Boson-boson effective nonrelativistic potential for higher-derivative electromagnetic theories in D dimensions

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(Received 1 July 2004; published 15 November 2004)

The problem of computing the effective nonrelativistic potential U_D for the interaction of charged-scalar bosons, within the context of D-dimensional electromagnetism with a cutoff, is reduced to quadratures. It is shown that U_3 cannot bind a pair of identical charged-scalar bosons; nevertheless, numerical calculations indicate that boson-boson bound states do exist in the framework of three-dimensional higher-derivative electromagnetism augmented by a topological Chern-Simons term.

DOI: 10.1103/PhysRevD.70.107705 PACS numbers: 11.10.St, 11.10.Kk, 12.20.Ds

We consider in this Brief Report the problem of determining the effective charged-scalar-boson—charged-scalar-boson low energy potential U_D arising from D-dimensional electromagnetism with a cutoff a. The Lagrangian concerning this theory can be written as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\lambda F_{\mu\lambda}, \tag{1}$$

where $F_{\mu\nu} \equiv \partial_\nu A_\mu - \partial_\mu A_\nu$ is the usual electromagnetic tensor field. Lagrangian (1) is gauge and Lorentz invariant; in addition, it leads to local field equations which are linear in the field quantities. At distances much larger than the cutoff, the fields described by Eq. (1) become essentially equivalent to the Maxwell fields.

We are motivated by two quite similar developments: In the first, we investigate whether U_3 can form "Cooper pairs."

Our second topic is related to three-dimensional Podolsky-Chern-Simons theory. Based on the interesting discussions from Jackiw [1] about the consistency of the nonrelativistic limit of certain relativistically invariant quantum field theories, it can be shown that the Chern-Simons term alone is unable to form boson-boson bound states [2]. Nonetheless, numerical calculations indicate that the Podolsky term provides a stabilizing mechanism allowing for the existence of Cooper pairs.

We use natural units throughout; our signature is $(+, -, -, \cdots, -)$.

Nonrelativistic quantum mechanics tells us that in the first Born approximation the cross section for the scattering of two indistinguishable massive particles, in the center-of-mass frame (CoM), is given by $\frac{d\sigma}{d\Omega} = |\frac{m}{4\pi} \times \int e^{-i\mathbf{p}'\cdot\mathbf{r}}V(r)e^{i\mathbf{p}\cdot\mathbf{r}}d^{D-1}\mathbf{r}|^2$, where $\mathbf{p}$ ($\mathbf{p}'$) is the initial (final) momentum of one of the particles in the CoM. In terms of the transfer momentum, $\mathbf{k} \equiv \mathbf{p}' - \mathbf{p}$, it reads

$$\frac{d\sigma}{d\Omega} = \left|\frac{m}{4\pi}\int V(r)e^{i\mathbf{k}\cdot\mathbf{r}}d^{D-1}\mathbf{r}\right|^2. \tag{2}$$

On the other hand, from quantum field theory we know that the cross section, in the CoM, for the scattering of two identical massive scalar bosons by an electromagnetic field, can be written as $\frac{d\sigma}{d\Omega} = |\frac{1}{16\pi E}\mathcal{M}|^2$, where E is the initial energy of one of the bosons and $\mathcal{M}$ is the Feynman amplitude for the process at hand, which in the nonrelativistic limit (NR) reduces to

$$\frac{d\sigma}{d\Omega} = \left|\frac{1}{16\pi m}\mathcal{M}_{\rm NR}\right|^2. \tag{3}$$

From Eqs. (2) and (3), we come to the conclusion that the expression that enables us to compute the D-dimensional effective nonrelativistic potential has the form

$$V(r) = \frac{1}{4m^2}\frac{1}{(2\pi)^{D-1}}\int d^{D-1}\mathbf{k}\,\mathcal{M}_{\rm NR}e^{-i\mathbf{k}\cdot\mathbf{r}}, \tag{4}$$

which clearly shows how the potential from quantum mechanics and the Feynman amplitude obtained via quantum field theory are related to each other.

