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Aspects of Supersymmetric Gauge Theories

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*"There is no exquisite beauty... without some **strangeness** in the proportion."*

Edgar Allan Poe, *Ligeia*.

Resumo

Esta dissertação tem como foco revisar alguns dos mais importantes avanços no estudo de Teorias de Campo Supersimétricas que ocorreram desde 1994. Mais especificamente, o foco será dado a teorias $d = 4, \mathcal{N} = 2$ super Yang-Mills e seu comportamento a baixas energias. Iremos apresentar uma revisão da teoria de Seiberg-Witten para o caso de Yang-Mills supersimétrico puro com grupo de gauge $SU(2)$, bem como uma generalização para teorias com grupo de gauge $SU(N)$. Também iremos fornecer uma revisão do procedimento de contagem de instantons que proporciona a derivação do prepotencial de Seiberg-Witten. Ao longo do trabalho, iremos encontrar diversas conexões interessantes com a Matemática, sendo exemplos notórios as Teorias de Campos Topológicas e Localização.

Palavras Chaves: Teorias de Campos Supersimétricas; Teoria de Seiberg-Witten; Localização; Contagem de Instantons.

Áreas do conhecimento: Física; Supersimetria; Teoria Quântica de Campos.

Abstract

This dissertation is aimed at reviewing some of the most important developments in the study of Supersymmetric Field Theories that took place since 1994. More specifically, the focus will be on $d = 4, \mathcal{N} = 2$ SYM theories and their low-energy behavior. We will provide a review of Seiberg-Witten theory for the case of pure $SU(2)$ supersymmetric Yang-Mills, as well as a generalization for theories with gauge group $SU(N)$. We will also review the instanton counting procedure which yields a derivation of the Seiberg-Witten prepotential. Throughout this work we will encounter many interesting connections with Mathematics, Topological Field Theory and Localization being the notable examples.

Keywords: Supersymmetric Field Theories; Seiberg-Witten Theory; Localization; Instanton counting.

Subject areas: Physics; Supersymmetry; Quantum Field Theory.

Contents

Introduction

Supersymmetric Gauge Theories have been some of the most interesting and productive settings to study Theoretical Physics for the past 40 years or so. The constraints imposed by supersymmetry allow for a deeper and more complete exploration of these theories, and therefore one can usually perform much more thorough investigations compared to less (or even non) supersymmetric theories. There are many well-known specific supersymmetric theories. As an example, we can mention $d = 4, \mathcal{N} = 4$ SYM theory which has been intensively studied over the last years due to its relation with Type IIB string theory on $AdS_5 \times S^5$ via the AdS/CFT correspondence. In fact, a lot of advances were possible due to the relationship of Supersymmetric Field Theories and String Theory in a plethora of different ways.

Mathematics plays a prominent role in this story. Its presence and importance in the development of Physics is undeniable and omnipresent, but we would argue that Supersymmetric Field Theories make up for the best playground for this interaction to take place. Indeed, in this case, Physics draws a lot from Mathematics, but the converse also outstandingly takes place. We can mention Topological Field Theories, which we will encounter further in this dissertation, as an illustration of quantum Physics leading to advances in Mathematics.

This dissertation will focus on the study of a particular type of Supersymmetric Gauge Theory: $d = 4, \mathcal{N} = 2$ supersymmetric Yang-Mills, and more specifically its low-energy behaviour. They have been extensively studied ever since Seiberg and Witten introduced an elegant and efficient way of describing their effective actions at low energies, and have instigated a rush of advances in the understanding of Supersymmetric Field Theories in different dimensions. Therefore, a good knowledge about its most important features is paramount in order to take on further research in the area. Providing this background and reviewing recent developments will thus be the purpose of this dissertation.

Outline

Chapter 1 starts with a swift, but necessary, overview of the classical description of $d = 4, \mathcal{N} = 2$ SYM. It is important to get acquainted with its classical action, symmetries and fields before proceeding to more involved matters. After making some general comments on the properties of the theory with generic gauge group G, we specialize to $SU(2)$ and review Seiberg-Witten theory [1], which provides the prepotential (and therefore the effective action) of the theory at low-energies. This goal is achieved through the construction of an auxiliary curve and a special one-form. We then extend the procedure to theories with gauge group $SU(N)$.

The Seiberg-Witten solution to the problem of determining the low-energy effective action may seem a bit "esoteric". Chapter 2 will be devoted to providing a derivation of this solution from first principles, i.e., by directly counting all the instanton contributions to the prepotential. This procedure, introduced by Nekrasov in 2002, will be referred to as "the instanton counting route". In order to fully understand it, we introduce the concepts of Localization from a mathematically formal point of view and then relate it to Physics by means of the so-called Matthai-Quillen representative in the form introduced by Atiyah and Jeffrey. The next step will be a review of Topological Field Theories as introduced by Witten in 1988, along which we show how the Localization concepts we just learned can be applied to a problem in Physics. We will then be ready to fully appreciate Nekrasov's derivation of the Seiberg-Witten prepotential [22]. We introduce the Ω-deformation and perform the Localization on the fixed points under the action of the maximal torus $\mathbb{T}_{SU(N)/\mathbb{Z}_N} \times \mathbb{T}_{SO(4)}$ of the moduli space of instantons by using the ADHM formalism. We will then show how to directly compute the instanton corrections to the partition function. Finally, we will review how to relate Nekrasov's partition function to the Seiberg-Witten curve and differential.

We conclude the work by summarizing the themes that we covered and by mentioning possible next steps in the study of Supersymmetric Gauge Theories.

Chapter 1

Seiberg-Witten theory

We start with an elementary introduction to $d = 4, \mathcal{N} = 2$ supersymmetric theories. We then move on to recent developments in the determination of their low-energy behavior. The path we take in this chapter is known as **Seiberg-Witten theory**. Developed in the nineties, it associates to each point of the quantum moduli space of vacua a genus one Riemann surface which allows the determination of the low-energy effective action of the theory. We start by introducing our notations and main concepts and then show the presence of $SL(2, \mathbb{Z})$ duality in the theory. This is already a nice hint of the relation to tori. Then we move on to study monodromies present in the quantum moduli space of vacua and how they are related to the existence of BPS particles which become massless at these points. Finally we introduce the Seiberg-Witten curve and differential which lead to a simple and elegant solution to the problem. There are many nice reviews on the subject, and we recommend [4, 5] in particular.

1.1 $d = 4, \mathcal{N} = 2$ SUSY crash-course

This section will be aimed at giving a brief introduction to $\mathcal{N} = 2$ SYM theory. As we have already mentioned in the introduction, the main goal of this chapter as a whole is to describe the solutions to their low-energy behavior. Therefore, the obvious first step is knowing what is the theory we are dealing with.

We start with the $d = 4, \mathcal{N} = 2$ algebra and its representations. There are 8 supercharges in total

$$Q^A_\alpha, \bar{Q}_{A\dot{\alpha}}, \text{with } \alpha, \dot{\alpha} = 1, 2 \text{ and } A = 1, 2, \tag{1.1.1}$$

which satisfy the usual extended supersymmetry algebra, namely

$$\{Q^A_\alpha, \bar{Q}_{B\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \delta^A_B, \qquad \{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} Z^{AB}. \tag{1.1.2}$$

In pure SYM we only consider the vector multiplet, which is the multiplet obtained when acting with the creation operators on the vacuum $|0\rangle$. The states thus obtained are:

$$A_a^\mu = (1/2,1/2,0)^0 \quad \psi_{a\alpha}^A = (1/2,0,1/2)^{+1} \quad \bar{\psi}_a^{A\dot{\alpha}} = (0,1/2,1/2)^{-1} \tag{1.1.3}$$

$$\phi^a = (0,0,0)^{+2} \quad \bar{\phi}^a = (0,0,0)^{-2} \tag{1.1.4}$$

where we have given their respective charges under

$$SU(2)_L \times SU(2)_R \times SU(2)_I \times U(1), \tag{1.1.5}$$

the global symmetry group of the theory. Here, the R-symmetry of the theory is denoted by $SU(2)_I \times U(1)_I$. The lower-case a indices are indices of the adjoint representation of the gauge group G, whereas the upper-case indices A on the fermions are $SU(2)$ doublet indices.

The next step is trying to find a $\mathcal{N}=2$ supersymmetric action for the theory. To this end, superfields are usually the way to go. Notice that the above classification of fields can be separated into two $\mathcal{N}=1$ multiplets: a vector multiplet with $\psi^1 \equiv \psi$ and A^μ denoted by V[1] and a massless chiral multiplet A to which belong ϕ and $\psi^2 \equiv \lambda$. The superscripts in the ψ's refer to the doublet index. This way of thinking will allow us to use what we know about constructing $\mathcal{N}=1$ supersymmetric actions to solve the problem. Indeed, at the end of the day, we only need to fix numerical factors such that the extra supersymmetry is preserved, and prohibit certain types of terms which would spoil it. Evidently, all the superfields should be in the adjoint representation and the superpotential must be zero. Indeed, if we introduced a superpotential in the action, the powers in V and A would never match, and they should in order to preserve the extra supersymmetries. These considerations constrain the action to be of the form

$$S = \mathfrak{Im}\,\mathrm{tr} \int d^4x \frac{\tau}{8\pi} \left\{ \int d^2\theta W^\alpha W_\alpha + 2 \int d^2\theta d^2\bar{\theta} A^\dagger e^{-2gV} A \right\} \tag{1.1.6}$$

in $\mathcal{N}=1$ superfield notation, where $W_\alpha \equiv -\frac{1}{4}\overline{DD}e^{-2V}D_\alpha e^{2V}$ and $\tau = \frac{\vartheta}{2\pi} + i\frac{4\pi}{g^2}$. However, manifest $\mathcal{N}=2$ supersymmetry is still missing. We thus generalize our superfield notation to include extra fermionic coordinates $\tilde{\theta}_\alpha, \bar{\tilde{\theta}}^{\dot{\alpha}}$, to define the

[1] We are using the Wess-Zumino gauge.

$\mathcal{N} = 2$ superfield

$$\Psi(y, \theta, \tilde{\theta}) = A(y, \theta) + \sqrt{2}\tilde{\theta}^\alpha W_\alpha(y, \theta) - \frac{1}{2}\tilde{\theta}^2 \int d^2\bar{\theta} A^\dagger e^{-2gV}. \tag{1.1.7}$$

In what follows, we denote $G(y, \theta) = -\frac{1}{2}\int d^2\bar{\theta} A^\dagger e^{-2gV}$. Then, renormalization and $\mathcal{N} = 2$ supersymmetry implies:

$$S = \mathfrak{Im}\frac{\tau}{16\pi}\int d^4x d^2\theta d^2\tilde{\theta}\mathrm{tr}\Psi^2. \tag{1.1.8}$$

The form of (1.1.8) is telling. First of all, we see that it is **holomorphic** in Ψ, i.e., it has no contribution of $\Psi^\dagger$. Indeed, any action of the type

$$\mathfrak{Im}\ \frac{1}{8\pi}\int d^4x d^2\theta d^2\tilde{\theta}\mathcal{F}(\Psi), \tag{1.1.9}$$

where the **prepotential** $\mathcal{F}(\Psi)$ is holomorphic in Ψ, is itself $\mathcal{N} = 2$ supersymmetric. The only constraint imposed by renormalizability is the power in Ψ. For low energy effective actions, this constraint is removed and we only need to worry about *holomorphicity*. The integration over $d^2\tilde{\theta}$ yields the expression in $\mathcal{N} = 1$ language, which is

$$S = \mathfrak{Im}\ \frac{1}{8\pi}\int d^4x \left\{ \int d^2\theta \mathcal{F}_{ab}(A) W^{a\alpha} W^b_\alpha + 2\int d^2\theta d^2\bar{\theta}\left(A^\dagger e^{-2gV}\right)\mathcal{F}_a(A)\right\}. \tag{1.1.10}$$

To see why, suppose that the prepotential can be given by a Taylor expansion in Ψ,i.e.,

$$\mathcal{F}(\Psi) = \sum_{n=0}^{\infty} f_n \Psi^n. \tag{1.1.11}$$

Then, at the term with power n, the only terms with the right number of $\tilde{\theta}$'s are the ones with $A^{n-2}W^2$ and $A^{n-1}G$. How do we get these? Well, to get the term with $A^{n-2}W^2$, there are $n(n-1)/2$ ways: we choose two of the n factors in the power to give W^2, then get the A^{n-2} contribution from the remaining $n-2$. For the $A^{n-1}G$ term we follow the same logic: there are n choices of factor to give the G contribution. All in all, we have:

$$\int d^2\tilde{\theta}\Psi^n = \frac{n(n-1)}{2}A^{n-2}W^2 + nA^{n-1}G = \frac{1}{2}\frac{\partial^2 A^n}{\partial A^2}W^2 + \frac{\partial A^n}{\partial A}G. \tag{1.1.12}$$

Going back to the Taylor expansion, we find equation (1.1.10). The determination of the exact prepotential for general QFTs is not straightforward. However, the constraints imposed by $\mathcal{N}=2$ supersymmetry allow us to be in a good position to tackle this problem. In the next section, we will describe one of the paths one can take to do so: Seiberg-Witten theory.

Finally, the explicit expression for the action (1.1.6) in terms of the components, after integration of the auxiliary fields, is given by[2]:

$$S = \frac{1}{g^2}\int d^4x \,\mathrm{tr}\Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g^2\frac{\vartheta}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} + (D_\mu\phi)^\dagger D^\mu\phi - \frac{1}{2}[\phi,\phi^\dagger]^2 \\ - i\lambda \not{D}_\mu\bar{\lambda} - i\bar{\psi}\not{D}_\mu\psi - i\sqrt{2}[\lambda,\psi]\phi^\dagger - i\sqrt{2}[\bar{\lambda},\bar{\psi}]\phi\Big]. \tag{1.1.13}$$

1.2 Seiberg-Witten Theory

Now that we have covered the basics of $\mathcal{N}=2$ supersymmetric gauge theories, we will study Seiberg-Witten theory [1] as promised. Before doing anything, though, it may be instructive to state precisely what we mean by Seiberg-Witten theory.

1.2.1 Setup

Seiberg-Witten theory is a framework dedicated to the study and determination of the low-energy effective action of $\mathcal{N}=2$ super Yang-Mills theories, which translates to the investigation of the moduli space of vacua of the theory. As we already mentioned before, the only constraint posed by $\mathcal{N}=2$ supersymmetry on the possible (effective) actions of the theory is holomorphicity of the prepotential. Therefore, we may start off from Equation (1.1.10).

First of all, let G be the gauge group of our theory. In the original papers, Seiberg and Witten considered the special simpler case of $G = SU(2)$. We will return to it later, namely when doing calculations, but for the initial remarks, it will be nice to be a little more general. As we have already seen, the $\mathcal{N}=2$ supersymmetric action had the following potential for the scalar fields:

$$V(\phi,\phi^\dagger) = \frac{1}{2g^2}[\phi,\phi^\dagger]^2. \tag{1.2.1}$$

[2]We define $\widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$

As we know, supersymmetry is spontaneously broken when we have a nonzero potential $V > 0$, whereas gauge symmetry is spontaneously broken when $\langle\phi\rangle \neq 0$. The potential, therefore, allows us to break gauge symmetry while simultaneously maintaining SUSY by choosing the vacuum of the theory to lie in a specific sector where the commutator $[\phi, \phi^\dagger]$ vanishes. How do we determine this sector? Since the fields in our multiplet, and $\phi, \phi^\dagger$ in particular, are in the adjoint representation of G, we need to look no further than the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g} = \mathrm{Lie}(G)$. We still have some left-over symmetry coming from the Weyl group $W(G)$ and therefore we conclude that the inequivalent choices of $\langle\phi\rangle$ are described by:

$$\mathcal{M}_{vac} = \mathfrak{t}/W(G). \tag{1.2.2}$$

In particular we notice that:

$$\dim \mathcal{M}_{vac} = \dim \mathfrak{t} = \mathrm{rank}\, G. \tag{1.2.3}$$

Now we come across the first difference between our sought after S^{eff}_{low} and Equation (1.1.10): the indices a, b will no longer be indices in the adjoint representation of the gauge group G. Instead, they will be indices of the adjoint representation of the leftover gauge symmetry group $G' = U(1)^{\mathrm{rank}\, G}$. The moduli space of the inequivalent non-vanishing vacua of scalars of the $\mathcal{N} = 2$ vector multiplet is known as the **Coulomb branch**. The quotient by $W(G)$ can be achieved by defining the physical coordinate:

$$u = \frac{1}{2}a^2. \tag{1.2.4}$$

The R-symmetry $SU(2)_I \times U(1)_I$ is broken to a smaller group. The $U(1)_I$ current is anomalous and therefore we have the global symmetry breaking $U(1)_I \to \mathbb{Z}_8$. This statement can be understood as coming from instanton contributions, the index theorem and the (also quantum) symmetry $\vartheta + 2\pi$. The scalars transform under this subgroup with charge ± 2 as we saw in the previous section and therefore there is a further breaking $\mathbb{Z}_8 \to \mathbb{Z}_4$ due to the vacuum expectation value. Since classically $u \propto a^2$, u has charge 4. Consequently, the group acting on $\mathcal{M}_{vac}$ is $\mathbb{Z}_2$. In other words, and this will be very important in the considerations to come, $u \to -u$ is a symmetry.

Before we move on, let us take some more time on the Higgs mechanism action in this particular theory. Recall Equation 1.1.13. Let us consider $SU(2)$ from now on and choose

$$\langle \phi \rangle = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}.$$

We know that the gauge bosons associated to the broken gauge symmetries gain their masses from the kinetic term for the Higgs field:

$$\begin{aligned}(D_\mu \phi)^\dagger D^\mu \phi &= \left(\partial_\mu \tilde{\phi} + ig[A_\mu, \langle \phi \rangle]\right)^\dagger \left(\partial^\mu \tilde{\phi} + ig[A^\mu, \langle \phi \rangle]\right) \\ &= \left(\partial_\mu \tilde{\phi}\right)^\dagger \partial^\mu \tilde{\phi} + \left(\partial_\mu \tilde{\phi}\right)^\dagger \left(ig[A^\mu, \langle \phi \rangle]\right) - ig[A_\mu, \langle \phi \rangle] \partial^\mu \tilde{\phi} \\ &\quad + g^2 |[A_\mu, \langle \phi \rangle]|^2. \end{aligned} \tag{1.2.5}$$

This last equation is really telling. We see that a choice of vacuum expectation value indeed leads to a specific “electromagnetism", since the commutator would vanish for the A_μ in $\mathfrak{t}(G)$ and this specific gauge boson remains massless. The way the spinors get massive is easy to see. For the scalars:

$$[\phi, \phi^\dagger]^2 = \left([\langle \phi \rangle, \langle \phi \rangle^\dagger] + [\langle \phi \rangle, \tilde{\phi}^\dagger] + [\tilde{\phi}, \langle \phi \rangle^\dagger] + [\tilde{\phi}, \tilde{\phi}^\dagger]\right)^2, \tag{1.2.6}$$

and we recall that $[\langle \phi \rangle, \tilde{\phi}^\dagger] \neq 0$ for the two $\tilde{\phi}$'s that are not in $\mathfrak{t}(G)$. Then, the $[\langle \phi \rangle, \tilde{\phi}^\dagger]^2$ leads to mass terms for these specific $\tilde{\phi}$'s. The same idea goes for the $[\tilde{\phi}, \langle \phi \rangle^\dagger]^2$ term.

It is important to make a remark here. We basically just saw that the Higgs mechanism leads to the breaking $SU(2) \to U(1)$ as well as the initial three massless $\mathcal{N} = 2$ vector multiplets becoming one massless $\mathcal{N} = 2$ vector multiplet together with other two massive $\mathcal{N} = 2$ vector multiplets. We know that massive representations of the supersymmetric algebra are generally bigger, i.e. have more states, than massless ones, unless we have central charges coming into play. In that case, we can have "short"representations, named BPS states, which saturate the BPS bound $m \geq \sqrt{2}|Z|$, where Z is the central charge. We thus see that the existence of the short massive representations indicates that we have BPS states in our system. These types of states, although not the ones we just saw, will play a prominent role in what is to come.

The classical action is given by (1.1.8) and Seiberg showed [2] that the only perturbative correction comes from 1-loop contributions, such that the perturbative effective action is

$$\mathcal{F}_{pert}^{1-loop} = \frac{i}{2\pi} \Psi^2 \ln\left(\frac{\Psi^2}{\Lambda^2}\right) \tag{1.2.7}$$

where Λ is the dynamically generated scale. The only other corrections are non-perturbative and come from instanton contributions. In all, we have

$$\mathcal{F}^{eff} = \frac{i}{2\pi}\Psi^2 \ln\left(\frac{\Psi^2}{\Lambda^2}\right) + \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{\Psi}\right)^{4k} \Psi^2, \tag{1.2.8}$$

where $\mathcal{F}_k$ are numbers which must be determined. We now explain this result. Start by considering a *perturbative* correction to the classical action. The change in the *Lagrangian* should be given by the anomaly of the $U(1)_I$, which is:

$$\delta\mathcal{L}_{pert} = -\frac{\delta\alpha}{4\pi^2} F_{\mu\nu}\widetilde{F}^{\mu\nu}, \tag{1.2.9}$$

where $\delta\alpha$ parametrizes the transformation. If one recalls Equation 1.1.10, one concludes that this equation can be translated to a change in the prepotential as:

$$\mathcal{F}''_{pert}(A + 2i\delta\alpha A) - \mathcal{F}''_{pert}(A) = -\frac{4\delta\alpha}{\pi} \Rightarrow \frac{\mathcal{F}''_{pert}(A + 2i\delta\alpha A) - \mathcal{F}''_{pert}(A)}{2i\delta\alpha A} = \frac{2i}{A\pi}. \tag{1.2.10}$$

On the right-hand side we have the definition of a derivative with respect to A and we have thus derived the following relation:

$$\frac{\partial^3 \mathcal{F}_{pert}(A)}{\partial A^3} = \frac{2i}{\pi A} \tag{1.2.11}$$

which can be integrated to

$$\mathcal{F}_{pert}(A) = \frac{i}{2\pi} A^2 \log\left(\frac{A}{\Lambda}\right)^2. \tag{1.2.12}$$

It is now easy to obtain the effective coupling constant $\tau(a)$ at 1-loop order:

$$\tau(a) = \frac{\partial^2 \mathcal{F}_{pert}(a)}{\partial a^2} = \frac{2i}{\pi} \log\left(\frac{a}{\Lambda}\right). \tag{1.2.13}$$

We will justify this last equation in the following subsection.

