficient groups, there exists a unique set of homomorphisms*

$$\mathrm{H}^{\varepsilon,\varepsilon'}(q, X, A) \colon \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A) \longrightarrow \overline{H}_q^{\varepsilon,\varepsilon'}(X, A)$$

defined for each filtered pair $((X, A), (F_X, F_A))$, for any integer q and real numbers $\varepsilon \le \varepsilon'$, such that:

1.

$$h_0^{\varepsilon,\varepsilon'} = \mathrm{H}^{\varepsilon,\varepsilon'}(0, P_0, \emptyset) \tag{2.4}$$

2. If $f \colon (X, A) \longrightarrow (X_1, A_1)$ is a filtration preserving map, where

$((X, A), (F_X, F_A))$ and $((X_1, A_1), (F_{X_1}, F_{A_1}))$ are filtered pair, then the following diagram is commutative for all q and $\varepsilon \leq \varepsilon'$:

$$\begin{tikzcd} \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A) \arrow[r, "\mathrm{H}^{\varepsilon,\varepsilon'}"] \arrow[d, "f_*:=f_*^{\varepsilon,\varepsilon'}"] & \overline{H}_q^{\varepsilon,\varepsilon'}(X, A) \arrow[d, "\overline{f}_*:=\overline{f}_*^{\varepsilon,\varepsilon'}"] \\ \mathrm{H}_q^{\varepsilon,\varepsilon'}(X_1, A_1) \arrow[r, "\mathrm{H}^{\varepsilon,\varepsilon'}"] & \overline{H}_q^{\varepsilon,\varepsilon'}(X_1, A_1) \end{tikzcd} \tag{2.5}$$

that is $\overline{f}_* \circ \mathrm{H}^{\varepsilon,\varepsilon'}(q, X, A) = \mathrm{H}^{\varepsilon,\varepsilon'}(q, X_1, A_1) \circ f_*$;

3. *The commutativity relation $\overline{\partial} \circ \mathrm{H}^{\varepsilon,\varepsilon'}(q, X, A) = \mathrm{H}^{\varepsilon,\varepsilon'}(q-1, A) \circ \partial$ holds in the diagram:*

$$\begin{tikzcd} \mathrm{H}_q^{\varepsilon,\varepsilon'}(X, A) \arrow[r, "\mathrm{H}^{\varepsilon,\varepsilon'}"] \arrow[d, "\partial"] & \overline{H}_q^{\varepsilon,\varepsilon'}(X, A) \arrow[d, "\overline{\partial}"] \\ \mathrm{H}_{q-1}^{\varepsilon,\varepsilon'}(A) \arrow[r, "\mathrm{H}^{\varepsilon,\varepsilon'}"] & \overline{H}_{q-1}^{\varepsilon,\varepsilon'}(A) \end{tikzcd} \tag{2.6}$$

If $h_0^{\varepsilon,\varepsilon'} : \mathrm{H}_0^{\varepsilon,\varepsilon'}(P_0, F_\alpha) \longrightarrow \overline{H}_0^{\varepsilon,\varepsilon'}(P_0, F_\alpha)$ is an isomorphism, then each $\mathrm{H}^{\varepsilon,\varepsilon'}(q, X, A)$ is also an isomorphism

Proof. Let $g \in G^{\varepsilon,\varepsilon'}$ and let $gA^0 \cdots A^q$ be a q-chain in $\mathrm{H}_q^{\varepsilon,\varepsilon'}(X)$. Define

$$\eta_q^{\varepsilon,\varepsilon'}(gA^0 \cdots A^q) = h_0^{\varepsilon,\varepsilon'}(g)A^0 \cdots A^q.$$

For each q we claim that $\eta_q^{\varepsilon,\varepsilon'} : C_q^{\varepsilon,\varepsilon'}(X) \longrightarrow \overline{C}_q^{\varepsilon,\varepsilon'}(X)$ is a homomorphism.

Indeed, if $q < 0$ or $q > \#X$, follows from Theorem 2.72 that both $C_q^{\varepsilon,\varepsilon'}(X, A) = 0$ or $\overline{C}_q^{\varepsilon,\varepsilon'}(X, A) = 0$ are zero, and then $\eta_q^{\varepsilon,\varepsilon'}$ is the null homomorphism.

If $0 \leq q \leq \#X$, $\eta_q^{\varepsilon,\varepsilon'}$ is defined to be a map that commutes the following diagram:

$$\begin{tikzcd} G^{\varepsilon,\varepsilon'} \arrow[r, "h_0^{\varepsilon,\varepsilon'}"] \arrow[dr] & \overline{G}^{\varepsilon,\varepsilon'} \arrow[r] & \overline{C}_q^{\varepsilon,\varepsilon'}(X) \\ & C_q^{\varepsilon,\varepsilon'}(X) \arrow[ur] & \end{tikzcd}$$

From Theorem 2.75, each q-chain c of (X, A) can be written uniquely in the form $c = \sum_{i=1}^{\alpha_q} g_i A_i^0 \cdots A_i^q, g_i \in G^{\varepsilon,\varepsilon'}$ and, from Theorem 2.74, the maps $G^{\varepsilon,\varepsilon'} \longrightarrow C_q^{\varepsilon,\varepsilon'}(X, A)$ and $\overline{G}^{\varepsilon,\varepsilon'} \longrightarrow \overline{C}_q^{\varepsilon,\varepsilon'}(X, A)$ are homomorphisms. Then, $\eta_q^{\varepsilon,\varepsilon'} : C_q^{\varepsilon,\varepsilon'}(X) \longrightarrow \overline{C}_q^{\varepsilon,\varepsilon'}(X)$ is a homomorphism.

From Theorems 2.76 and 2.79 it follows that $\eta_q^{\varepsilon,\varepsilon'}$ commutes with $f_q^{\varepsilon,\varepsilon'}$ and $\partial_q^{\varepsilon,\varepsilon'}$.

Consequently, $\eta_q^{\varepsilon,\varepsilon'}$ defines homomorphisms

$$Z_q^{\varepsilon,\varepsilon'}(X, A) \longrightarrow \overline{Z}_q^{\varepsilon,\varepsilon'}(X, A) \text{ and } B_q^{\varepsilon,\varepsilon'}(X, A) \longrightarrow \overline{B}_q^{\varepsilon,\varepsilon'}(X, A)$$

and thus induces homomorphisms of their quotient groups

$$\mathrm{H}^{\varepsilon,\varepsilon'}(q,X,A):\mathcal{H}_q^{\varepsilon,\varepsilon'}(X,A)\longrightarrow\overline{\mathcal{H}}_q^{\varepsilon,\varepsilon'}(X,A)$$

that satisfies (1)-(3).

We will prove now that the homomorphisms $\mathrm{H}^{\varepsilon,\varepsilon'}(q,X,A)$ that satisfy (1)-(3) are unique.

We claim that, if (s^q,F_0) is an ordered q-simplex and $g\in G^{\varepsilon,\varepsilon'}$, then

$$\mathrm{H}^{\varepsilon,\varepsilon'}(q,s^q,\dot{s}^q)(gs^q)=h_0^{\varepsilon,\varepsilon'}(g)s^q. \tag{2.7}$$

Indeed, let us prove by induction in q. Suppose $q=0$, since s^0 is a 0-simplex, it has empty boundary and, by (2.4), $h_0^{\varepsilon,\varepsilon'}=\mathrm{H}^{\varepsilon,\varepsilon}(0,P_0,\emptyset)$ and we have that

$$\mathrm{H}^{\varepsilon,\varepsilon'}(0,s^0,\dot{s}^0)(gs^0)=\mathrm{H}^{\varepsilon,\varepsilon'}(0,P_0,\emptyset)(gs^0)=h_0^{\varepsilon,\varepsilon'}(g)s^0.$$

Now suppose $q>0$ and that (2.7) holds for $q-1$. Let $A^0<\cdots<A^q$ be the set of vertices of s^q and let s^{q-1} be the face whose vertices are $A^1<\cdots<A^q$. By, Corollary 2.61, we have that $[s^q:s^{q-1}]gs^q=gs^{q-1}$. In Definition 2.60, we define $[s^q:s^{q-1}]$ to be the incidence isomorphism Δ of the Theorem 2.58, then explicitly, it may be defined from the diagram

$$\mathrm{H}_p^{\varepsilon,\varepsilon'}(s^q,\dot{s}^q)\xrightarrow{\partial}\mathrm{H}_{p-1}^{\varepsilon,\varepsilon'}(\dot{s}^q)\xrightarrow{i_*}\mathrm{H}_{p-1}^{\varepsilon,\varepsilon'}(\dot{s}^q,c^{q-1})\xleftarrow{j_*}\mathrm{H}_{p-1}^{\varepsilon,\varepsilon'}(s^{q-1},\dot{s}^{q-1}),$$

where $i\colon(\dot{s}^q)\longrightarrow(\dot{s}^q,c^{q-1})$ and $j\colon(s^{q-1},\dot{s}^{q-1})\longrightarrow(\dot{s}^q,c^{q-1})$ are inclusions, by $[s^q:s^{q-1}]=(j_*^{\varepsilon,\varepsilon'})^{-1}\circ i_*^{\varepsilon,\varepsilon'}\circ\partial$.

Then, by (2.5) and (2.6), we have that

$$\mathrm{H}^{\varepsilon,\varepsilon'}(q-1,s^{q-1},\dot{s}^{q-1})[s^q:s^{q-1}]=\overline{[s^q:s^{q-1}]}\mathrm{H}^{\varepsilon,\varepsilon'}(q,s^q,\dot{s}^q).$$

Therefore,

$$\begin{aligned}\overline{[s^q:s^{q-1}]}\mathrm{H}^{\varepsilon,\varepsilon'}(q,s^q,\dot{s}^q)=\mathrm{H}^{\varepsilon,\varepsilon'}(q-1,s^{q-1},\dot{s}^{q-1})[s^q:s^{q-1}]=&\\ h_0^{\varepsilon,\varepsilon'}(g)s^{q-1}=\overline{[s^q:s^{q-1}]}h_0^{\varepsilon,\varepsilon'}(g)s^q&\end{aligned}$$

and, since $\overline{[s^q:s^{q-1}]}$ is an isomorphism by definition, the result follows.

In addition, let $gA^0\cdots A^q\in C_q^{\varepsilon,\varepsilon'}(X,A)$. We claim that

$$\mathrm{H}^{\varepsilon,\varepsilon'}(q,X,X)(gA^0\cdots A^q)=h_0^{\varepsilon,\varepsilon'}(g)A^0\cdots A^q. \tag{2.8}$$

By Definition 2.73, we have that $gA^0\cdots A^q=f_*^{\varepsilon,\varepsilon'}(gs^q)$, where s^q is an ordered q-simplex and $f\colon(s^q,\dot{s}^q)\longrightarrow(X,X)$ is the filtered map of the definition. Thus, by (2.5) and (2.7), we have

$$\begin{aligned}\mathrm{H}^{\varepsilon,\varepsilon'}(q,X,X)(gA^0\cdots A^q)&=\mathrm{H}^{\varepsilon,\varepsilon'}(q,X,X)(f_*^{\varepsilon,\varepsilon'}(gs^q))\\&\overset{(2.5)}{=}\overline{f}_*^{\varepsilon,\varepsilon'}(\mathrm{H}^{\varepsilon,\varepsilon'}(q,X,X))(gs^q)\\&\overset{(2.7)}{=}h_0^{\varepsilon,\varepsilon'}(g)A^0\cdots A^q.\end{aligned}$$

Now let $H^{\varepsilon,\varepsilon'}$ and $h'^{\varepsilon,\varepsilon'}$ two families of homomorphisms satisfying (2.4)-(2.6).

Let $((X,A),(F_X,F_A))$ be a filtered pair. We will use abbreviations, one associated to a filtered pair $((X,X),(F^{(q)},F^{(q-1)}),$

$$h_1^{\varepsilon,\varepsilon'}=\mathrm{H}^{\varepsilon,\varepsilon'}(q,X,X)$$

and another associated with the filtered pair $((X,A),(F^{(q)},F_A))$,

$$h_2^{\varepsilon,\varepsilon'} = \mathrm{H}^{\varepsilon,\varepsilon'}(q,X,A).$$

Similarly we define $h_1'^{\varepsilon,\varepsilon'}$ and $h_2'^{\varepsilon,\varepsilon'}$.

By (2.8), we have that $\mathrm{H}_1^{\varepsilon,\varepsilon'} = h_1'^{\varepsilon,\varepsilon'}$.

