Absorption cross section of electromagnetic waves for Schwarzschild black holes

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The absorption cross section of black holes has been investigated for various fields. Nevertheless, the absorption cross section of Schwarzschild black holes for the electromagnetic field has been only calculated in the low- and high-frequency approximations until now. Here we compute it numerically for arbitrary frequencies.

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The absorption cross section of Schwarzschild black holes in four dimensions has been computed for the massless scalar [1] and fermion [2] fields for arbitrary frequencies and for the electromagnetic field in the low- and high-frequency regimes [3–5]. Some of these cases also have been extended to spherically symmetric black holes in higher dimensions using standard field theory in curved spacetimes [6–8] and effective string model [9]. However, the absorption cross section of Schwarzschild black holes for the electromagnetic field has not been computed for arbitrary frequencies so far. This may be because of additional difficulties associated with the gauge freedom of the Maxwell field, which are not present in the Klein-Gordon and Dirac fields. In this paper, we obtain the absorption cross section for Schwarzschild black holes in four dimensions for electromagnetic waves with arbitrary frequencies through numerical computations.

Let us start by analyzing the classical solutions of the Maxwell field in the Schwarzschild spacetime characterized by the line element

$$ds^2 = f(r)dt^2 - f(r)^{-1}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2,$$

where $f(r) = 1 - 2M/r$. (Here we let $c = G = 1$.)

The Lagrangian density of the electromagnetic field in the modified Feynman gauge is

$$\mathcal{L} = \sqrt{-g}[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}H^2],$$

with $g = \det(g_{\mu\nu})$, $H \equiv \nabla^\mu A_\mu + K^\mu A_\mu$, and $K^\mu = (0, df/dr, 0, 0)$. The corresponding Euler-Lagrange equations are

$$\nabla_\nu F^{\mu\nu} + \nabla^\mu H - K^\mu H = 0. \tag{1}$$

Schwarzschild spacetime is endowed with the timelike Killing field ∂_t in the region $r > 2M$. Thus, it is convenient to look for solutions of Eq. (1) in the form

$$A_\mu^{(\varepsilon n\omega lm)} = \zeta_\mu^{\varepsilon n\omega lm}(r, \theta, \phi)e^{-i\omega t}, \qquad \omega > 0.$$

Modes incoming from the past horizon $\mathcal{H}^-$ are labeled by $n = \rightarrow$, whereas modes incoming from the past infinity $\mathcal{J}^-$ are labeled by $n = \leftarrow$. Here $\varepsilon = G, I, II, NP$ represent the four different polarization modes. Modes with $\varepsilon = G$ are called pure gauge modes. They satisfy the gauge condition $H = 0$ and in addition can be written as $A_\mu^{(Gn\omega lm)} = \nabla_\mu \Lambda$ with Λ being a scalar field. The physical modes, with $\varepsilon = I, II$, also satisfy $H = 0$ but are not pure gauge modes. The label $\varepsilon = NP$ is for the nonphysical modes, i.e., the ones which do not satisfy the gauge condition. We shall only be concerned with the two physical modes. (Further discussions on pure gauge and nonphysical modes can be found elsewhere [8,10].)

The physical modes can be written as

$$A_\mu^{(In\omega lm)} = \left(0, \frac{\varphi_{\omega l}^{In}(r)}{r}Y_{lm}, \frac{f}{l(l+1)}\frac{d}{dr}[r\varphi_{\omega l}^{In}(r)]\partial_\theta Y_{lm}, \frac{f}{l(l+1)}\frac{d}{dr}[r\varphi_{\omega l}^{In}(r)]\partial_\phi Y_{lm}\right)e^{-i\omega t}, \tag{2}$$

$$A_\mu^{(IIn\omega lm)} = (0, 0, r\varphi_{\omega l}^{IIn}(r)Y_\theta^{lm}, r\varphi_{\omega l}^{IIn}(r)Y_\phi^{lm})e^{-i\omega t}, \tag{3}$$

with $l \geq 1$. For $l = 0$ there are no modes satisfying the gauge condition which are not pure gauge. The functions Y_{lm} and Y_a^{lm} are the scalar and vector spherical harmonics [11], respectively.