As for the non-perturbative corrections, instantonic contributions corresponding to instanton number k are proportional to $e^{-8\pi^2 k/g^2}$. If we now use the β-function[3] to express:

$$\frac{1}{g^2} = \frac{1}{2\pi^2} \log\left(\frac{A}{\Lambda}\right) \tag{1.2.14}$$

[3]The β-function of pure $\mathcal{N} = 2, d = 4$ SYM with gauge group $SU(2)$ is $\beta(g) = -\frac{1}{4\pi^2} g^3$.

we find that

$$e^{-8\pi^2 k/g^2} = \left(\frac{\Lambda}{A}\right)^{4k}. \tag{1.2.15}$$

By assigning to Λ the same charge $[2]_{U(1)_I}$ as A (which makes sense since the scaling is related to a), the instantonic contribution becomes neutral and the full $U(1)_I$ symmetry is restored. Since the prepotential has charge $[4]_{U(1)_I}$ in the globally $U(1)_I$-symmetric theory, the instanton contributions should be proportional to A^2 in order to guarantee the proper transformation laws.

1.2.2 $SL(2,\mathbb{Z})$ Duality

Now that we have the coordinates $a, \bar{a}$, we can start to investigate the structure of the moduli space of vacua. We look back once again at our trusty Equation (1.1.10). The Kähler potential is easily identified:

$$K(A,\bar{A},V) = \mathfrak{Im}\left(\bar{A}\frac{\partial\mathcal{F}(A)}{\partial A}\right) \tag{1.2.16}$$

It gives the metric for the space of fields by

$$K_{a\bar{a}} = \frac{\partial^2 K}{\partial a \partial \bar{a}} = \frac{\partial^2}{\partial a \partial \bar{a}}\mathfrak{Im}\left(\bar{A}\frac{\partial\mathcal{F}(A)}{\partial A}\right) = \mathfrak{Im}\left(\frac{\partial^2\mathcal{F}(a)}{\partial a^2}\right) \tag{1.2.17}$$

since, as we know, $\mathcal{F}(A)$ is holomorphic in the chiral superfield. Thus, we conclude that the Kähler metric for the space of fields, and consequently the moduli space of vacua, is given by:

$$ds^2 = K_{a\bar{a}} da d\bar{a} = \mathfrak{Im}\left(\frac{\partial^2\mathcal{F}(a)}{\partial a^2}\right) da d\bar{a}. \tag{1.2.18}$$

We finally notice that

$$\mathfrak{Im}\left(\frac{\partial^2\mathcal{F}(a)}{\partial a^2}\right) = \mathfrak{Im}\ \tau(a) \tag{1.2.19}$$

simply by reading the first term in Equation (1.1.10). Therefore, $\mathcal{N} = 2$ SUSY implies that the metric of the moduli space of vacua is given by the effective gauge coupling constant $\tau(a)$:

$$ds^2 = \mathfrak{Im}\tau(a) da d\bar{a}. \tag{1.2.20}$$

Since this new effective coupling constant is a holomorphic function on a, $\mathfrak{Im}\tau(a)$ is a harmonic function and therefore cannot have a minimum. The metric on the moduli space of vacua is therefore not positive definite. This in turn implies, after we recall that

$$\tau(a) = \frac{\vartheta}{2\pi} + i\frac{4\pi}{g^2(a)}, \tag{1.2.21}$$

that $g^2(a) < 0$ for some values of a, which leads to an unstable theory, as the potential is $\propto \frac{1}{g^2}[\phi, \phi^\dagger]^2$ and as such would not have a minimum. Therefore, if the theory is to be stable, these coordinates must be valid only locally. We will see that this is indeed the case.

What about other parametrizations for $\mathcal{M}_{vac}$? Start with the component field expanded Lagrangian

$$\frac{1}{32\pi}\mathfrak{Im}\int \tau(a)\big[F + i\star F\big]^2 = \frac{1}{16}\mathfrak{Im}\int \tau(a)\big[F^2 + i\star F\wedge F\big]. \tag{1.2.22}$$

We now impose the Bianchi identity $dF = 0$ as a constraint by adding a term with Lagrange multiplier V_D to the Lagrangian. Instead of treating F as the curvature coming from the gauge fields A, we treat it as a dynamical field of its own and integrate over it in the path integral. The constraint term contributes as

$$\begin{aligned}\frac{1}{8\pi}\int V_D dF &= \frac{1}{8\pi}\int d^4x V_{D\mu}\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}\\ &= -\frac{1}{8\pi}\int d^4x\epsilon^{\mu\nu\rho\sigma}\partial_\nu V_{D\mu}F_{\rho\sigma}\\ &= \frac{1}{8\pi}\int d^4x\epsilon^{\nu\mu\rho\sigma}\partial_\nu V_{D\mu}F_{\rho\sigma}\\ &= \frac{1}{8\pi}\int F\wedge\star F_D\\ &= -\frac{1}{16\pi}\mathfrak{Im}\int \big(F_D - i\star F_D\big)\wedge\big(F - i\star F\big).\end{aligned} \tag{1.2.23}$$

Thus, we have something like:

$$\int \mathcal{D}F \exp\left[-\frac{1}{16\pi}\mathfrak{Im}\int\Big(\tau(a)[F^2 + i\star F\wedge F] + [F_D - i\star F_D]\wedge[F - i\star F]\Big)\right] \tag{1.2.24}$$

which leads to the similar lagrangian for the dual field

$$-\mathfrak{Im}\frac{1}{16\pi}\frac{1}{\tau(a)}[F_D^2 + i\star F_D\wedge F_D]. \tag{1.2.25}$$

Without much more work, this can be recalculated for the superspace notation, which yields:

$$\frac{1}{8\pi}\mathfrak{Im}\int d^2\theta \frac{-1}{\tau(A)} W_D^2. \tag{1.2.26}$$

In order to have the whole theory in the dual description, we still need to transform the chiral multiplet A kinetic term:

$$\mathfrak{Im}\int d^4\theta h(A)\bar{A} = \mathfrak{Im}\int d^4\theta A_D\bar{A} = -\mathfrak{Im}\int d^4\theta A\bar{A}_D = \mathfrak{Im}\int d^4\theta h_D(A_D)\bar{A}_D, \tag{1.2.27}$$

where we have defined $A_D = h(A)$ and $A_D = \partial\mathcal{F}(A)/\partial A$. Then

$$h_D(h(A)) = -A \Rightarrow \frac{dh_D}{dh}\frac{dh}{dA} = -1 \Rightarrow h'_D(A_D) \equiv \tau_D(A_D) = \frac{-1}{\tau(A)} \tag{1.2.28}$$

which means that the duality transformation $A_D = h(A)$ and $h_D = -A$, given by the S generator of $SL(2,\mathbb{Z})$:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.2.29}$$

has the role of inverting the $\tau(a)$ coupling and exchanges strong coupling with weak coupling and vice-versa. Another symmetry comes from the periodicity of the ϑ angle, leading to $\tau(a) \mapsto \tau(a) + 1$ and is given by the generator:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{1.2.30}$$

We are considering the action of $SL(2,\mathbb{Z})$ on $\tau(a)$ as

$$\tau(a) \mapsto \frac{a\tau + b}{c\tau + d} \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}). \tag{1.2.31}$$

In terms of the coordinates (a_D, a) defined from the moduli space, the duality group acts as:

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} aa_D + ba \\ ca_D + da \end{pmatrix}. \tag{1.2.32}$$

Let us go back to the metric

$$ds^2 = \mathfrak{Im}\left(\frac{\partial^2\mathcal{F}(a)}{\partial a^2}\right) da d\bar{a} \to \mathfrak{Im}\frac{\partial a_D}{\partial a} da d\bar{a} = \mathfrak{Im} da_D d\bar{a} \tag{1.2.33}$$

and introduce some mathematical structure. As we know, the "true" local coordinates of the moduli space $\mathcal{M}_{vac}$ are $u, \bar{u}$. Therefore, we introduce an auxiliary complex space X locally described by $(a_D(u), a(u))$ such that we have a symplectic form $\omega = da_D \wedge d\bar{a}$. The relation between these distinct spaces is given simply by a map

$$\begin{aligned} f : \mathcal{M}_{vac} &\to X \\ u &\mapsto (a_D(u), a(u)) \end{aligned} \tag{1.2.34}$$

from which we can take the Kähler form on $\mathcal{M}_{vac}$ to be $f^*\omega$, yielding:

$$ds^2 = \mathfrak{Im}\,\frac{da}{du}\frac{d\bar{a}_D}{d\bar{u}}dud\bar{u} = -\frac{i}{2}\Big(\frac{da_D}{du}\frac{d\bar{a}}{d\bar{u}} - \frac{da}{du}\frac{d\bar{a}_D}{d\bar{u}}\Big)dud\bar{u}. \tag{1.2.35}$$

which makes $SL(2,\mathbb{Z})$ invariance manifest. As a matter of fact, from the metric we have $Sp(2,\mathbb{R}) \times \mathbb{C} \simeq SL(2,\mathbb{R}) \times \mathbb{C}$ invariance, but further considerations, which we will shortly investigate, constrain the field $\mathbb{R}$ to $\mathbb{Z}$ and invalidate the translational symmetry.

Before we move on, a final remark: we have introduced an auxiliary space X, which is still mysterious, where a $SL(2,\mathbb{Z})$ symmetry acts. Notice also that this space can be thought of as being related to a bundle over $\mathcal{M}_{vac}$. As of right now, it may not seem really important, but it is a pivotal actor in the solution, as we shall soon learn.

1.2.3 BPS states and singularities

As promised, in this section we go back to the matter of BPS states in the theory and how they may help us solve it. For $u \neq 0$, we observe the Higgs mechanism taking place in the theory. We end up with massive particles, more precisely, two out of the three $\mathcal{N} = 2$ vector multiplets, as we already saw. At a first glance, this seems odd. In supersymmetric theories, massless representations have $2^{\mathcal{N}}$ states, whereas massive representations have $2^{2\mathcal{N}}$. Since the Higgs mechanism does not produce any new particles, there seems to be a problem, since we started with a massless multiplet. We need not be worried, though: the assertion above only holds for theories without a central extension, as one should recall. For theories with a central extension, massive states in a "small" representation are the BPS states.

Taking the gauge boson related to the unbroken $U(1)$ symmetry and endowing it with the interpretation of electromagnetism, the BPS states have masses

$$m^2 = 2|Z|^2 = 2|a(n_e + \tau_{cl} n_m)|^2, \tag{1.2.36}$$

i.e., the non-vanishing Higgs expectation value introduces a central extension in the theory. The electric charge is denoted by n_e and the magnetic charge by n_m in the previous equation. Since this computation has been extensively redone in many different places, we refer the reader to the original paper [3].

Quantum effects lead to a slight, but important, change in Equation (1.2.36). Indeed, we saw that the S-duality of the effective theory relates a to a_D, which already indicates that we should have something different. To check how we should explicitly write the expression for the masses of BPS states, we suppose we have some hypermultiplet $M, \bar{M}$ with electric charge n_e. In order to couple it to the $\mathcal{N} = 1$ vector multiplet in a $\mathcal{N} = 2$ supersymmetric way, one should also add a superpotential as follows:

$$\sqrt{2} n_e A M \bar{M}. \tag{1.2.37}$$

We easily see that such a term indeed gives a mass to M when $\langle A \rangle \neq 0$. More precisely, the mass is equal to $m = a n_e$. The same reasoning, when applied to a monopole with magnetic charge n_m followed by a generalization to dyons leads to

$$Z = a n_e + a_D n_m. \tag{1.2.38}$$

We see that the corrected expression for Z now has manifest duality, as well as the right semi-classical limit, where $a_D \sim \tau_{cl} a$.

This realization will end up restraining the duality studied above. Indeed, the first thing we should notice is that Z should be preserved under monodromies, after all it gives the masses of particles. Therefore, the freedom we had before, namely $SL(2,\mathbb{R}) \times \mathbb{C}^2$ on X should be reduced to $SL(2,\mathbb{R})$, since adding constants to (a_D, a) does not preserve Z. Furthermore:

$$Z = (n_e, n_m) \cdot \begin{pmatrix} a_D \\ a \end{pmatrix} \mapsto (n_e, n_m) T M \begin{pmatrix} a_D \\ a \end{pmatrix} \tag{1.2.39}$$

imposes the condition $T = M^{-1}$. Also, $M, M^{-1} \in SL(2,\mathbb{Z})$ as opposed to $SL(2,\mathbb{C})$, since the electric and magnetic charges are integers, and should remain as such. Therefore the symmetry group of X should indeed be $SL(2,\mathbb{Z})$.

Now let us study the monodromies observed in the u-plane. From the one-loop correction to the prepotential (1.2.7) at the limit of large a

$$\mathcal{F}_{pert}^{1-loop} = \frac{i}{2\pi} a^2 \ln\left(\frac{a^2}{\Lambda^2}\right) \tag{1.2.40}$$

we can calculate the behavior of the dual coordinate a_D

$$a_D \equiv \frac{\partial \mathcal{F}_{pert}^{1-loop}}{\partial a} = \frac{2i}{\pi} a \ln\frac{a}{\Lambda} + \frac{i}{\pi} a. \tag{1.2.41}$$

Here, a_D is clearly not a single-valued function of a at this limit. Going back to the reasoning of X as a bundle over $\mathcal{M}_{vac}$, we can analyze the monodromy at large a, or "infinity", by using the classical mapping $u = \frac{1}{2}a^2$ which is valid in this limit. Then, let us consider how $(a_D(u), a(u))$ change along the loop $u \to e^{2\pi i}u$.

$$a = \sqrt{2u} \mapsto a' = \sqrt{2e^{2\pi i}u} = e^{\pi i} a \Rightarrow a \mapsto -a \tag{1.2.42}$$

$$a_D \mapsto -\frac{2i}{\pi} a \ln\frac{-a}{\Lambda} - \frac{i}{\pi} a = -\frac{2i}{\pi} a \Big(\ln\frac{a}{\Lambda} + \pi i \Big) - \frac{i}{\pi} a = -a_D + 2a \Rightarrow a_D \mapsto -a_D + 2a. \tag{1.2.43}$$

and the monodromy transformation is given by the matrix

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}. \tag{1.2.44}$$

It is interesting to note that the classical Weyl reflection $a \mapsto -a$, and consequently $a_D \mapsto -a_D$, is indeed included in this transformation, while the perturbative quantum correction can be isolated. It is given by

$$M_\infty^q = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}. \tag{1.2.45}$$

Another interesting thing to see is how to extract the transformation of the magnetic and electric charges of general (n_m, n_e) dyons under M_∞. Masses shouldn't change under monodromies, and therefore the transformation which acts on the charge vector should be M_∞^{-1}. A quick computation shows that:

$$M_\infty^{-1} = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix} \tag{1.2.46}$$

and we have the following transformation $(n_m, n_e) \mapsto (-n_m, -n_e - 2n_m)$.

The reader may have two questions: are there more monodromies in the strong coupling region of u and why did the author write the plural singularities in the title of this section? The answer to both questions is tied and the time has come to tackle them. Points of singularity arise when some aspect of the theory does not work properly near that particular place. In our case, at strong coupling the integration of massive particle states (the ones which lead to problems are assumed to belong to spin$\leq 1/2$ massive multiplets) to obtain the low-energy effective theory may become *illegal* at specific points of $\mathcal{M}_{vac}$. Why is that so? Recall that the masses of BPS states are given by

$$m = \sqrt{2}|an_e + a_D n_m| \tag{1.2.47}$$

and therefore depend on the coordinates $(a_D(u), a(u))$ of X. But why should this matter? What guarantees that these multiplets are BPS? The answer is simple: $\mathcal{N} = 2$ supersymmetry. Multiplets with spin$\leq 1/2$ have 4 states in total (two Weyl fermions and two complex scalars). At other points in $\mathcal{M}_{vac}$, these multiplets should be massive (they are only massless at the singularity which they generate). Now, massive multiplets have $2^{2\mathcal{N}}$ states in general and $2^{\mathcal{N}}$ if they are BPS. Substituting $\mathcal{N} = 2$, we see that the type of multiplets we are considering (with four states in total) must be BPS in a $\mathcal{N} = 2$ supersymmetric theory. Furthermore, this means that there are indeed certain values of $(a_D(u), a(u))$ where dyons and monopoles become massless. Namely

$$\begin{aligned} \text{dyons with } n_e = -n_m &\Rightarrow a_D - a = 0 \\ \text{monopoles} &\Rightarrow a_D = 0, \text{any } a \end{aligned} \tag{1.2.48}$$

are examples of such points. Therefore, to complete the explanation, at these points, the integration over massive states becomes faulty, as we are integrating over extra states which should be included locally. It is precisely this phenomenon that leads to extra monodromies. By now, hopefully we have been able to justify the title of this section. It is also important to note that, due to the symmetry of $u \to -u$, these monodromies at strong coupling come in pairs.

Let us start with the analysis of a monopole. Since we are looking for a u_0 such that $a_D(u_0) = 0$, this should be easier than the one for a general dyon. Before we proceed, notice that we need to perform a duality transformation in order to be able to correctly study the monopole. Near the point of interest, the theory is a

$U(1)$ gauge theory with a hypermultiplet. Knowledge of the β-function for this theory combined with the feature that the scalars and spinors in the hypermultiplet have the same charge under $U(1)$, as well as the fact that the energy scale μ of the theory is naturally given by a_D, led Seiberg and Witten to

$$\tau_D(a_D) = -\frac{i}{\pi}\ln a_D. \tag{1.2.49}$$

Near that point, we may Taylor expand and express a_D as $a_D \approx c_0(u-u_0)$. Now, as we saw earlier,

$$\tau_D(a_D) = -\frac{1}{\tau(a)} = -\frac{da}{da_D}, \tag{1.2.50}$$

and therefore

$$\begin{aligned} a(a_D) &= \int da_D \frac{i}{\pi}\ln a_D \\ &\Rightarrow a = a_0 - a_D + \frac{i}{\pi}a_D\ln a_D \approx a_0 - c_0(u-u_0) + \frac{i}{\pi}c_0(u-u_0)\ln c_0(u-u_0). \end{aligned} \tag{1.2.51}$$

Let us investigate how a changes under $u \mapsto e^{2\pi i}u$. Since we are taking a loop of u around u_0, the logarithm changes by $+2\pi i$. More precisely:

$$a \mapsto a_0 - c_0(u-u_0) + \frac{i}{\pi}\Big(2\pi i c_0(u-u_0) + c_0(u-u_0)\ln c_0(u-u_0)\Big) = a - 2a_D. \tag{1.2.52}$$

This means that the monodromy at u_0 is given by

$$M_{u_0} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}. \tag{1.2.53}$$

Note that $(1,0)M_{u_0} = (1,0)$, i.e., the charge of the monopole is not changed by the action of the corresponding monodromy.

Now, the analysis of the third singularity is easy, since we can use the fact that $M_\infty = M_{u_0}M_{-u_0}$. A quick calculation reveals that

$$M_{-u_0} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}. \tag{1.2.54}$$

It is really nice to find the missing monodromy this easily. It would be even better to know what type of particle generates it. Notice that the charges of the particles that generate a given monodromy should remain unchanged by the action of this monodromy. This means that

$$\big(n_e, n_m\big)M_{-u_0} = \big(-n_e - 2n_m, 2n_e + 3n_m\big) = \big(n_e, n_m\big) \Rightarrow n_e = -n_m \qquad (1.2.55)$$

and, in particular, $(1, -1)$. We conclude that the $(1, -1)$ dyon generates the third monodromy.

Note that, through the action of the monodromy at infinity, we may reach other dyons. Indeed, the $(1, -1)$ dyon is transformed to:

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}^n = \Big((-1)^n, \;\; (-1)^{n+1}(2n+1)\Big)\,. \qquad (1.2.56)$$

By doing the same with the dyon $(-1, 1)$, we arrive at

$$\big((-1)^{n+1}, (-1)^n(2n+1)\big).$$

Similarly, for the monopole $(1, 0)$, we have

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}^n = \Big((-1)^n \;\;\; (-1)^{n+1}2n\Big)\,. \qquad (1.2.57)$$

Finally, starting with the monopole $(-1, 0)$ we arrive at

$$\big((-1)^{n+1}, (-1)^n 2n\big).$$

This exhausts the possibilities of stable dyons in the theory and therefore we conclude that indeed we are not missing anything by considering just two monodromies.