Now, in the Lorentz gauge Podolsky's scalar QED is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\alpha F_{\mu\alpha} - \frac{1}{2\lambda}(\partial_\nu A^\nu)^2 + (D_\mu\phi)^*D^\mu\phi - m^2\phi^*\phi, \tag{5}$$

where $D_\mu \equiv \partial_\mu + iQA_\mu$. Therefore, the interaction Lagrangian to order Q for the process $S + S \longrightarrow S + S$, where S denotes a spinless boson of mass m and charge Q, is $\mathcal{L}_{\rm int} = iQA^\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi)$. The Feynman rule for the elementary vertex is shown in Fig. 1. Accordingly, the Feynman amplitude for the interaction of two charged spinless bosons of equal mass via a Podolskian photon exchange (see Fig. 2) is

$$\mathcal{M} = V^\mu(p, p')D_{\mu\nu}(k)V^\nu(q, q'), \tag{6}$$

where $D_{\mu\nu}(k)$ designates the Podolskian photon propagator.

We propose now an algorithm for computing the propagator for electromagnetic theories with higher deriva-

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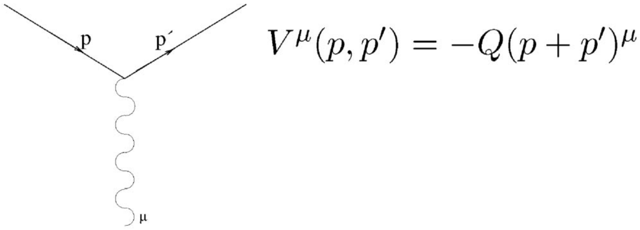

FIG. 1. The relevant vertex for boson-boson interaction.

tives, based on the usual transverse and longitudinal vector projector operators, namely, $\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box}$, $\omega_{\mu\nu} = \frac{\partial_\mu \partial_\nu}{\Box}$, which satisfy the relations $\theta_{\mu\rho}\theta^{\rho}_{\nu} = \theta_{\mu\nu}$, $\omega_{\mu\rho}\omega^{\rho}_{\nu} = \omega_{\mu\nu}$, $\theta_{\mu\rho}\omega^{\rho}_{\nu} = 0$, where $\eta_{\mu\nu}$ is the Minkowski metric. The set of operators $\{\theta, \omega\}$ is a complete set of projector operators for rank-one tensors. Indeed, they are idempotent, mutually orthogonal, and satisfy the completeness relation $[\theta + \omega]_{\mu\nu} = \eta_{\mu\nu} \equiv I_{\mu\nu}$.

Let $\bar{\mathcal{L}}$ be the Lagrangian for electromagnetism with higher derivatives. Since $\bar{\mathcal{L}}$ is a gauge-invariant Lagrangian, we add to it a gauge-fixing Lagrangian $\mathcal{L}_{gf}$, which implies that $\mathcal{L} \equiv \bar{\mathcal{L}} + \mathcal{L}_{gf}$ can be written as $\mathcal{L} = \frac{1}{2}A^{\mu}O_{\mu\nu}A^{\nu}$. Expanding O in the basis $\{\theta, \omega\}$ yields $O = x_1\theta + x_2\omega$. Accordingly, $O^{-1} = y_1\theta + y_2\omega$, where O^{-1} is the propagator and y_1 and y_2 are parameters to be determined. Now, taking into account that $OO^{-1} = I$, we promptly obtain

$$O^{-1} = \frac{1}{x_1}\theta + \frac{1}{x_2}\omega,$$

where we are supposing that both x_1 and x_2 are nonvanishing. Note that the procedure we have just outlined is quite straightforward: On the one hand, it reduces the work of calculating the propagator to a trivial algebraic exercise; on the other hand, it greatly simplifies calculations involving the contraction of conserved currents ($\partial_\nu J^{\nu} = 0$) with the propagator since in this case the alluded contraction simply gives

$$O^{-1}_{\mu\nu}J^{\mu} = \frac{J_\nu}{x_1}. \tag{7}$$

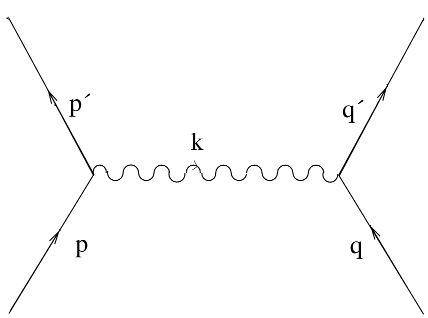

FIG. 2. One-Podolskian photon-exchange contribution to the scattering of two identical massive charged bosons.

