Pion interactions in the $X(3872)$

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We consider pion interactions in an effective field theory of the narrow resonance $X(3872)$, assuming it is a weakly bound molecule of the charm mesons $D^0\bar{D}^{*0}$ and $D^{*0}\bar{D}^0$. Since the hyperfine splitting of the D^0 and D^{*0} is only 7 MeV greater than the neutral pion mass, pions can be produced near threshold and are nonrelativistic. We show that pion exchange can be treated in perturbation theory and calculate the next-to-leading-order correction to the partial decay width $\Gamma[X \to D^0\bar{D}^0\pi^0]$.

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I. INTRODUCTION

The idea that the recently discovered $X(3872)$ is a shallow molecular bound state of $D^{*0}\bar{D}^0$ and $\bar{D}^{*0}D^0$ mesons is extremely attractive and has motivated numerous calculations of $X(3872)$ properties using effective-range theory, for a review see Ref. [1]. Going beyond this approximation requires including effects from dynamical pion exchange. The goal of this paper is to develop an effective theory of nonrelativistic D mesons and pions that can be used to compute properties of the $X(3872)$ systematically at low energies. Because of the accidental nearness of the D^*-D hyperfine splitting and the pion mass, pion exchanges are characterized by an anomalously small scale compared to what is usually the case in nuclear physics [2]. We argue in this paper that, unlike in conventional nuclear physics, these effects can be treated using perturbation theory and compute the decay $X \to D^0\bar{D}^0\pi^0$ to next-to-leading order (NLO) in the effective theory.

We begin by reviewing the current experimental understanding of the $X(3872)$. The $X(3872)$ is a narrow resonance discovered by the Belle Collaboration [3] in electron-positron collisions through the decay $B^\pm \to XK^\pm$ followed by the decay $X \to J/\psi\pi^+\pi^-$. Its existence has been confirmed by the CDF and D0 Collaborations through its inclusive production in proton-antiproton collisions [4,5] and by the *BABAR* Collaboration through the discovery mode $B^\pm \to XK^\pm$ [6]. The combined averaged mass of the $X(3872)$ measured by these experiments is [7]

$$m_X = 3871.2 \pm 0.5 \text{ MeV}. \tag{1}$$

Note that the mass of the $X(3872)$ is quite close to the $D^0\bar{D}^{*0}$ threshold at 3871.81 ± 0.36 MeV [8]. The Belle Collaboration has placed an upper limit on the width of the $X(3872)$ [3]:

$$\Gamma_X < 2.3 \text{ MeV (90\% C.L.)}. \tag{2}$$

The $X(3872)$ has also been observed in the decays $X \to J/\psi\pi^+\pi^-\pi^0$ and $X \to J/\psi\gamma$ [9]. The ratio of branching fractions for the three- and two-pion final states is [9]

$$\frac{\text{Br}[X \to J/\psi\pi^+\pi^-\pi^0]}{\text{Br}[X \to J/\psi\pi^+\pi^-]} = 1.0 \pm 0.4 \pm 0.3. \tag{3}$$

Since these decays are thought to proceed through $J/\psi\rho$ for the $J/\psi\pi^+\pi^-$ final state, and through $J/\psi\omega$ for the $J/\psi\pi^+\pi^-\pi^0$ final state, the ratio in Eq. (3) indicates a large violation of isospin invariance. A near-threshold enhancement in $D^0\bar{D}^0\pi^0$ has been observed in $B \to D^0\bar{D}^0\pi^0K$ decays [10]. This is the first evidence for the decay $X \to D^0\bar{D}^0\pi^0$, though the peak of the observed resonance in Ref. [10] is at $3875.2 \pm 0.7^{+0.3}_{-1.6} \pm 0.8$ MeV, which is 2σ above the world-averaged $X(3872)$ mass. The branching ratio for $X \to D^0\bar{D}^0\pi^0$ observed in Ref. [10] is $8.8^{+3.1}_{-3.6}$ larger than the discovery mode $X \to J/\psi\pi^+\pi^-$. The *BABAR* Collaboration has established $\text{Br}[X \to J/\psi\pi^+\pi^-] > 0.042$ at 90% C.L. [11,12]. Various upper limits have been placed on the product of $\text{Br}[B^\pm \to XK^\pm]$ and other branching fractions of the $X(3872)$ including $D^0\bar{D}^0$, D^+D^- [13], $\chi_{c1}\gamma$, $\chi_{c2}\gamma$, $J/\psi\pi^0\pi^0$ [14], and $J/\psi\eta$ [15]. Upper limits have also been placed on the partial widths for the decay of $X(3872)$ into e^+e^- [16,17] and into $\gamma\gamma$ [17].