1.2.4 Seiberg-Witten curve and solution

We are now finally ready to study the solution given by Seiberg and Witten. Let us first recall that, although the actual quantum moduli space of vacua $\mathcal{M}_{vac}$ is the u-plane with the already familiar monodromies

$$M_{-u_0} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix},\; M_{u_0} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix},\; M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \qquad (1.2.58)$$

at the singularities $-u_0, u_0$ and ∞, deeper knowledge of coordinates $\big(a_D(u), a(u)\big)$ is what allows us to solve the theory. This leads us to consider a flat principal $SL(2,\mathbb{Z})$-bundle $\mathcal{V}$ over $\mathcal{M}_{vac}$ such that $a_D(u), a(u) \in \Gamma(\mathcal{M}_{vac}, \mathcal{V})$. To each point $u \in \mathcal{M}_{vac}$, we may assign a curve

$$\Sigma_u : y^2 = (x-u_0)(x+u_0)(x-u) \tag{1.2.59}$$

by introducing extra complex coordinates (y, x). This surface is a real two-

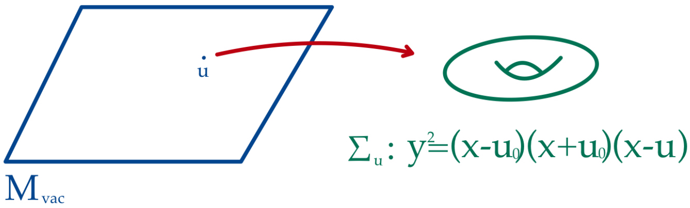

Figura 1.1: Seiberg-Witten curve association.

dimensional (complex one-dimensional) Riemann surface with four branch points and describes a torus. We regard x as parametrizing the Riemann sphere, such that Σ is a double-cover of $\mathbb{C} \cup \{\infty\}$. Now that we have built this curve, we may shed more light on the bundle we just mentioned. We have the duality group (or gauge group of the bundle) $SL(2,\mathbb{Z})$ and the curve Σ_u which is basically a torus. We know that $SL(2,\mathbb{Z})$ is the symmetry group of the torus, and particularly $H_1(\Sigma_u, \mathbb{C})$. Therefore, a natural realization of the fibers of $\mathcal{V}$ is precisely $H_1(\Sigma_u, \mathbb{C})$. Notice that, by Poincaré duality:

$$H_1(\Sigma_u, \mathbb{C}) \simeq H^1(\Sigma_u, \mathbb{C}) \tag{1.2.60}$$

since $\dim \Sigma_u = 2$. This means that, if we have a local trivialization of the bundle given by one-cycles $\gamma_A, \gamma_B \in H_1(\Sigma_u, \mathbb{C})$ with intersection number

$$\gamma_A \cdot \gamma_B = \int_{\Sigma_u} \eta_A \wedge \eta_B = 1,$$

we can get an equivalent description using meromorphic $(1,0)$-forms on Σ_u which we denote by λ_1 and λ_2. Also, any section $\sigma \in \Gamma(\mathcal{M}_{vac}, \mathcal{V})$ can now be written as

$$\sigma(u) = \sum_{i=2}^{2} c_i(u)\lambda_i \in \mathcal{V}_u \simeq H^1(\Sigma_u, \mathbb{C}). \tag{1.2.61}$$

The suitable choice for these base forms is:

$$\lambda_1 = \frac{dx}{y}, \quad \lambda_2 = \frac{xdx}{y}, \tag{1.2.62}$$

such that the τ-parameter of the torus is given by

$$b_i = \oint_{\gamma_i} \lambda_1 \text{ and } \tau_u = \frac{b_A}{b_B}, \text{such that } \mathfrak{Re}\ \tau_u, \mathfrak{Im}\ \tau_u > 0. \tag{1.2.63}$$

Let us now stop and recapitulate. We just saw that we may construct an auxiliary object, the Seiberg-Witten curve Σ_u, based on the monodromies found in $\mathcal{M}_{vac}$ by field-theoretic considerations, for each $u \in \mathcal{M}_{vac}$. The first cohomology (and equivalently the first homology) group of Σ_u gives the fiber $\mathcal{V}_u$ of a principal $SL(2,\mathbb{Z})$-bundle. Furthermore, and most importantly, we managed to associate $u \in \mathcal{M}_{vac}$ to the τ-parameter of the associated SW-curve, which satisfies $\mathfrak{Im}\ \tau_u > 0$. If we could find a way to, within this framework, associate $a_D(u), a(u)$ to the τ-parameter while also satisfying the correct monodromies, we would have found the explicit solution to the problem. One way to do that is by defining:

$$a_D = \oint_{\gamma_A} \lambda_{SW} \qquad a = \oint_{\gamma_B} \lambda_{SW}, \tag{1.2.64}$$

for a special one-form λ_{SW} which satisfies

$$\frac{d\lambda_{SW}}{du} = f(u)\lambda_1. \tag{1.2.65}$$

This particular differential gets the name of *Seiberg-Witten differential*. If this is true, then it follows that:

$$\tau(a) = \frac{\partial a_D}{\partial a} = \frac{da_D/du}{da/du} = \frac{\oint_{\gamma_A} \frac{d\lambda_{SW}}{du}}{\oint_{\gamma_B} \frac{d\lambda_{SW}}{du}} = \frac{f(u)\oint_{\gamma_A}\lambda_1}{f(u)\oint_{\gamma_B}\lambda_1} = \tau_u \Rightarrow \mathfrak{Im}\ \tau(a) = \mathfrak{Im}\ \tau_u > 0, \tag{1.2.66}$$

which is precisely what we wanted! This is promising, and the problem now boils down to finding λ_{SW} with the correct monodromy behavior for a_D, a.

So, what is the one-form λ_{SW} with the correct behavior? It turns out that the answer is

$$\lambda_{SW} = \frac{\sqrt{2}}{2\pi}\frac{\sqrt{x-u}dx}{\sqrt{x^2-u_0^2}} = \frac{\sqrt{2}}{2\pi}\frac{ydx}{x^2-u_0^2}. \tag{1.2.67}$$

Indeed:

$$\frac{d\lambda_{SW}}{du} = -\frac{\sqrt{2}}{2\pi}\frac{dx}{2\sqrt{(x-u)(x^2-u_0^2)}} = -\frac{\sqrt{2}}{4\pi}\frac{dx}{y} = -\frac{\sqrt{2}}{4\pi}\lambda_1 \tag{1.2.68}$$

satisfies Equation 1.2.65 for $f(u) = -\frac{\sqrt{2}}{4\pi}$. Taking the A-cycle of the torus to loop around $-u_0$ and u_0 and the B-cycle to wind around u_0 and u, the explicit expressions for the $(a_D(u), a(u))$ are

$$a = \frac{\sqrt{2}}{\pi}\int_{-u_0}^{u_0} dx \frac{\sqrt{x-u}}{\sqrt{x^2-u_0^2}} \qquad a_D = \frac{\sqrt{2}}{\pi}\int_{u_0}^{u} dx \frac{\sqrt{x-u}}{\sqrt{x^2-u_0^2}}. \tag{1.2.69}$$

Now we check whether or not the monodromies are indeed recovered. We start with the monodromy at ∞. By taking $u \to \infty$, the expression for a can be approximated by

$$a \approx \frac{\sqrt{2}}{\pi}\int_{-u_0}^{u_0} dx \frac{\sqrt{-u}}{\sqrt{x^2-u_0^2}} = \frac{\sqrt{-2u}}{\pi}\int_{-u_0}^{u_0} \frac{dx}{\sqrt{x^2-u_0^2}} = \sqrt{-2u} \tag{1.2.70}$$

since in the integral we have $-u_0 < x < u_0$ and therefore $x - u \to -u$. This means that a loop $u \mapsto e^{2\pi i}u$ implies $a \to -a$, and we recover what we found in the previous section. What about a_D?

$$a_D = \lim_{u\to\infty} \frac{\sqrt{2}}{\pi}\int_{u_0}^{u} dx \frac{\sqrt{x-u}}{\sqrt{x^2-u_0^2}} = \lim_{u\to\infty} i\frac{\sqrt{-2u}\ln u}{\pi}. \tag{1.2.71}$$

Then, taking $u \mapsto e^{2\pi i}u$:

$$i\frac{\sqrt{2u}}{\pi}\ln u \mapsto -i\frac{\sqrt{2u}}{\pi}\ln u - 2\frac{\sqrt{2u}}{\pi}2\pi i = -a_D + 2a. \tag{1.2.72}$$

In all, we find:

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \tag{1.2.73}$$

which is precisely what we had found before. This is an auspicious start!

Let's go to the strong coupling cases and consider the monodromy at u_0. For a

we need only evaluate the integral directly:

$$a = \frac{\sqrt{2}}{\pi} \int_{-u_0}^{u_0} dx \frac{\sqrt{x-u_0}}{\sqrt{x^2-u_0^2}} = \frac{\sqrt{2}}{\pi} \int_{-u_0}^{u_0} dx \frac{1}{\sqrt{x+u_0}} = \frac{4}{\pi}. \tag{1.2.74}$$

In order to find the behavior with respect to u locally around u_0, we can Taylor expand. This means we need to analyze the first derivative:

$$\frac{\partial a}{\partial u} = -\frac{1}{\sqrt{2}\pi} \int_{-u_0}^{u_0} \frac{dx}{\sqrt{x^2-u_0^2}\sqrt{x-u}} \approx -\frac{(u-u_0)\ln(u-u_0)}{2\pi} + \text{non div.} \tag{1.2.75}$$

which is in agreement with the calculations in the previous section. As expected, $a_D = 0$, since:

$$a_D = \frac{\sqrt{2}}{\pi} \int_{u_0}^{u_0} dx \frac{\sqrt{x-u}}{\sqrt{x^2-u_0^2}} = 0. \tag{1.2.76}$$

Nonetheless, let us expand the result around $u = u_0$. First, for general u:

$$a_D(u) = \sqrt{\frac{2}{-u_0}} \Big(2u_0 E\Big(\frac{u_0-u}{2u_0}\Big) - (u_0+u)K\Big(\frac{u_0-u}{2u_0}\Big)\Big). \tag{1.2.77}$$

If we expand this expression around $u = u_0$, we find the following behavior:

$$a_D(u) \propto \big(u-u_0\big) + O(u^2) \tag{1.2.78}$$

which precisely recovers the $a_D \mapsto a_D$ behavior found previously.

The monodromy at $-u_0$ can be found, as before, by making use of the two we already recovered. This means that we have indeed retrieved the monodromies found using field theory. Since that was the last condition the proposed solution had to satisfy in order to be right, we conclude that it is truly correct! Now, we just need to integrate the equation:

$$\frac{\partial \mathcal{F}(a)}{\partial a} = a_D \tag{1.2.79}$$

to find the prepotential!

Before we end the section, a few comments about the somewhat "artificial" objects Σ_u and λ_{SW} are in order. The previous pages introduced them without any compelling reason besides being elegant and giving the right answer. However, they do have physical meaning if one is willing to consider String Theory/M-Theory [6]. In this framework, one should regard the field theory as the low-energy limit theory of NS-5 brane configurations in type IIA String Theory, or a single M5-brane in M-theory. Then, the Seiberg-Witten curve is realized as the particular wrapping of said brane around an actual four-dimensional space.

We just described the Seiberg-Witten solution for pure $d = 4, \mathcal{N} = 2$ SYM with $SU(2)$ gauge group as done in [1]. In the remainder of the chapter we will quickly show how to generalize the formalism to study theories with gauge group $SU(N)$. The Seiberg-Witten curve Σ_u and Seiberg-Witten differential λ_{SW} will feature prominently as one should expect, although with a few modifications.

1.3 Seiberg-Witten Theory: other groups

Apart from the early considerations up until Subsection 1.2.2, we have only considered $d = 4, \mathcal{N} = 2$ SYM with gauge group $SU(2)$. By that, we mean that we have only explicitly analyzed the monodromies, BPS spectrum and SW curve and differential for this particular theory. This section will be aimed at providing such a more complete description for theories with $SU(N)$ gauge groups, as developed in [7, 8] soon after the original papers by Seiberg and Witten were published. Namely, the generalization is obtained by considering genus $N - 1$ Riemann surfaces and their periods. These hyperelliptic curves are parametrized in a different way than that of Equation (1.2.59). We should finally mention that, besides the physical realization of the Seiberg-Witten curve and prepotential, reference [6] also provides a generalization of the Seiberg-Witten solution to $d = 4, \mathcal{N} = 2$ SYM with quiver gauge groups, i.e., $\Pi_{i=1}^{n} SU(N_i)$.

1.3.1 Reviewing basics

Recall from our discussion in Subsection 1.2.1 that a generic vacuum expectation value for the scalar $\langle \phi \rangle \neq 0$ breaks the original gauge group from G to its maximal torus $U(1)^{\text{rank } G}$. In particular, if we start with $SU(N)$, the generic gauge symmetry breaking would be

$$SU(N) \to U(1)^{N-1}. \tag{1.3.1}$$

There are, thus, $N-1$ coordinates a_i and $N-1$ dual coordinates $a_{D,i}$. The original a_i may be combined in gauge-invariant coordinates given by

$$u_{n-1} \equiv \frac{1}{n}\langle \mathrm{tr}(\phi^n)\rangle, \quad n = 2, \cdots, N \tag{1.3.2}$$

or even by symmetric polynomials[4] in the a_i's, which we denote by s_n. These are the true coordinates of the moduli space of vacua:

$$\mathcal{M}_{vac} = \mathbb{C}^{\mathrm{rank}\ G}/W(G). \tag{1.3.3}$$

From the previous discussion for $SU(2)$, we should expect that each of the original coordinates is related to the integral over the periods of a given curve of a meromorphic one-form λ_{SW}. Thus, we should at least expect that the Seiberg-Witten curve related to these higher rank theories has genus $N-1$.

1.3.2 Finding the Seiberg-Witten toolkit

Although the realization that we should have a genus $N-1$ Riemann surface for our Seiberg-Witten curve is important, it is still very far from fully determining the correct candidates. In order to do so, we need to further investigate expected behaviors.

The first analysis one should make concerns specific values of a_i such that the symmetry breaking in Equation (1.3.1) is not completely carried out. More specifically, for special values of a_i, we can have more than just $N-1$ massless gauge bosons, i.e., some of the gauge bosons associated to generators of $\mathfrak{su}(n)$ which do not belong to the Cartan subalgebra may remain massless. The directions where this happens are singular (by the same reasoning about the appearance of extra massless states that we discussed earlier). Recall that the discriminant of the Seiberg-Witten curve encodes information about these directions or special regions. Indeed, one may characterize them as the regions described by u_n or s_n such that the discriminant of the SW curve associated to that specific theory vanishes. In the $SU(2)$ case, although the form of the curve is different from what we will introduce next, we observed that the tori were singular when we had $u = \pm u_0$. Note that *quantum* considerations were already taken into account when providing the explicit expression for the SW curve in Equation (1.2.59), and therefore the

[4]The symmetric condition is expected, as we know that the Weyl group of $SU(N)$ is the permutation group S_N.

massless states associated to singular curves correspond to the special monopoles and dyons that we have already considered. Right now, we are dealing only with *classical* considerations, and we stress that the extra massless states referred to above are non-abelian gauge bosons. Once again, knowledge of the previous sections reminds us that these are prohibited by the same quantum corrections alluded to above[5]. Nonetheless, we hope that by recalling this discussion through the already familiar example of $G = SU(2)$ we will have satisfactorily motivated the discussion about discriminants. We conclude that the curve should be given by a power p of the degree N polynomial $P_N(x, u_n)$ whose discriminant Δ_P encodes the information about regions of weaker symmetry breaking when taking the classical limit $\Lambda \to 0$.

In order to have the right amount of roots ($2N$) to yield a genus $N-1$ Riemann surface, the power of the polynomial should be $p = 2$. Furthermore, the quantum corrections should appear through the Λ, as already advertised. The power of Λ is dictated by the $\mathbb{Z}_{2N}$-charge assignment to (y, x). Assigning $[y]_{\mathbb{Z}_{2N}} = N$ and $[x]_{\mathbb{Z}_{2N}} = 1$, and knowing that $[\Lambda]_{\mathbb{Z}_{2N}} = 2$, we conclude that Λ should appear as Λ^{2N}. Thus, we conclude that the Seiberg-Witten curve for $SU(N), d = 4, \mathcal{N} = 2$ SYM should be given by:

$$y^2 = P_N^2(x, u_n) - \Lambda^{2N}. \tag{1.3.4}$$

For completion and in order to hopefully clarify the previous paragraph, the discriminant of *this* curve is the one which encodes quantum contributions and, therefore, the *correct* regions corresponding to the *actual* extra massless particles that may appear.

Now that we have defined the SW curve, we should declare the periods and the SW differential. The periods can be declared by choosing a basis of cycles satisfying the intersection conditions:

$$\gamma_i \cdot \gamma_{D,j} = \delta_{ij} \quad \gamma_i \cdot \gamma_j = 0. \tag{1.3.5}$$

The Seiberg-Witten differential given by:

$$\lambda_{SW} = \frac{x}{y}\frac{dP_N(x, u_n)}{dx}dx. \tag{1.3.6}$$

[5]Quantum considerations introduce powers of Λ into the discriminant, as mentioned in [7].

passes the non-trivial checks performed in the references. Finally, it is important to mention that the positivity condition on the metric $\mathfrak{Im}\tau_{ij}$ of the moduli space is ensured by the imaginary positive definite condition of the period matrix of the surface, as it coincides with τ_{ij}. This is nothing more than a generalization of the argument presented for the $SU(2)$ case.

Chapter 2

The instanton counting route

So far, we have studied the main features of $d = 4, \mathcal{N} = 2$ supersymmetric Yang-Mills theory, as well as one particular way of obtaining the Low-Energy Effective theory for the gauge group $SU(2)$ called Seiberg-Witten theory. The present chapter is devoted to another path to obtaining the Low-Energy prepotential of $d = 4$, $\mathcal{N} = 2$ SYM. Such path approaches the problem in a much more grounded way, as it focuses on the direct computation of the non-perturbative instanton contributions from Equation (1.2.8). Nevertheless, it is also very involved as it needs a working knowledge of Localization Techniques and Topological QFT's.

We will start by giving an introduction to the concept of Localization and to how it can be useful in Physics. In particular, we will focus on the Mathai-Quillen formalism, which provides the bridge between Mathematics and Physics. Next, we move on to connect the idea of Localization with TQFT's. The relation between them is very straightforward and particularly interesting since, as we shall also see, $d = 4, \mathcal{N} = 2$ SYM can be twisted to provide a theory in this class. We will then take these concepts even further by introducing the Ω-deformation [22]. Finally, we will see how this allows us to count the contribution of the instantons to the prepotential, deriving the Seiberg-Witten solution from a field-theoretic point of view.

2.1 Localizing Integrals

In this section we cover some mathematical background behind the Localization techniques used in Topological QFTs in Physics. We hope that by including this preamble the considerations that will follow in the subsequent sections will be more easily digested. Furthermore, we believe that the right amount of mathematical credentials are helpful in organizing arguments and conclusions. We will try, however, to maintain a palatable presentation for Physics audiences.

2.1.1 The Poincaré dual

A mathematical realization of Localization can be thought of as the following: suppose one has an n-manifold M, a closed oriented k-submanifold $S \subset M$ and a closed k-form $\omega \in \Omega_c^k(M)$ with compact support. Since the dimensions match, we can integrate ω, or more precisely[1], $i^*\omega$ on S. The question we may ask is: can we relate this integral to an integral over the bigger manifold M? Conversely, can an integration over M be reduced to an integration over S? It turns out that the answers for both questions are respectively yes and yes (in some cases)! To see why, we need to understand what **Poincaré duality** is.

The first step is to keep in mind the situation described in the beginning of the previous paragraph. If we further assume some nice properties[2] about M, then integration gives a non-degenerate pairing between two cohomologies of the manifold. More precisely, we have that:

$$\int \cdot \wedge \cdot : H_c^k(M) \otimes H^{n-k}(M) \to \mathbb{R} \tag{2.1.1}$$

is non-degenerate. The existence of such pairing induces a famous isomorphism:

$$\left(H_c^k(M)\right)^* \simeq H^{n-k}(M) \tag{2.1.2}$$

which is called *Poincaré duality*. It states that that integration over a k-dimensional manifold is dual, or corresponds, to an unique cohomology class of degree $(n-k)$. A notable consequence of this isomorphism, and in particular the one which will be most important to the considerations to come, is that there exists a unique[3] closed form $\eta_S \in \Omega^{n-k}(M)$ such that:

$$\int_S i^*\omega = \int_M \omega \wedge \eta_S. \tag{2.1.3}$$

The form η_S is called the **Poincaré dual** of S. Note that η_S defines a cohomology class $[\eta_S] \in H^{n-k}(M)$. Indeed, consider, instead of η_S, another differential given by $\eta_S + d\alpha \in \Omega^{n-k}(M)$. Then:

$$\int_M \omega \wedge (\eta_S \wedge d\alpha) = \int_M (\omega \wedge \eta_S + \omega \wedge d\alpha) = \int_M \left(\omega \wedge \eta_S + (-1)^n d(\omega \wedge \alpha)\right). \tag{2.1.4}$$

[1]Here we denote by $i : S \hookrightarrow M$ the inclusion of S into M.
[2]Namely that it is orientable and has a finite good cover.
[3]Up to an exact form, of course.

Now we use Stoke's Theorem to find:

$$\int_M \omega \wedge \eta_S + (-1)^n \int_{\partial M} \omega\alpha = \int_M \omega \wedge \eta_S. \tag{2.1.5}$$

Of course, we have used the fact that M is a closed manifold[4].

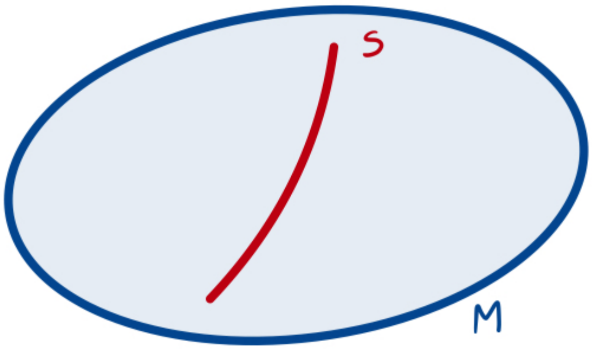

Figura 2.1: Schematic drawing for a 2-dimensional manifold with 1-dimensional submanifold. The Poincaré dual to S is a 1-form.

The main takeaway from the exposition above is that the Poincaré dual η_S can be seen as a "specially tailored δ-function" for the submanifold S. We mean this in the sense that the integration of ω times this "function" over the whole manifold can be *localized* to a subspace of M.

2.1.2 The Thom isomorphism and the Thom class

In this section we expand the notion of Poincaré dual and allow it to be useful in cases with some "verticality", i.e., in cases where M becomes the base manifold for a vector bundle E. In what follows, we take E to be an orientable n-bundle. We are thus taking the next step towards Physics, since vector bundles are common place in field theory, for example.

The first idea that needs to be introduced is that of integration along the fibers. In order to do that, we will consider the complex $\Omega^\bullet_{cv}(E)$ of the so-called differential forms with compact support in the vertical. As the name suggests, these are the differential forms defined on the vector bundle E which have the special property that the intersection of their support with the lift $\pi^{-1}(K)$ of a

[4]Compact and without boundary.

compact subspace $K \subset M$ is also compact. What makes this property interesting is that it guarantees that integration along fibers is a well-behaved operation. Indeed, notice that if $\omega \in \Omega^{\bullet}_{cv}(E)$, its support along the fibers is compact since $\pi^{-1}(p)$ is compact for every $p \in M$. Most importantly, the fact that we can integrate them means that integration along the fibers defines a map

$$\pi_* : \Omega^{\bullet}_{cv}(E) \to \Omega^{\bullet - n}(M) \tag{2.1.6}$$

which decreases the degree of the form by the dimension of the fibers. This map can be extended to the cohomology and in that case it becomes an *isomorphism* (after requiring extra properties about the vector bundle). It is denoted by:

$$\mathcal{T}^{-1} : H^{\bullet}_{cv}(E) \simeq H^{\bullet - n}(M). \tag{2.1.7}$$

The inverse of this isomorphism is famously know as the **Thom isomorphism** $\mathcal{T}$.