Let $l\colon ((X,A),(F^{(q)},F_A)) \longrightarrow ((X,X),(F^{(q)},F^{(q-1)})$ be the inclusion map. Then we have, by (2),

$$\overline{l}_*^{\varepsilon,\varepsilon'} \circ h_2^{\varepsilon,\varepsilon'} = h_1^{\varepsilon,\varepsilon'} \circ l_*^{\varepsilon,\varepsilon'} = h_1'^{\varepsilon,\varepsilon'} \circ l_*^{\varepsilon,\varepsilon'} = \overline{l}_*^{\varepsilon,\varepsilon'} \circ h_2'^{\varepsilon,\varepsilon'}$$

Since the kernel of $\overline{l}_*$ is zero, it follows that $h_2^{\varepsilon,\varepsilon'} = h_2'^{\varepsilon,\varepsilon'}$. Now considering the inclusion map $j\colon ((X,A),(F^{(q)},F_A)) \longrightarrow ((X,A),(F_X,F_A))$, we have

$$\mathrm{H}^{\varepsilon,\varepsilon'}(q,X,A) \circ j_* = \overline{j}_* \circ h_2 = \overline{j}_* \circ h_2' = h'(q,X,A) \circ j_*.$$

Since j_* is onto, it follows that $h^{\varepsilon,\varepsilon'}(q,X,A) = h'^{\varepsilon,\varepsilon'}(q,X,A)$. Thus, $h^{\varepsilon,\varepsilon'} = h'^{\varepsilon,\varepsilon'}$ and uniqueness has been proved.

Now let $h_0^{\varepsilon,\varepsilon'} : G^{\varepsilon,\varepsilon'} \approx \overline{G}^{\varepsilon,\varepsilon'}$ be an isomorphism and let $\overline{h}_0^{\varepsilon,\varepsilon'} : \overline{G}^{\varepsilon,\varepsilon'} \approx G^{\varepsilon,\varepsilon'}$ be the inverse isomorphism. Let $\overline{h}(q,X,A)\colon \overline{\mathcal{H}}_q^{\varepsilon,\varepsilon'}(X,A) \longrightarrow \mathcal{H}_q^{\varepsilon,\varepsilon'}(X,A)$ be the homomorphism satisfying (1)-(3) from Theorem 2.87 relative to $\overline{h}_0$. Then

$$\overline{h}^{\varepsilon,\varepsilon'}(q,X,A) \circ \mathrm{H}^{\varepsilon,\varepsilon'}(q,X,A) : \mathcal{H}_q^{\varepsilon,\varepsilon'}(X,A) \longrightarrow \mathcal{H}_q^{\varepsilon,\varepsilon'}(X,A)$$

are homomorphisms satisfying (1)-(3) relative to the identity map $G^{\varepsilon,\varepsilon'} \longrightarrow G^{\varepsilon,\varepsilon'}$.

Therefore, by the uniqueness property, we have $\overline{h}^{\varepsilon,\varepsilon'}(q,X,A) \circ \mathrm{H}^{\varepsilon,\varepsilon'}(q,X,A)$ is the identity.

Similarly, $\mathrm{H}^{\varepsilon,\varepsilon'}(q,X,A) \circ \overline{h}^{\varepsilon,\varepsilon'}(q,X,A)$ is also the identity. Thus $\mathrm{H}^{\varepsilon,\varepsilon}(q,X,A)$ is a isomorphism with $\overline{h}^{\varepsilon,\varepsilon}(q,X,A)$ as its inverse. The results follow using the isomorphisms of Theorem 2.85. $\square$

3 Persistent Cohomology Invariants

3.1 Persistent Cohomology

Let X be a finite set. We already know different ways to construct a filtration of X, that we called $\{K^\varepsilon\}$, for example using Vietoris-Rips or filtered maps. So, after this construction we also can construct a chain complex

$$C\colon \cdots C_{n+1}(K^\varepsilon) \xrightarrow{\partial^\varepsilon_{n+1}} C_n(K^\varepsilon) \xrightarrow{\partial^\varepsilon_n} C_{n-1}(K^\varepsilon) \xrightarrow{\partial^\varepsilon_{n-1}} \cdots$$

In this chapter, we are interested in to define the persistent cohomology of X. In order to do it, we need to construct the cochain complex.

Definition 3.1. $C^q(K^\varepsilon) = Hom(C_q(K^\varepsilon), \mathbb{F})$, where $\mathbb{F}$ is a field. Thus, a q-cochaim is an $\mathbb{F}$-homomorphism $\alpha\colon C_q(K^\varepsilon) \longrightarrow \mathbb{F}$.

Remark 3.2. The cochains are totally defined by the values in each q-simplex, in other words, is isomorphic to a direct product of copies of $\mathbb{F}$.

Now fix $n > 0$. Let us define the homomorphism between $C^n(K^\varepsilon)$ and $C^{n+1}(K^\varepsilon)$, that we are going to call δ^n_ε.

Definition 3.3. δ^n_ε is the unique homomorphism that satisfies $\alpha(\partial^\varepsilon_{n+1}(z)) = \delta^n_\varepsilon(\alpha)(z)$, for all $(q+1)$-chains z and q-cochains α.

Explaining, let $\alpha \in C^n(K^\varepsilon)$.

$\delta^n_\varepsilon(\alpha) = \beta$ such that

$$\beta(\{x_0, \ldots, x_n\}) = \sum_{i=0}^{n+1} (-1)^i \alpha(x_0, \ldots \widehat{x_i}, \ldots, x_n),$$

where $\{x_0, \ldots, x_n\}$ represents any $n+1$ simplex in $C_{n+1}(K^\varepsilon)$ and $\widehat{x_i}$ means that we omit this vertex of our simplex.

In order to define the cohomology, let us define what is a cocycle and a coboundary.

Definition 3.4. If $\alpha \in C^n(K^\varepsilon)$ and $\delta^n_\varepsilon(\alpha) = 0$, then we say that α is a n-cocycle.

If $\exists\beta$ such that $\beta \in C^{n-1}(K^\varepsilon)$ and $\delta^{n-1}_\varepsilon(\beta) = \alpha$, then we call α a n-coboundary.

Proposition 3.5. $\delta^n_\varepsilon \circ \delta^{n-1}_\varepsilon = 0$, *for all* $n \geq 1$.

Proof. Let $\{x_0, \ldots, x_{n-1}\} \in C_{n-1}(K^\varepsilon)$ an arbitrary $(n-1)$-simplex and $f \in C^{n-1}(K^\varepsilon)$.

$$\begin{aligned}\delta^n_\varepsilon \circ \delta^{n-1}_\varepsilon(f)(\{x_0, \ldots, x_{n-1}\}) &= \delta^n_\varepsilon(\delta^{n-1}_\varepsilon(f)(\{x_0, \ldots, x_{n-1}\})) \\ &= \delta^n_\varepsilon(\sum_{i=0}^{n-1}(-1)^i \alpha(\{x_0, \ldots \widehat{x_i}, \ldots, x_n\})) \\ &= \sum_{j<i}(-1)^i(-1)^j \alpha(\{x_0, \ldots, \widehat{x_j}, \ldots, \widehat{x_i}, \ldots, x_n\}) \\ &\quad + \sum_{i<j}(-1)^i(-1)^{j-1} \alpha(\{x_0, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_n\}).\end{aligned}$$

Note that each $\alpha(\{x_0, \ldots, \widehat{x_j}, \ldots, \widehat{x_i}, \ldots, x_n\})$ appears two times, but with opposite sign, which gives us the result. □

So, what we have now is that, for every $\varepsilon \in \mathbb{R}$ we can construct the cochaim complex

$$C\colon \cdots \longrightarrow C^{n-1}(K^\varepsilon) \xrightarrow{\delta^{n-1}_\varepsilon} C^n(K^\varepsilon) \xrightarrow{\delta^n_\varepsilon} C^{n+1}(K^\varepsilon) \xrightarrow{\delta^{n+1}_\varepsilon} \cdots$$

Note that C^n is a contravariant functor, since it is the composite of C_n, that is variant, with $Hom(\star, \mathbb{F})$, that is contravariant. Then, if $f\colon X \longrightarrow Y$, we can define $C^q(f)\colon C^q(Y) \longrightarrow C^q(X)$ by the formula

$$C^q(f)(\alpha)(z) = \alpha(C_q(f)z).$$

This map will be important, sincefor every pair $(\varepsilon, \varepsilon')$ we have an inclusion $\nu^{\varepsilon,\varepsilon'} : K^\varepsilon \hookrightarrow K^{\varepsilon'}$ and, using the last statement, we have the induced

$$\nu^q_{\varepsilon,\varepsilon'} := C^q(\nu^{\varepsilon,\varepsilon'}) : C^q(K^{\varepsilon'}) \longrightarrow C^q(K^\varepsilon).$$

Now, we can define the persistent cohomology associated to $\varepsilon, \varepsilon'$, that is,

$$H^n_{\varepsilon,\varepsilon'}(X) = \frac{\nu^n_{\varepsilon,\varepsilon'}(\mathrm{Ker}(\delta^n_{\varepsilon'}))}{\mathrm{Im}(\delta^{n-1}_\varepsilon) \cap \nu^n_{\varepsilon,\varepsilon'}(\mathrm{Ker}(\delta^n_{\varepsilon'}))}.$$

3.1.1 Barcodes

Let $\mathbb{X}$ be a filtration of a topological space, that is,

$$\mathbb{X}\colon X^0 \subset X^1 \subset \cdots \subset X^n.$$

We can associate to each integer i a real number a_i, satisfying $a_0 < a_1 < \cdots < a_n$.

Applying the cohomology functor over a field $\mathbb{F}$ to this filtration we obtain a **persistent module**, a diagram of finite dimensional vector spaces and linear maps.

$$H(\mathbb{X}) : H^*(X^0) \longleftarrow H^*(X^1) \longleftarrow \cdots \longleftarrow H^*(X^n).$$

A good point here is that we can well define another operation in these spaces to give to this structures as graded rings and obtain another objects and ideas, as can be noted in this chapter. This kind of persistent module also decomposes as direct sum of interval modules [ZC05].

If we have an interval $[a_i, a_j]$ in the decomposition, it means that we have a feature, or a class of a cocycle, that survived during this time in the filtration.

The **barcode** is the multiset of pairs $[i, j]$ in this decomposition, as discussed in the previous chapter.

$$\mathcal{B}(H^*(\mathbb{X})) = \mathrm{Barc}(H^*(\mathbb{X})) = \{[i_0, j_0], \ldots, [i_m, j_m]\}$$

3.1.2 Algorithm for persistent cohomology

Let F_K denote a filtration of a simplicial complex K and $\mathbb{F}$ a field. Let us assume that $F_K$is such that $K_i = K_{i-1} \cup \{\sigma\}$.

Each simplex σ_i will be seen as a column matrix with nonzero entry just at $a_{i1} = 1$ and each cochain $\alpha = \sum a_i \sigma_1^*$ will be seen as a row matrix with entries a_i.

$$\sigma_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\sigma_1^* + \sigma_3^* = \begin{bmatrix} 1 & 0 & 1 & \cdots & 0 \end{bmatrix}$$

The boundary operator is a $n \times n$ matrix where each column i is the $\partial(\sigma_i)$.

The following algorithm is a modified version of the one found in [DSMVJ11b]. The input is the boundary operator matrix.

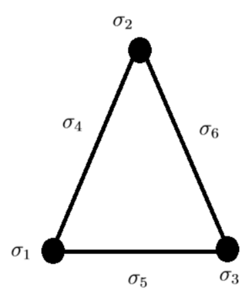
Algorithm 3: Compute Persistent Cohomology.

Input : The boundary operator matrix.
Output : The intervals and representatives cochains for each time ε in our filtration.

$H_0 \leftarrow \emptyset$
for $\sigma_i \in F_K$ **do**
  $S_i \leftarrow \{\gamma \in H_{i-1} \mid v_\alpha := (\delta\gamma)(\sigma_i) = \gamma(\partial\sigma_i) \neq 0\}$
  Let α be the element of greatest index in S_i
  if *α exists ($S_i \neq \emptyset$)* **then**
    Remove α from H_{i-1}
    For all $\beta \in H_{i-1}$ s.t. $\beta \in S_i$ replace $\beta = \beta - \frac{v_\beta}{v_\alpha}\alpha$
    $H_i \leftarrow H_{i-1}$
    Print out $[i_\alpha, i)$
  else
    Add σ_i^* to H_{i-1}
    $H_i \leftarrow H_{i-1}$

This algorithm produces two important things for us, the collection of bars, barcode, and the representatives cocycles for each of these bars.

Example 3.6. Let $\mathbb{F} = \mathbb{Z}_2$. The following space is simple, but can illustrate for us the whole power of the algorithm.