From the above we find that the propagator for Podolsky's electrodynamics in the Lorentz gauge assumes the form

$$D_{\mu\nu}(k) = \frac{M^2}{k^2(k^2 - M^2)}\theta_{\mu\nu} - \frac{\lambda}{k^2}\omega_{\mu\nu}, \tag{8}$$

where $M^2 \equiv 1/a^2$.

From Eqs. (6)–(8), we get immediately $\mathcal{M} = \frac{M^2Q^2(2p-k)(2q+k)}{k^2(k^2-M^2)}$, which implies

$$\mathcal{M}_{\rm NR} = \frac{4m^2M^2Q^2}{\mathbf{k}^2(\mathbf{k}^2 + M^2)}. \tag{9}$$

Inserting Eq. (9) into Eq. (4), we obtain

$$V(r) = \int_0^\infty f(|\mathbf{k}|)|\mathbf{k}|^{n-1}d|\mathbf{k}| \int_0^{2\pi} d\theta_1 \int_0^\pi \sin\theta_2 d\theta_2 \int_0^\pi \times \sin^2\theta_3 d\theta_3 \ldots \int_0^\pi e^{-i|\mathbf{k}|r\cos\theta_{n-1}} \sin^{n-2}\theta_{n-1}d\theta_{n-1},$$

where $2 < n \equiv D - 1$ and $f(|\mathbf{k}|) \equiv \frac{Q^2}{(2\pi)^n}(\frac{1}{\mathbf{k}^2} - \frac{1}{\mathbf{k}^2 - M^2})$. Now, taking into account that

$$\int_0^\pi \sin^m\theta d\theta = \frac{\sqrt{\pi}\Gamma(\frac{m-1}{2})}{\Gamma(\frac{m+2}{2})},$$

$$\int_0^\pi e^{-i|\mathbf{k}|r\cos\theta_{n-1}} \sin^{n-2}\theta_{n-1}d\theta_{n-1} = \frac{2^{(n-2)/2}\Gamma(\frac{1}{2})}{(|\mathbf{k}|r)^{(n-2)/2}}\Gamma\left(\frac{n-1}{2}\right)J_{(n-2)/2}(|\mathbf{k}|r),$$

where J denotes the Bessel function, we arrive at the following expression for the potential:

$$U_D(r) = \frac{Q}{(2\pi)^{(D-1)/2}r^{(D-3)/2}} \int_0^\infty \left(\frac{1}{\mathbf{k}^2} - \frac{1}{\mathbf{k}^2 - M^2}\right) \times |\mathbf{k}|^{(D-1)/2}J_{(D-3)/2}(|\mathbf{k}|r)d|\mathbf{k}|, \tag{10}$$

where $U_D(r) \equiv \frac{V(r)}{Q}$ and $D > 3$.

On the other hand, it is trivial to show that U_3 can be evaluated from the expression

$$U_3(r) = \frac{Q}{2\pi} \int_0^\infty \left(\frac{1}{\mathbf{k}^2} - \frac{1}{\mathbf{k}^2 - M^2}\right)|\mathbf{k}|J_0(|\mathbf{k}|r)d|\mathbf{k}|,$$

which allows us to conclude that Eq. (10) can also be applied to the case $D = 3$. Hence, the problem of computing the effective nonrelativistic potential for $D > 2$ was reduced to quadratures.

If Eq. (10) is correct, it must reproduce the Podolskian potential in 3 + 1 dimensions. Performing the computation for $D = 4$, we get

$$U_4(r) = \frac{Q}{4\pi}\frac{1 - e^{-(r/a)}}{r},$$

which is just the same result as that obtained in Podolsky's electromagnetic theory.

For $D = 3$, Eq. (10) yields

$$U_3(r) = -\frac{Q}{2\pi}\left[\ln\frac{r}{r_0} + K_0(Mr)\right], \tag{11}$$

where r_0 is an infrared regulator and K is the modified Bessel function.