Notice that $\mathcal{T}$ takes $[1] \in H^0(M)$ to an unique cohomology class $[\Phi] \in H^n_{cv}(E)$, called the **Thom class**. A particular representative Φ is called the Thom representative. More generally, we have the following explicit form of the isomorphism:

$$\mathcal{T}\Big(\omega\Big) = \pi^* \omega \wedge \Phi, \omega \in \Omega^{\bullet}(M). \tag{2.1.8}$$

In particular, notice that integration of Φ along a fiber E_p should be such that $\int_{E_p} \Phi = 1$. Although we have been using the condition of vertical compactness, the same considerations can be translated to the analogously defined rapid-decay or gaussian-decay cohomology. Since the latter is more suitable to be readily applied to Physics, we consider it as our cohomology of our choice in what follows.

This is all fine, but how are these new constructions related to the Poincaré dual? Well, it so happens that the Poincaré dual class of the zero locus of a generic section $s : M \to E$ and the cohomology class of the pullback of the Thom representative $[s^*(\Phi)]$ by said section are one and the same! More precisely, for $s : M \to E$ generic and $\mathcal{Z}(s) = \{p \in M | s(p) = 0 \in E_p\}$:

$$\int_{\mathcal{Z}(s)} i^* \omega = \int_M \omega \wedge s^* \big(\Phi(E)\big). \tag{2.1.9}$$

By definition, if $s : M \to E$ is a generic section, then $s(M)$ intersects transversally with $0_E(M)$, the image of M under the zero section 0_E. In turn, this means that:

$$\begin{aligned}
&T_s + T_{0_E} = T_E \Rightarrow \dim\left(T_s + T_{0_E}\right) = \dim T_E \\
&\Rightarrow \dim T_s + \dim T_{0_E} - \dim I = \dim M + \text{rank } E \Rightarrow \dim I = \dim M - \text{rank } E \\
&\Rightarrow \dim I = \dim M - \text{rank } E
\end{aligned} \tag{2.1.10}$$

where $I \equiv s(M) \cap 0_E(M)$. Therefore, acting with s^{-1}, which preserves dimensions, on I means that the zero locus $\mathcal{Z}(s)$ is such that codim $\mathcal{Z}(s)$ = rank E. We conclude that the dimension of the tubular neighborhood of $\mathcal{Z}(s)$ in M is rank E. This first check on the dimensions is nice, as it means that $s^*(\Phi)$ and the Poincaré dual are both (rank E)-forms. It is, however, nothing more than a check.

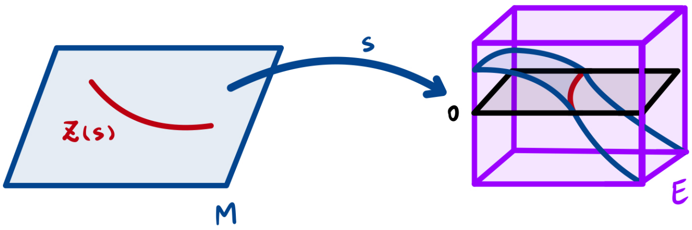

Figura 2.2: Schematic drawing of the zero locus $\mathcal{Z}(s)$ of a generic section s.

In order to see that both cohomology classes coincide, we will need to consider a specific representative of the cohomology, named **Mathai-Quillen representative**, which will be introduced shortly. We can give a taste of the reason, though. The main idea is that the representative will be an exact form with respect to some equivariant differential. Furthermore, it has a term $-|s|^2$ which appears on an exponent. Since the representative is exact (and furthermore closed), it can be rescaled to $-t|s|^2$ for some real number t, on which it does not depend (again due to exactness). Therefore, we may take the limit $t \to \infty$, which takes contributions only from the zero locus.

2.1.3 Equivariant Cohomology

The final ingredient needed for the establishment of the bridge between the mathematical idea of Localization and the Physical intuition is **Equivariant Cohomology**. This is the natural cohomology to consider when one is dealing with the action of a symmetry group G on a manifold M.

The main objective behind the definition of equivariant cohomology is to study the cohomology of G-manifolds without the ambiguities that arise when a symmetry is present. Of course, the first instinctive reaction would be considering the cohomology of M/G. However, this is a naïve guess since such quotient spaces usually suffer from singularities (for example, when G is a finite group and its action is not free, we get orbifolds). One should, therefore, tread more carefully.

There are many different models of equivariant cohomology, each one equivalent to the other. Physics thrives in algebraic settings, so it is only natural to seek out an algebraic model, as opposed to the more abstract topological one. There are many such algebraic models, differing by the complex and the differential used to define the cohomology. We shall use the **Cartan model**, which is comprised of the complex

$$\Omega_G^C(M) \equiv \left(S(\mathfrak{g}^*) \otimes \Omega^\bullet(M)\right)^G \tag{2.1.11}$$

and the differential

$$d_C = 1 \otimes d - \phi \otimes \iota_\xi, \tag{2.1.12}$$

where $S(\mathfrak{g}^*)$ is the symmetric algebra of $\mathfrak{g}^*$ and $\phi \in S(\mathfrak{g}^*)$. In the definition of $\Omega_G^C(M)$, the superscript G on the right-hand side denotes G-invariant forms, i.e., $\omega \in \Omega(M)^G \Leftrightarrow \mathcal{L}_\xi \omega = 0 \; \forall \; \xi \in \mathfrak{g}$. Notice that this requirement is needed in order for d_C^2 to be zero, since:

$$d_C^2 = -\phi \otimes d\iota_\xi - \phi \otimes \iota_\xi d = -\phi \otimes \left(d\iota_\xi + \iota_\xi d\right) = -\phi \otimes \mathcal{L}_\xi \tag{2.1.13}$$

by Cartan's magic formula. Unless we consider forms in $\Omega(M)^G$, d_C is not nilpotent and therefore cannot define a cohomology.

The important takeaway is that, in the Cartan model, the differential used gets contributions from the Lie algebra of the gauge group and it squares to the Lie derivative (or gauge transformation) of the differential forms. Therefore, it successfully fulfills the original objective of getting rid of symmetry ambiguities.

2.2 Localization: the Physics

Now that we have seen how to relate the Poincaré dual of the zero locus $\mathcal{Z}(s)$ of a generic section $s : M \to E$ to the Thom class of the bundle, we are able to apply these tools to Physics. In order to do that, we must introduce a particular representative of the Thom class through the Mathai-Quillen formalism. It turns out that the Mathai-Quillen representative has a very evocative form, namely that of a path integral! We shall, however, use a slightly modified version of the Mathai-Quillen representative developed by Atiyah and Jeffrey [12] in order to correctly study systems with gauge symmetry. The Cartan model of equivariant cohomology we just saw will play a central role in the understanding of this representative.

2.2.1 Mathai-Quillen representative

The Mathai-Quillen representative [10] is responsible for the bridge between the mathematical formulation of Localization we just saw and its physical application. It realizes the gaussian Thom form as an universally defined $SO(n)$-equivariant differential form. We shall not spend a lot of time on this matter, saving the more developed presentation to the modified case introduced by Atiyah and Jeffrey. Instead, we state the explicit form of the representative and comment on its significance.

The main result of [10] for us is that the element:

$$U = \int_{V^* \times \Pi V^*} \prod_{i=1}^{n} \frac{dB_i}{\sqrt{2\pi}} \frac{d\chi_i}{\sqrt{2\pi}} e^{Q_W \Psi_{g.f.}}, \tag{2.2.1}$$

is a universal Thom form, i.e., it satisfies:

1. $\int_V U = 1$.
2. U is basic.
3. U is closed.

The differential Q_W is a differential in the Weyl model of equivariant cohomology (which is equivalent to the Cartan model we just saw) and $\Psi_{g.f.}$ is the gauge-fixing fermion explicitly given by

$$\Psi = -i\langle \chi, x \rangle + \frac{1}{4}(\chi, \theta\chi)_{V^*} - \frac{1}{4}(\chi, B)_{V^*}. \tag{2.2.2}$$

The coordinates x are the coordinates on the fibers and χ, B are auxiliary coordinates of ΠV^* and $T^*\Pi V^*$ respectively. They were introduced in order to make an integral representation possible and can be seen as antighosts from a physical point of view. As expected, we observe that these antighosts decouple in the sense that their definition is arbitrary since the measure in U does not change for a redefinition $\chi \mapsto A\chi$ and $B \mapsto BA^{-1}$ for any $A \in GL(V)$. Furthermore, along these lines, the differential Q_W can be regarded as a BRST-like operator.

The first and third requirements listed above are expected: the former being a consequence of the definition of the Thom isomorphism, which is induced by integration along the fibers, while the latter is the most basic requisite in order to define a cohomology class. The second requirement is due to the choice of the Weyl model for equivariant cohomology and the definition of the differential complex in this framework.

In fact, the explicit expression found in [10] is not the one above. The expression in Equation (2.2.1) takes advantage of auxiliary spaces which have a BRST connotation as to have all of the enumerated conditions manifest. This is in fact the final expression found in [11]. We can already see why we mentioned that the Mathai-Quillen representative can be seen as a path integral. However, Equation (2.2.1) is not enough for our purposes, as it is not suitable to be applied to gauge theories. The necessary modifications where first done by Atiyah and Jeffrey in [12], whose work we review below.

Nevertheless, we can already see the significance of the Mathai-Quillen formalism in its full glory: the representative is a path integral with some anti-ghosts, as well as a BRST-like operator. Also, note that the action in the path integral is exact with respect to this BRST-like operator. This last property will be pivotal in the examination of Topological Field Theories and makes evident the proof of $\mathcal{Q}_C$-closedness of the representative, as well as that $s^*(U)$ is the Poincaré dual to the zero locus of some generic section s along the lines advertised in the last paragraph of Subsection 2.1.2.

2.2.2 The Atiyah-Jeffrey representative

We have already advertised the need of a slightly modified version of the Mathai-Quillen representative when dealing with systems with gauge symmetry. The corrected version of the representative was developed by Atiyah and Jeffrey [12] soon after the original work done by Mathai and Quillen was published and was motivated by the Topological Quantum Field Theories proposed by Witten [15], which we will review in the next section.

Start with the following setup: let $P \to M$ be a G-principal bundle. In addition, assume that P is a Riemannian manifold (with coordinates p_i) and that the G-action on it is carried out freely by isometries. This allows for the definition of horizontal and vertical subspaces H_p and V_p respectively, which in turn offer a connection which we denote by θ following the original paper's notation. We also define the map $C : \mathfrak{g} \to TP$. Now let V be the $2m$-dimensional vector space with inner product given by an orthogonal representation of G. The vector bundle we will be interested in will be the associated vector bundle

$$E \equiv \big(P \times V\big)/G \to M. \tag{2.2.3}$$

We may perform some changes on the Mathai-Quillen form above since we will be restricting the original representative to be horizontal by the inclusion of a projection form. This means that, for our purposes, we may use the θ-defined curvature Ω as

$$\Omega = d\theta + \frac{1}{2}\big[\theta, \theta\big] \Rightarrow \Omega|_{H_p} = d\theta \tag{2.2.4}$$

since the commutator term vanishes identically when evaluated in the horizontal subspace by the definition of connection. One can change things even further by defining $\nu \in \Omega^1(P) \otimes \mathfrak{g}^*$ as:

$$\nu_\xi(\alpha) := \langle C\xi, \alpha \rangle,\ \forall\ \xi \in \mathfrak{g}, \alpha \in TP. \tag{2.2.5}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the metric-induced inner-product in TP. In particular, for $\eta \in \mathfrak{g}$, it follows that

$$\nu_\xi(C\eta) = \langle C\xi, C\eta \rangle = \langle C^*C\xi, \eta \rangle \equiv \big(R\xi, \eta\big), \tag{2.2.6}$$

where $\big(\,\cdot\,,\,\cdot\,\big)$ denotes the Killing form in $\mathfrak{g}$ and $R \equiv C^*C$. Comparatively, the evaluation of the connection is simply $\theta_\xi(\eta) = \big(\xi, \eta\big)$. Going back to the evaluation with α:

$$\theta_\xi(\alpha) = \big(\xi, C^{-1}\alpha\big) \text{ and } \nu_\xi(\alpha) = \big(R\xi, C^{-1}\alpha\big) \tag{2.2.7}$$

from which we conclude that

$$\nu = R\theta \tag{2.2.8}$$

leading to

$$d\nu = (dR)\theta + Rd\theta \Rightarrow d\nu|_{H_p} = Rd\theta|_{H_p} \tag{2.2.9}$$

since we can throw away the vertical contribution of θ when doing the horizontal projection. Finally, we conclude that we may perform the following substitution:

$$d\theta \to R^{-1} d\nu. \tag{2.2.10}$$

The horizontal projection, which we have advertised a lot already, can be obtained by exhibiting a form which behaves[5] as if it is the representative of the Thom class for the principal bundle. The needed result is:

$$\det R \int dg = \int e^{\left\langle R^{-1} dp, C\eta \right\rangle} d\eta \tag{2.2.11}$$

where η is a fermionic Lie algebra variable. Although the determinant of R may seem undesirable, it is in fact very useful in order to present the proper Mathai-Quillen representative without the R^{-1} factor explicitly. Indeed, after Fourier transforming the familiar expression for the MQ representative -with the aforementioned substitutions of $\Omega \to R^{-1}\nu$- to the Lie algebra variables $\phi, \overline{\phi}$, we end up with an extra factor of $\det R$, which is precisely the one absorbed by the identity in Equation (2.2.11).

In all, we obtain:

$$\begin{aligned} s^*(\mathcal{U}) = \frac{1}{(2\pi)^{d+m}} \int d\eta d\phi d\overline{\phi} \exp \Big\{ -\frac{1}{2}|s|^2 + \frac{1}{2}\chi^a \omega(\phi)_{ab}\chi^b + i\chi_a ds_a - i\langle d\nu, \overline{\phi}\rangle \\ + i\langle \phi, R\overline{\phi}\rangle + \langle ds, C\overline{\phi}\rangle \Big\}, \end{aligned} \tag{2.2.12}$$

which is final expression found in [12].

We can go even further with the help of the Cartan model of equivariant cohomology. Let us denote the Cartan model differential by $\mathcal{Q}_C$ from now on. After the inclusion of the antighost auxiliary fields H_a and defining the action:

$$\mathcal{Q}_C \overline{\phi} = \eta, \qquad \mathcal{Q}_C \eta = +i[\phi, \overline{\phi}], \tag{2.2.13}$$

$$\mathcal{Q}_C \chi_a = H_a, \qquad \mathcal{Q}_C H_a = \omega(\phi)_{ab}, \tag{2.2.14}$$

$$\mathcal{Q}_C x_\mu = \psi_\mu, \qquad \mathcal{Q}_C \psi_\mu = D_\mu \phi \tag{2.2.15}$$

on the new Lie algebra-valued variables, Equation (2.2.12) can be rewritten as a

[5] We do not call it a proper Thom class, because it is only defined for vector bundles.

$\mathcal{Q}_C$-exact form [16]:

$$s^*(\mathcal{U}) = \frac{1}{(2\pi)^{d+m+\dim G}} \int d\chi dH d\eta d\phi d\overline{\phi} \, \exp\left\{ \mathcal{Q}_C\left(\Psi_{g.f.} + \Psi_{proj.}\right)\right\} \quad (2.2.16)$$

where the $\Psi_{g.f.}$ is inherited from the original Mathai-Quillen representative:

$$\Psi_{g.f.} = i\langle \chi, s\rangle - \left(\chi, H\right)_{V^*} \quad (2.2.17)$$

and the projection fermion $\Psi_{proj.}$ is explicitly given by:

$$\Psi_{proj.} = i\langle \overline{\phi}, \nu\rangle. \quad (2.2.18)$$

Note that here we have also chosen to present the representative in a $\mathcal{Q}_C$-exact form.

2.3 General Localization

We must clarify a point before we go any further. Although we have been explicitly considering $\mathcal{Q}_C$-exact forms throughout the last sections, one should not assume that they are important because *only* they allow Localization. We chose to work with them above because they make $\mathcal{Q}_C$-invariance of the Atiyah-Jeffrey and Mathai-Quillen representatives, as they were introduced, trivial. Furthermore, $\mathcal{Q}_C$-exactness of the action will be pivotal in the next section, when we analyse Witten's work on Topological QFTs. Finally, we will encounter another $\mathcal{Q}_C$-exact action when studying Nekrasov's instanton counting.

As a matter of fact, Localization depends **only** on $\mathcal{Q}_C$-closedness of the equivariant form one is integrating. Indeed, let α be a $\mathcal{Q}_C$-closed form, and let equivariance be understood with respect to a Lie group G with Lie algebra $\mathfrak{g}$. Then, Localization is the claim that

$$\int_M \alpha = \int_F \frac{i^*\alpha}{e(\nu_F)}. \quad (2.3.1)$$

The space F is the set of fixed points of the G-action generated by $\mathfrak{g}$, and $e(\nu_F)$ is the equivariant Euler class of the normal bundle ν_F of F to M. In Physics language, this is the 1-loop determinant one obtains from integrating with respect to the space of linearized fluctuations ν_F around the classical solutions (fixed points space). It is easy to see that this is correct if one multiplies α by a term $e^{-t\mathcal{Q}_C\mathcal{O}}$ for some suitable $\mathcal{Q}_C$-equivariant operator $\mathcal{O}$ such that $\mathcal{Q}_C\mathcal{O}$ is positive definite, and then takes the limit $t \to \infty$. Two great references on this general framework are [13, 14].

One nice example which will provide us with some important intuition for later is the Duistermaat-Heckman formula from symplectic geometry (see Appendix B for the necessary definitions). It is given by:

$$\int_M \frac{e^{\omega-\mu(\xi)}}{n!} = \sum_{f:V_\xi(f)=0} \frac{e^{-\mu(f)}}{\prod_i w_i^\xi(f)}. \tag{2.3.2}$$

It is the special case of 2.3.1 for a Hamiltonian $\mathbb{T}_G$-action on a symplectic manifold with moment map μ and generators $\xi \in \mathfrak{t}_G$. The points f are the fixed points under this action, and $w_i^\xi(f)$ denotes the weights of the action on the tangent space (physically speaking, fluctuation space) of the fixed point f.

2.4 Twisted $d = 4, \mathcal{N} = 2$ pure super Yang-Mills

In order to use Localization in $d = 4, \mathcal{N} = 2$ super Yang-Mills, we need to recast the theory in a different form. Namely, we need to perform the so-called **topological twist** introduced by Witten [15]. In a general manifold, this procedure is carried out by introducing a background gauge field corresponding to one of the extra global symmetries (the R-symmetries) as to offset the coefficients of the spin connection and allows the existence of an unbroken supersymmetry generator. In the simpler case of $\mathbb{R}^4$, however, it is simply a redefinition of the spins of the fermionic fields. We shall see that the resulting scalar generator is precisely the differential of the Cartan model of Equivariant cohomology used in the construction of the Atiyah-Jeffrey Thom representative. In fact, we will observe that the partition function of the twisted theory is precisely an Atiyah-Jeffrey representative. We will follow closely the original reference.