Consider that, for each step i, the simplex σ_i is added at the filtration. Applying the algorithm, we obtain:

1. a bar $[1, \infty)$, with the representative cocycle $\sigma_1^* + \sigma_2^* + \sigma_3^*$
2. $[2, 4)$ with the representative cocycle σ_2^*
3. $[2, 5)$ with the representative cocycle σ_3^*
4. $[6, \infty)$ with the representative cocycle σ_6^*

Example 3.7. Let $\mathbb{F} = \mathbb{Z}_2$. Let us use this algorithm to compute the persistent cup length associated to the Torus. For this we will take the following triangulation.

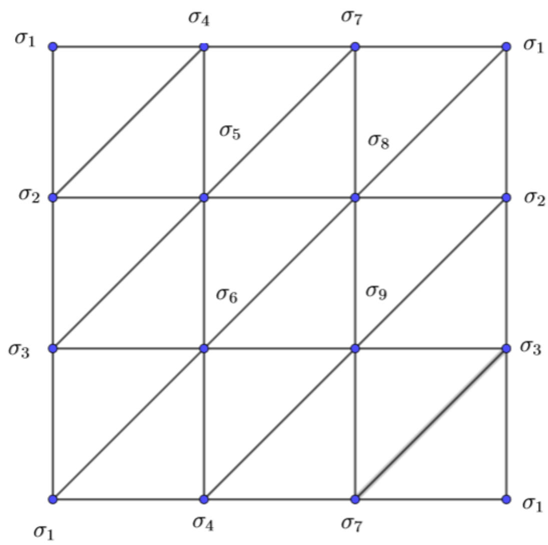


Figure 3.1: Triangulation of a Torus

We will start in the first step with the 0-simplex σ_1. The following step will add the σ_2 and so on, until the step 9. After this the filtration will add the 1-simplex that connects σ_1 to σ_2, that will be the σ_{10}. After this the 1-simplex that connect σ_1 to σ_3 and so on. After all the 1-simplices we will add the 2-simplices. Then, our filtration will have 54 steps.

We will have that our boundary matrix is a 54×54 matrix and, applying the algorithm we will obtain the following barcode:

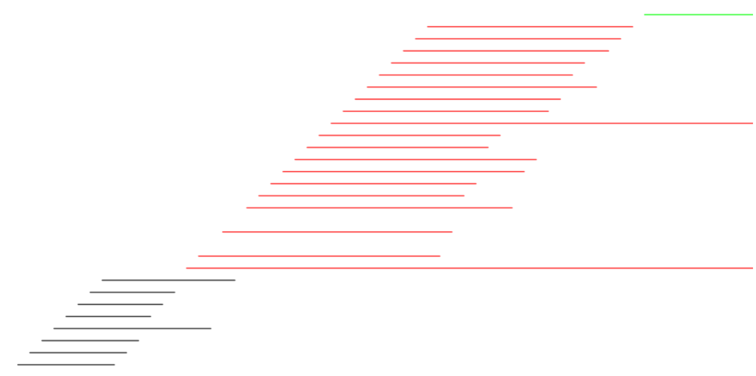


Figure 3.2: Barcode associated to our filtration of the torus.

In the barcode, the black bars correspond to 0-degree classes, the red bars to the 1-degree classes and the green one to the cohomological class of degree 2. We can see that the bars that has no end points are the four bars that we were expecting to survive, since the final space is a torus. If for our problem we need to have a representative cocycle for each bar, the algorithm gives it for us as well.

3.1.3 Comparison between the persistent homology and cohomology

Due to the Universal Coefficient Theorem [Hat00], we have that $H_n(X, \mathbb{F}) \cong H^n(X, \mathbb{F})$, with $\mathbb{F}$ a field. Then, our barcodes associated with persistent homology and persistent cohomology are the same. Since $H_n^{\varepsilon,\varepsilon'}(X)$ and $H^n_{\varepsilon,\varepsilon'}(X)$ counts the number of bars in the barcode that contains the interval $[\varepsilon, \varepsilon']$, we have that

$$H_n^{\varepsilon,\varepsilon'}(X) = H^n_{\varepsilon,\varepsilon'}(X)$$

Comparing Algorithm 2 and 1, we can note that Algorithm 1 has a problem when compared with Algorithm 2, while it cannot forget the dead cycles, because in some moment they may be necessary again for a operation, while the row algorithm is able to, once the cycles is dead it will be not used again, so it can be forgotten. So, in the second algorithm, we have less information to store in each step, which makes the performance really better comparing with the first one. Algorithm 3 also does that, it is able to, after find a bar, drop one of the cocycles that for sure will be not necessary to store anymore.

This is very clear, when in [DSMVJ11a], the authors compare the performance of the Algorithm 1 and Algorithm 3. The algorithm that computes the persistent cohomology was really faster, almost 40 times in the example given in the paper. It is important to note this behavior didn't happen just in this single example. The work to try to find a filtration when the Algorithm 1 is faster still alive.

If you have a choice and just the barcode is enough for what you need in your work, both algorithms will work, with Algorithm 3 being the faster. But, as in [DSMVJ11b], there are some problems where is necessary the explicit cocycle associated to the bars of the barcode. If this is of the necessities, the Algorithm 3 is the one that needs to be used. Besides that, we know that cohomology is a very richer structure, since we can give it a structure of ring, with a second operation. Due to this, we can study in this spaces evolution of products and decomposition during filtration. This is one of the motivations to the next section.

3.2 Persistent invariants

In Linear Algebra, an *invariant* is a number that remains invariant under a linear isomorphism of vector spaces. In a classical topology sense, it is a number associated to a given topological space that keeps invariant under a homeomorphism.

In order to study and extract information about the TDA structures, we can lift these notions of *invariants* to a 'persistence' setting in TDA and study the *persistent invariants*.

A good example of it is *rank invariant* for persistent vector spaces in [CZ09, Car09, KM18]. We will develop the *persistent cup-length invariant* of persistent graded rings.

For general definitions and results in category theory, we refer to [Awo10, Lei14, ML13].

Definition 3.8. Let C be any category. We call a **persistent object in** C any functor $\mathbf{F}\colon (\mathbb{R}, \leq) \to C$, in other words, a persistent object in C, $\mathbf{F}\colon (\mathbb{R}, \leq) \to C$, consists of

- for each $t \in \mathbb{R}$, an object $\mathbf{F}_t$ of C,

- for each $t \leq s$ in $\mathbb{R}$, an arrow or morphism $f_t^s : \mathbf{F}_t \to \mathbf{F}_s$, such that

  a) $f_t^t = \mathbf{id}_{\mathbf{F}_t}$

  b) $f_s^r \circ f_t^s = f_t^r$, for all $t \leq s \leq r$.

Definition 3.9. Let $\mathbf{F}, \mathbf{G} \colon (\mathbb{R}, \leq) \to C$ be two persistent objects in C. A **natural transformation from F to G**, $\varphi \colon \mathbf{F} \Rightarrow \mathbf{G}$, consists of an $\mathbb{R}$-indexed family $(\varphi_t : \mathbf{F}_t \to \mathbf{G}_t)_{t \in \mathbb{R}}$ of morphisms in C, such that the following diagram commutes for all $t \leq s$:

$$\begin{tikzcd}
\mathbf{F}_t \arrow[r, "\varphi_t"] \arrow[d, "f_t^s"'] & \mathbf{G}_t \arrow[d, "g_t^s"] \\
\mathbf{F}_s \arrow[r, "\varphi_s"] & \mathbf{G}_s
\end{tikzcd}$$

Example 3.10. Applying the p-th (co)homology functor to a persistent topological space $\mathbf{X}$, for each $t \in \mathbb{R}$ we obtain the vector space $\mathbf{H}_p(\mathbf{X}_t)$ (or $\mathbf{H}^p(\mathbf{X}_t)$, resp.) and for each pair of parameters $t \leq s$ in $\mathbb{R}$, we have the linear map in (co)homology induced by the inclusion $\mathbf{X}_t \hookrightarrow \mathbf{X}_s$. This is another example of a persistent object, namely a *persistent vector space* $\mathbf{H}_p(\mathbf{X}) \colon (\mathbb{R}, \leq) \to \mathbf{Vec}$ (or $\mathbf{H}^p(\mathbf{X}) \colon (\mathbb{R}, \leq) \to \mathbf{Vec}^{op}$, resp.).

We recall the notions of invariant and define an epi-mono invariant in a general sense.

Definition 3.11. Let C be any category. An **invariant in** C is any map $J \colon \mathbf{Ob}(C) \to \mathbb{N}$ such that: if $X \cong Y$ in C, then $J(X) = J(Y)$.

Example 3.12. We recall the following examples of invariants:

- For the category of finite sets, **Set**, whose morphisms are functions between finite sets, we consider $J \colon \mathbf{Ob}(\mathbf{Set}) \to \mathbb{N}$ to be the **cardinality invariant**.

- For the category of smooth manifolds, **Man**, whose morphisms are continuously differentiable maps, we consider $J \colon \mathbf{Ob}(\mathbf{Man}) \to \mathbb{N}$ to be the **genus invariant**.

- For the category of finite dimensional vector spaces over $\mathbb{K}$, **Vec**, whose morphisms are linear maps, we consider $J \colon \mathbf{Ob}(\mathbf{Vec}) \to \mathbb{N}$ to be the **dimension invariant**.

Definition 3.13. Let C be any category and let $f \colon A \to B$ be a morphism of C. We say that a monomorphism $g \colon I \to B$ is the **image of** f, and we denote I by $\mathbf{Im} f$ or $f(A)$, if there exists a morphism $h \colon A \to I$ such that $f = g \circ h$. We want these morphisms to satisfy the universal property that: for a monomorphism $g' \colon I' \to B$ and a morphism $h' \colon A \to I'$ such that $f = g' \circ h'$, there is a unique morphism $k \colon I \to I'$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "h"] \arrow[ddr, "h'"'] & & B \\
& I \arrow[ur, "g"] \arrow[d, "k"] & \\
& I' \arrow[uur, "g'"'] &
\end{tikzcd}$$

Such a factorization, if it exists, is unique up to isomorphism. Moreover, this unique object, I, will be denoted by $\mathbf{Im}(f)$, the unique morphism h is necessarily an epimorphism which will be denoted by q_f, and the unique morphism g is necessarily a monomorphism which will be denoted by j_f. We will denote the unique epi-mono factorization of f as

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "q_f"'] & & B \\ & \mathbf{Im}(f) \arrow[ur, hook, "j_f"'] & \end{tikzcd}$$

If every morphism in C has an image, then we say that C is a *category with images*.

Definition 3.14. Let C be a category with images. We will call a map

$$J\colon \mathbf{Ob}(C) \to \mathbb{N}$$

to be an **epi-mono invariant in** C if it sends monomorphisms to inequalities and epimorphisms to reverse inequalities, i.e.

$$X \hookrightarrow Y \mapsto J(X) \leq J(Y)$$
$$X \twoheadrightarrow Y \mapsto J(X) \geq J(Y).$$

Example 3.15. Let **Vec** be the category of finite dimensional vector spaces over the field $\mathbb{K}$ with $\mathbb{K}$-linear maps. The invariant $\mathbf{dim}\colon \mathbf{Ob}(\mathbf{Vec}) \to \mathbb{N}$, that assigns to a vector space its dimension, is an example of an epi-mono invariant.

Remark 3.16. We have that any epi-mono invariant in C is, in particular, an invariant in C, but the converse is not true in general. A good and simple counter example is the invariant that counts the number of connected components of a topological space.

Remark 3.17. Any epi-mono invariant $J\colon \mathbf{Ob}(C) \to \mathbb{N}$ of C is also an epi-mono invariant in the opposite category C^{op} of C, the category whose objects are the same, but whose arrows are the arrows of C with the reverse direction.

This becomes clear when we realize that the epimorphisms $X \twoheadrightarrow Y$ in C^{op} are exactly the monomorphisms $Y \hookrightarrow X$ in C and the monomorphisms $X \hookrightarrow Y$ in C^{op} are exactly the epimorphisms $Y \twoheadrightarrow X$ in C and then we have that

$$(X \twoheadrightarrow Y \text{ in } C^{op}) \Rightarrow (Y \hookrightarrow X \text{ in } C) \Rightarrow (J(X) \geq J(Y))\,.$$

$$(X \hookrightarrow Y \text{ in } C^{op}) \Rightarrow (Y \twoheadrightarrow X \text{ in } C) \Rightarrow (J(X) \leq J(Y))\,,$$

We now develop a technique to turn an epi-mono invariant into a persistent invariant.