The angular components of the physical modes in Eqs. (2) and (3) can be written as

$$A_a^{(\lambda n\omega lm)} = \partial_a \Upsilon^{(\lambda n\omega lm)} + \epsilon_{ab}\partial^b \Xi^{(\lambda n\omega lm)}, \tag{4}$$

where the functions $\Upsilon^{(\lambda n\omega lm)}$ and $\Xi^{(\lambda n\omega lm)}$ ($\lambda = I, II$) are

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$$\Upsilon^{(In\omega lm)} = \frac{f}{l(l+1)}\frac{d}{dr}[r\varphi^{In}_{\omega l}(r)]Y_{lm}e^{-i\omega t}, \quad (5)$$

$$\Xi^{(IIn\omega lm)} = \frac{-1}{\sqrt{l(l+1)}} r\varphi^{IIn}_{\omega l}(r)Y_{lm}e^{-i\omega t}, \quad (6)$$

and $\epsilon_{\theta\theta} = \epsilon_{\phi\phi} = 0$, $\epsilon_{\theta\phi} = -\epsilon_{\phi\theta} = \sin\theta$. The indices a and b denote angular variables on the unit 2-sphere S^2 with metric $\tilde{\eta}_{ab}$ and inverse metric $\tilde{\eta}^{ab}$ [with signature $(--)$].

The radial functions $\varphi^{\lambda n}_{\omega l}(r)$ satisfy the following differential equation

$$(\omega^2 - V_S)[r\varphi^{\lambda n}_{\omega l}(r)] + f\frac{d}{dr}\left(f\frac{d}{dr}[r\varphi^{\lambda n}_{\omega l}(r)]\right) = 0, \quad (7)$$

where the scattering potential is $V_S \equiv fl(l+1)/r^2$.

The conjugate momenta associated with the field modes $A^{(i)}_\mu$ above are

$$\Pi^{(i)\mu\nu} = -[F^{\mu\nu} + g^{\mu\nu}H]|_{A_\mu = A^{(i)}_\mu},$$

where the label (i) stands for the set $(\varepsilon n\omega lm)$.

The modes can be orthonormalized using the generalized Klein-Gordon inner product

$$(A^{(i)}, A^{(j)}) \equiv \int_\Sigma d\Sigma^{(3)} n_\mu W^\mu[A^{(i)}, A^{(j)}].$$

Here $d\Sigma^{(3)}$ is the invariant 3-volume element of the Cauchy surface Σ, n^μ is the future pointing unit vector orthogonal to Σ and $W^\mu[A^{(i)}, A^{(j)}]$ is the conserved current given by

$$W^\mu[A^{(i)}, A^{(j)}] \equiv i[A^{(i)*}_\nu \Pi^{(j)\mu\nu} - \Pi^{(i)\mu\nu*}A^{(j)}_\nu]. \quad (8)$$

The physical modes will be orthonormalized by imposing

$$(A^{(\lambda n\omega lm)}, A^{(\lambda' n'\omega' l'm')}) = \delta_{\lambda\lambda'}\delta_{nn'}\delta_{ll'}\delta_{mm'}\delta(\omega - \omega'). \quad (9)$$

The solutions of Eq. (7) are functions whose properties are not well known. We can, however, obtain their analytic form (i) in the asymptotic regions for any frequency and (ii) everywhere in the low-frequency approximation. In order to study the asymptotic behavior of the physical modes we use the Wheeler coordinate $x = r + 2M\ln(r/2M - 1)$ and rewrite Eq. (7) as

$$(\omega^2 - V_S)[r\varphi^{\lambda n}_{\omega l}(x)] + \frac{d^2}{dx^2}[r\varphi^{\lambda n}_{\omega l}(x)] = 0.$$