We discuss now the existence of boson-boson bound states in the context of planar quadratic electromagnetism. The corresponding time-independent Schrödinger equation can be written as

$$\begin{aligned}\mathcal{H}_l\mathcal{R}_{nl} &= -\frac{1}{m}\left(\frac{d^2}{dr^2}\mathcal{R}_{nl} + \frac{1}{r}\frac{d}{dr}\mathcal{R}_{nl}\right) + V_l^{\mathrm{eff}}\mathcal{R}_{nl}\\ &= E_{nl}\mathcal{R}_{nl},\\ V_l^{\mathrm{eff}} &\equiv \frac{l^2}{mr^2} + QU_3(r)\\ &= \frac{l^2}{mr^2} - \frac{Q^2}{2\pi}\left[\ln\frac{r}{r_0} + K_0(Mr)\right],\end{aligned}$$

where $\mathcal{R}_{nl}$ is the nth normalizable eigenfunction of the radial Hamiltonian $\mathcal{H}_l$ whose corresponding eigenvalue is E_{nl} and V_l^{eff} is the lth partial wave effective potential. On the other hand,

$$\frac{d}{dr}V_l^{\mathrm{eff}} = -\frac{2l^2}{m}\frac{1}{r^3} - \frac{Q^2}{2\pi}\frac{1}{r} + \frac{Q^2M}{2\pi}K_1(Mr),$$

which allows us to conclude that $\frac{d}{dr}V_l^{\mathrm{eff}} < 0$ in the interval $0 < r < \infty$, implying that V_l^{eff} is strictly decreasing in this interval. Consequently, in the framework of planar quadratic electromagnetism, no bound state concerning the two charged-scalar bosons system exists.

Since boson-boson bound states do not show up in Podolsky planar electromagnetism, we investigate here whether the effective boson-boson low energy potential related to Podolsky-Chern-Simons (PCS) planar theory can bind a pair of identical charged-scalar bosons. The Lagrangian for PCS scalar QED, in the Lorentz gauge, can be written as

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\alpha F_{\mu\alpha} - \frac{1}{2\lambda}(\partial_\nu A^\nu)^2\\ &+ (D_\mu\phi)^* D^\mu\phi - m^2\phi^*\phi + \frac{s}{2}\varepsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho,\end{aligned} \tag{12}$$

where $s > 0$ is the topological mass.

In the basis $\{\theta, \omega, S\}$, where $S_{\mu\nu} \equiv \varepsilon_{\mu\rho\nu}\partial^\rho$, the propagator assumes the form

$$O^{-1} = \frac{(a^2k^4 - k^2)\theta}{(a^2k^4 - k^2)^2 - s^2k^2} - \frac{\lambda\omega}{k^2} - \frac{sS}{(a^2k^4 - k^2)^2 - s^2k^2}.$$

Now, in the nonrelativistic limit the Feynman amplitude for the process shown in Fig. 2 reduces to

$$\mathcal{M}_{\mathrm{NR}} = \left[\frac{(a^2\mathbf{k}^4 + \mathbf{k}^2)4Q^2m^2}{(a^2\mathbf{k}^4 + \mathbf{k}^2)^2 + s^2\mathbf{k}^2} + \frac{8ismQ^2\mathbf{k}\wedge\mathbf{P}}{(a^2\mathbf{k}^4 + \mathbf{k}^2)^2 + s^2\mathbf{k}^2}\right],$$

where $\mathbf{P} \equiv \frac{1}{2}(\mathbf{p} - \mathbf{q})$ is the relative momentum of the incoming charged-scalar bosons in the CoM. On the other hand, it is trivial to show that if $a < (2\sqrt{3})/(9s)$ the equation

$$x^3 + \frac{2x^2}{a^2} + \frac{x}{a^4} + \frac{s^2}{a^4} = 0, \tag{13}$$

where $x \equiv \mathbf{k}^2$ has three distinct negative real roots. In this case $\mathcal{M}_{\mathrm{NR}}$ can be rewritten as

$$\mathcal{M}_{\mathrm{NR}} = \frac{8ismQ^2\mathbf{k}\wedge\mathbf{P}}{a^4}\left[\sum_{j=1}^{3}\frac{B_j}{\mathbf{k}^2 - x_j} + \frac{a^4}{s^2\mathbf{k}^2}\right] + \frac{4Q^2m^2}{a^4}\times\sum_{j=1}^{3}\frac{A_j}{\mathbf{k}^2 - x_j},$$

where x_1, x_2, and x_3 are the roots of Eq. (13) and $A_1 \equiv \frac{1+a^2x_1}{(x_1-x_2)(x_1-x_3)}$, $A_2 \equiv \frac{1+a^2x_2}{(x_2-x_1)(x_2-x_3)}$, $A_3 \equiv \frac{1+a^2x_3}{(x_3-x_1)(x_3-x_2)}$, $B_1 \equiv \frac{-(1+a^2x_1)^2}{s^2(x_1-x_2)(x_1-x_3)}$, $B_2 \equiv \frac{-(1+a^2x_2)^2}{s^2(x_2-x_1)(x_2-x_3)}$, and $B_3 \equiv \frac{-(1+a^2x_3)^2}{s^2(x_3-x_1)(x_3-x_2)}$.