Definition 3.18. Let $J\colon \mathbf{Ob}(C) \to \mathbb{N}$ be an epi-mono invariant of a category C with images. For any given persistent object $\mathbf{F}\colon (\mathbb{R}, \leq) \to C$, we associate the map

$$J(\mathbf{F}) : \mathbf{Int} \to \mathbb{N}$$
$$[a, b] \mapsto J(\mathbf{Im}(f_a^b)),$$

and we call $J(\mathbf{F})$ the persistent invariant associated to $\mathbf{F}$. From now on, we will not have problems to denote $J(f(X))$ and $J(\mathbf{Im}(f))$ simply by $J(f)$, for any morphism $f\colon X \to Y$ in C.

Lemma 3.19. *Let C be a category with images and let J be an epi-mono invariant. Then we have*

$$J(g \circ f) \leq \min\{J(f), J(g)\},$$

for any two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$.

Proof. We have a canonical monomorphism $g(f(R)) \hookrightarrow g(S)$. Since J is an epi-mono invariant, this monomorphism yields a inequality $J(g(f(R))) \leq J(g(S))$, or simply $J(g \circ f) \leq J(g)$.

On other hand, we have a canonical epimorphism $f(R) \twoheadrightarrow g(f(R))$ that yields the inequality $J(f(R)) \geq J(g(f(R)))$, or simply $J(g \circ f) \leq J(f)$. □

The previous Lemma is simple but extremely important for us, since it is a key tool to the proof of the following proposition that states the functoriality of persistent invariants.

Proposition 3.20 (Functoriality of persistent invariants)**.** *Let C be a category with images, let $J\colon \mathbf{Ob}(C) \to \mathbb{N}$ be an epi-mono invariant and let $\mathbf{F}\colon (\mathbb{R}, \leq) \to C$ be a persistent object. The associated persistent invariant $J(\mathbf{F})$ forms a functor $J(\mathbf{F})\colon (\mathbf{Int}, \subset) \to (\mathbb{N}, \geq)$, i.e.*

$$[a, b] \subseteq [c, d] \Rightarrow J(f_a^b) \geq J(f_c^d).$$

Proof. Suppose $[a, b] \subseteq [c, d]$. Since $\mathbf{F}\colon (\mathbb{R}, \leq) \to C$ is a persistent object, we can decompose f_c^d as $f_b^d \circ f_a^b \circ f_c^a$. Then,

$$\begin{aligned} J(f_c^d) &= J(f_b^d \circ f_a^b \circ f_c^a) \\ &\overset{3.19}{\leq} \min\{J(f_b^d), J(f_a^b \circ f_c^a)\} \\ &\leq J(f_a^b \circ f_c^a) \\ &\overset{3.19}{\leq} \min\{J(f_a^b), J(f_c^a)\} \\ &\leq J(f_a^b). \end{aligned}$$

□

3.2.1 Persistent cup-length

In persistence theory, given a collection of spaces $\mathbf{X} = \{\mathbf{X}_t\}_{t \in \mathbb{R}}$ such that $\mathbf{X}_t \subset \mathbf{X}_s$ for all $t \leq s$, called a *filtration* of space, we study the *p-th persistent homology* vector space $\mathbf{H}_p(\mathbf{X})\colon (\mathbb{R}, \leq) \to \mathbf{vec}$, $t \mapsto \mathbf{H}_p(\mathbf{X}_t)$ ([EH08]), which encrypt the interval lifespans of the *p-dimensional holes* (*p-cycles* that are not *p-boundaries*) in $\mathbf{X}$. We already know that the collection of these intervals is called *the p-th barcode of* $\mathbf{X}$, and its elements are called *bars*, we will denote it by $\mathcal{B}^p(\mathbf{X})$. Even being a powerful tool, there are cases where $\mathcal{B}^p(\mathbf{X})$ cannot distinguish two different pair of filtrations as we will see in Example 3.32.

If we consider the dual notion of persistent cohomology $\mathbf{H}^p(\mathbf{X})$ and attach to it the *cup-product* operation on cocycles, we have an enriched structure of a persistent graded ring $\mathbf{H}^*(\mathbf{X})$.

3.2.2 Graded ring structure and cup-length

For a space $\mathbb{X}$, the cup-product yields a linear map $\smile\colon \mathbf{H}^p(\mathbb{X}) \otimes \mathbf{H}^q(\mathbb{X}) \to \mathbf{H}^{p+q}(\mathbb{X})$ of vector spaces and defining the total cohomology vector space as $\mathbf{H}^*(\mathbb{X}) := \bigoplus_{p \in \mathbb{N}} \mathbf{H}^p(\mathbb{X})$, the cup-product enriches the old vector space into a graded ring $(\mathbf{H}^*(\mathbb{X}), +, \smile)$.

The *cohomology ring map* $\mathbb{X} \mapsto \mathbf{H}^*(\mathbb{X})$ defines a functor from the category of spaces, **Top**, to the opposite category of graded rings, $\mathbf{Ring}^{op}$. (see [Hat00, §3]). Now we will recall some classical definitions and results of ring theory.

Definition 3.21. A **ring** $(R, +, \cdot)$ is a set R equipped with an operation $+$ (addition) and an operation $\cdot$ (multiplication), such that

(i) $(R, +)$ is an abelian group,

(ii) $(R, \cdot)$ is a monoid,

(iii) multiplication is distributive with respect to addition, that is $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$.

Definition 3.22. A ring $(R, +, \cdot)$ is called **graded ring** if there exists a family of subgroups $\{R_p\}_{p \in \mathbb{N}}$ of R such that

1. $R = \bigoplus_{p \in \mathbb{N}} R_p$ (as abelian groups),
2. $R_a \cdot R_b \subseteq R_{a+b}$ for all $a, b \in \mathbb{N}$.

A ring homomorphism $\varphi \colon R \to S$, where R and S are two graded rings, is called to be a **graded homomorphism** if $\varphi(R_p) \subset S_p$, for all $p \in \mathbb{N}$.

Unfortunately it is really hard to compute graded ring structures in a machine, instead of doing it we will recall and study a computable invariant of graded rings, called *length* of a ring.

Definition 3.23. The **length** of a graded ring R is the largest non-negative integer ℓ such that there exist elements $\eta_1, \ldots, \eta_\ell \in R$ with nonzero degrees (which means that $\eta_1, \ldots, \eta_\ell \in \bigcup_{p \geq 1} R_p$), such that $\eta_1 \bullet \cdots \bullet \eta_\ell \neq 0$. We declare that the length of R is zero if $\bigcup_{p \geq 1} R_p = 0$.

We denote the length of a graded ring R by $\mathbf{len}(R)$. The map

$$\mathbf{len} \colon \mathbf{Ob}(\mathbf{Ring}) \to \mathbb{N}, \text{ with } R \mapsto \mathbf{len}(R)$$

is called the **length invariant**.

Considering the cohomology ring, when $R = (\mathbf{H}^*(\mathbb{X}), +, \smile)$ for some space $\mathbb{X}$, we denote $\mathbf{cup}(\mathbb{X}) := \mathbf{len}(\mathbf{H}^*(\mathbb{X}))$ and call it the **cup-length of** $\mathbb{X}$. The map

$$\mathbf{cup} \colon \mathbf{Ob}(\mathbf{Top}) \to \mathbb{N}, \text{ with } X \mapsto \mathbf{cup}(\mathbb{X})$$

is called the **cup-length invariant**.

Now, we state and prove some properties of the (cup-)length invariant that we will use further.

Proposition 3.24. *The length invariant* $\mathbf{len} \colon \mathbf{Ob}(\mathbf{Ring}) \to \mathbb{N}$ *is an epi-mono invariant in* **Ring**.

Proof. First, we need to prove that if $f \colon R \hookrightarrow S$ is a monomorphism, then $\mathbf{len}(R) \leq \mathbf{len}(S)$. Let $f \colon R \hookrightarrow S$ be a graded ring monomorphism and suppose $\mathbf{len}(R) = \ell$. Then, by definition, there exist $\eta_1, \ldots, \eta_\ell \in \bigcup_{p \geq 1} R_p$, such that $\eta := \eta_1 \bullet \cdots \bullet \eta_\ell \neq 0 \in R$. Then, $\mathbf{len}(S) \geq \ell$, since

$$\begin{aligned} f(\eta_1) \bullet \cdots \bullet f(\eta_\ell) &= f(\eta_1 \bullet \cdots \bullet \eta_\ell) \\ &= f(\eta) \neq 0 \quad \text{(since } \eta \neq 0 \text{ and } f \text{ is a monomorphism).} \end{aligned}$$

Second, if $f \colon R \twoheadrightarrow S$ is an epimorphism, then $\mathbf{len}(R) \geq \mathbf{len}(S)$. Indeed, let $f \colon R \twoheadrightarrow S$ be a graded ring epimorphism, and suppose $\mathbf{len}(S) = \ell$. Since f is surjective, there are $f(\alpha_i) \in \bigcup_{p \geq 1} S_p$, such that $f(\alpha_1) \bullet \cdots \bullet f(\alpha_\ell) \neq 0 \in S$. Since f is a ring homomorphism, then $f(\alpha_1 \bullet \cdots \bullet \alpha_\ell) \neq 0 \in S$. Then we have $\alpha_1 \bullet \cdots \bullet \alpha_\ell \neq 0 \in R$ and, by definition, $\mathbf{len}(R) \geq \ell$. $\square$

Proposition 3.25. *Let R be a graded ring and suppose $B = \bigcup_{p\geq 1} B_p$, where each B_p generates R_p as a monoid under addition. Then* $\mathbf{len}(R) = \sup\left\{\ell \geq 1 \mid B^\ell \neq \{0\}\right\}$.

For the cohomology ring case, let B_p be a linear basis for $\mathbf{H}^p(\mathbb{X})$ for each $p \geq 1$ and let $B := \bigcup_{p\geq 1} B_p$. Then $\mathbf{cup}(\mathbb{X}) = \sup\left\{\ell \geq 1 \mid B^\ell \neq \{0\}\right\}$, *where* $B^\ell = \underbrace{B \cdot B \cdots B}_{\ell \text{ times}}$.

Proof. By definition we have that $\mathbf{len}(R) = \sup\left\{\ell \geq 1 \mid (\bigcup_{p\geq 1} R_p)^\ell \neq \{0\}\right\}$. To prove the proposition we need to show that $\left(\bigcup_{p\geq 1} R_p\right)^\ell \neq \{0\}$ iff $B^\ell \neq \{0\}$.

Note that if $\eta_1 \bullet \cdots \bullet \eta_\ell \neq 0$, with each $\eta_i \in \bigcup_{p\geq 1} R_p$, every η_i can be written as a linear sum of elements in B and then η can be written as a linear sum of elements in the form of $r_1 \bullet \cdots \bullet r_\ell$, where each $r_j \in B$. Since $\eta \neq 0$, there exist at least one summand $r_1 \bullet \cdots \bullet r_\ell \neq 0$. Therefore, $B^\ell \neq \{0\}$. □

The following result states some properties about the length of graded rings.

Proposition 3.26. *Let $f: R \to R'$, $g: S \to S'$ be morphisms in* $\mathbf{Ring}^{op}$, *and $\mathbb{X}, \mathbb{Y}$ be two spaces. Then: and*

$$\begin{aligned} \mathbf{len}(R \otimes S) &= \mathbf{len}(R) + \mathbf{len}(S) & \mathbf{len}(R \times S) &= \max\{\mathbf{len}(R), \mathbf{len}(S)\} \\ \mathbf{len}(f \otimes g) &= \mathbf{len}(f) + \mathbf{len}(g) & \mathbf{len}(f \times g) &= \max\{\mathbf{len}(f), \mathbf{len}(g)\} \\ \mathbf{cup}(\mathbb{X} \times \mathbb{Y}) &= \mathbf{cup}(\mathbb{X}) + \mathbf{len}(\mathbb{Y}) & \mathbf{cup}(\mathbb{X} \amalg \mathbb{Y}) &= \max\{\mathbf{cup}(\mathbb{X}), \mathbf{cup}(\mathbb{Y})\}. \end{aligned}$$

Proof.

- $\mathbf{len}(R \otimes S) = \mathbf{len}(R) + \mathbf{len}(S)$.

  $\mathbf{len}(R \otimes S) \geq \mathbf{len}(R) + \mathbf{len}(S)$. Let $B_p := \bigcup_{j=0}^{p} R_j \otimes S_{p-j}$ for $p \geq 1$, and let $B := \bigcup_{p\geq 1} B_p$. We have that $\mathbf{len}(R \otimes S) = \sup\{\ell \geq 1 \mid B^\ell \neq 0\}$, by Proposition 3.25. Assuming that $B^\ell \neq 0$, there exists elements $r_1, \ldots, r_\ell \in R$ and $s_1, \ldots, s_\ell \in S$ such that $\deg(r_i) + \deg(s_i) \geq 1$ for each i, and $(r_1 \otimes s_1) \bullet \cdots \bullet (r_\ell \otimes s_\ell) \neq 0$. Then we have that $r_1 \bullet \cdots \bullet r_\ell \neq 0$ and $s_1 \bullet \cdots \bullet s_\ell \neq 0$. For every i we have $\deg(r_i) \geq 1$ or $\deg(s_i) \geq 1$, and thus there are at least ℓ elements in the set $\{r_i\}_{i=1}^\ell \cup \{s_i\}_{i=1}^\ell$ with positive degree. It follows that $\mathbf{len}(R) + \mathbf{len}(S) \geq \ell$. The inequality $\mathbf{len}(R \otimes S) \leq \mathbf{len}(R) + \mathbf{len}(S)$ is trivial.