Since we are interested in computing the absorption cross section, we only need to deal with the modes incoming from $\mathcal{J}^-$. The asymptotic forms of these modes are

$$r\varphi^{\lambda\leftarrow}_{\omega l}(r) \approx \begin{cases} B^{\lambda\leftarrow}_{\omega l}T^{\lambda\leftarrow}_{\omega l}e^{-i\omega x} & (r \approx 2M), \\ B^{\lambda\leftarrow}_{\omega l}((-i)^{l+1}\omega x h^{(1)*}_l(\omega x) + R^{\lambda\leftarrow}_{\omega l} i^{l+1}\omega x h^{(1)}_l(\omega x)) & (r \gg 2M). \end{cases} \quad (10)$$

Here $h^{(1)}_l(x)$ are the spherical Bessel functions of the third kind [12], $B^{\lambda\leftarrow}_{\omega l}$ are normalization constants, $|R^{\lambda\leftarrow}_{\omega l}|^2$ and $|T^{\lambda\leftarrow}_{\omega l}|^2$ are the reflexion and transmission coefficients, respectively, satisfying the usual probability conservation equation $|R^{\lambda\leftarrow}_{\omega l}|^2 + |T^{\lambda\leftarrow}_{\omega l}|^2 = 1$. Using the generalized Klein-Gordon product (9) we get $|B^{I\leftarrow}_{\omega l}| = \sqrt{l(l+1)/(4\pi\omega^3)}$ and $|B^{II\leftarrow}_{\omega l}| = 1/\sqrt{4\pi\omega}$.

Let us now find the analytic expressions of the physical modes in the low-frequency approximation. This is going to be useful as a consistency check for our arbitrary frequency numerical calculation. For this purpose we rewrite Eq. (7) as

$$\frac{d}{dz}\left[(1-z^2)\frac{d\varphi^{\lambda n}_{\omega l}(z)}{dz}\right] + \left[l(l+1) - \frac{2}{z+1} - \omega^2 M^2\frac{(z+1)^3}{z-1}\right]\varphi^{\lambda n}_{\omega l}(z) = 0, \quad (11)$$

where $z \equiv r/M - 1$. In the low-frequency regime, the solutions of Eq. (11) for modes incoming from $\mathcal{J}^-$, with $l \geq 1$, are given by

$$\varphi^{\lambda\leftarrow}_{\omega l}(z) \approx C^{\lambda\leftarrow}_{\omega l}\left[\mathrm{P}_l(z) - \frac{(z-1)}{l(l+1)}\frac{d\mathrm{P}_l(z)}{dz}\right], \quad (12)$$

where $\mathrm{P}_l(z)$ are the Legendre functions of the first kind [13] and $C^{\lambda\leftarrow}_{\omega l}$ are normalization constants. Now, by comparing the asymptotic form of Eq. (12) for $r \gg 2M$ with Eq. (10) in the low-frequency limit, we find that the normalization constants are (up to arbitrary phases)

$$C^{I\leftarrow}_{\omega l} = \frac{1}{\sqrt{\pi l(l+1)}}\frac{2^l((l+1)!)^2M^l}{(2l)!(2l+1)!!}\omega^{l-1/2},$$

$$C^{II\leftarrow}_{\omega l} = \frac{1}{\sqrt{\pi}}\frac{2^l(l+1)(l!)^2M^l}{l(2l)!(2l+1)!!}\omega^{l+1/2}.$$