We start by recalling the field content and global symmetries of pure $d = 4, \mathcal{N} = 2$ pure super Yang-Mills (with some suitable adequation to the notation of the main reference). We have a gauge boson A_μ, Weyl fermions $\psi_{\alpha A}$ and their conjugates, and the complex scalars $\phi, \overline{\phi}$. The global symmetry is:

$$SU(2)_L \times SU(2)_R \times SU(2)_I \times U(1)_I, \tag{2.4.1}$$

where the first two factors are the Lorentz symmetry and $SU(2)_I \times U(1)_I$ is the

R-symmetry. The Weyl fermions form a doublet under the $SU(2)_I$ R-symmetry, while the other particles are singlets, as can be seen in:

$$A^{\mu} = (1/2, 1/2, 0)^{0} \tag{2.4.2}$$

$$\psi_{\alpha A} = (1/2, 0, 1/2)^{+1} \quad \overline{\psi}^{\dot{\alpha} A} = (0, 1/2, 1/2)^{-1} \tag{2.4.3}$$

$$\phi^{a} = (0,0,0)^{+2} \quad \bar{\phi}^{a} = (0,0,0)^{-2}. \tag{2.4.4}$$

The twist will change these charge assignments. It is defined as follows: instead of considering the usual $SU(2)_R \times SU(2)_I$, we take the action to be that of $\Delta(SU(2)_R \times SU(2)_I)$. Then, the initial Lorentz spins and R-symmetry charges combine in the following way:

$$0 \otimes 0 = 0 \quad 0 \otimes 1/2 = 1/2 \quad 1/2 \otimes 1/2 = 0 \oplus 1. \tag{2.4.5}$$

Therefore, after the twisting, the bosons remain the same: a vector A_{μ} and the complex scalars $\phi, \overline{\phi}$. The left-handed Weyl fermion's spin becomes $(1/2, 1/2)^{+1}$, which means:

$$\psi_{\alpha A} \mapsto \psi_{\mu}. \tag{2.4.6}$$

Finally, the left-handed Weyl fermion splits up into a scalar and an self-dual 2-form:

$$\overline{\psi}^{\dot{\alpha} A} \mapsto \eta \oplus \chi^{+}_{\mu\nu}. \tag{2.4.7}$$

Another way to see the same phenomenon at work is identifying $A \rightarrow \dot{\alpha}$, i.e., identifying the $SU(2)_I$ index to the $SU(2)_R$ index. We work out the transformation of the supersymmetry generators in this new language. We get:

$$Q_{\alpha A} \mapsto Q_{\alpha \dot{\beta}} \tag{2.4.8}$$

$$\bar{Q}_{\dot{\alpha} B} \mapsto \bar{Q}_{\dot{\alpha}\dot{\beta}}. \tag{2.4.9}$$

Therefore, we may define a vector generator G_{μ}, a scalar generator $\mathcal{Q}$ and a self-dual 2-form generator $\bar{Q}^{+}_{\mu\nu}$ as:

$$G_{\mu} = \frac{i}{4} \bar{\sigma}_{\mu}^{\dot{\alpha}\beta} Q_{\beta\dot{\alpha}}, \tag{2.4.10}$$

$$\bar{Q}^{+}_{\mu\nu} \propto \bar{\sigma}_{\mu\nu}^{\dot{\alpha}\dot{\beta}} \bar{Q}_{\dot{\alpha}\dot{\beta}}, \tag{2.4.11}$$

$$\mathcal{Q} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{Q}_{\dot{\alpha}\dot{\beta}}. \tag{2.4.12}$$

Notice that $\bar{\sigma}_{\mu\nu}$ is self-dual, so the second definition above makes sense. After this procedure, we end up with eight generators in total (1 scalar + 4 from vector + 3 from self-dual 2-form), as we should. The new scalar supercharge $\mathcal{Q}$ acts on the fields by:

$$\delta A_\mu = i\varepsilon\psi_\mu, \quad \delta\psi_\mu = -\varepsilon D_\mu\phi, \quad \delta\phi = 0, \tag{2.4.13}$$

$$\delta\overline{\phi} = 2i\varepsilon\eta, \quad \delta\eta = \frac{1}{2}\varepsilon\left[\phi,\overline{\phi}\right], \quad \delta\chi^+_{\mu\nu} = \varepsilon\left(F_{\mu\nu} + \widetilde{F}_{\mu\nu}\right), \tag{2.4.14}$$

and squares to a gauge-transformation with gauge-parameter[6] $i\varepsilon_1\varepsilon_2\phi$. Since this charge will be very important for us right now, and the vector charge G_μ will be important in instanton counting, we give two results involving both below, which we prove in Appendix A.

$$\mathcal{Q}^2 = 0, \tag{2.4.15}$$

$$\{\mathcal{Q}, G_\mu\} = \partial_\mu. \tag{2.4.16}$$

Then, taking usual action for pure $d = 4, \mathcal{N} = 2$ SYM with topological term, rewriting it in terms of the twisted fields and adding the $\mathcal{Q}$-exact term $-\frac{1}{4}\mathcal{Q}\mathrm{tr}\left(\eta\left[\phi,\overline{\phi}\right]\right)$, we obtain the following expression:

$$S = \int d^4x\sqrt{g}\,\mathrm{tr}\Big[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}F_{\mu\nu}\widetilde{F}^{\mu\nu} - \frac{1}{2}D_\mu\phi D^\mu\overline{\phi} - i\eta D_\mu\psi^\mu + iD_\mu\psi_\nu\chi^{+\mu\nu} \\ - \frac{1}{8}\left[\phi,\overline{\phi}\right]^2 - \frac{i}{8}\phi\left[\chi^+_{\mu\nu},\chi^{+\mu\nu}\right] - \frac{i}{2}\overline{\phi}\left[\psi_\mu,\psi^\mu\right] - \frac{i}{2}\phi\left[\eta,\eta\right]\Big] \tag{2.4.17}$$

It may not seem very telling in any special way, but in fact, once we calculate[7]:

$$\{\mathcal{Q}, V\} = \mathcal{Q}\int d^4x\sqrt{g}\,\mathrm{tr}\left[\frac{1}{4}F_{\mu\nu}\chi^{+\mu\nu} + \frac{1}{2}\psi_\mu D^\mu\overline{\phi} - \frac{1}{4}\eta[\phi,\overline{\phi}]\right] \tag{2.4.18}$$

we conclude that the action is $\mathcal{Q}$-exact[8]. This realization is extremely important. The main consequence is that the partition function is a **topological invariant**. To see why, we first realize that the correlation function of $\mathcal{Q}$-exact operators is zero,

[6] Do note that, since ϕ is in the adjoint, this makes sense.

[7] This computation can be found in Appendix A.

[8] Recall that the addition of $\mathcal{Q}$-exact gauge-invariant terms is permitted since it does not spoil $\mathcal{Q}$-invariance of the theory.

since we assume that the $\mathcal{Q}$-symmetry is non-anomalous. Indeed:

$$\delta_\varepsilon \mathcal{Z}(\mathcal{O}) = \int [\mathcal{D}\Phi] e^{\varepsilon \mathcal{Q}} e^{\frac{-S}{g^2}} \mathcal{O} = \int [\mathcal{D}\Phi] e^{-\frac{S}{g^2}} (\mathcal{O} + \varepsilon \{\mathcal{Q}, \mathcal{O}\}). \tag{2.4.19}$$

Independence on ε then implies

$$\langle \{\mathcal{Q}, \mathcal{O}\} \rangle = 0, \tag{2.4.20}$$

i.e., the expectation value of $\mathcal{Q}$-exact terms is zero. Then, since the energy-momentum tensor is $\mathcal{Q}$-exact

$$\begin{aligned} T_{\mu\nu} &= \mathcal{Q}\,\mathrm{tr}\left(F_{(\mu|\rho|}\chi^{+\rho}_{\nu)} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma}\chi^{\rho\sigma} + \psi_{(\mu} D_{\nu)}\overline{\phi} - \frac{1}{2} g_{\mu\nu}\psi_\mu D^\mu \overline{\phi} + \frac{1}{4} g_{\mu\nu}\eta\,[\phi, \overline{\phi}] \right) \\ &\equiv \{\mathcal{Q}, \mathcal{Y}_{\mu\nu}\}, \end{aligned} \tag{2.4.21}$$

as it is straightforward to see from Equation 2.4.18, it follows that:

$$\begin{aligned} \delta_g \mathcal{Z} &= \delta_g \int [\mathcal{D}\Phi] e^{-\frac{S}{g^2}} = -\frac{1}{g^2} \int [d\Phi] e^{-\frac{S}{g^2}} \Big(\int d^4x \delta g^{\mu\nu} T_{\mu\nu} \Big) \\ &= -\frac{1}{g^2} \Big\langle \int d^4x \delta g^{\mu\nu} T_{\mu\nu} \Big\rangle = -\frac{1}{g^2} \Big\langle \{\mathcal{Q}, \int d^4x \delta g^{\mu\nu} \mathcal{Y}_{\mu\nu}\} \Big\rangle = 0 \end{aligned} \tag{2.4.22}$$

This is the reason why this theory is called **topological**. As a matter of fact, any correlator of $\mathcal{Q}$-closed and metric-independent operators, which are called **topological operators**, is also metric-independent.

Furthermore, Localization is at play, because of the $\mathcal{Q}$-symmetry. Note that in writing Equations 2.4.13, 2.4.17 and 2.4.18 we have implicitly integrated over the auxiliary variable $H_{\mu\nu}$ associated with $\chi^+_{\mu\nu}$. We have also added the $\mathcal{Q}$-exact term $-\frac{1}{4}\mathcal{Q}\Big(\eta\,[\phi, \overline{\phi}]\Big)$ to the action. We can recover the auxiliary field by considering the expression [16, 17]:

$$\mathcal{Q} \int d^4x \sqrt{g}\,\mathrm{tr}\Big[2\chi^+_{\mu\nu} (F^{+\mu\nu} - \frac{1}{2} H^{\mu\nu}) - \frac{1}{2}\eta\,[\phi, \overline{\phi}] + \frac{1}{2}\psi_\mu D^\mu \overline{\phi} \Big] \tag{2.4.23}$$

and amending the transformations:

$$\delta\chi^+_{\mu\nu} = \varepsilon H_{\mu\nu}, \qquad \delta H_{\mu\nu} = i\varepsilon \big[\chi_{\mu\nu}, \phi\big]. \tag{2.4.24}$$

There is a general change in the numerical factors, but they are not important for the point we are trying to make. If we now compare Equation 2.4.23 with the Atiyah-Jeffrey representative of the Thom class we saw in a previous section, it becomes very clear that we are dealing with a specific example. Note that:

$$\langle \overline{\phi}, \nu \rangle \propto \psi_\mu D^\mu \overline{\phi}. \tag{2.4.25}$$

We can even identify the base manifold, the fiber bundle and the section which gives the zero locus whereon the localization takes place. Indeed, we conclude that:

$$M = \mathcal{A} \tag{2.4.26}$$

$$E = P \times_G F^+ \tag{2.4.27}$$

$$s : A \in \mathcal{A} \mapsto F_A^+ \in \Omega_+^2(X, \mathfrak{g}). \tag{2.4.28}$$

In other words, the base manifold is the space of connections on E, the fibers of the vector bundle are the vector space of self-dual 2-forms $\Omega_+^2(X, \mathfrak{g})$ and the section associates the corresponding self-dual curvature to each connection. Then, localization on the zero locus of s means that we are localizing over the subspace $\mathcal{Z}(s) \subset M$ of anti-self-dual connections, i.e., anti-self-dual instantons. This statement relies on the fact that in $d = 4$, $\star : \Omega^2(M) \to \Omega^2(M)$, $\star^2 = 1$ and therefore $\Omega^2(M) = \Omega_+^2(M) \oplus \Omega_-^2(M)$.

Before we move on, we state that we can arrive at the same conclusion about localization from the independence of the partition function on the gauge coupling constant. Indeed, this should not come as a surprise given everything we saw above, but it is instructive to see that the formal Mathematics matches the intuitive Physics. The inclusion of $\mathcal{Q}$-exact terms in the action should not change the partition function, since it amounts to taking the correlation function of powers of the same $\mathcal{Q}$-exact term. Therefore, the careful choice of the right term multiplied by some parameter t allows us to take the limit were the contributions are peaked at the classical solutions of the equations of motion derived from the action, i.e., the minimizing solutions. Since we are in an $\mathcal{N} = 2$ theory and, therefore, we include a $F_{\mu\nu}\widetilde{F}^{\mu\nu}$ term which picks up the instanton contributions, the minima of the action happen precisely for anti-self-dual gauge field solutions. We have thus arrived at the same conclusion as in the last paragraph.

As noted before, the action of the equivariant derivative, or supercharge, $\mathcal{Q}$

on the fields of the theory is very reminiscent of the action of a BRST charge. Furthermore, the Atiyah-Jeffrey construction of the Mathai-Quillen representative, and even the Mathai-Quillen construction itself, are very keen in introducing auxiliary fields which are not present in the initial problem. One may ask, thus, if there is any intuition to be gained by trying to relate these objects to a BRST interpretation. Indeed, there is, and soon after Witten proposed the topologically twisted theory, many papers sought out to do just that. We will comment quickly on one of these interpretations, namely that of Baulieu and Singer [18]. They started with the topological action:

$$\int \mathrm{tr} F \wedge F, \tag{2.4.29}$$

which, due to being topological, has a larger gauge symmetry than the usual Yang-Mills one. In fact, the connection has the following symmetry:

$$\delta A_\mu = D_\mu \xi + \tilde{\xi}_\mu \tag{2.4.30}$$

and the gauge parameter $\tilde{\xi}_\mu$ has itself the symmetry

$$\delta \tilde{\xi}_\mu = D_\mu \omega. \tag{2.4.31}$$

Of course, we can BRST gauge fix all these symmetries by introducing the corresponding ghosts and auxiliary fields. It is clear, however, that in order to find Witten's theory one needs to BRST-fix only some of the symmetries, as the theory is still a G gauge theory. The $\tilde{\xi}_\mu$ is fixed by introducing a one-form ψ_μ of ghost number $+1$. The corresponding anti-ghosts and auxiliary fields are, respectively, $\chi_{\mu\nu}$ with ghost number -1 and $H_{\mu\nu}$. The gauge-fixing condition chosen is precisely the projection of F onto its self-dual or anti-self-dual solutions and effectively determines whether $\chi_{\mu\nu}$ is self-dual or anti-self-dual as well. The weird symmetry in (2.4.31) also needs to be gauge fixed. The gauge-fixing ghost is ϕ, while the antighost is $\overline{\phi}$ and η is the auxiliary field. The gauge-fixing term is $D_\mu \psi^\mu$.

The choice of notation in the last paragraph, as the reader may have noticed, was not done arbitrarily. It makes the connection between the BRST-fixing ghosts, anti-ghosts and auxiliary fields, and the fields in Witten's theory very clear. This procedure also makes some things more evident. The first is the fact that the localization of the theory over the (anti-)instantons can be seen as a consequence of the fixing of the extra symmetries. Second, the topological nature of the theory should not come as a surprise, given the fact that we started with a topological action.

2.5 Instantons: moduli space and the ADHM construction

The previous considerations have all corroborated the already widespread acknowledgement of instantons as important objects in the study of Quantum Field Theories. Indeed, we have seen that they are pivotal in the determination of the low-energy prepotential, as well as giving the zero locus of the section which localizes twisted $d = 4, \mathcal{N} = 2$ super Yang-Mills. As the title of the chapter suggests, we will proceed with the counting of the instanton contributions to the prepotential. Therefore, a solid knowledge of these solutions as well as the space where they live in will be very important. As such, we devote the present section to this goal.

2.5.1 The moduli space $\mathfrak{M}_k$

The first solution to the instanton equations found by Belavin, Polyakov, Schwartz and Tyupkin [19] for $SU(2)$ gauge group is given by (in singular gauge):

$$A_\mu = \bar{\eta}^i_{\mu\nu} \frac{\rho^2 (x-X)_\nu}{(x-X)^2\big((x-X)^2+\rho^2\big)} g \sigma^i g^{-1} \tag{2.5.1}$$

where $\bar{\eta}^i_{\mu\nu}$ are the 't Hooft symbols. The reason we chose to start the section with this example is that it explicitly shows the existence of coordinates which parametrize instanton solutions. Indeed, in the expression above we find eight of them: the four center coordinates X^μ, the scale size ρ and the three parameters of the global $SU(2)$ gauge transformation g. Therefore, we may think of the moduli space of instantons as being spanned by these coordinates. Another important remark is that the particular instanton solution above has instanton charge 1, and this is the reason why the moduli space has dimension 8. As we will see, the dimension of the moduli space depends on the instanton charge k.

The k-instanton moduli space $\mathfrak{M}_k$ is therefore defined as the space of inequivalent solutions to the instanton equations with instanton charge k. Here, equivalence is to be understood as equivalence up to **local** gauge transformations.

Let us now study $\mathfrak{M}_k$. We start by listing some of its properties:

1. $\dim \mathfrak{M}_k = 4kN$, for gauge group $SU(N)$.
2. $\mathfrak{M}_k$ is a Riemannian manifold with singularities.
3. $\mathfrak{M}_k$ is a hyperKähler manifold.

In order to show that $\dim \mathfrak{M}_k = 4kN$ for a $SU(N)$ Yang-Mills theory, we first need to introduce the concept of **instanton zero modes**. We follow the approach of [20]. Suppose one has a self-dual connection A_μ and takes a perturbation $A_\mu + \delta_i A_\mu$. The subscript i is a collective coordinate index of the moduli space. What are the conditions that δA_μ should satisfy in order that the perturbed connection also be self-dual? First of all, up to linear order in $\delta_i A_\mu$:

$$\hat{F} = \star\hat{F} \Rightarrow F_{\mu\nu} + \partial_\mu \delta_i A_\nu - \partial_\nu \delta_i A_\mu + [A_\mu, \delta_i A_\nu] + [\delta_i A_\mu, A_\nu] = \epsilon_{\mu\nu\rho\sigma}\left(F^{\rho\sigma} + \delta_i F^{\rho\sigma}\right) \tag{2.5.2}$$

Notice that $\partial_\mu \delta_i A_\nu + [A_\mu, \delta_i A_\nu] \equiv D_\mu \delta_i A_\nu$ and therefore, what we have is:

$$D_\mu \delta_i A_\nu - D_\nu \delta_i A_\mu = \epsilon_{\mu\nu\rho\sigma} D^\rho \delta_i A^\sigma. \tag{2.5.3}$$

This is a good first condition, but it is not sufficient to discard the possibility of local gauge transformations generating $\delta_i A_\mu$ since, of course, gauge transformations trivially satisfy it. Therefore, we may take advantage of the fact that $\mathfrak{M}_k$ has a Riemannian metric and require $\delta_i A_\mu$ to be orthogonal to general gauge transformations $D_\mu \xi$, $\forall\, \xi$. This requirement is described by:

$$\int D_\mu \xi \left(\delta_i A_\mu\right) = 0 \Rightarrow D_\mu \delta_i A_\mu = 0 \tag{2.5.4}$$

where we have performed an integration by parts. Therefore, collective coordinates are related to variations which satisfy:

$$\begin{cases} D_\mu \delta_i A_\nu - D_\nu \delta_i A_\mu = \epsilon_{\mu\nu\rho\sigma} D^\rho \delta_i A^\sigma, \\ D_\mu \delta_i A_\mu = 0. \end{cases} \tag{2.5.5}$$

We now use these requirements to construct a sequence of mappings

$$0 \xrightarrow{d_{-1}} M^0 \xrightarrow{d_0} M^1 \xrightarrow{d_1} M^2 \xrightarrow{d_2} 0. \tag{2.5.6}$$

The vector spaces M_i are spaces of fields of interest in our considerations: M_0 is the space of scalar fields, M_1 is the space of vector fields and M_2 is the space of anti-self-dual rank-two tensor fields. The mappings act as follows:

$$d_{-1}(0) \mapsto 0 \tag{2.5.7}$$

$$\left(d_0(\lambda)\right)_\mu \mapsto D_\mu \lambda \tag{2.5.8}$$

$$\left(d_1(A)\right)_{\mu\nu} \mapsto D_\mu A_\nu - D_\nu A_\mu - \epsilon_{\mu\nu\rho\sigma} D^\rho A^\sigma \tag{2.5.9}$$

$$d_2(F) \mapsto 0. \tag{2.5.10}$$

By noticing that $d_i d_{i-1} = 0$, we may define a cohomology:

$$H^i = \frac{\ker d_i}{\mathrm{im}\, d_{i-1}} \tag{2.5.11}$$

and denote $\dim H^i \equiv h^i$. It is then clear that we want to find h^1, which is precisely the dimension of the space of variations which satisfy condition (2.5.3), are not gauge transformations themselves (they are not $D_\mu \lambda$ for some scalar field λ) and are locally gauge inequivalent. If we define an elliptic operator $\hat{d}$ by:

$$\left(\widehat{d}(\lambda, F)\right)_\mu = \left(d_0(\lambda)\right)_\mu + \left(d_1^*(F)\right)_\mu \tag{2.5.12}$$

we may use the Atiyah-Singer index theorem to yield:

$$\mathfrak{I}(\widehat{d}) = \sum_i (-1)^i \dim H^i = h^0 - h^1 + h^2 = 4C(G)k + \dim(G). \tag{2.5.13}$$

It turns out that for S^4 the dimension $h^2 = 0$ and, for $SU(N)$, $C(G) = N$ and $h^0 = 0$ for $k \geq \frac{1}{2}N$. One then concludes that:

$$\dim \mathfrak{M}_k = h^1 = 4C(G)k - (N^2 - 1). \tag{2.5.14}$$

We must include the global gauge freedoms and, after that, we reach the promised result:

$$\dim \mathfrak{M}_k = 4kN. \tag{2.5.15}$$

For $k \leq \frac{1}{2}N$, there are technicalities which require a correction to the formula above, but we they will not interfere in the considerations of this dissertation. Indeed, if we focus on $SU(2)$, this condition is trivially satisfied.

2.5.2 The ADHM construction

Although we have exhibited an instanton solution in Equation (2.5.1) and have enumerated several important facts about the moduli space, we still don't have a general method for finding instanton solutions for arbitrary charge k. As a matter of fact, the instanton equations

$$F = \star F \text{ or } F = - \star F \tag{2.5.16}$$

are a series of coupled partial differential equations which are, at first, worthy opponents. It turns out that in 1978, Atiyah, Hitchin, Drinfeld and Manin [21] provided a simple way to obtain *all* the instanton solutions in S^4.

The setup for the construction for the gauge group $U(N)$ is as follows: consider two complex vector spaces V and W such that:

$$\dim_{\mathbb{C}} V = k \qquad \dim_{\mathbb{C}} W = N. \tag{2.5.17}$$

Then, consider the following linear maps:

$$B_1, B_2 \in \mathrm{End}(V) \quad I \in \mathrm{Hom}(W \to V) \quad J \in \mathrm{Hom}(V \to W). \tag{2.5.18}$$

The transformations $\mathfrak{D}_{ADHM} = \{B_1, B_2, I, J\}$ are the so-called **ADHM data** and

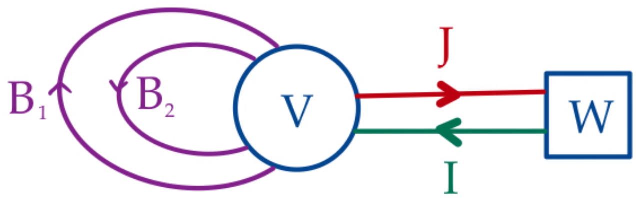

Figura 2.3: ADHM data and vector spaces.

span a vector space $\mathfrak{V}_{ADHM} \equiv \mathrm{span}\{(B_1, B_2, I, J)\}$. Furthermore, as it will prove important later, we shall assign the following transformation laws for the ADHM data under $U(k) \times U(N) \times \mathbb{T}_{SO(4)}$: B_i are scalars under $U(N)$, are in the fundamental with respect to the generator ϵ_i of $\mathbb{T}_{SO(4)}$ and are in the adjoint of $U(k)$. The datum I is in the antifundamental of $U(N)$, is a scalar under $\mathbb{T}_{SO(4)}$ and is in the fundamental under $U(k)$. Finally, J is in the fundamental of $U(N)$, in the fundamental of both ϵ_1 and ϵ_2 of $\mathbb{T}_{SO(4)}$ and in the antifundamental of $U(k)$. Now, consider the so-called **ADHM equations** given by:

$$\mu_{\mathbb{R}} = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0 \quad (2.5.19)$$

$$\mu_{\mathbb{C}} = [B_1, B_2] + IJ = 0. \quad (2.5.20)$$

Then, the ADHM construction claims that:

$$\mathfrak{M}_k = \vec{\mu}^{-1}(0)/U(k), \quad (2.5.21)$$

i.e., that the moduli space of $U(N)$ instantons with charge k is given by the hyperKähler quotient of $\mathfrak{V}_{ADHM}$ by the $U(k)$ action. Indeed, the maps $\vec{\mu}$ are precisely the moment maps of the $U(k)$ action on $\mathfrak{V}_{ADHM}$. Also, the moment maps are consistent with the $U(k) \times U(N) \times \mathbb{T}_{SO(4)}$.