- $\mathbf{len}(R \times S) = \max\{\mathbf{len}(R), \mathbf{len}(S)\}$.

  This equality follows directly since each $\eta \in R \times S$ has the form $\eta = (r, s)$ for $r \in R$ and $s \in S$. Given $\eta_i = (r_i, s_i)$ for $i = 1, \ldots, \ell$, $\eta_1 \bullet \cdots \bullet \eta_\ell \neq 0$ iff either $r_1 \bullet \cdots \bullet r_\ell \neq 0$ or $s_1 \bullet \cdots \bullet s_\ell \neq 0$.

- From ring theory, we have that $(f \otimes g)(R \otimes S) = f(R) \otimes g(S)$, $(f \times g)(R \times S) = f(R) \times g(S)$, $\mathbf{H}^*(\mathbb{X} \amalg \mathbb{Y}) \cong \mathbf{H}^*(\mathbb{X}) \times \mathbf{H}^*(\mathbb{Y})$ and $\mathbf{H}^*(\mathbb{X} \times \mathbb{Y}) \cong \mathbf{H}^*(\mathbb{X}) \otimes \mathbf{H}^*(\mathbb{Y})$ and all the other cases follow from these equalities. □

3.2.3 Cup-length functions

A functor $\mathbf{R}\colon (\mathbb{R}, \leq) \to \mathbf{Ring}^{op}$ is said to be a **persistent graded ring**. Let $\mathbf{H}^*\colon \mathbf{Top} \to \mathbf{Ring}^{op}$ be the cohomology ring functor. Given a persistent space $\mathbf{X}\colon (\mathbb{R}, \leq) \to \mathbf{Top}$, the composition $\mathbf{H}^*\mathbf{X}\colon (\mathbb{R}, \leq) \to \mathbf{Ring}^{op}$ is said to be the **persistent cohomology ring of X**.

Let $\mathbf{X} := \{\mathbf{X}_t\}_{t\in\mathbb{R}}$ be a filtration of topological spaces and for each $t \in \mathbb{R}$, let $[\cdot]_t$ denote the cohomological class of a cocycle in $\mathbf{X}_t$ and let $p \in \mathbb{N}$.

A $\mathcal{B}^p(\mathbf{X})$-indexed collection $\boldsymbol{\sigma}^p := \{\sigma_I\}_{I\in\mathcal{B}^p(\mathbf{X})}$ of p-cocycles in $\mathbf{X}$ is called **a family of representative p-cocycles for X**, if:

- For any $t \in \mathbb{R}$, $\{[\sigma_I]_t\}_{t\in I\in\mathcal{B}^p(\mathbf{X})}$ forms a basis for $\mathbf{H}^p(\mathbf{X}_t)$, and
- For any $t \leq s$, $\mathbf{H}^p(\iota_t^s) : \mathbf{H}^p(\mathbf{X}_s) \to \mathbf{H}^p(\mathbf{X}_t)$ is given by $[\sigma_I]_s \mapsto \left[\sigma_I|_{C_p(\mathbb{X}_t)}\right]_t$.

The disjoint union $\boldsymbol{\sigma}$ of $\boldsymbol{\sigma}^p$, over all $p \in \mathbb{N}$, is called **a family of representative cocycles for X**.

Remark 3.27. Let $\mathcal{B}(\mathbf{X}) := \sqcup_{p\in\mathbb{N}}\mathcal{B}^p(\mathbf{X})$ be the disjoint union of $\mathcal{B}^p(\mathbf{X})$, which we call *the total cohomology barcode of* **X** and let σ_I be a representative cocycle of an interval $I \in \mathcal{B}(\mathbf{X})$. Then, from the definition of representative cocycles, we have that for any $t \leq s$, $\mathbf{H}^*(\iota_t^s)([\sigma_I]_s) \neq 0 \iff [t, s] \subset I$.

We proved that **len** is an epi-mono invariant, Proposition 3.24, and then we are able to lift it into a persistent invariant. This lifting gives rise to what we will call **persistent length function of R**. We will prove two statements with some properties of this function: We use Proposition 3.29 to compute the cohomology images of a persistent cohomology ring and Proposition 3.30 allows us to distribute the computation of cup-length functions.

Definition 3.28. Given a persistent graded ring $\mathbf{R}\colon (\mathbb{R}, \leq) \to \mathbf{Ring}^{op}$ we define the **persistent length function of R** as

$$\mathbf{len}(\mathbf{R})\colon \mathbf{Int} \to (\mathbb{N}, \geq) \text{ with } [t, s] \mapsto \mathbf{len}(\mathrm{Im}(\mathbf{R}_s \to \mathbf{R}_t)).$$

If $\mathbf{R} = \mathbf{H}^*\mathbf{X}$ is the persistent cohomology ring of a given persistent space $\mathbf{X}\colon (\mathbb{R}, \leq) \to \mathbf{Top}$, then we will call

$$\mathbf{len}(\mathbf{H}^*\mathbf{X}) : \mathbf{Int} \to (\mathbb{N}, \geq) \text{ with } [t, s] \mapsto \mathbf{len}\left(\mathrm{Im}(\mathbf{H}^*(\mathbf{X}_s) \to \mathbf{H}^*(\mathbf{X}_t))\right),$$

the **cup-length function of X**, and we will denote it by $\mathbf{cup}(\mathbf{X})\colon \mathbf{Int} \to (\mathbb{N}, \geq)$.

Proposition 3.29. *Let* $\mathbf{X} = \{\mathbf{X}_t\}_{t\in\mathbb{R}}$ *be a filtration, together with a family of representative cocycles* $\boldsymbol{\sigma} = \{\sigma_I\}_{I\in\mathcal{B}(\mathbf{X})}$ *for* $\mathbf{H}^*\mathbf{X}$*. Let* $t \leq s$ *in* $\mathbb{R}$*. Then* $\mathrm{Im}(\mathbf{H}^*(\mathbf{X}_s) \to \mathbf{H}^*(\mathbf{X}_t)) = \langle [\sigma_I]_t : [t, s] \subset I \in \mathcal{B}(\mathbf{X})\rangle$*, generated as a graded ring.*

Proof. Given a space $\mathbb{X}$, considering the cup products, the cohomology ring $\mathbf{H}^*(\mathbb{X}) \in \mathbf{Ring}$ is a graded ring generated by the graded cohomology vector space $\mathbf{H}^*(\mathbb{X}) \in \mathbf{Vec}$. For this reason, a basis of $\mathbf{H}^*(\mathbb{X})$ generates the ring $\mathbf{H}^*(\mathbb{X})$, under the cup product.

Considering induced cohomology ring morphism $f := \iota^*\colon \mathbf{H}^*(\mathbb{Y}) \to \mathbf{H}^*(\mathbb{X})$ of an inclusion $\mathbb{X} \xhookrightarrow{\iota} \mathbb{Y}$ and let M be a basis for $\mathbf{H}^*(\mathbb{Y})$. We have that the image $f(M)$ generates $f(\mathbf{H}^*(\mathbb{Y}))$ as a ring.

Considering $\mathbf{H}^*(\iota_t^s)\colon \mathbf{H}^*(\mathbf{X}_s) \to \mathbf{H}^*(\mathbf{X}_t)$ as the cohomology map induced by the inclusion $\iota_t^s\colon \mathbf{X}_t \hookrightarrow \mathbf{X}_s$, we obtain that $\mathbf{H}^*(\mathbf{X}_s)$ has the set $M := \{[\sigma_I]_s : s \in I \in \mathcal{B}(\mathbf{X})\}$ as a linear basis. Therefore, $\mathbf{H}^*(\iota_t^s)(M)$ generates $\mathrm{Im}(\mathbf{H}^*(\iota_t^s))$ as a ring.

Furthermore, using Remark 3.27, we have that

$$\mathbf{H}^*(\iota_t^s)(M) = \left\{\mathbf{H}^*(\iota_t^s)\left([\sigma_I]_s\right) : [t,s] \subset I \in \mathcal{B}(\mathbf{X})\right\} = \{[\sigma_I]_t : [t,s] \subset I \in \mathcal{B}(\mathbf{X})\}. \qquad \square$$

Proposition 3.30. *Let* $\mathbf{X}, \mathbf{Y}\colon (\mathbb{R}, \leq) \to \mathbf{Top}$ *be two persistent spaces. Then:*

- $\mathbf{cup}\,(\mathbf{X} \times \mathbf{Y}) = \mathbf{cup}(\mathbf{X}) + \mathbf{cup}(\mathbf{Y})$,
- $\mathbf{cup}\,(\mathbf{X} \amalg \mathbf{Y}) = \max\{\mathbf{cup}(\mathbf{X}), \mathbf{cup}(\mathbf{Y})\}$, *and*
- $\mathbf{cup}\,(\mathbf{X} \vee \mathbf{Y}) = \max\{\mathbf{cup}(\mathbf{X}), \mathbf{cup}(\mathbf{Y})\}$.

Proof. By functoriality of products, disjoint unions, and wedge sums, we can define the persistent spaces: $\mathbf{X} \times \mathbf{Y} := (\{\mathbb{X}_t \times \mathbb{Y}_t\}_{t\in\mathbb{R}}, \{f_t^s \times g_t^s\})$, $\mathbf{X} \amalg \mathbf{Y} := (\{\mathbb{X}_t \amalg \mathbb{Y}_t\}_{t\in\mathbb{R}}, \{f_t^s \amalg g_t^s\})$, and $\mathbf{X} \vee \mathbf{Y} := (\{\mathbb{X}_t \vee \mathbb{Y}_t\}_{t\in\mathbb{R}}, \{f_t^s \vee g_t^s\})$. Let $[a, b]$ be any interval in $\mathbf{Int}$. Utilizing the contravariance property of the cohomology ring functor $\mathbf{H}^*$, we obtain:

$$\begin{aligned}
\mathbf{cup}\,(\mathbf{X} \times \mathbf{Y})\,([a, b]) &= \mathbf{len}\left(\mathbf{H}^*(f_a^b \times g_a^b)\right) \\
&= \mathbf{len}\left(\mathbf{H}^*(f_a^b) \otimes \mathbf{H}^*(g_a^b)\right) \\
&= \mathbf{len}\left(\mathbf{H}^*(f_a^b)\right) + \mathbf{len}\left(\mathbf{H}^*(g_a^b)\right) \\
&= \mathbf{cup}(\mathbf{X})([a, b]) + \mathbf{cup}(\mathbf{Y})([a, b]), \\
\mathbf{cup}\,(\mathbf{X} \amalg \mathbf{Y})\,([a, b]) &= \mathbf{len}\left(\mathbf{H}^*(f_a^b \amalg g_a^b)\right) \\
&= \mathbf{len}\left(\mathbf{H}^*(f_a^b) \times \mathbf{H}^*(g_a^b)\right) \\
&= \max\left\{\mathbf{len}\left(\mathbf{H}^*(f_a^b)\right), \mathbf{len}\left(\mathbf{H}^*(g_a^b)\right)\right\} \\
&= \max\left\{\mathbf{cup}(\mathbf{X})([a, b]), \mathbf{cup}(\mathbf{Y})([a, b])\right\}, \text{ and} \\
\mathbf{cup}\,(\mathbf{X} \vee \mathbf{Y})\,([a, b]) &= \mathbf{len}\left(\mathbf{H}^*(f_a^b \vee g_a^b)\right) \\
&= \mathbf{len}\left(\mathbf{H}^*(f_a^b) \times \mathbf{H}^*(g_a^b)\right) \\
&= \max\left\{\mathbf{len}\left(\mathbf{H}^*(f_a^b)\right), \mathbf{len}\left(\mathbf{H}^*(g_a^b)\right)\right\} \\
&= \max\left\{\mathbf{cup}(\mathbf{X})([a, b]), \mathbf{cup}(\mathbf{Y})([a, b])\right\}.
\end{aligned}$$

□

3.2.4 Examples and visualization

Each interval $[a, b]$ in $\mathbf{Int}$ is visualized as a point (a, b) in the half-plane above the diagonal (See Figure 3.3). In order to visualize the cup-length function of a filtration $\mathbf{X}$, we assign to each point (a, b) the integer value $\mathbf{cup}(\mathbf{X})([a, b])$, if it is positive and if $\mathbf{cup}(\mathbf{X})([a, b]) = 0$ we do not assign any value.