It follows that the effective nonrelativistic potential can be calculated from the expression

$$\begin{aligned}U_3(r) = \frac{isQ}{\pi ma^4}&\left[\frac{a^4}{s^2}\lim_{\sigma\to 0}\int_0^\infty\frac{(\mathbf{k}\wedge\mathbf{P})J_0(|\mathbf{k}|r)|\mathbf{k}|d|\mathbf{k}|}{\mathbf{k}^2 + \sigma^2}\right.\\ &\left.+\sum_j\int_0^\infty\frac{(\mathbf{k}\wedge\mathbf{P})B_jJ_0(|\mathbf{k}|r)|\mathbf{k}|d|\mathbf{k}|}{\mathbf{k}^2 - x_j}\right]\\ &+\frac{Q}{2\pi a^4}\sum_j\int_0^\infty\frac{A_j}{\mathbf{k}^2 - x_j}J_0(|\mathbf{k}|r)|\mathbf{k}|d|\mathbf{k}|.\end{aligned}$$

Performing the computations, we obtain

$$\begin{aligned}U_3(r) = &-\frac{sQ}{\pi ma^4}\left[\frac{a^4}{s^2}\frac{1}{r^2} + \frac{1}{r}\sum_j B_j\sqrt{|x_j|}K_1(\sqrt{|x_j|}r)\right]\mathbf{L}\\ &+\frac{Q}{2\pi a^4}\left[\sum_j A_jK_0(\sqrt{|x_j|}r)\right],\end{aligned} \tag{14}$$

where $\mathbf{L} \equiv \mathbf{r}\wedge\mathbf{P}$ is the orbital angular momentum.

Using Jackiw's arguments [1], one can show that the topological term alone is unable to bind the charged scalar bosons [2].

We return now to the problem of probing whether "Cooper pairs" exist in the framework of PCS scalar QED. In this case, the radial Schrödinger equation is

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right]\mathcal{R}_{nl} + m[E_{nl} - V_l^{\mathrm{eff}}]\mathcal{R}_{nl} = 0, \tag{15}$$

where

$$V_l^{\rm eff}(r) = -\frac{sQ^2}{\pi m a^4}\left[\frac{a^4}{s^2}\frac{1}{r^2} + \frac{1}{r}\sum_j B_j\sqrt{|x_j|}K_1(\sqrt{|x_j|}r)\right]l + \frac{Q^2}{2\pi a^4}\left[\sum_j A_j K_0(\sqrt{|x_j|}r)\right] + \frac{l^2}{mr^2}.$$

Employing the dimensionless parameters $y \equiv sr$, $\alpha \equiv \frac{Q^2}{\pi s}$, $b_j \equiv \frac{s^2}{a^4}B_j$, $X_j \equiv \frac{|x_j|}{s}$, $\beta \equiv \frac{m}{s}$, $a_j \equiv \frac{A_j}{a^4}$, and $\tilde{E}_{nl} \equiv \frac{mE_{nl}}{s^2}$, we can rewrite Eq. (15) as

$$\left[\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy}\right]\mathcal{R}_{nl} + [\tilde{E}_{nl} - \tilde{V}_l^{\rm eff}]\mathcal{R}_{nl} = 0, \qquad (16)$$

with

$$\tilde{V}_l^{\rm eff} \equiv -\frac{l(\alpha - l)}{y^2} + \frac{\alpha\beta}{2}\sum_j a_j K_0(X_j y) - \frac{\alpha l}{y}\sum_j b_j X_j K_1(X_j y).$$

Note that $\tilde{V}_l^{\rm eff}$ behaves as $\frac{l^2}{y^2}$ at the origin and as $\frac{l(l-\alpha)}{y^2}$ asymptotically. On the other hand, the derivative of this potential with respect to y is given by

$$\frac{d}{dy}\tilde{V}_l^{\rm eff} = \frac{2l(\alpha - l)}{y^3} + \alpha\sum_j\left[\frac{2l}{y^2}b_j - \frac{\beta a_j}{2}\right]X_j K_1(X_j y) + \frac{\alpha l}{y}\sum_j b_j X_j^2 K_0(X_j y).$$