Now, let us calculate the black hole absorption cross section. Since the Schwarzschild spacetime is asymptotically flat, we consider an incident electromagnetic circularly polarized plane wave propagating in the z direction:

$$A_x = \exp[i\omega(z-t)], \quad (13)$$

$$A_y = i\exp[i\omega(z-t)]. \quad (14)$$

Expanding Eqs. (13) and (14) in terms of Legendre functions, we may write the spherical polar components of this incident electromagnetic plane wave as

$$A_r = \sum_{l=1}^{\infty} i^{l+1}(2l+1)\left[\frac{j_l(\omega r)}{\omega r}\right]\mathrm{P}^1_l(\cos\theta)e^{i\phi}e^{-i\omega t}, \quad (15)$$

$$A_\theta = \sum_{l=0}^{\infty} i^l (2l+1)[r j_l(\omega r)] \mathrm{P}_l(\cos\theta) \cos\theta e^{i\phi} e^{-i\omega t}, \quad (16)$$

$$A_\phi = \sum_{l=0}^{\infty} i^{l+1} (2l+1)[r j_l(\omega r)] \mathrm{P}_l(\cos\theta) \sin\theta e^{i\phi} e^{-i\omega t}, \quad (17)$$

where $\mathrm{P}_l^1(\cos\theta)$ are the associated Legendre functions [12,13] and $j_l(\omega r)$ are the spherical Bessel functions of the first kind [12].

Let us now write the angular components of this incident plane wave as

$$A_a = \partial_a \Phi + \epsilon_{ab} \partial^b \Psi. \quad (18)$$

To determine Φ and Ψ we recall that far away from the black hole the gauge condition $H = 0$ reduces to the usual Lorenz condition in flat spacetime, namely,

$$\nabla^\mu A_\mu = \partial_t A_t - \frac{1}{r^2} \partial_r (r^2 A_r) + \frac{1}{r^2} \tilde{\nabla}^a A_a = 0,$$

where $\tilde{\nabla}_a$ is the associated covariant derivative on S^2, $\tilde{\nabla}^a \equiv \tilde{\eta}^{ab} \tilde{\nabla}_b$, and $\tilde{\nabla}^2 \equiv \tilde{\eta}^{ab} \tilde{\nabla}_a \tilde{\nabla}_b$. From Eq. (18), it follows that $\epsilon^{ab} \tilde{\nabla}_a A_b = -\tilde{\nabla}^2 \Psi$ and $\tilde{\nabla}^a A_a = \tilde{\nabla}^2 \Phi$. We use these relations together with Lorenz condition to obtain from Eqs. (15)–(17) that

$$\Phi = \sum_{l=1}^{\infty} \frac{i^{l+1}}{\omega} \frac{(2l+1)}{l(l+1)} \frac{d}{dr} [r j_l(\omega r)] \mathrm{P}_l^1(\cos\theta) e^{i\phi} e^{-i\omega t},$$

$$\Psi = \sum_{l=1}^{\infty} i^{l-1} \frac{(2l+1)}{l(l+1)} [r j_l(\omega r)] \mathrm{P}_l^1(\cos\theta) e^{i\phi} e^{-i\omega t}.$$

In the limit $r \to \infty$ the scalar functions Φ and Ψ can be written as

$$\Phi \approx \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{2\omega^2} \frac{(2l+1)}{l(l+1)} \frac{d}{dr} [e^{-i\omega r} - (-1)^l e^{i\omega r}] \times \mathrm{P}_l^1(\cos\theta) e^{i\phi} e^{-i\omega t}, \quad (19)$$

$$\Psi \approx \sum_{l=1}^{\infty} \frac{(-1)^l}{2\omega} \frac{(2l+1)}{l(l+1)} [e^{-i\omega r} - (-1)^l e^{i\omega r}] \times \mathrm{P}_l^1(\cos\theta) e^{i\phi} e^{-i\omega t}. \quad (20)$$

Equations (19) and (20) will be used as a guide for the asymptotic form of the incident plane wave in the black hole spacetime. Substituting Eq. (10) in Eqs. (4)–(6) we can write

$$\Phi_{\mathrm{bh}} \approx \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{2\omega^2} \frac{(2l+1)}{l(l+1)} \frac{d}{dr} [e^{-i\omega r} + R_{\omega l}^{I\leftarrow} e^{i\omega r}] \times \mathrm{P}_l^1(\cos\theta) e^{i\phi} e^{-i\omega t}, \quad (21)$$

$$\Psi_{\mathrm{bh}} \approx \sum_{l=1}^{\infty} \frac{(-1)^l}{2\omega} \frac{(2l+1)}{l(l+1)} [e^{-i\omega r} + R_{\omega l}^{II\leftarrow} e^{i\omega r}] \times \mathrm{P}_l^1(\cos\theta) e^{i\phi} e^{-i\omega t}, \quad (22)$$

which correspond to Φ and Ψ given in Eqs. (19) and (20), respectively.