The first check we can make is to calculate the dimension of this quotient. We have

$$\dim \mathfrak{V}_{ADHM} = 2 \times \dim_{\mathbb{R}} End(V) + \dim_{\mathbb{R}} Hom(V \to W) + \dim_{\mathbb{R}} Hom(W \to V)$$

minus $3k^2$ constraints coming from the ADHM equations and finally k^2 constraints coming from the $U(k)$ quotient. Therefore, the total number of leftover d.o.f. is:

$$4k^2 + 2kN + 2kN - 3k^2 - k^2 = 4kN$$

and we conclude that $\dim_{\mathbb{R}} \mathfrak{M}_k = 4kN$.

In order to show that the claim is indeed correct, we need to exhibit an instanton solution from the ADHM data. To do so, define the operator

$$D^\dagger : V \otimes \mathbb{C}^2 \oplus W \to V \otimes \mathbb{C}^2$$

$$D : V \otimes \mathbb{C}^2 \to V \otimes \mathbb{C}^2 \oplus W$$

$$D^\dagger = \begin{pmatrix} B_1 - z_1 & B_2 - z_2 & I \\ -B_2^\dagger + \bar{z}_2 & B_1^\dagger - \bar{z}_1 & -J^\dagger \end{pmatrix} \text{ and } D = \begin{pmatrix} B_1^\dagger - \bar{z}_1 & -B_2 + z_2 \\ B_2^\dagger - \bar{z}_2 & B_1 - z_1 \\ I^\dagger & -J \end{pmatrix}. \quad (2.5.22)$$

A simple computation shows that:

$$D^\dagger D = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} : V \otimes \mathbb{C}^2 \to V \otimes \mathbb{C}^2 \tag{2.5.23}$$

where $\Delta : K \to K$ is given by

$$\Delta = B_1 B_1^\dagger + B_2 B_2^\dagger + I I^\dagger - z_1 B_1^\dagger - \bar{z}_1 B_1 - z_2 B_2^\dagger - \bar{z}_2 B_2. \tag{2.5.24}$$

The off-diagonal terms in Equation (2.5.23) are zero because of the $\mu_{\mathbb{C}}$ ADHM equations and the explicit form of Δ is a consequence of $\mu_{\mathbb{R}}$. Now consider a matrix Θ satisfying:

$$D^\dagger \Theta = 0, \qquad \Theta^\dagger D = 0, \qquad \Theta^\dagger \Theta = \mathbb{1}_W \tag{2.5.25}$$

where we expand

$$\Theta = \begin{pmatrix} \alpha_+ \\ \alpha_- \\ \beta \end{pmatrix} \tag{2.5.26}$$

and the triad of maps which make up Θ are such that:

$$\alpha_\pm : W \to V, \qquad \beta : W \to W. \tag{2.5.27}$$

In particular, this also means that the map:

$$\Theta^\dagger d\Theta : W \to W \tag{2.5.28}$$

is and endomorphism of the vector space W. Note that since Θ is defined to span the kernel of the operator $D^\dagger$, it also depends on the space-time coordinates $(z_1, \bar{z}_1, z_2, \bar{z}_2)$. Now consider the one-form:

$$A = \Theta^\dagger d\Theta \tag{2.5.29}$$

with associated curvature $F_A = dA + [A, A]$ explicitly given by:

$$F_A = d\Theta^\dagger d\Theta + \Theta^\dagger d\Theta \Theta^\dagger d\Theta. \tag{2.5.30}$$

The main result regarding the ADHM construction is that F_A above is an anti-self-dual $U(N)$ instanton with instanton charge k. To show that, we start by noticing that[9]:

$$A^{\dagger} = d\Theta^{\dagger}\Theta = -\Theta^{\dagger}d\Theta \tag{2.5.31}$$

i.e., the connection is skew-Hermitian. Since connections take values in the Lie algebras of the gauge group and skew-Hermitian $N \times N$ matrices generate $\mathfrak{u}(N)$, we conclude that we are indeed dealing with $U(N)$. Now, to prove that F_A is anti-self-dual we note that:

$$\begin{aligned} F_A &= d\Theta^{\dagger}d\Theta + \Theta^{\dagger}d\Theta\Theta^{\dagger}d\Theta \\ &= d\Theta^{\dagger}d\Theta - d\Theta^{\dagger}\Theta\Theta^{\dagger}d\Theta \\ &= d\Theta^{\dagger}\left(1 - \Theta\Theta^{\dagger}\right)d\Theta. \end{aligned} \tag{2.5.32}$$

We can, however, define two operators:

$$D\frac{1}{\Delta}D^{\dagger}, \quad \Theta\Theta^{\dagger} \tag{2.5.33}$$

such that

$$\left(D\frac{1}{\Delta}D^{\dagger}\right)^2 = D\frac{1}{\Delta}D^{\dagger}D\frac{1}{\Delta}D^{\dagger} = D\frac{1}{\Delta}\Delta\frac{1}{\Delta}D^{\dagger} = D\frac{1}{\Delta}D^{\dagger} \tag{2.5.34}$$

$$\left(\Theta\Theta^{\dagger}\right)^2 = \Theta\left(\Theta^{\dagger}\Theta\right)\Theta^{\dagger} = \Theta\Theta^{\dagger} \tag{2.5.35}$$

$$D\frac{1}{\Delta}\left(D^{\dagger}\Theta\right)\Theta^{\dagger} = 0 \tag{2.5.36}$$

$$\tag{2.5.37}$$

i.e., they are orthogonal projection operators. We conclude that:

$$1 - \Theta\Theta^{\dagger} = D\frac{1}{\Delta}D^{\dagger} \tag{2.5.38}$$

and we end up with

$$F_A = d\Theta^{\dagger}D\frac{1}{\Delta}D^{\dagger}d\Theta \tag{2.5.39}$$

and finally

$$F_A = \Theta^{\dagger}\left(dD\right)\frac{1}{\Delta}\left(dD^{\dagger}\right)\Theta \tag{2.5.40}$$

[9] The following equation is a consequence of $\Theta^{\dagger}\Theta = \mathbb{1}_W$.

where we have used

$$dD^\dagger\Theta = -D^\dagger d\Theta, \qquad d\Theta^\dagger D = -\Theta^\dagger dD. \tag{2.5.41}$$

In coordinates, the final expression can be written as

$$F_{\mu\nu} = \Theta^\dagger \left(\partial_{[\mu} D\right) \frac{1}{\Delta} \left(\partial_{\nu]} D^\dagger\right) \Theta. \tag{2.5.42}$$

Choosing

$$z_1 \equiv -x_4 + ix_3, \qquad z_2 \equiv x_2 + ix_2 \tag{2.5.43}$$

we realize that

$$\partial_\mu D = i\sigma_\mu, \qquad \partial_\mu D^\dagger = i\overline{\sigma}_\mu \tag{2.5.44}$$

and therefore

$$F_A = -2\sigma_{\mu\nu} \otimes \frac{1}{\Delta}. \tag{2.5.45}$$

Since $\sigma_{\mu\nu}$ is an anti-self-dual two-form in four dimensions, we conclude that $F_{\mu\nu}$ is an anti-self-dual $U(N)$ instanton. The proof of why the instanton charge is k involves some abstract mathematics, namely regarding the space $\ker D^\dagger$ and its complement as a bundle over S^4 and using Chern theory to calculate its second Chern class. Since it will not bring important insights to the remainder of our discussion, we refrain from presenting it here.

2.6 Nekrasov's solution

Now we are finally ready to study the other solution to the problem of determining the low-energy effective action of $d = 4, \mathcal{N} = 2$ SYM. As we will see, we will make use of the knowledge about Topological Quantum Field Theories (of the cohomological type), Localization and the ADHM construction in order to be able to count the instanton contribution. The procedure we review here was introduced by Nekrasov in [22] and fully developed in [23] together with Okounkov. We also point the reader to references [24, 25, 26, 27, 28, 29, 30] which are helpful in understanding the general idea behind the construction.

2.6.1 The Ω-deformation

In the preceding sections of this chapter we saw two facts about Donaldson-Witten theory defined on a general four-manifold M: it localizes onto the moduli space of instantons $\mathfrak{M}_k$ for each instanton number k, and computes topological invariants of said manifold. Our main objective is directly computing the non-perturbative contributions to the prepotential of $d = 4, \mathcal{N} = 2$ $U(N)$ SYM on $\mathbb{R}^4$. If we recall Equation 1.2.8, it seems that computing it directly from the partition function is a promising path. Simultaneously, the mathematical idea behind Localization expressed in Equations 2.1.3, 2.1.9 and 2.3.1, we conclude that computing the partition function of the theory is closely related to computing the equivariant volume of the submanifold over which we localize. Thus, computing the partition function as it stands in Donaldson-Witten theory on $\mathbb{R}^4$ amounts to computing the $U(N)$-equivariant volume of the moduli space of instantons for each instanton number k. This is not that straightforward, however, as the instanton moduli spaces are non-compact. Therefore, in order to get a sensible result, some sort of regularization is in order. This is the reason behind the Ω-deformation [22].

We will take the point of view of [34] to motivate the result found by Nekrasov. Starting from the ADHM construction and the HyperKähler nature of $\mathfrak{M}_k$, the volume of $\mathfrak{M}_k$ can be written as:

$$\begin{aligned} &\int_{\mathfrak{M}_k=\vec{\mu}^{-1}(0)/U(k)} e^{\vec{\omega}} = \\ &= \int_{\mathfrak{u}(k)} \frac{\mathcal{D}\phi}{\mathrm{Vol}(K)} \int \mathcal{D}\vec{\chi}\mathcal{D}\vec{H}\mathcal{D}\eta\mathcal{D}\overline{\phi} e^{\mathcal{Q}\left(\Psi_{loc.}+\Psi_{proj.}\right)} \int_{\mathfrak{D}_{ADHM}\oplus\Psi_{ADHM}} e^{\vec{\omega}-\langle\vec{\mu},\phi\rangle}. \end{aligned} \tag{2.6.1}$$

Note that in this case the supercharge $\mathcal{Q}$ is equivariant with respect to $U(k)$. The expression on the left-hand side is yet to be renormalized and therefore diverges as it stands. This is the reason why we must introduce the regularization[10].

2.6.2 Partition function

Consider now two isometries V^μ and $\overline{V}^\mu$ defined by:

$$V^\mu \equiv \Omega^\mu_\nu x^\nu, \qquad \overline{V}^\mu = \overline{\Omega}^\mu_\nu x^\nu \tag{2.6.2}$$

[10]There is also non-compactness due to singularities arising from small instantons, but we assume that these are implicitly taken care of [22]. Concretely, this is achieved by deforming the real moment map $\mu_\mathbb{R} = \zeta_\mathbb{R}\mathbb{1}_V$.

where the Ω's are generators of the Cartan subalgebra of $SO(4)$ $\mathfrak{t}_{SO(4)} \subset \mathfrak{so}(4)$. Explicitly, we take:

$$\Omega^\mu_\nu = \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_2 \\ 0 & 0 & -\epsilon_2 & 0 \end{pmatrix}, \qquad \overline{\Omega}^\mu_\nu = \begin{pmatrix} 0 & \bar{\epsilon}_1 & 0 & 0 \\ -\bar{\epsilon}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\epsilon}_2 \\ 0 & 0 & -\bar{\epsilon}_2 & 0 \end{pmatrix}. \tag{2.6.3}$$

Of course they commute and, since rank $SO(4) = 2$, they generate the entire $\mathfrak{t}_{SO(4)}$. The parametrization above is chosen such that the generators are related to two-dimensional rotations on:

$$\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2 \sim \mathbb{C} \oplus \mathbb{C} \tag{2.6.4}$$

by (ϵ_1, ϵ_2) in each plane. Now, starting from the $\mathcal{Q}$ supercharge in 2.6.1, which is equivariant with respect to $U(k)$, we define a new supercharge, or new equivariant differential, by making use of the isometries and the vector supercharge from undeformed Donaldson-Witten theory (remember we are still considering the twisted theory with its $SU(2)_I$ twisted supercharges), as well as modifying it to be $\mathcal{G} \equiv SU(N)/\mathbb{Z}_N$ equivariant. The latter group is the subgroup of $U(N)$ whose action on the moduli space of instantons is non-trivial [22]. This means that the new equivariant differential is modified to:

$$\tilde{\mathcal{Q}} = \mathcal{Q} - \iota_{V_{\vec{a}}} - \iota_{V_{\epsilon_1,\epsilon_2}} = d - \iota_{V_\phi} - \iota_{V_{\vec{a}}} - \iota_{V_{\epsilon_1,\epsilon_2}}, \tag{2.6.5}$$

where $\vec{a}$ with $\sum_{l=1}^N a_l = 0$ parametrizes $\mathbb{T}_{\mathcal{G}}$. This will be the Cartan differential which will provide us with our new deformed Atiyah-Jeffrey Thom class representative.

From the point of view of [34], the deformation of the equivariant differential is equivalent to a regularization of the volume of the instanton moduli space. Indeed, since both $U(N)$ and $\mathbb{T}_{SO(4)}$ act consistently on the $\mathfrak{M}_k$, we may define the HyperKähler regularized volume as:

$$\begin{aligned} \mathrm{Vol}_{(\vec{a},\epsilon_1,\epsilon_2)}(\mathfrak{M}) &\equiv \int_{\mathfrak{M}_k} e^{\vec{\overline{\omega}} - \mathcal{H}_{\vec{a}} - \mathcal{H}_{\epsilon_1,\epsilon_2}} \\ &= \int_{\mathfrak{u}(k)} \frac{\mathcal{D}\phi}{\mathrm{Vol}(K)} \int \mathcal{D}\vec{\chi}\mathcal{D}\vec{H}\mathcal{D}\eta\mathcal{D}\bar{\phi} e^{\tilde{\mathcal{Q}}\left(\Psi_{loc.}+\Psi_{proj.}\right)} \int_{\mathfrak{D}_{ADHM} \oplus \Psi_{\mathfrak{D}_{ADHM}}} e^{\vec{\omega} - \langle \vec{\mu}, \phi \rangle - \mathcal{H}_{\vec{a}} - \mathcal{H}_{\epsilon_1,\epsilon_2}}. \end{aligned} \tag{2.6.6}$$

The $\mathcal{H}_j$ denote the Hamiltonian which generates the symmetries corresponding to j. Note that we have $\mathcal{Q} \to \widetilde{\mathcal{Q}}$, and now the whole "action" is equivariant[11] with respect to the group $U(k) \times \mathbb{T}_{\mathcal{G}} \times \mathbb{T}_{SO(4)}$. Since we are only integrating equivariant forms, it follows that Localization techniques can be applied. We have already seen that this leads to summing over the contribution of fixed points under the action of the group which gives the notion of equivariance. In this sense, including new groups and deforming the equivariant differential implies in restricting even further the space of points which are taken into account. Therefore, the "deformed" integral is indeed a regularization if compared to the original one. Another way to see this is that the full original manifold may be non-compact, but the fixed points may make up a discrete set. In the following, we will consider a scaling of $\vec{\omega}$ and $\mathcal{H}_j$ to zero and will end up with a $\widetilde{Q}$-exact action [24].

From a physical point of view, we have deformed the action by $\Omega, \overline{\Omega}$-dependent terms in order to have $\widetilde{\mathcal{Q}}$ as a conserved supersymmetry. The starting point of $\mathcal{Q}$ being a $U(k)$-equivariant differential is simply due to the fact that we chose to mention the idea of regularized volumes of HyperKähler quotients, which makes the use of the ADHM construction natural. As a matter of fact, one can start with the usual supercharge $\mathcal{Q}$ and twisted Donaldson-Witten theory on $\mathbb{R}^4$ and deform its action to allow for the conservation of $\widetilde{\mathcal{Q}}$, where now the correction is performed by coupling the isometries with the vector supercharge $V^{i\mu}G_\mu$ (this is the same thing as including $\iota_{V_{\epsilon_1,\epsilon_2}}$). One way to obtain this deformation/background is by compactifying $d = 6$, $\mathcal{N} = 1$ SYM with gauge group $U(N)$ on a background described by a non-trivial metric and a Wilson loop on the R-symmetry group $SU(2)_I$ [23].

2.6.3 Comment on conserved supersymmetries on curved manifolds

Before we move on to the evaluation of the path integral, we take a break and briefly discuss the conservation of supersymmetries on curved manifolds. In doing so, we will be able to motivate both the topological twisting procedure and Ω-deformation procedures from a physical perspective.

In a general curved manifold $\mathcal{M}$, the condition for the existence of preserved

[11] Indeed, we have a $\widetilde{\mathcal{Q}}$-exact form + $\widetilde{\mathcal{Q}}$-closed form.

rigid supersymmetries is the existence of Killing spinors satisfying

$$\nabla_\mu \xi^\alpha = 0. \tag{2.6.7}$$

This condition is related to the variation of the gravitino in the supermultiplet containing the metric. Unfortunately, it is too strong and there are few manifolds where such spinors exist. This shortcoming may be circumvented if there is extra structure in the theory. More precisely, the extra structure may be used to "deform" Equation 2.6.7, rendering it more tolerant. Such procedure is systematically introduced in [31] and reviewed in [32, 33]. In the case of topological twisting, the extra structure is the R-symmetry arising from extended supersymmetry as already mentioned previously, whereas the structure involved in the Ω-deformation is the existence of isometries.

We have seen in Section 2.4 that twisting was equivalent to redefining the spins of the different fields in the theory by considering a different action of the $SO(4)$ group. There is, however, a more general way to understand it. Given any manifold $\mathcal{M}$ and a theory with a continuous R-symmetry, one can couple the original theory to a background R-symmetry gauge-connection. By appropriately choosing its classical value, i.e., choosing a background, the modified Killing spinor equation leads to a conserved supersymmetry. More concretely, the modified Killing spinor equation becomes:

$$\nabla_\mu \overline{\xi}^{\dot{\alpha}A} + \widetilde{A}_{\mu}{}^{A}_{B} \overline{\xi}^{\dot{\alpha}B} = 0. \tag{2.6.8}$$

If we choose $\widetilde{A}_{\mu}{}^{A}_{B} = -\Omega_{\mu}{}^{A}_{B}$, then the constant spinor $\overline{\xi}^{\dot{\alpha}A} = \delta^{\dot{\alpha}A}$ generates as conserved supersymmetry.

The Ω-deformation takes this idea one step further. Consider now an already twisted theory on a manifold with isometries V^i_μ. In this framework, the Ω-background is the choice of classical values for the bosonic fields in the metric supermultiplet such that the spinors generating the supercharge

$$\widetilde{\mathcal{Q}} = \mathcal{Q} + c_i V^{i\mu} G_\mu \tag{2.6.9}$$

satisfy a generalized Killing spinor equation.

2.6.4 Evaluating the Partition Function

Now we elaborate further on this new construction, making its connection with the previous sections clearer. We may represent the partition function of the deformed theory as [22]:

$$\mathcal{Z}_{inst}(\vec{a},\epsilon_1,\epsilon_2) = \sum_k \Lambda^{2kN} \mathcal{Z}_k(\vec{a},\epsilon_1,\epsilon_2) \tag{2.6.10}$$

$$\mathcal{Z}_k(\vec{a},\epsilon_1,\epsilon_2) = \int_{\mathfrak{u}(k)} \frac{\mathcal{D}\phi}{\mathrm{Vol}(U(k))} \int \mathcal{D}\vec{A}\mathcal{D}\vec{H}\mathcal{D}\vec{\chi}\mathcal{D}\vec{\Psi}\mathcal{D}\overline{\phi}\mathcal{D}\eta \; e^{\widetilde{\mathcal{Q}}\left(\vec{\chi}\cdot\vec{\mu}+\Psi\cdot V(\overline{\phi})+\eta[\phi,\overline{\phi}]\right)}\Big|_{\vec{a}} \tag{2.6.11}$$

where now the ADHM data and the associated 1-forms appear explicitly. Although we have not manifestly mentioned it, do note that the moment maps $\vec{\mu}$ (which are really the ADHM equations) give the sections whose zero loci we localize over, and therefore it is easy to see that this form is precisely the usual form of the Mathai-Quillen-Atiyah-Jeffrey representative. This concurs with the intepretation of the representative as a "fancy δ-function" as mentioned in earlier sections. Indeed, the new deformed supercharge $\widetilde{\mathcal{Q}}$ acts by:

$$\widetilde{\mathcal{Q}}B_i = \Psi_i, \quad \widetilde{\mathcal{Q}}\Psi_i = [\phi, B_i] + \epsilon_i B_i \tag{2.6.12}$$

$$\widetilde{\mathcal{Q}}I = \Psi_I, \quad \widetilde{\mathcal{Q}}\Psi_I = \phi I - Ia, \tag{2.6.13}$$

$$\widetilde{\mathcal{Q}}J = \Psi_J, \quad \widetilde{\mathcal{Q}}\Psi_J = -J\phi + Ja - (\epsilon_1+\epsilon_2)J, \tag{2.6.14}$$

$$\widetilde{\mathcal{Q}}\chi_{\mathbb{R}} = H_{\mathbb{R}}, \quad \widetilde{\mathcal{Q}}H_{\mathbb{R}} = [\phi, \chi_{\mathbb{R}}], \tag{2.6.15}$$

$$\widetilde{\mathcal{Q}}\chi_{\mathbb{C}} = H_{\mathbb{C}}, \quad \widetilde{\mathcal{Q}}H_{\mathbb{C}} = [\phi, \chi_{\mathbb{C}}] + (\epsilon_1+\epsilon_2)\chi_{\mathbb{C}}, \tag{2.6.16}$$

$$\widetilde{\mathcal{Q}}\phi = 0, \quad \widetilde{\mathcal{Q}}\overline{\phi} = \eta, \quad \widetilde{\mathcal{Q}}\eta = [\phi, \overline{\phi}], \tag{2.6.17}$$

and

$$\Psi\cdot V(\overline{\phi}) = \left(\Psi_i[\overline{\phi}, B_i^\dagger] + \Psi_I[\overline{\phi}, I^\dagger] - \Psi_J[\overline{\phi}, J^\dagger] + c.c.\right). \tag{2.6.18}$$

These formulas agree with the expected behavior of the set of fields in the framework we have been studying in this chapter. To each coordinate (each ADHM datum) we associate a 1-form (the fermion $\Psi_{\mathfrak{D}}$). Furthermore, to each fermion χ (here $\vec{\chi}$) multiplying the section in the gauge fermion we associate an auxiliary variable H (here $\vec{H}$). Finally, we include the projection multiplet $(\eta, \overline{\phi})$ associated to the Lie algebra-valued boson ϕ by which we quotient the zero locus of the sections s (again, here $\vec{\mu}$). The transformation under the equivariant differential/supercharge

$\widetilde{\mathcal{Q}}$ is consistent with the fact that it squares to a $U(k) \times \mathbb{T}_{\mathcal{G}} \times \mathbb{T}_{SO(4)}$ gauge transformation and that it involves only one pair of associated fields at a time.