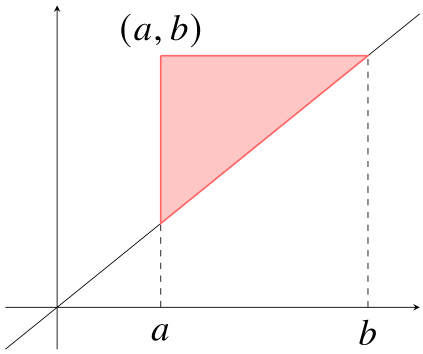


Figure 3.3: The interval $[a, b]$ in **Int** corresponds to the point (a, b) in $\mathbb{R}^2$. Source of the image: [CMSZ21].

We introduce several examples of how cup-length functions can be visualized in the upper-diagonal plane.

Example 3.31. Consider the filtration $\mathbf{X} = \{\mathbf{X}_t\}_{t\geq 0}$ of a Klein bottle with a 2-cell attached, as shown in Figure 3.4. Consider the persistent cohomology $\mathbf{H}^*\mathbf{X}$ in $\mathbb{Z}_2$-coefficients. Let v be the 0-cocycle (vertex) born at $t = 0$ and staying alive, let α be the 1-cocycle born at $t = 1$ in $\mathbb{X}_1$ and died at $t = 3$, and let β be the 1-cocycle born at time $t = 2$ and staying alive. Let $\gamma := \beta \smile \beta$, which is then a non-trivial 2-cocycle born at time $t = 2$ and staying alive, like β. Then the cohomology barcodes of $\mathbf{X}$ are: $\mathcal{B}^0(\mathbf{X}) = \{[0, \infty)\}$, $\mathcal{B}^1(\mathbf{X}) = \{[1, 3), [2, \infty)\}$, and $\mathcal{B}^2(\mathbf{X}) = \{[2, \infty)\}$. See the last figure of Figure 3.4 for the cohomology barcode. Using the formula in Proposition 3.29 and Figure 3.5, for any $t \leq s$,

$$\mathbf{Im}(\mathbf{H}^*(\mathbb{X}_s) \to \mathbf{H}^*(\mathbb{X}_t)) = \begin{cases} \langle [v]_t, [\beta]_t, [\gamma]_t \rangle, & \text{if } 2 \leq t < 3 \text{ and } s \geq 3 \\ \langle [v]_t, [\alpha]_t, [\beta]_t, [\gamma]_t \rangle, & \text{if } 2 \leq t \leq s < 3 \\ \langle [v]_t, [\alpha]_t \rangle, & \text{if } 1 \leq t < 2 \text{ and } s < 3 \\ \langle [v]_t \rangle, & \text{otherwise.} \end{cases}$$

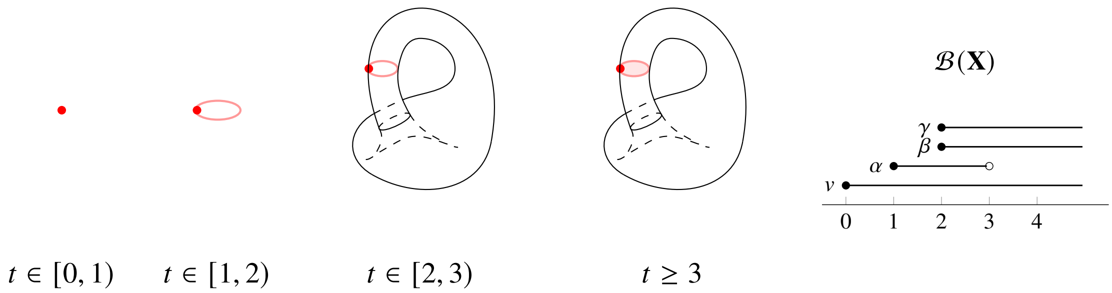


Figure 3.4: The filtration $\mathbf{X}$ of a Klein bottle with a 2-cell attached and its total barcode $\mathcal{B}(\mathbf{X})$, see Example 3.31.

Thus, the cup-length function of $\mathbf{X}$ is given by

$$\mathbf{cup}(\mathbf{X})([t, s]) = \begin{cases} 2, & \text{if } t \geq 2 \\ 1, & \text{if } 1 \leq t < 2 \text{ and } s < 3 \\ 0, & \text{otherwise.} \end{cases}$$

The Figure 3.5 shows the cup-length function $\mathbf{cup}(\mathbf{X})$ in the half-plane above the diagonal.

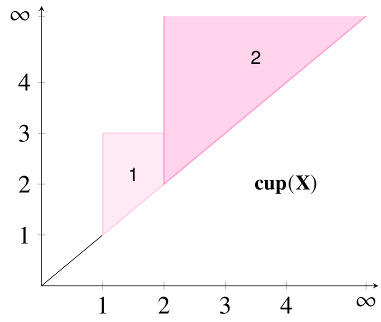


Figure 3.5: Visualization of the cup-length function $\mathbf{cup}(\mathbf{X})(\bullet)$. See Example 3.31. Source of the image: [CMSZ21].

The following example is one of the reason to work with this invariant. In the Example 3.32, the cup-length function is able to distinguish a pair of filtrations that the total cohomology barcode is not able to.

Example 3.32. Consider the filtration $\mathbf{X} = \{\mathbf{X}_t\}_{t\geq 0}$ of a 2-torus $\mathbb{T}^2$ and the filtration $\mathbf{Y} = \{\mathbf{Y}_t\}_{t\geq 0}$ of the space $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$ as shown in Figure 3.6. Given the fact that $\mathbb{T}^2$ and $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$ have the same (co)homology vector spaces in all dimensions, the persistent (co)homology vector spaces associated to $\mathbf{X}$ and $\mathbf{Y}$ are the same. However, for $t = 2$, the cohomology ring structure of $\mathbf{X}_2$ is different from that of $\mathbf{Y}_2$, since the torus $\mathbf{X}_3$ has an intersection of 1-cycles and $\mathbf{Y}_3$ does not. This means that in the cohomology ring of the torus, $\mathbf{X}_2$, there is a non-zero cup product of cocycles, which do not occur in the cohomology ring of $\mathbf{Y}_2 = \mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$. This non-trivial information of these two filtration can be extracted and also quantified by the cup-length functions of filtration. See Figure 3.6.

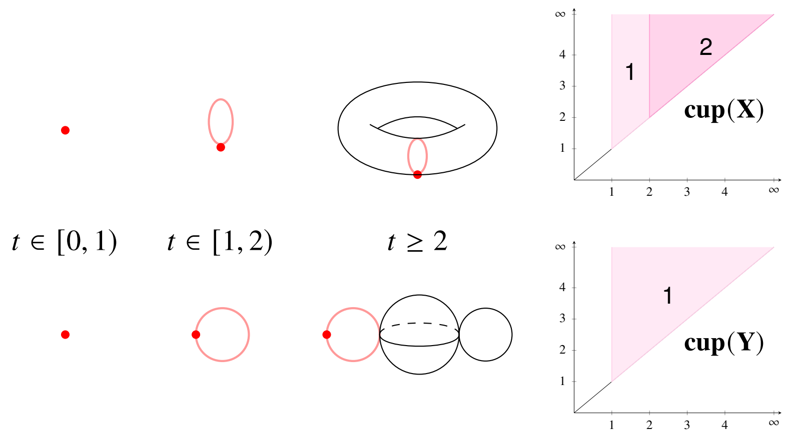


Figure 3.6: The filtration $\mathbf{X}$ of a 2-torus $\mathbb{T}^2$ and $\mathbf{cup}(\mathbf{X})$ (top), and the filtration $\mathbf{Y}$ of $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$ and $\mathbf{cup}(\mathbf{Y})$ (bottom), see Example 3.32. Source of the image: [CMSZ21].

3.3 Cup-length diagram

Here we introduce the notion of the *cup-length diagram of a filtration*, by utilizing the cup product operation on cocycles. The motivation of the study of cup-length diagram, another invariant related to cup-products, was born first when we realize that simply taking the Möbius inversion of the cup-length function does not yield a valid notion of diagram, as we can see in §3.3.1. In §3.3.2, we state some important results of [CMSZ21] of how the cup-length diagram can be used to compute the cup-length function (see Theorem 3.42).

3.3.1 The Möbius inversion of the cup-length function

Given a persistence module $\mathbf{M}\colon (\mathbb{Z}, \leq) \to \mathbf{Vec}$ and a dimension $p \in \mathbb{N}$, we can recover the barcode $\mathcal{B}(\mathbf{M})$ of $\mathbf{M}$, from the rank invariant $\mathbf{rank}(\mathbf{M})\colon \mathbf{Int}(\mathbb{Z}) \to \mathbb{N}$ of $\mathbf{M}$, using the process called the *Möbius inversion* [Pat18]. The Möbius inversion of $\mathbf{rank}(\mathbf{M})\colon \mathbf{Int}(\mathbb{Z}) \to \mathbb{N}$ is a function $\mathbf{dgm}(\mathbf{M})\colon \mathbf{Int}(\mathbb{Z}) \to \mathbb{N}$ satisfying, element-wisely, the equation:

$$\mathbf{rank}(\mathbf{M})([a, b]) = \sum_{[c,d]\supset[a,b]} \mathbf{dgm}(\mathbf{M})([c, d]). \tag{3.1}$$

To compute the Möbius inversion $\mathbf{dgm}(\mathbf{M})$ we use the persistent rank function and the formula

$$\begin{aligned}\mathbf{dgm}(\mathbf{M})([a, b]) := \ &\mathbf{rank}(\mathbf{M})([a, b]) - \mathbf{rank}(\mathbf{M})([a-1, b]) \\ &- \mathbf{rank}(\mathbf{M})([a, b+1]) + \mathbf{rank}(\mathbf{M})([a-1, b+1]).\end{aligned}$$

Definition 3.33. The Möbius inversion $\mathbf{dgm}(\mathbf{M})\colon \mathbf{Int}(\mathbb{Z}) \to \mathbb{N}$ of the rank invariant of $\mathbf{M}$ is called **the persistence diagram of M**.

It is also valid for our length function $\mathbf{L}\colon \mathbf{Int}(\mathbb{Z}) \to (\mathbb{N}, \geq)$ a notion of Möbius inversion, $\mathbf{L}^\star$, of $\mathbf{L}$, in the same way as we said before. However, the function $\mathbf{L}^\star$ may have non-positive values, meaning that it may simply be a general integer-valued function $\mathbf{L}^\star\colon \mathbf{Int}(\mathbb{Z}) \to \mathbb{Z}$ which does not satisfy the Möbius inversion formula.

The reason why the Möbius inversion of the rank invariant of $\mathbf{M}$ is always non-negative is due to the fact that any persistence module $\mathbf{M}\colon \mathbf{Int}(\mathbb{Z}) \to \mathbf{Vec}$ is interval decomposable (see [CB15]).

In Example 3.34, we have a filtration such that the Möbius inversion of its cup-length function is negative for an interval and, because of this example, this well-defined notion of a persistent diagram will need to be defined differently, as we will see in next section.

Example 3.34. Let $\mathbf{X} = \{\mathbf{X}_t\}_{t\geq 0}$ be a filtration of a disjoint union of two (2-dim) disks, as shown in Figure 3.7. The total barcode $\mathcal{B}(\mathbf{X})$ and the cup-length function $\mathbf{cup}(\mathbf{X})(\bullet)$ of $\mathbf{X}$, are shown in Figure 3.8.