The absorption cross section is given by $\sigma \equiv -\mathcal{F}/J$, where $\mathcal{F}$ is the integrated flux of the modes with $n = \leftarrow$ and J is the incident current density. The incident current associated to the plane wave (13) and (14) is $J = 4\omega$. The incoming ($n = \leftarrow$) flux of the current (8) associated to the physical modes given by Eqs. (2) and (3) is

$$\begin{aligned} \mathcal{F} &= i \oint_{r\to\infty} d\Omega r^2 W^r \\ &= i \oint_{r\to\infty} d\Omega [(\tilde{\nabla}^2 \Phi_{\mathrm{bh}}) \partial_r \Phi_{\mathrm{bh}}^* - \Phi_{\mathrm{bh}}^* \partial_r (\tilde{\nabla}^2 \Phi_{\mathrm{bh}}) \\ &\quad + (\tilde{\nabla}^2 \Psi_{\mathrm{bh}}) \partial_r \Psi_{\mathrm{bh}}^* - \Psi_{\mathrm{bh}}^* \partial_r (\tilde{\nabla}^2 \Psi_{\mathrm{bh}})], \end{aligned} \quad (23)$$

where $d\Omega$ is the solid angle element. We have used the fact that $A_r^{(\lambda n\omega lm)}$ falls as r^{-2} for large r. Substituting Eqs. (21) and (22) in Eq. (23) we obtain the black hole absorption cross section for the electromagnetic field:

$$\sigma_E = \sum_{l=1}^{\infty} \sigma_E^{(l)} = \frac{\pi}{2\omega^2} \sum_{l=1}^{\infty} \sum_{\lambda=I,II} (2l+1) |T_{\omega l}^{\lambda\leftarrow}|^2. \quad (24)$$

Using the low-frequency approximation of $\varphi_{\omega l}^{\lambda\leftarrow}$ given by Eq. (12) for $r \to 2M$ and comparing it with Eq. (10) for $r \to 2M$ and $\omega \approx 0$, we find that $|T_{\omega 1}^{I\leftarrow}|^2 = |T_{\omega 1}^{II\leftarrow}|^2 \approx 64(M\omega)^4/9$, which gives the leading ω contribution in Eq. (24). Hence, to lowest order in ω, the absorption cross section is [4,5]

$$\sigma_E^{(\omega\approx 0)} \approx \tfrac{4}{3} \pi R_S^4 \omega^2, \quad (25)$$

where $R_S = 2M$ is the Schwarzschild radius.

Now, we compute the absorption cross section numerically, using Eq. (24). The numerical method used here is described in Ref. [14].

In Fig. 1 we plot our results for the partial absorption cross section $\sigma_E^{(l)}$ for $l = 1$ up to $l = 6$. We see that, for each value of l, the corresponding partial absorption cross section starts from zero, reaches a maximum value $\sigma_E^{(l)\max}$, and then falls to zero for $\omega^2 \gg V_S^{\max}$, where $V_S^{\max}$ is the maximum of the effective scattering potential. We also see that the larger the value of $l (\geq 1)$ is (i) the smaller is the corresponding value of $\sigma_E^{(l)\max}$ and (ii) the larger is the value of ω associated with $\sigma_E^{(l)\max}$. This is all compatible with the fact that the scattering potential V_S is larger for larger values of l. (We recall that modes with $l = 0$ are not physical modes in the electromagnetic case.)