The calculation of the above expression has been done [34] and yields the following contour integral[12]:

$$\mathcal{Z}_k(a, \epsilon_1, \epsilon_2) = \frac{1}{k!} \oint \prod_{l=1}^{k} \left(\frac{\epsilon_1 + \epsilon_2}{2\pi i \epsilon_1 \epsilon_2} \frac{d\phi_l}{P(\phi_l) P(\phi_l + \epsilon)} \right) \prod_{m \neq n} \frac{\phi_{mn}(\phi_{mn} + \epsilon)}{(\phi_{mn} + \epsilon_1)(\phi_{mn} + \epsilon_2)} \tag{2.6.19}$$

where we denote $\phi_{mn} \equiv \phi_m - \phi_n$, $\epsilon = \epsilon_1 + \epsilon_2$ and

$$P(x) \equiv \prod_{i=1}^{N} (x - a_i). \tag{2.6.20}$$

It is easy to motivate this result if we recall how the $\mathbb{T}_{SO(4)} \times U(N) \times U(k)$ acts on the ADHM data. Up to linear order we have:

$$B_i \mapsto (\phi_m - \phi_n + \epsilon_i) B_{i,mn} \tag{2.6.21}$$

$$I \mapsto (\phi_m - a_l) I_{m,l} \tag{2.6.22}$$

$$J \mapsto (\phi_m + \epsilon - a_l) J_{m,l}. \tag{2.6.23}$$

for general $\phi \in \mathfrak{u}(k), \epsilon_i \in \mathfrak{so}(4), a \in \mathfrak{u}(N)$. This means that the conditions for fixed-points[13] are given by requiring that the following equations:

$$(\phi_m - \phi_n + \epsilon_i) B_{i,mn} = 0, \tag{2.6.24}$$

$$(\phi_m - a_l) I_{m,l} = 0, \tag{2.6.25}$$

$$(\phi_m + \epsilon - a_l) J_{m,l} = 0, \tag{2.6.26}$$

have non-trivial solutions. It is important to remember that these conditions obviously take into account the $U(k)$ acting on the data, i.e., the action of the group $\mathbb{T}_{SO(4)} \times \mathcal{G}$ can change a datum only up to a $U(k)$ transformation. Now, one should note that the conditions for non-trivial solutions precisely label the poles of the contour integral in Equation 2.6.19.

Now that we have some intuition, let us give a precise derivation of the result.

[12] We define the pole prescription $\epsilon_1, \epsilon_2 \in \mathbb{R} + i0$

[13] We are basically making use of the Duistermaat-Heckman formula, which we have already seen, with some extra new ingredients.

The integral over the Lie algebra of $U(k)$ can be rewritten as:

$$\int_{\mathfrak{u}(k)} \frac{d\phi}{\mathrm{Vol}(U(k))} = \frac{1}{k!}\int_{\mathfrak{t}_{U(k)}} \prod_{l=1}^{k} d\phi_l \prod_{m\neq n} \phi_{mn}. \tag{2.6.27}$$

Then, the action of the equivariant differential $\widetilde{\mathcal{Q}}$ over the bosonic ADHM data (B_i, I, J) leads to a determinant[14]:

$$\begin{aligned} \frac{1}{\det_{\mathfrak{D}_{ADHM}}(\widetilde{\mathcal{Q}})} &= \prod_{l=1}^{k} \frac{1}{\prod_{i=1}^{N}(\phi_l - a_i)\prod_{j=1}^{N}(\phi_l + \epsilon - a_j)} \prod_{m,n} \frac{1}{(\phi_{mn}+\epsilon_1)(\phi_{mn}+\epsilon_2)} \\ &= \prod_{l=1}^{k} \frac{1}{P(\phi_l)P(\phi_l+\epsilon)} \prod_{m,n} \frac{1}{(\phi_{mn}+\epsilon_1)(\phi_{mn}+\epsilon_2)} \end{aligned} \tag{2.6.28}$$

The first product refers to the action on I, J respectively, whereas the second product refers to the action on the B_i. Finally, the action of the supercharge $\widetilde{\mathcal{Q}}$ on the fermionic variables $(\chi_{\mathbb{C}}, H_{\mathbb{C}})$ leads to the determinant:

$$\det_{\chi_{\mathbb{C}}, H_{\mathbb{C}}}(\widetilde{\mathcal{Q}}) = \prod_{m,n} (\phi_{mn} + \epsilon). \tag{2.6.29}$$

Putting all the terms together and rewriting the product:

$$\prod_{m,n} \frac{(\phi_{mn}+\epsilon)}{(\phi_{mn}+\epsilon_1)(\phi_{mn}+\epsilon_2)} = \left(\frac{\epsilon}{\epsilon_1\epsilon_2}\right)^k \prod_{m\neq n} \frac{(\phi_{mn}+\epsilon)}{(\phi_{mn}+\epsilon_1)(\phi_{mn}+\epsilon_2)}, \tag{2.6.30}$$

we reach Equation 2.6.19. This is the main Theorem of [34]. Note that, as pointed out in the same reference, the integration over the fields $(\eta, \overline{\phi})$ and $(\chi_{\mathbb{R}}, H_{\mathbb{R}})$ is trivial because of the freedom to add $\widetilde{\mathcal{Q}}$ terms.

We are now ready to analyse the contour integral. First of all, notice that it is divergent in the $\epsilon_1, \epsilon_2 \to \infty$ limit, which is consistent with the fact that these numbers parametrize the regularization of the volume. The poles of the contour integral in 2.6.19 are neatly organized into partitions of Young tableaux. For each instanton charge k, we associate a partition $\vec{Y}_i$ of N Young tableaux. The l'th Young

[14]These are the Euler class of the nomal bundle from Equation 2.3.1.

Tableaux of the i'th partition has $k_{i,l} \geq 0$ boxes. Therefore:

$$\sum_{l=1}^{N} k_{i,l} = k. \tag{2.6.31}$$

Furthermore, we require that, for a given column of a Young tableaux, the column immediately to its right has either the same number or less boxes, i.e., we use the "French notation".

Then, the partition $\vec{Y}_i$ corresponds to the i'th pole. More precisely, it describes the pole associated to the ϕ's[15] given by:

$$\phi_{l,\rho,\sigma} = a_l + (\rho - 1)\epsilon_1 + (\sigma - 1)\epsilon_2. \tag{2.6.32}$$

In other words, each ϕ is tied to a box at position[16] $(\rho, \sigma)_l$ of the $k_{m,l}$ Young tableau.

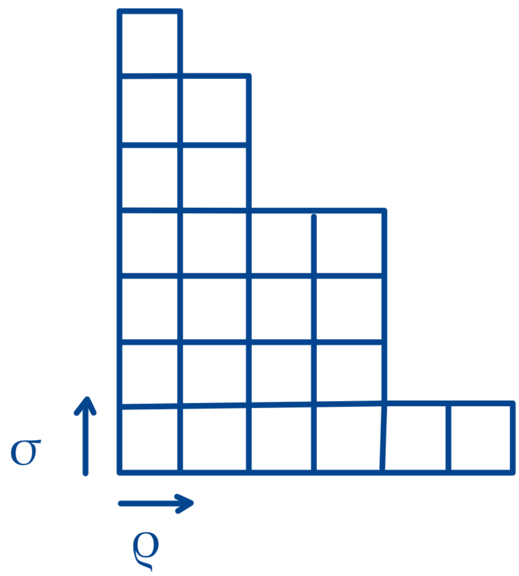

Figura 2.4: Example of Young Tableau used. To each box one associates a ϕ with value given by Equation 2.6.32.

We provide the simplest example for the sake of clarity: we will calculate the $\mathcal{Z}_1$ instantonic contribution to the partition function for the gauge group $U(N)$.

[15]The ordering in the assignment of boxes to the ϕ's does not matter because of the factor $1/k!$ before the contour integral.

[16]Here $(\rho, \sigma)_l$ denotes the σ-th box in the ρ-th column of the partition Y_l.

The general expression 2.6.19 reduces to:

$$\left(\frac{\epsilon}{\epsilon_1\epsilon_2}\right)\int\frac{d\phi}{2\pi i}\prod_{i=1}^{N}\frac{1}{(\phi-a_i)(\phi+\epsilon-a_i)}. \tag{2.6.33}$$

The reasoning above tells us that $k=1$ implies N partitions, each with 1 box distributed along N tableaux. The poles are located at $\phi=a_i$. This allows us to evaluate the integral:

$$\sum_{i=1}^{N}\left(\frac{\epsilon}{\epsilon_1\epsilon_2}\right)\frac{1}{\epsilon\prod_{j\neq i}a_{ij}(\epsilon+a_{ij})}=\frac{1}{\epsilon_1\epsilon_2}\sum_{i=1}^{N}\prod_{j\neq i}\frac{1}{a_{ij}(a_{ij}+\epsilon)}, \tag{2.6.34}$$

where we have defined $a_{ij}\equiv a_i-a_j$ and the product implies running over j with i fixed. We have thus calculated the $k=1$ instanton contribution to the partition function:

$$\mathcal{Z}_1=\frac{1}{\epsilon_1\epsilon_2}\sum_{i=1}^{N}\prod_{j\neq i}\frac{1}{a_{ij}(a_{ij}+\epsilon)}. \tag{2.6.35}$$

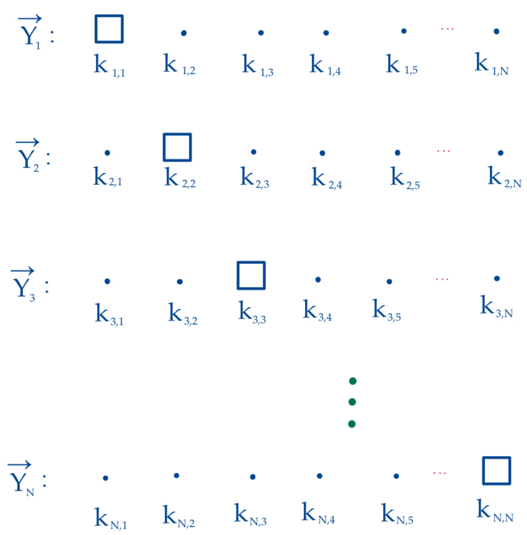

Figura 2.5: Partitions and Young tableaux of the $U(N), k=1$ case.

2.7 The Seiberg-Witten rendez-vous

Now consider again Equation 2.6.19. We see that, in the limit $\epsilon_1, \epsilon_2 \to 0$, it can be re-expressed as

$$\Lambda^{2kN} \mathcal{Z}_k(\vec{a}, \epsilon_1, \epsilon_2) \sim \exp\left(\frac{1}{\epsilon_1 \epsilon_2} \mathbf{E}_\Lambda(\rho)\right), \tag{2.7.1}$$

where we have introduced the *eigenvalue density function* $\rho(x)$ defined by:

$$\rho(x) \equiv \epsilon_1 \epsilon_2 \sum_{l=1}^{k} \delta(x - \phi_l). \tag{2.7.2}$$

and denote[17]:

$$\mathbf{E}_\Lambda(\rho) \equiv -\widehat{\int} dx dy \frac{\rho(x)\rho(y)}{(x-y)^2} - 2\int dx \rho(x) \log\left(\frac{P(x)}{\Lambda^N}\right). \tag{2.7.3}$$

Defining the new notation:

$$\widetilde{\mathcal{Z}}_k(a, \epsilon_1, \epsilon_2) = \prod_{l=1}^{k} \frac{1}{(\phi_l - a_i)(\phi_l + \epsilon - a_i)} \prod_{m,n} \frac{\phi_{mn}^{1-\delta_{mn}}(\phi_{mn} + \epsilon)}{(\phi_{mn} + \epsilon_1)(\phi_{mn} + \epsilon_2)}, \tag{2.7.4}$$

note that

$$\begin{aligned}
\widetilde{\mathcal{Z}}_k(a, \epsilon_1, \epsilon_2) &= \exp\left(\log \widetilde{\mathcal{Z}}_k\right) \\
&= \exp\left\{\log\left[\prod_{l=1}^{k} \frac{1}{(\phi_l - a_i)(\phi_l + \epsilon - a_i)} \prod_{m,n} \frac{\phi_{mn}^{1-\delta_{mn}}(\phi_{mn} + \epsilon)}{(\phi_{mn} + \epsilon_1)(\phi_{mn} + \epsilon_2)}\right]\right\} \\
&= \exp\left\{-\sum_{l=1}^{k} \log\left[P(\phi_l)P(\phi_l + \epsilon)\right] + \sum_{m,n} \log\left[\frac{\phi_{mn}^{1-\delta_{mn}}(\phi_{mn} + \epsilon)}{(\phi_{mn} + \epsilon_1)(\phi_{mn} + \epsilon_2)}\right]\right\}.
\end{aligned} \tag{2.7.5}$$

[17] The integral $\widehat{\int}$ includes a prescription to avoid the singular loci of the integrand [23].

Now we enforce the $\epsilon_1, \epsilon_2 \to 0$ limit. Taylor expansion of both terms in the exponential above yields:

$$\begin{aligned}\text{first term} &= \sum_{l=1}^{k}\left(2\log P(\phi_l) - \frac{\epsilon_1 \frac{d}{d\epsilon_1}P(\phi_l+\epsilon)|_{\epsilon_1,\epsilon_2=0}}{P(\phi_l)} - \frac{\epsilon_2 \frac{d}{d\epsilon_2}P(\phi_l+\epsilon)|_{\epsilon_1,\epsilon_2=0}}{P(\phi_l)}\right) \\ &\quad + o(\epsilon_1\epsilon_2) \\ &= 2\sum_{l=1}^{k}\log P(\phi_l)\end{aligned} \tag{2.7.6}$$

$$\text{second term} = -\epsilon_1\epsilon_2 \sum_{m,n}\frac{1}{\phi_{mn}^2} + o(\epsilon_1^2\epsilon_2, \epsilon_1\epsilon_2^2) \tag{2.7.7}$$

Multiplying and dividing by $\epsilon_1\epsilon_2$ leads to:

$$\widetilde{\mathcal{Z}}_k(a,\epsilon_1,\epsilon_2) = \exp\left\{\frac{1}{\epsilon_1\epsilon_2}\Big(-2\epsilon_1\epsilon_2\sum_{l=1}^{k}\log P(\phi_l) - \epsilon_1^2\epsilon_2^2\sum_{m,n}\frac{1}{\phi_{mn}^2}\Big)\right\}. \tag{2.7.8}$$

When $\epsilon_1, \epsilon_2 \to 0$, there is a value for which contributes most to the sum. This is due [27] to the factor of $1/k!$ in Equation 2.6.19, and therefore we only need to worry about the term with $k \sim \frac{1}{\epsilon_1\epsilon_2}$. Finally we incorporate Λ^{2kN} to find the desired result. Thus, we do conclude that:

$$\Lambda^{2kN}\mathcal{Z}_k(\vec{a},\epsilon_1,\epsilon_2) \sim \exp\Big(\frac{1}{\epsilon_1\epsilon_2}\mathbf{E}_\Lambda(\rho)\Big). \tag{2.7.9}$$

By varying the "action" $\mathbf{E}_\Lambda(\rho)$ with respect to the eigenvalue density function:

$$\frac{\delta\mathbf{E}_\Lambda(\rho)}{\delta\rho(x)} = -\int dydz\frac{\delta(x-y)\rho(z)}{(y-z)^2} - \int dydz\frac{\rho(y)\delta(x-z)}{(y-z)^2} - 2\int dy\delta(x-y)\log\Big(\frac{P(y)}{\Lambda^N}\Big) \tag{2.7.10}$$

we conclude that the minimizing density $\rho^\star(x)$ should satisfy:

$$\int dy\frac{\rho^\star(y)}{(x-y)^2} + \log\Big(\frac{P(x)}{\Lambda^N}\Big) = 0. \tag{2.7.11}$$

Instead of finding a solution to $\rho^\star$, we change our language to continuous Young diagrams and follow the original derivation of [23].

The first step is defining what **continuous Young diagrams** $f(x)$ are:

$$f(x) \equiv -\rho(x) + \sum_{l=1}^{N} |x - a_l|. \tag{2.7.12}$$

Their name stems from the fact that they are the generalization of the profile of a Young tableau in the "Russian notation" when one takes $k \sim \frac{1}{\epsilon_1\epsilon_2}$ in the $\epsilon_1, \epsilon_2 \to 0$ limit. In this situation, the sharp edges of the profile may be smoothed out and

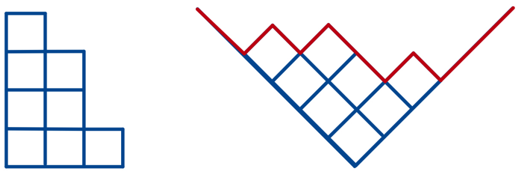

Figura 2.6: French notation on the left and Russian notation on the right. The red line is the profile for this partitions.

consequently it satisfies the following conditions:

$$|f(x) - f(y)| \leq |x - y|, \tag{2.7.13}$$

$$f(x) \geq |x|, \tag{2.7.14}$$

$$\widehat{\int} dx\, f'(x) = 0, \tag{2.7.15}$$

$$f(x) = |x|, \text{ for } |x| >> 0. \tag{2.7.16}$$

It is not hard to see that these conditions are a relaxation of the conditions one would get from the "jagged" profiles like the ones in the previous Figure. For example, the Lipschitz condition above is weaker than requiring $f'(x) = \pm 1$, as is the case of the "jagged" profiles. Also note that they satisfy:

$$\int_{\alpha_l^-}^{\alpha_l^+} x f''(x) dx = a_l, \tag{2.7.17}$$

$$\int f''(x) dx = N. \tag{2.7.18}$$

If we make the substitution 2.7.12 and then integrate by parts the expression for $\mathbf{E}_\Lambda(\rho)$, the "action" in Equation 2.7.3 can be recast in terms of the new continuous

Young diagrams. We get:

$$\mathbf{E}_{\Lambda}(f) = \frac{1}{4}\widehat{\int}_{y<x} dxdy\, f''(x)f''(y)(x-y)^2\left(\log\left(\frac{x-y}{\Lambda}\right) - \frac{3}{2}\right) \quad (2.7.19)$$

plus the perturbative contribution:

$$\mathbf{E}_{pert} = \frac{1}{2}\sum_{m\neq n}(a_m - a_n)^2\Big(\log\frac{a_m - a_n}{\Lambda} - \frac{3}{2}\Big). \quad (2.7.20)$$

We will show below that the prepotential will be obtained from:

$$\mathcal{F}(a,\Lambda) = -\mathrm{Crit}_{f\in\mathcal{CY}}\mathbf{E}_{\Lambda}(f), \quad (2.7.21)$$

where $\mathcal{CY}$ is the set of continuous Young diagrams. Before we find the limiting shapes, we remind ourselves that the a_l should be fixed for the particular choice of vacuum expectation value of the scalar. This motivates the introduction of Lagrange multipliers which we denote by ξ_l. We label them as $\xi_1 > \cdots > \xi_N$. They are introduced by defining:

$$\sum_{l=1}^{N}\xi_l a_l = -\frac{1}{2}\widehat{\int}_{\mathbb{R}}\sigma(f'(x))dx, \quad (2.7.22)$$

where the function $\sigma(x)$ satisfies the following two conditions

$$\sigma'(y) = \xi_l, \quad y\in\big[-N+2(l-1), -N+2l\big], \quad (2.7.23)$$

$$\sigma(-N) = -\sigma(N) = -\sum_{l=1}^{N}\xi_l. \quad (2.7.24)$$

Intuitively, it is a piecewise-linear function that splits the interval $[-N,N]$ into l subintervals, where its slope is given by ξ_l. The goal now is to find the limiting shape for the action:

$$\mathbf{S}_{\Lambda} = -\mathbf{E}_{\Lambda} + \frac{1}{2}\widehat{\int}\sigma(f'(x))dx. \quad (2.7.25)$$

In particular, this means that, by construction:

$$\frac{\partial\mathbf{S}_{\Lambda}(f_\star)}{\partial\xi_l}\Big|_{f_\star} = a_l. \quad (2.7.26)$$

We take the standard route:

$$\frac{\delta \mathbf{S}_\Lambda(f)}{\delta f(z)} = -\widehat{\int}_{x<y} dy \log\left(\frac{x-y}{a}\right) f''(y) + \frac{1}{2}\sigma'(f'(x)) = 0 \tag{2.7.27}$$

If we integrate the first term once by parts, we find:

$$\widehat{\int}_{y\neq x} dy (y-x)\left(\log\left|\frac{y-x}{\Lambda}\right| - 1\right) f''(y) = \sigma'(f'(x)). \tag{2.7.28}$$

Defining

$$\mathbb{X} f(x) \equiv \widehat{\int}_{y\neq x} dy (y-x)\left(\log\left|\frac{y-x}{\Lambda}\right| - 1\right) f''(y), \tag{2.7.29}$$

we conclude that condition 2.7.28 can be rephrased as follows: $f_\star(x)$ is a limit shape of the action if and only if

$$\mathbb{X} f_\star(x) = \xi_l, \qquad \text{for } -N+2l-2 < f'_\star(x) < -N+2l, \tag{2.7.30}$$

$$\xi_l > \mathbb{X} f_\star(x) > \xi_{l+1}, \qquad \text{for } f'_\star(x) = -N+2l, l = 0, \cdots, N. \tag{2.7.31}$$

In particular, note that in the first case the function $\mathbb{X} f_\star(x)$ is constant. We can now define a conformal map $\Phi(z)$ from the Upper Half Plane to the half-strip such that its extension to the real line is given by:

$$\varphi(x) \equiv f'_\star(x) + \frac{1}{\pi i}\left[\mathbb{X} f_\star\right]'(x). \tag{2.7.32}$$

In particular, note that it has vertical slits that are distant from each other by a distance of 2. More precisely:

$$\Delta = \left\{ w \in \mathbb{C} | -N < \mathfrak{Re} w < N,\ \mathfrak{Im} w > 0 \right\} \tag{2.7.33}$$

with vertical slits along

$$\left\{ \mathfrak{Re}\, w = -N+2l, \quad \mathfrak{Im}\, w \in [0, \eta_l] \right\}, \quad l = 1, \cdots, N-1. \tag{2.7.34}$$

We also impose the condition:

$$\Phi(z) = N + \frac{2N}{\pi i}\log\left(\frac{\Lambda}{z}\right) + o(1/z), z \to \infty. \tag{2.7.35}$$

We will achieve the construction of this map by composing a series of more elementary maps. Start with a monic real polynomial denoted by $P_N(z) = z + \cdots$ such that all the roots of the equation:

$$P_N^2(z) - 4\Lambda^{2N} = \prod_{l=1}^{N} \left(z - \alpha_l^+\right)\left(z - \alpha_l^-\right) \tag{2.7.36}$$

are real. Now let $\widetilde{w}(z)$ denote the smaller (in absolute value) root of the equation[18]:

$$\Lambda^N \left(w + \frac{1}{w}\right) = P_N(z). \tag{2.7.37}$$

Finally, take the logarithm of $\widetilde{w}$ to define:

$$\Phi(z) = \frac{2}{\pi i} \log \widetilde{w} + N. \tag{2.7.38}$$

It is easy to see that this recovers the required $z \to \infty$ behavior, as well as behavior of the region Δ. In particular, notice that we find $P_N(\alpha_l^\pm) = 2\Lambda^N$ for any l. The labelling is such that, as a consequence, $z \in \left[\alpha_l^-, \alpha_l^+\right]$ satisfies $|z| < 2\Lambda^N$ and therefore is mapped to the base of the half-strip. Now we have an indirect association between the limit shape $f_\star(x)$ and a map from the UHP to the region Δ.