In order to find out whether or not boson-boson bound states could be formed, we shall analyze how $\frac{d}{dy}\tilde{V}_l^{\rm eff}$ behaves for small values of the cutoff a. Indeed, only if $a \lll 1$ will the well-recognized properties of QED$_3$ be preserved. In this limit, we get

$$\frac{d}{dy}\tilde{V}_l^{\rm eff} \sim \frac{2l(\alpha - l)}{y^3} - \left[\frac{2\alpha l}{y^2} + \frac{\alpha\beta}{2}\right]K_1(y) - \frac{\alpha l}{y}K_0(y).$$

We assume from now on $a \lll 1$ and $l > 0$, without any loss of generality. It is trivial to see that, if $l > \alpha$, the potential is strictly decreasing, which precludes the existence of bound states. The remaining possibility is $l < \alpha$. In this interval $\tilde{V}_l^{\rm eff}$ approaches $+\infty$ at the origin and 0^- for $y \to +\infty$, which is indicative of a local minimum. Therefore, the existence of Cooper pairs is subordinated to the conditions $a \lll 1$ and $0 < l < \alpha$.

Of course, it is impossible to solve Eq. (16) analytically; however, it can be solved numerically. To do that, we rewrite beforehand the radial function as $\mathcal{R}_{nl} \equiv u_{nl}/\sqrt{y}$. As a consequence, Eq. (16) takes the form

$$\left[\frac{d^2}{dy^2} + \frac{1}{4y^2}\right]u_{nl} + [\tilde{E}_{nl} - \tilde{V}_l^{\rm eff}]u_{nl} = 0. \qquad (17)$$

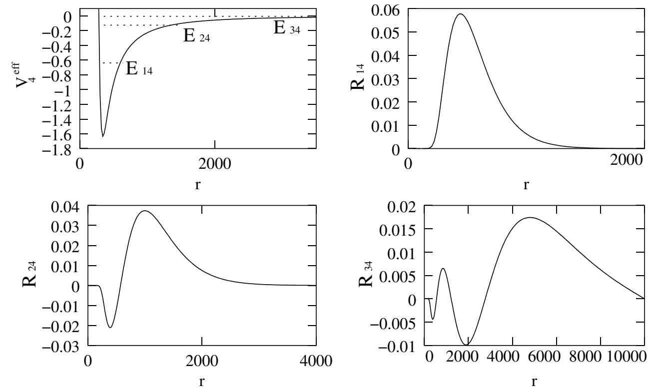

FIG. 3. $V_4^{\rm eff}$ with the lowest three allowed energies and the corresponding energy eigenfunctions. Here $[V_4^{\rm eff}] = {\rm eV}$, $[r] = {\rm MeV}^{-1}$, $\alpha = 8$, $\beta = 2000$, and $a = 0.009\,52\ {\rm MeV}^{-1}$.

Using the Numerov algorithm [3], we solved Eq. (17) numerically for several values of the parameters α, β, and l, keeping the cutoff a fixed. The latter was chosen equal to $\frac{2}{3}r_e = 9.520\,33 \times 10^{-3}\ {\rm MeV}^{-1}$, where r_e is the four-dimensional classical radius of the electron. It is worth mentioning that the anomalous factor of $\frac{4}{3}$ in the inertia related to the Abraham-Lorentz model for the electron does not show up if $a > \frac{1}{2}r_e$ [4,5].

In Fig. 3 we present our numerical results for the potential and the corresponding radial eigenfunctions concerning the first three bound states in the specific case of $l = 4$. The associated energies are $E_{14} = -6.375\,01 \times 10^{-7}$ MeV, $E_{24} = -1.2536 \times 10^{-7}$ MeV, $E_{34} = -5.224\,41 \times 10^{-9}$ MeV.

The graphics shown in Fig. 3 exhibit, in a sense, the generic features of the potential and of the radial eigenfunctions, although they have been composed using particular values of the parameters α, β, l, and a. A detailed study of the modifications of the effective potential induced by radioactive corrections, as well as the corresponding alterations to the eigenvalue structure, will be published elsewhere [2].

A. A. thanks CNPq-Brazil and M. D. thanks CAPES-Brazil for financial support.

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