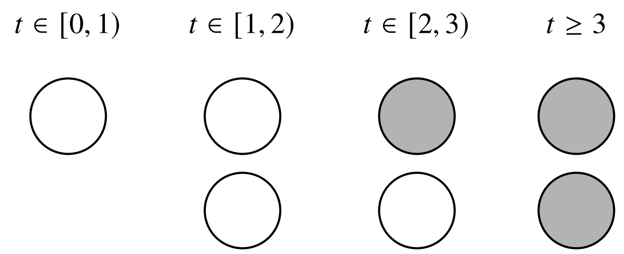


Figure 3.7: A filtration $\mathbf{X}$ of a disjoint union of two (2-dim) disks, see Example 3.34

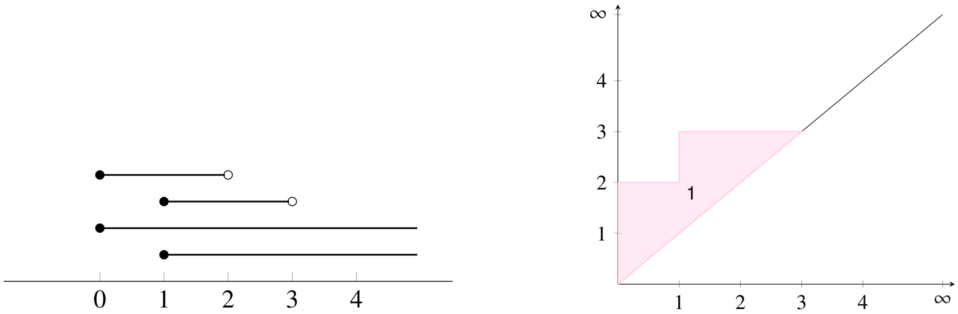


Figure 3.8: The total barcode $\mathcal{B}(\mathbf{X})$ (left) and the cup-length function $\mathbf{cup}(\mathbf{X})(\bullet)$ (right) of $\mathbf{X}$, where $\mathbf{X}$ is the filtration of two disjoint disks, given in Example 3.34. Source of the image: [CMSZ21].

The critical points of the filtration $\mathbf{X}$ are $\{0, 1, 2, 3\}$. Thus, we can equivalently think of the cup-length function $\mathbf{cup}(\mathbf{X}) \colon \mathbf{Int} \to \mathbb{N}$, as a length function of the form $\mathbf{L} \colon \mathbf{Int}(\mathbb{Z}) \to \mathbb{N}$, where $\mathbf{L}$ is simply the restriction of $\mathbf{cup}(\mathbf{X})(\bullet)$ to the integers. If we consider the interval $[1, 1] = \{1\}$, then the Möbius inversion of $\mathbf{L}$ applied to $[1, 1]$ is negative, i.e.

$$\mathbf{L}^\star([1, 1]) = \mathbf{L}([1, 1]) - \mathbf{L}([0, 1]) - \mathbf{L}([1, 2])) + \mathbf{L}([0, 2]) = 1 - 1 - 1 + 0 = -1 < 0.$$

Example 3.35. If we consider the filtration $\mathbf{X}$ of the pinched Klein bottle as in Example 3.31 we will have another example where our Möbius idea assumes negative values. By definition, the critical points of the filtration $\mathbf{X}$ are $\{0, 1, 2, 3\}$. As before, we can think of $\mathbf{cup}(\mathbf{X}) \colon \mathbf{Int} \to \mathbb{N}$ as a length function of the form $\mathbf{L} \colon \mathbf{Int}(\mathbb{Z}) \to \mathbb{N}$, where $\mathbf{L}$ is simply the restriction of $\mathbf{cup}(\mathbf{X})(\bullet)$ to the integers. Consider the interval $[2, 2] = \{2\}$, and notice that the Möbius inversion of $[2, 2]$ is negative, i.e.

$$\mathbf{L}^\star([2, 2]) = \mathbf{L}([2, 2]) - \mathbf{L}([1, 2]) - \mathbf{L}([2, 3])) + \mathbf{L}([1, 3]) = 2 - 1 - 2 + 0 = -1 < 0.$$

With these examples we can see that using the Möbius inversion to associate a diagram to the cup-length function will not work. Now, we introduce a new way to define a diagram arising from the cup-length invariant, which can also be used to compute the cup-length function (see Theorem 3.42, [CMSZ21]). This diagram will be called the cup-length diagram.

3.3.2 Cup-length diagram and cup-length function

We will define the notion of cup-length diagram using a family of representative cocycles. This notion will be fascinating because even using a certain family to be defined it is independent of the choice of a family, as we will see in Proposition 3.41. As another important feature, in Theorem 3.42 we see that the cup-length function can be obtained from the cup-length diagram, but not vice versa (see Example 3.44), making the cup-length diagram a stronger invariant. All these results can be founded in [CMSZ21].

Definition 3.36. Let $\boldsymbol{\sigma}$ be a family of representative cocycles for $\mathbf{H}^*\mathbf{X}$. Let $\ell \in \mathbb{N}^*$ and let $I_1, \ldots, I_\ell$ be a sequence of elements in $\mathcal{B}(\mathbf{X})$ with representative cocycles $\sigma_{I_1}, \ldots, \sigma_{I_\ell} \in \boldsymbol{\sigma}$, respectively. Let $[a_k, b_k] := \overline{I_k}$, $k = 1, \ldots, \ell$. We define the **ℓ-fold $*_\sigma$-product of** $[a_1, b_1], \ldots, [a_\ell, b_\ell]$, as

$$[a_1, b_1] *_\sigma \cdots *_\sigma [a_\ell, b_\ell] := \begin{cases} \emptyset, & \text{if } [\sigma_{I_1} \smile \cdots \smile \sigma_{I_\ell}]_t = [0]_t, \forall t \in \mathbb{R} \\ [\max\{a_1, \ldots, a_\ell\}, \min\{b_1, \ldots, b_\ell\}], & \text{otherwise.} \end{cases}$$

We associate to each interval $[\max\{a_1, \ldots, a_\ell\}, \min\{b_1, \ldots, b_\ell\}]$, which is in fact the intersection $\bigcap_{i=1}^{\ell}[a_i, b_i]$, the representative cocycle $\sigma_{I_1} \smile \cdots \smile \sigma_{I_\ell}$.

Remark 3.37. Because of the nature of the representative cocycles, $[\sigma_{I_1} \smile \cdots \smile \sigma_{I_\ell}]_t = [0]_t$ holds for any t if, and only if, it holds for t equal to $\min\{d_1, \ldots, d_\ell\} - \delta$ for a $\delta > 0$ chosen sufficiently small.

It is not hard to prove that the (2-fold) $*_{\sigma}$-product is associative and, using Proposition 3.38, that it is commutative.

Proposition 3.38. *Let $[a_k, b_k]$, $k = 1, \ldots, \ell$, be as in Definition 3.36. The ℓ-fold $*_{\sigma}$-product of $[a_1, b_1], \ldots, [a_\ell, b_\ell]$ is symmetric, i.e. for any permutation τ of $\{1, 2, \ldots, \ell\}$,*

$$[b_1, d_1] *_{\sigma} \cdots *_{\sigma} [b_\ell, d_\ell] = [b_{\tau(1)}, d_{\tau(1)}] *_{\sigma} \cdots *_{\sigma} [b_{\tau(\ell)}, d_{\tau(\ell)}].$$

Proof. The proof follows because of the symmetry of max and min, and by the following property of cup product: for any pair α, β of cochains, $\alpha \smile \beta = (-1)^m \beta \smile \alpha$, for some integer m, which implies that

$$[\sigma_{I_1} \smile \cdots \smile \sigma_{I_\ell}]_{\min\{d_1,\ldots,d_\ell\}} = [0] \Leftrightarrow [\sigma_{I_{\tau(1)}} \smile \cdots \smile \sigma_{I_{\tau(\ell)}}]_{\min\{d_1,\ldots,d_\ell\}} = [0],$$

for any permutation τ of $\{1, 2, \ldots, \ell\}$. □

Definition 3.39 (Cup-length diagram)**.** Let $\mathbf{X}$ be a filtration and let $\mathcal{B}^{\geq 1}(\mathbf{X})$ be the total cohomology barcode of $\mathbf{X}$ over positive dimensions. Let $\boldsymbol{\sigma} = \{\sigma_I\}_{I \in \mathcal{B}(\mathbf{X})}$ be a family of representative cocycles of $\mathbf{X}$. The **persistent cup-length diagram of X associated to $\boldsymbol{\sigma}$** or simply the **cup-length diagram of X** is the map $\mathbf{dgm}_{\boldsymbol{\sigma}}^{\smile}(\mathbf{X}) : \mathbf{Int} \to \mathbb{N}$, given by:

$$\mathbf{dgm}_{\boldsymbol{\sigma}}^{\smile}(\mathbf{X})(I) := \max\{\ell \in \mathbb{N}^* \mid I = \overline{I_1} *_{\sigma} \cdots *_{\sigma} \overline{I_\ell}, \text{ where each } I_k \in \mathcal{B}^{\geq 1}(\mathbf{X})\},$$

with the convention that $\max \emptyset = 0$.

One of the most important things about this diagram is proved in the following results: The diagram $\mathbf{dgm}_{\boldsymbol{\sigma}}^{\smile}(\mathbf{X})$ is independent of the choice of the family of representative cocyles $\boldsymbol{\sigma}$.

Lemma 3.40. *Let $\mathbf{X}$ be a filtration and let $\mathcal{B}^{\geq 1}(\mathbf{X}) = \{I_1, \ldots, I_n\}$ be the total barcode of $\mathbf{X}$ in positive dimensions. Let $\boldsymbol{\sigma} := \{\sigma_{I_j}\}_{j=1}^{n}, \boldsymbol{\tau} := \{\tau_{I_j}\}_{j=1}^{n}$ be two families of representative cocycles for $\mathcal{B}^{\geq 1}(\mathbf{X})$. Then, for any $j = 1, \ldots, n$ and any $t \in \mathbb{R}$:*

$$[\tau_{I_j}]_t = \sum_{I_k = I_j} \lambda_{j,k} \cdot [\sigma_{I_k}]_t, \textit{ for some } \lambda_{j,k} \in \mathbb{K}.$$

Proof. The proof of this Lemma can be found in [CMSZ21] □

Proposition 3.41. *Let $\mathbf{X}$ be a filtration. The cup-length diagram of $\mathbf{X}$ is independent of the choice of a family $\boldsymbol{\sigma}$ of representative cocycles of $\mathcal{B}(\mathbf{X})$.*

Proof. Proof in [CMSZ21]. □

From now on we will denote the cup-length diagram of $\mathbf{X}$ simply by $\mathbf{dgm}^{\smile}(\mathbf{X})$.

In standard persistence theory, for each interval $[a, b]$ the rank invariant of a persistence module $\mathbf{M}$ counts the **sum** of the multiplicities of the intervals in the barcode of $\mathbf{M}$ containing $[a, b]$. This multiplicity function is what we call the persistent diagram of $\mathbf{M}$.

In our case, we will follow the same ideas of it, but understanding that the cup-length function counts the **maximum** number, not the sum, of non-zero cup-products of cocycles.

The following result proves that replacing 'sum' by 'max' operation, the cup-length function can be obtained from the cup-length diagram.

Theorem 3.42. *Let* $\mathbf{X}$ *be a filtration. The cup-length function* $\mathbf{cup}(\mathbf{X})\colon \mathbf{Int} \to \mathbb{N}$ *can be retrieved from the cup-length diagram* $\mathbf{dgm}^{\smile}(\mathbf{X})\colon \mathbf{Int} \to \mathbb{N}$*, element-wisely as:*

$$\mathbf{cup}(\mathbf{X})([a,b]) = \max_{[c,d]\supseteq[a,b]} \mathbf{dgm}^{\smile}(\mathbf{X})([c,d]). \tag{3.2}$$

Proof. Let $[a,b]$ be an interval in $\mathbf{Int}$. Then, we compute:

$$\begin{aligned}
&\mathbf{cup}(\mathbf{X})([a,b]) \\
&= \mathbf{len}\left(\mathbf{Im}\left(\mathbf{H}^*(\mathbb{X}_b) \to \mathbf{H}^*(\mathbb{X}_a)\right)\right) \\
&= \mathbf{len}\left(\langle[\sigma_J] : \mathcal{B}(\mathbf{X}) \ni J \supseteq [a,b]\rangle\right) && \text{(by Proposition 3.29)} \\
&= \max\left\{\ell \in \mathbb{N}^* \mid [\sigma_{J_1}] \smile \cdots \smile [\sigma_{J_\ell}] \neq [0],\ \mathcal{B}^{\geq 1}(\mathbf{X}) \ni J_k \supseteq [a,b],\ \forall\, k \leq \ell\right\} \cup \{0\} \\
& && \text{(by Proposition 3.25)} \\
&= \max\left\{\ell \in \mathbb{N}^* \mid [\sigma_{J_1} \smile \cdots \smile \sigma_{J_\ell}] \neq [0],\ \bigcap_{k=1}^{\ell} J_k \supseteq [a,b],\ J_k \in \mathcal{B}^{\geq 1}(\mathbf{X})\right\} \cup \{0\} \\
& && \text{(by } \mathbf{cup} \text{ and } \mathbf{Int} \text{ properties)} \\
&= \max_{[c,d]\supseteq[a,b]} \left\{\max\left\{\ell \in \mathbb{N}^* \mid [\sigma_{J_1} \smile \cdots \smile \sigma_{J_\ell}] \neq [0],\ \bigcap_{k=1}^{\ell} \overline{J_k} = [c,d]\right\} \cup \{0\}\right\} \\
& && \text{(by } \mathbf{Int} \text{ property)} \\
&= \max_{[c,d]\supseteq[a,b]} \left\{\max\left\{\ell \in \mathbb{N}^* \mid [c,d] = \overline{J_1} *_{\boldsymbol{\sigma}} \cdots *_{\boldsymbol{\sigma}} \overline{J_\ell},\ \text{where } J_k \in \mathcal{B}^{\geq 1}(\mathbf{X})\right\} \cup \{0\}\right\} \\
& && \text{(by Definition 3.36)} \\
&= \max_{[c,d]\supseteq[a,b]} \left\{\mathbf{dgm}^{\smile}(\mathbf{X})([c,d])\right\} && \text{(by Definition 3.39)}
\end{aligned}$$

which completes the proof. □

Now we have several examples of cup-length diagrams of some filtrations, utilizing the Equation 3.2 for computing the cup-length functions, as we proved in Theorem 3.42

In Example 3.43 we will see that the cup-length diagram is reacher than standard persistence diagram in positive dimensions and in Example 3.44 we have a case when two different filtrations have the same cup-length function, but different cup-length diagrams, showing that this diagram is a powerful tool. For the following examples, $[1,\infty], [2,\infty], \ldots$ will be assumed as intervals in $\mathbf{Int}$.