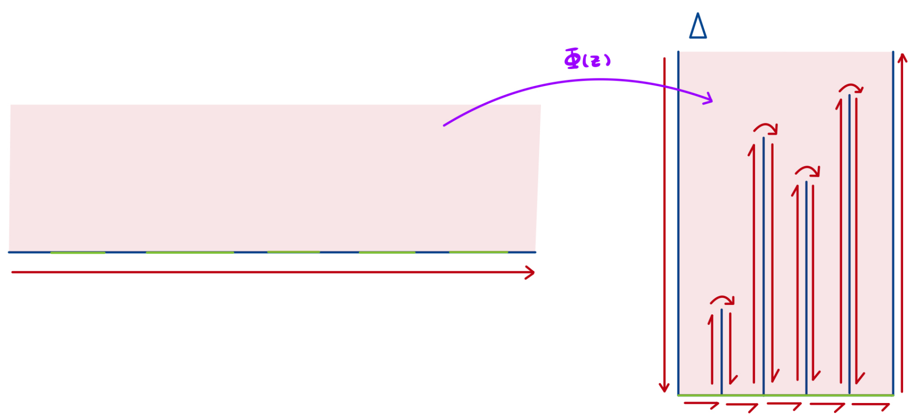

Figura 2.7: Schematic representation of the map $\Phi(z)$ for $N = 5$.

Now, consider extending Φ to $\mathbb{C} - \bigcup_{l=1}^{N} [\alpha_l^-, \alpha_l^+]$ such that we obtain a covering

[18]Note that for real $|z| < 2\Lambda^N$, $\widetilde{w} = \frac{\frac{z}{\Lambda^N} \pm \sqrt{\frac{z^2}{\Lambda^N} - 4}}{2}$ is mapped to the contour of the unit disk. Real $|z| > 2\Lambda^N$ are mapped to the real line.

map with cuts along $[\alpha_l^-, \alpha_l^+]$. Then,

$$\Phi'(x) = \varphi'(x) = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f''(z)}{(z-w)} dz. \tag{2.7.39}$$

Note that we can also write:

$$\Phi'(x) = \frac{2}{\pi i} \frac{dw}{w}. \tag{2.7.40}$$

Now, if we compare the continuous partition properties of Equation 2.7.17 with Equations 2.7.39 and 2.7.40 above, we conclude that:

$$a_l = \frac{1}{2} \int_{\alpha_l^-}^{\alpha_l^+} x f''(x) dx = \oint_{\gamma_{B,l}} \frac{1}{2\pi i} z \frac{dw}{w} \tag{2.7.41}$$

It then follows that the Seiberg-Witten differential is given by:

$$\lambda_{SW} = \frac{1}{2\pi i} z \frac{dw}{w}. \tag{2.7.42}$$

In turn, this also means that:

$$\xi_{l+1} - \xi_l = -\pi \int_{\alpha_l^+}^{\alpha_{l+1}^-} \mathfrak{Im}\varphi(x) dx = 2\pi i \oint_{\gamma_{A,l} - \gamma_{A,l+1}} \lambda_{SW}. \tag{2.7.43}$$

In the previous equations, the cycles $\gamma_{A_l}, \gamma_{B,l}$ are the cycles of the Riemann surface obtained from the covering map mentioned above. Finally, if we compare Equation 2.7.26 with Equation 1.2.79 we conclude that, after a Legendre transformation:

$$\mathbf{S}_\Lambda(f_\star) = \mathcal{F}(a, a_D). \tag{2.7.44}$$

We have thus shown that:

$$\mathcal{F}(a, a_D) = -\mathbf{E}_\Lambda(f_\star) = -\lim_{\epsilon_1, \epsilon_2 \to 0} \epsilon_1 \epsilon_2 \log \mathcal{Z}(\vec{a}, \epsilon_1, \epsilon_2), \tag{2.7.45}$$

ultimately recovering the Seiberg-Witten structures from instanton counting!

Conclusion

In this dissertation we have provided a review of some basic aspects of Supersymmetric Gauge Theories, with main focus on pure $d = 4, \mathcal{N} = 2$ super Yang-Mills. More precisely, special focus was given in understanding the low-energy behavior of this theory through two different paths: Seiberg-Witten Theory and Instanton Counting. Both approaches aim to compute the coefficients of the non-perturbative corrections to the effective prepotential.

The first path focuses on the moduli space of vacua $\mathcal{M}_{vac}$ parametrized by the Weyl group invariant coordinates $u, \overline{u}$ and involves the construction of two auxiliary objects: the Seiberg-Witten curve Σ_u and the Seiberg-Witten differential λ_{SW}. To each point $u \in \mathcal{M}_{vac}$ we associate a curve Σ_u obtained from the behavior of the sections $(a_D(u), a(u))$ under monodromies determined by the perturbative correction to the prepotential and singularities where the BPS states (monopoles and dyons) become massless. One defines a bundle over $\mathcal{M}_{vac}$ whose fibers are the first cohomology/homology group of the curve associated to each point. Then, the integration of the differential λ_{SW} along the cycles of Σ_u is related to $(a_D(u), a(u))$. Since this set of dual coordinates is related to one another by the prepotential through the equation:

$$\frac{\partial \mathcal{F}(a)}{\partial a} = a_D, \tag{2.7.46}$$

the expression for $\mathcal{F}(a)$ *including* non-perturbative corrections can be obtained from Σ_u and λ_{SW}. Note that this description follows the consistency "constraints" imposed by the known perturbative corrections, as well as the known behavior of (a_D, a) at the strong coupling regions where the mass of BPS states vanish.

The second path, on the other hand, focuses on directly computing the non-perturbative instanton contributions to the prepotential. The main idea follows from Donaldson-Witten theory, where the topological sector of the theory computes observables which are localized onto the space of anti-self-dual instantons. In order to be able to compute the prepotential from localization, however, one needs to enlarge the group of symmetries whose equivariant differential one uses to establish the notion of "topological operator". Indeed, one must include the maximal torus of $SO(4)$ and localize with respect to $\mathbb{T}_{\mathcal{G}} \times \mathbb{T}_{SO(4)}$, where $\mathcal{G} = SU(N)/\mathbb{Z}_N$ is the subgroup of $U(N)$ whose action is non-trivial in the moduli space of instantons. In this sense, we are regularizing the volume of the instanton moduli space. This is achieved by Ω-deforming the original $d = 4, \mathcal{N} = 2$ SYM action, a procedure which can be understood as a coupling to commuting isometry vector fields parametrized by (ϵ_1, ϵ_2). The partition function of the deformed theory becomes the Atiyah-Jeffrey version of the Mathai-Quillen representative of the Poincaré dual to the field configurations which are anti-self-dual instantons fixed under the action of the Cartan subalgebra $\mathbb{T}_{\mathcal{G}} \times \mathbb{T}_{SO(4)}$. By looking at the limit shapes of the partitions that label such field configurations, one can extract the Seiberg-Witten prepotential, as well as the objects Σ_u, (a_D, a) and λ_{SW} which are central players in Seiberg-Witten theory.

As already mentioned in the Introduction, the subjects presented on this dissertation make up some of the fundamentals of current research in supersymmetric gauge theories. Indeed, many interesting recent developments have the Nekrasov partition function and Seiberg-Witten theory as starting points, and provide good next steps after the study of the topics presented here. We can mention a few examples: the Localization methods utilized by Pestun to study supersymmetric Wilson-loops [37], connections to integrable systems [38], and the AGT correspondence [39].

Apêndice A

Computations

A.1 Algebra

The $\mathcal{N}=2$ Supersymmetry algebra without central charges in 4 dimensions is given by the following commutation relations:

$$\{Q_{\alpha A},\overline{Q}_{\dot\alpha B}\}=2\epsilon_{AB}\sigma^{\mu}_{\alpha\dot\alpha}P_\mu,\quad \{Q_{\alpha A},Q_{\beta B}\}=\{\overline{Q}_{\dot\alpha A},\overline{Q}_{\dot\beta B}\}=0, \tag{A.1.1}$$

$$[P_\mu,Q_{\alpha A}]=[P_\mu,\overline{Q}_{\dot\alpha A}]=0, \tag{A.1.2}$$

$$[M_{\mu\nu},Q_{\alpha A}]=-(\sigma_{\mu\nu})_{\alpha}{}^{\beta}Q_{\beta A},\quad [M_{\mu\nu},\overline{Q}^{\dot\alpha A}]=-(\overline{\sigma}_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\overline{Q}^{\dot\beta A}, \tag{A.1.3}$$

$$[Q_{\alpha A},R^a]=-\frac{1}{2}(T^a)_A{}^B Q_{\alpha B},\quad [\overline{Q}_{\dot\alpha}{}^A,R^a]=\frac{1}{2}\overline{Q}_{\dot\alpha}{}^B(T^a)_B{}^A, \tag{A.1.4}$$

$$[Q_{\alpha A},\widetilde{R}]=Q_{\alpha A},\quad [\overline{Q}_{\dot\alpha A},\widetilde{R}]=-\overline{Q}_{\dot\alpha A}. \tag{A.1.5}$$

We have denoted the translation generators by P_μ, the Lorentz generators by $M_{\mu\nu}$, the $SU(2)_I$ R-symmetry generators by R^a and the $U(1)_I$ generators by $\widetilde{R}$. The generators of the $\mathfrak{su}(2)$ algebra are denoted by T^a, where $a=1,\cdots,3$.

With these relations in mind, we are able to demonstrate straightforwardly that

$$\mathcal{Q}^2=0 \tag{A.1.6}$$

$$\{\mathcal{Q},G_\mu\}=\partial_\mu \tag{A.1.7}$$

as promised. Start with:

$$\{\mathcal{Q},\mathcal{Q}\}=\epsilon^{\dot\alpha\dot\beta}\epsilon^{\dot\gamma\dot\delta}\{\overline{Q}_{\dot\alpha\dot\beta},\overline{Q}_{\dot\gamma\dot\delta}\}=0. \tag{A.1.8}$$

Next, consider:

$$\begin{aligned}\{\mathcal{Q}, G_\mu\} &= \frac{i}{4}\epsilon^{\dot\alpha\dot\beta}\overline{\sigma}_\mu^{\dot\gamma\delta}\{\overline{Q}_{\dot\alpha\dot\beta}, Q_{\delta\dot\gamma}\} \\ &= \frac{i}{4}\frac{i}{4}\epsilon^{\dot\alpha\dot\beta}\overline{\sigma}_\mu^{\dot\gamma\delta}(\epsilon_{\dot\beta\dot\gamma}\sigma^\nu_{\dot\alpha\delta}P_\nu) \\ &= -\frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\epsilon_{\dot\beta\dot\gamma}\overline{\sigma}_\mu^{\dot\gamma\delta}\sigma^\nu_{\dot\alpha\delta}\partial_\nu \\ &= -\frac{1}{2}\delta^{\dot\alpha}_{\dot\gamma}\overline{\sigma}_\mu^{\dot\gamma\delta}\sigma^\nu_{\dot\alpha\delta}\partial_\nu \\ &= -\frac{1}{2}\mathrm{Tr}(\overline{\sigma}_\mu\sigma_\nu)\partial^\nu = \eta_{\mu\nu}\partial^\nu = \partial_\mu.\end{aligned} \tag{A.1.9}$$

A.2 Donaldson-Witten action $\mathcal{Q}$-exactness

We start by showing that the action 2.4.17 is $\mathcal{Q}$-exact. Indeed,

$$\mathcal{Q}F_{\mu\nu} = \mathcal{Q}(\partial_{[\mu}A_{\nu]} + [A_\mu, A_\nu]) = D_{[\mu}\psi_{\nu]} \tag{A.2.1}$$

and therefore

$$\frac{1}{4}\mathcal{Q}(F_{\mu\nu}\chi^{+\mu\nu}) = \frac{1}{4}\Big(2iD_\mu\psi_\nu\chi^{+\mu\nu} + F_{\mu\nu}F^{\mu\nu} + F_{\mu\nu}\widetilde{F}^{\mu\nu}\Big). \tag{A.2.2}$$

Also, since

$$\mathcal{Q}(D_\mu\varphi) = \partial_\mu\mathcal{Q}\varphi + i[\psi_\mu, \varphi] + [A_\mu, \mathcal{Q}\varphi] = D_\mu\mathcal{Q}\varphi + i[\psi_\mu, \varphi] \tag{A.2.3}$$

for any field φ, we conclude that

$$\frac{1}{2}\mathcal{Q}(\psi_\mu D^\mu\overline{\phi}) = -\frac{1}{2}\Big(D_\mu\phi D^\mu\overline{\phi} + 2i\psi_\mu D^\mu\eta + i\psi_\mu[\psi_\mu, \overline{\phi}]\Big). \tag{A.2.4}$$

Finally

$$-\frac{1}{4}\mathcal{Q}(\eta[\phi,\overline{\phi}]) = -\frac{1}{4}\Big(\frac{1}{2}[\phi,\overline{\phi}]^2 - 2i\eta[\phi,\eta]\Big). \tag{A.2.5}$$

Now we recall

$$V = \int d^4x\mathrm{tr}\left[\frac{1}{4}F_{\mu\nu}\chi^{+\mu\nu} + \frac{1}{2}\psi_\mu D^\mu\overline{\phi} - \frac{1}{4}\eta[\phi,\overline{\phi}]\right] \tag{A.2.6}$$

and find that

$$\begin{aligned}\{\mathcal{Q}, V\} = \int d^4x \operatorname{tr}\Big[&\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}F_{\mu\nu}\widetilde{F}^{\mu\nu} + \frac{i}{2}D_\mu\psi_\nu\chi^{+\mu\nu} \\ &- \frac{1}{2}D_\mu\phi D^\mu\overline{\phi} - i\eta D_\mu\psi^\mu - \frac{i}{2}\overline{\phi}[\psi_\mu, \psi^\mu] \\ &- \frac{1}{8}[\phi, \overline{\phi}]^2 - \frac{i}{2}\phi[\eta, \eta]\Big].\end{aligned} \tag{A.2.7}$$

To reach the final expression, we make use of the equation of motion for $\chi^+_{\mu\nu}$ which states that:

$$iD_\mu\psi_\nu = \frac{i}{4}\phi\chi^+_{\mu\nu}. \tag{A.2.8}$$

We then sum $\frac{i}{2}D_\mu\psi_\nu$ and subtract $\frac{i}{8}\phi\chi^+_{\mu\nu}$ in A.2.7. We arrive at:

$$\begin{aligned}\{\mathcal{Q}, V\} = \int d^4x \operatorname{tr}\Big[&\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}F_{\mu\nu}\widetilde{F}^{\mu\nu} - \frac{1}{2}D_\mu\phi D^\mu\overline{\phi} - i\eta D_\mu\psi^\mu + iD_\mu\psi_\nu\chi^{+\mu\nu} \\ &- \frac{1}{8}[\phi, \overline{\phi}]^2 - \frac{i}{8}\phi[\chi^+_{\mu\nu}, \chi^{+\mu\nu}] - \frac{i}{2}\overline{\phi}[\psi_\mu, \psi^\mu] - \frac{i}{2}\phi[\eta, \eta]\Big]\end{aligned} \tag{A.2.9}$$

which is precisely what we wanted to show.

Apêndice B

Symplectic/Kähler geometry toolkit

As the title suggests, this appendix will serve as a toolkit where we will store some assorted facts about Symplectic/Kähler/HyperKähler geometry. In particular, we will focus on what is needed to motivate the language used in the ADHM construction and Nekrasov's solution sections of this dissertation. We will be brief and therefore refer the reader to [34, 36] for more detailed accounts.

A **symplectic manifold** is a pair $(\mathcal{M}, \omega)$ comprised of a manifold $\mathcal{M}$ and a closed non-degenerate 2-form ω. The **symplectic form** ω is intimately tied to Hamiltonian Mechanics, as it associates a unique sense of time-evolution to any smooth function $\mathcal{H} : \mathcal{M} \to \mathbb{R}$. More precisely, start with $d\mathcal{H}$. Then, because of the non-degeneracy of the symplectic form, we can associate a unique vector field $V_{\mathcal{H}}$ to $d\mathcal{H}$ by:

$$\iota_{V_{\mathcal{H}}} \omega = d\mathcal{H}. \tag{B.0.1}$$

Then, the evolution generated by $\mathcal{H}$ is associated to the one-parameter family of diffeomorphisms generated by $V_{\mathcal{H}}$. As one may guess, the vector field $V_{\mathcal{H}}$ is called the **Hamiltonian vector field** with **Hamiltonian function** $\mathcal{H}$. To recover the notion of Poisson brackets, note that for any other function $F \in C^{\infty}(\mathcal{M})$, its variation along $V_{\mathcal{H}}$ is given by:

$$\mathcal{L}_{V_{\mathcal{H}}} F = \iota_{V_{\mathcal{H}}} dF = \iota_{V_{\mathcal{H}}} \iota_{V_F} \omega = \omega(V_F, V_{\mathcal{H}}) \equiv \{F, \mathcal{H}\} \tag{B.0.2}$$

where we have defined the **Poisson bracket** through:

$$\{F, \mathcal{H}\} = \omega(V_F, V_{\mathcal{H}}). \tag{B.0.3}$$

In Physics, we generally have Lie groups acting on systems through their Lie algebras. Symplectic geometry realizes this through and object called **moment map**. The moment map associated to the action of a Lie group G on a symplectic manifold $\mathcal{M}$ is denoted by $\mu : \mathcal{M} \to \mathfrak{g}^*$ and is defined such that, for every $\xi \in \mathfrak{g}$, the following equation holds:

$$d\langle \mu, \xi \rangle = \iota_{V_\xi} \omega, \tag{B.0.4}$$

where $V_\xi \in T\mathcal{M}$ denotes the vector field associated to ξ. In other words, the moment map is responsible for assigning a Hamiltonian function to the elements of the Lie algebra of G.

Finally, there is an extremely important theorem by Marsden-Weinstein-Meyer which formalizes the idea of quotienting a subset of a symplectic manifold by its symmetry group. It goes as follows: let $(\mathcal{M}, \omega, G, \mu)$ be a symplectic manifold with a G-action with moment map μ. Denote by $i : \mu^{-1}(0) \hookrightarrow \mathcal{M}$ the inclusion map and assume that the action of G on $\mu^{-1}(0)$ is free. Then, the quotient space $\mathcal{M}_{red} = \mu^{-1}(0)/G$ is a symplectic manifold whose symplectic form $\overline{\omega}$ satisfies $i^*\omega = \pi^*\overline{\omega}$, and $\pi^* : \mu^{-1}(0) \to \mathcal{M}_{red}$ is a principal G-bundle.

Kähler and HyperKähler manifolds are symplectic manifolds with extra structure. More precisely, a **Kähler manifold** is a symplectic manifold that also has a Riemannian structure g and a complex structure I, such that the three structures are compatible. This means that:

$$\omega(\cdot, \cdot) = g(I\cdot, \cdot), \qquad \nabla I = 0. \tag{B.0.5}$$

The definition for HyperKähler manifolds is analogous, but now one must have a triplet complex structures I, J, K and another triplet of symplectic forms satisfying:

$$\nabla I = 0, \qquad \omega^{\mathbb{R}} = g(I\cdot, \cdot) \tag{B.0.6}$$

$$\nabla J = 0, \qquad \omega^{\mathbb{C}} = g(J\cdot, \cdot) + ig(K\cdot, \cdot) \tag{B.0.7}$$

$$\nabla K = 0, \qquad \omega^{\overline{\mathbb{C}}} = g(J\cdot, \cdot) - ig(K\cdot, \cdot). \tag{B.0.8}$$

The previous considerations about moment maps and reductions follow straightforwardly with the appropriate modifications:

$$d\langle \mu, \xi \rangle = \iota_{V_\xi} \omega, \quad \mu^{-1}(0)/G \to \text{Kahler} \tag{B.0.9}$$

$$d\langle \vec{\mu}, \xi \rangle = \iota_{V_\xi} \vec{\omega}, \quad \vec{\mu}^{-1}(0)/G \to \text{HyperKahler}. \tag{B.0.10}$$

Now, an important fact for our considerations about the equivariant regularization of the volume of $\mathfrak{M}_k$ is that the form $\omega - \langle \mu, \xi \rangle$ is equivariantly closed with respect to the equivariant differential $Q = d - \iota_{V_\xi}$. Indeed:

$$\begin{aligned} Q\big(\omega - \langle \mu, \xi \rangle\big) &= \big(d - \iota_{V_\xi}\big)\big(\omega - \langle \mu, \xi \rangle\big) \\ &= d\omega - \iota_{V_\xi}\omega - d\langle \mu, \xi \rangle + \iota_{V_\xi}\langle \mu, \xi \rangle \\ &= -\iota_{V_\xi}\omega - d\langle \mu, \xi \rangle = 0. \end{aligned} \tag{B.0.11}$$

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