The possible J^{PC} quantum numbers of the $X(3872)$ have been examined. The observation of $X \to J/\psi\gamma$ establishes $C = +$. This is consistent with the shape of the $\pi^+\pi^-$ invariant mass distributions [3,6,18]. Belle's angular distribution analysis of $X \to J/\psi\pi^+\pi^-$ favors $J^{PC} = 1^{++}$ [19]. A recent CDF analysis [20] finds that $J/\psi\pi^+\pi^-$ angular distributions are only consistent with $J^{PC} = 1^{++}$ and 2^{-+}.

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The quantum numbers $J^{PC} = 1^{++}$ arise if the $X(3872)$ is a $C = +$, S-wave molecular bound state of $D^0\bar{D}^{*0} + \bar{D}^0 D^{*0}$. The possibility of a shallow molecular state is motivated by the proximity of the $X(3872)$ to the $D^0\bar{D}^{*0}$ threshold and naturally explains the large isospin violation observed in pion decays and the dominance of the $D^0\bar{D}^0\pi^0$ decay mode. The narrow width and the nonobservation of decays such as $X \to \chi_c\gamma$ are highly unusual for a conventional charmonium state above the $D\bar{D}$ threshold. From the mass in Eq. (1) and the recent measurement of the D^0 mass in Ref. [8] one infers a binding energy

$$E_X = m_D + m_{D^*} - m_X = 0.6 \pm 0.6 \text{ MeV}. \qquad (4)$$

This favors a bound-state interpretation of the $X(3872)$; however, because of the large uncertainty, the mass alone cannot rule out a resonance or "cusp" near the $D^0\bar{D}^{*0}$ threshold [21]. In this paper we will assume the $X(3872)$ is a molecular bound state, though our method can be extended to the case where the $X(3872)$ is a shallow resonance. For other interpretations, see Refs. [22–37]. A recent review can be found in Ref. [38].

The interpretation as a DD^* molecule is particularly predictive because the small binding energy implies that the molecule has universal properties that are determined by the binding energy [39–44]. The small binding energy can be further exploited through factorization formulae for production and decay rates of the $X(3872)$ [45,46]. Voloshin calculated the decays $X \to D^0\bar{D}^0\pi^0$ [39] and $X \to D^0\bar{D}^0\gamma$ [40] using the universal wave function of the molecule.

The main purpose of this paper is to consider the effect of π^0 exchange on the properties of the $X(3872)$. Consider the one-pion-exchange contribution to $D^{*0}\bar{D}^0 \to D^0\bar{D}^{*0}$ scattering depicted in Fig. 1. This leads to an amplitude

$$\frac{g^2}{2f_\pi^2}\frac{\vec{\epsilon}^*\cdot\vec{q}\,\vec{\epsilon}\cdot\vec{q}}{\vec{q}^2 - \mu^2}, \qquad (5)$$

where g is the D-meson axial (transition) coupling, f_π is the pion decay constant, $\vec{\epsilon}$ and $\vec{\epsilon}^*$ are the polarization vectors of the incoming and outgoing D^* mesons, respectively, and $\vec{q}$ is the momentum transfer. The scale μ appearing in the propagator denominator is given by $\mu^2 = \Delta^2 - m_\pi^2$, where Δ is the D^*-D hyperfine splitting and m_π is the neutral pion mass. The hyperfine splitting, Δ, appears in the pion propagator because the exchanged pion carries energy $q^0 \simeq \Delta$ as well as momentum $\vec{q}$. Note that μ is anomalously small, $\mu \approx 45$ MeV, because of the nearness of $\Delta = 142$ MeV and $m_\pi = 135$ MeV. This suggests that pions generate anomalously long-range effects and should be included as explicit degrees of freedom in the description of the molecule, if the binding energy in Eq. (4) is not much smaller than its upper limit.