Example 3.43. Consider the filtration $\mathbf{X} = \{\mathbf{X}_t\}_{t\geq 0}$ of a Klein bottle with a 2-cell attached, as in Example 3.31. The cohomology barcodes of $\mathbf{X}$ in positive dimensions are given by $\mathcal{B}^1(\mathbf{X}) = \{[1,3),[2,\infty)\}$, $\mathcal{B}^2(\mathbf{X}) = \{[2,\infty)\}$ and the corresponding representative cocycles by α, β and γ, respectively. Let $\boldsymbol{\sigma} := \{\alpha,\beta,\gamma\}$. Because $\mathbf{H}^*(\mathbf{X})$ has dimension at most 2, we know that $\mathbf{dgm}^{\smile}(\mathbf{X})(I) \leq 2$ for any I.

Since $[\alpha \smile \alpha]_3 = 0$, then we have that $\mathbf{dgm}^{\smile}(\mathbf{X})([1,3]) = 1$. Because $[\alpha \smile \beta]_3 = [\gamma]_3 \neq 0$, we have $[2,3] = [1,3] *_{\boldsymbol{\sigma}} [2,\infty]$, implying that $\mathbf{dgm}^{\smile}(\mathbf{X})([2,3]) = 2$. A similar argument holds for $[2,\infty]$, using the fact that $[\beta \smile \beta]_\infty = [\gamma]_\infty \neq 0$. Therefore,

$$\mathbf{dgm}^{\smile}(\mathbf{X})(I) = \begin{cases} 1, & \text{if } I = [1,3] \\ 2, & \text{if } I = [2,3] \text{ or } I = [2,\infty] \\ 0, & \text{otherwise.} \end{cases}$$

Applying Theorem 3.42, we obtain the cup-length function $\mathbf{cup}(\mathbf{X})$ that we saw in Figure 3.5.

Comparing the cup-length diagram and the standard persistence diagram (obtained from the rank invariant) in positive dimensions, we see that the cup-length diagram contains more information, see Figure 3.9.

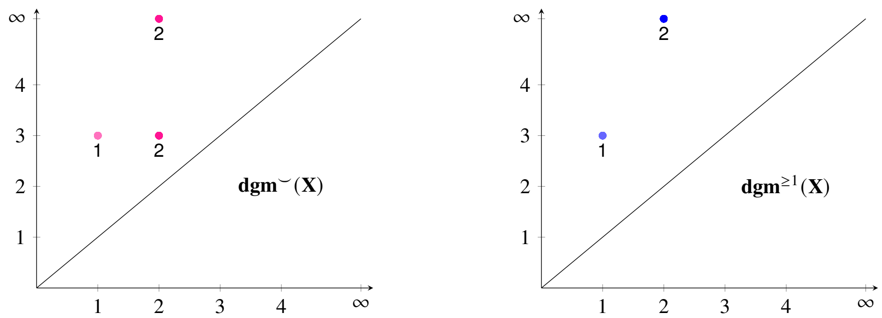


Figure 3.9: The cup-length diagram $\mathbf{dgm}^{\smile}(\mathbf{X})$ (left) and the union of positive-degree persistence diagrams $\mathbf{dgm}^{\geq 1}(\mathbf{X})$ (right), where $\mathbf{X}$ is the filtration of a Klein bottle with a 2-cell attached. See Ex. 3.43. Source of the image: [CMSZ21].

Example 3.44. Consider another filtration $\mathbf{Y} = \{\mathbf{Y}_t\}_{t\geq 0}$ of the disjoint union of a torus and a (2-dim) disks, as shown in Figure 3.10. Let α be the 1-cocycle born at $t = 1$ and dead when the disk is filled at $t = 3$. Consider β and δ the 1-cocycles and γ the 2-cocyle born at $t = 2$ when the torus appears. Then the cohomology barcodes of $\mathbf{Y}$ in positive dimensions are:

$\mathcal{B}^1(\mathbf{Y}) = \{[1,3), [2,\infty), [2,\infty)\}$ and $\mathcal{B}^2(\mathbf{Y}) = \{[2,\infty)\}$. The only non-trivial cup-product at the death times is $[\beta \smile \delta]_\infty$ and thus,

$$\mathbf{dgm}^{\smile}(\mathbf{Y})(I) = \begin{cases} 1, & \text{if } I = [1,3] \\ 2, & \text{if } I = [2,\infty] \\ 0, & \text{otherwise.} \end{cases}$$

Using the Equation (3.2) of Theorem 3.42 we compute the cup-length function of $\mathbf{Y}$, and obtain that $\mathbf{cup}(\mathbf{Y}) = \mathbf{cup}(\mathbf{X})$.

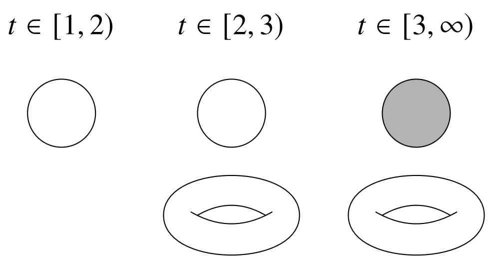


Figure 3.10: A filtration $\mathbf{Y}$ of the disjoint union of a (2-dim) disk with a 2-torus, see Example 3.44.

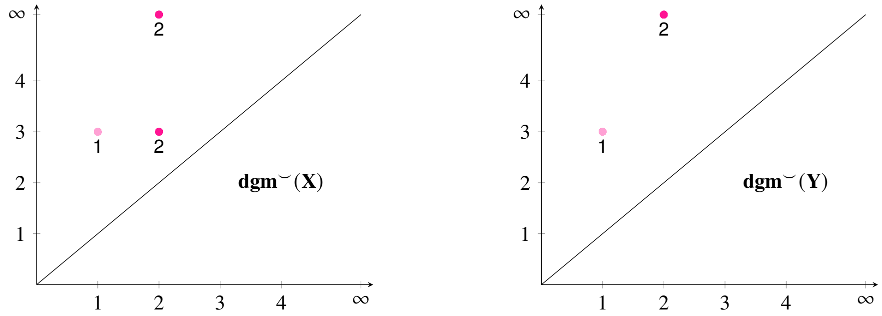


Figure 3.11: The cup-length diagrams $\mathbf{dgm}^{\smile}(\mathbf{X})$ (left) and $\mathbf{dgm}^{\smile}(\mathbf{Y})$ (right). See Example 3.43.

Example 3.45. Consider the filtration $\mathbf{X} = \{\mathbf{X}_t\}_{t\in\mathbb{R}}$ of a disjoint union of two (2-dim) disks, as in Example 3.34.The cohomology barcodes of $\mathbf{X}$ in positive dimensions are $\mathcal{B}^1(\mathbf{X}) = \{[0,2),[1,3)\}$ and let $\boldsymbol{\sigma} := \{\alpha,\beta\}$ denote the representative cocycles with α and β corresponding to the top and bottom circle in Figure 3.7, respectively. Then we compute

$$\mathbf{dgm}^{\smile}(\mathbf{X})(I) := \begin{cases} 1, & \text{if } I=[0,2] \text{ or } I=[1,3] \\ 0, & \text{otherwise.} \end{cases}$$

In Figure 3.12, we present the cup-length diagram $\mathbf{dgm}^{\smile}(\mathbf{X})$ on the left and the cup-length function $\mathbf{cup}(\mathbf{X})$, computed from the cup-length diagram and Theorem 3.42, on the right.

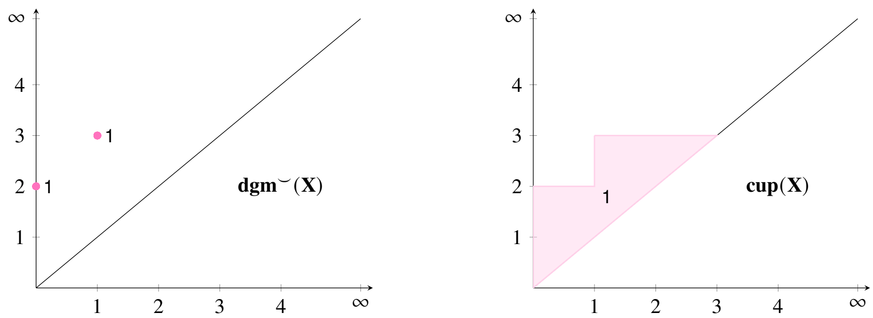


Figure 3.12: The cup-length diagram $\mathbf{dgm}^{\smile}(\mathbf{X})$ (left) and the cup-length function $\mathbf{cup}(\mathbf{X})$ (right), where $\mathbf{X}$ is the filtration of a disjoint union of two (2-dim) disks. See Example 3.45. Source of the image: [CMSZ21].

Example 3.46. Let $\mathbf{X} = \{\mathbf{X}_t\}_{t\geq 0}$ be a filtration given by $\mathbf{X}_t := \underbrace{\mathbb{S}^1\times\mathbb{S}^1\times\cdots\times\mathbb{S}^1}_{\lfloor t\rfloor \text{ times}}$, for $t\geq 0$. For any $t\leq s$, $\mathbf{X}_t \hookrightarrow \mathbf{X}_s$ is given by the natural inclusion. When $t<1$, $\mathbf{H}^*(\mathbf{X}_t)=0$. When $t\geq 1$, there are $\lfloor t\rfloor$ linearly independent 1-cocycles $\eta_1,\ldots,\eta_{\lfloor t\rfloor}$. Notice that the set $\boldsymbol{\sigma} := \{\eta_{i_1}\smile\cdots\smile\eta_{i_j}\}_{1\leq i_1\leq\cdots\leq i_j\leq\lfloor t\rfloor}$ forms a family of representative cocycles for $\mathbf{H}^*\mathbf{X}$ in positive degrees, with which we compute the cup-length diagram $\mathbf{dgm}^{\smile}(\mathbf{X})$ and present it in left figure of Figure 3.13. Then we apply Theorem 3.42 to obtain that $\mathbf{cup}(\mathbf{X})([t,s]) = \mathbf{cup}(\mathbf{X}_t) = \lfloor t\rfloor$, if $1\leq t\leq s$ and 0 otherwise, and we plot the cup-length function $\mathbf{cup}(\mathbf{X})$ in the right figure of Figure 3.13.

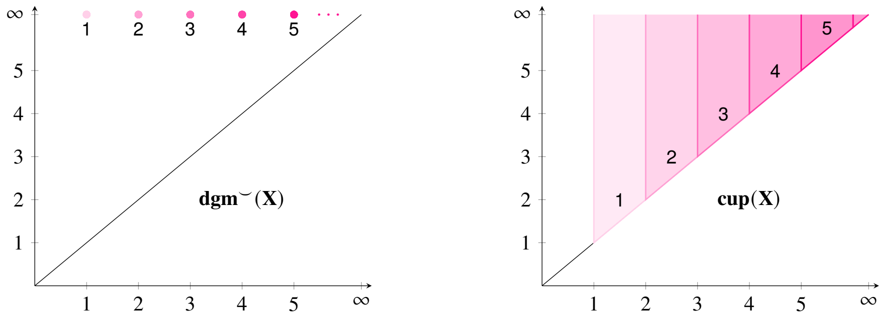


Figure 3.13: The cup-length diagram $\mathbf{dgm}^{\smile}(\mathbf{X})$ (left) and the cup-length function $\mathbf{cup}(\mathbf{X})$ (right). See Example 3.46. Source of the image: [CMSZ21].

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