In Fig. 2 we plot our results for the total absorption cross section σ_E with the summation in Eq. (24) performed up to $l = 6$. We note that for large ω, the total absorption cross

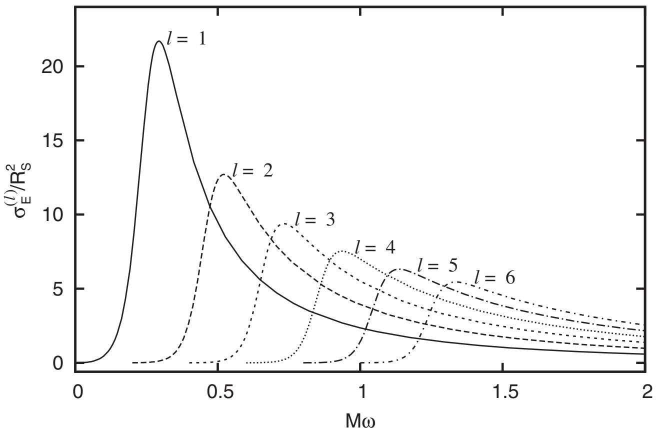

FIG. 1. The partial electromagnetic absorption cross section $\sigma_E^{(l)}$ is plotted as a function of the wave frequency ω for $l = 1$ up to $l = 6$. We note that the larger l is, the smaller is the maximum of the corresponding $\sigma_E^{(l)}$.

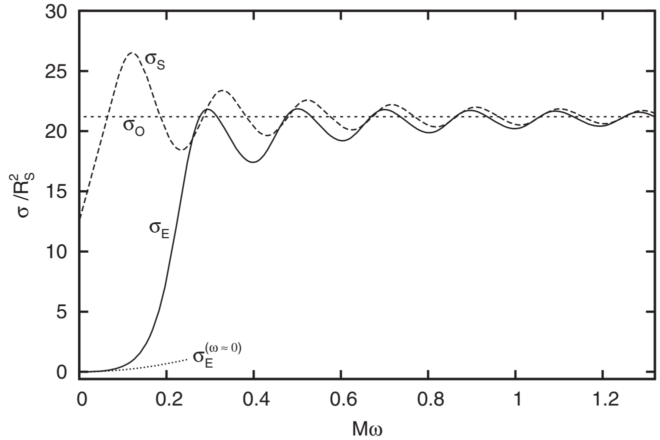

FIG. 2. The total absorption cross sections for the electromagnetic case σ_E and massless scalar case σ_S are plotted as functions of the frequency ω. The summation in the angular momentum is performed up to $l = 6$. We see that for high frequencies they oscillate around the limit of geometrical optics $\sigma_O \equiv 27\pi M^2$. We also plot $\sigma_E^{(\omega\approx 0)}$ calculated analytically [see Eq. (25)] to show that this is in agreement with our numerical calculation.

section σ_E oscillates around the limit of geometrical optics $\sigma_O \equiv 27\pi M^2$. This is also verified by the total absorption cross section for the massless scalar field, σ_S, which was originally computed by Sanchez [1]. From Fig. 2 we can see also that for very low frequencies, corresponding to large wavelengths compared to the Schwarzschild radius, σ_E (as well as σ_S) behaves as expected. While for the electromagnetic case the low-energy total absorption cross section goes to zero, in the massless scalar case it tends to the area of the black hole (see, e.g., Ref. [6]). We also plot $\sigma_E^{(\omega\approx 0)}$ given by Eq. (25), which has been analytically calculated. This makes it clear that our numerical computation of σ_E agrees with its analytic value in the low-frequency limit. It is interesting to note that σ_E and σ_S have similar oscillatory behaviors in the high-frequency regime. In both cases the total absorption cross sections have local maxima corresponding to the maximum of each partial absorption cross section.

In summary we have computed numerically the total Schwarzschild black hole absorption cross section of electromagnetic waves for arbitrary frequencies. We have checked our result in the low- and high-frequency limits and found that its high-frequency behavior is very similar to that for the massless scalar field.

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