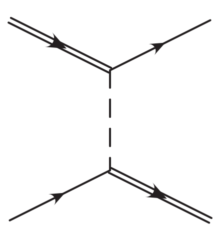


FIG. 1. One-pion-exchange diagram for $D^{*0}\bar{D}^0 \to D^0\bar{D}^{*0}$ scattering. The single and double lines represent the spin-0 and spin-1 D mesons, respectively. The dashed line represents the π^0.

The pion interactions in the D and D^* system were quantitatively analyzed using a one-pion-exchange potential model by Tornqvist [47], who actually predicted a $D\bar{D}^*$ bound state (deuson) with a mass close to the observed $X(3872)$. After the discovery of the $X(3872)$, Swanson [26] considered a potential model that includes both a one-pion-exchange potential and a quark-exchange potential and found a weakly bound state in the S-wave $J^{PC} = 1^{++}$ channel. These authors worked in the isospin limit and used isospin-averaged pion masses and hyperfine splittings and obtained long-range Yukawa-like potentials [47]. Note that the effective mass term in the propagator in Eq. (5) has the opposite sign from what one typically obtains from meson exchange. This leads to a π^0-exchange potential in position space which is oscillatory rather than Yukawa-like, as pointed out by Suzuki [2].

A central point of this paper is that the effect of π^0 exchange can be dealt with using perturbation theory. Naive dimensional analysis of the relative size of two-pion- and one-pion-exchange graphs yields the ratio

$$\frac{g^2 M_{DD^*}\mu}{4\pi f_\pi^2} \approx \frac{1}{20} - \frac{1}{10}, \qquad (6)$$

where M_{DD^*} is the reduced mass of the D and D^* and we have set $g = 0.5$–0.7 [48–50]. This is in contrast with two-nucleon systems where a similar estimate yields [51,52]

$$\frac{g_A^2 M_N m_\pi}{8\pi f_\pi^2} \approx \frac{1}{2}, \qquad (7)$$

where $g_A = 1.25$ is the nucleon axial coupling and M_N is the nucleon mass. A perturbative treatment of pions fails in the $^3S_1$ channel where iteration of the spin-tensor force yields large corrections at next-to-next-leading order (NNLO) [53,54]. This is in part due to the large expansion parameter in Eq. (7) and in part due to large numerical coefficients appearing in the NNLO calculation. The amplitude in Eq. (5) also gives rise to a spin-tensor force and one may worry that the perturbative treatment of pions will fail. However, even if large NNLO coefficients like those found in Refs. [53,54] appear in similar diagrams for the $X(3872)$, the expansion parameter in Eq. (6) is small enough that one can reasonably expect perturbation theory to work.

In this paper, we derive an effective field theory of the $D^0\bar{D}^{*0}$ and $D^{*0}\bar{D}^0$ interacting with neutral pions near the $D^0\bar{D}^{*0}$ threshold. This theory is very similar in structure to the Kaplan-Savage-Wise (KSW) theory of NN interactions in Refs. [51,52] where a leading-order (LO) contact interaction is summed to all orders in perturbation theory to produce a bound state at LO and pion exchange is treated perturbatively.[1] A novel feature of the effective theory for the $X(3872)$ is that the hyperfine splitting of the D^0 and D^{*0} is only 7 MeV above the π^0 mass and thus the pions are included as nonrelativistic particles. In this paper we focus on the decay $X \to D^0\bar{D}^0\pi^0$. Our results are easily extended to $X \to D^0\bar{D}^0\gamma$. At LO our theory reproduces Voloshin's calculations using effective-range theory [39,40]. We then compute the NLO corrections to the decay width. These include effective-range corrections as well as calculable nonanalytic corrections from π^0 exchange. We find that nonanalytic calculable corrections from pion exchange are negligible and the NLO correction is dominated by contact interaction contributions.

This paper is organized as follows. In Section II, we give the Lagrangian and discuss power counting in our theory. In Section III, we describe our calculation of the partial width $\Gamma[X \to D^0\bar{D}^0\pi^0]$. In Section IV, we summarize and conclude. Appendix A describes how our Lagrangian is derived by integrating out the scales m_π and Δ from heavy-hadron chiral perturbation theory (HHχPT) [57–59]. Appendix B gives the results of evaluating the individual NLO diagrams for the decay amplitude and the wave function renormalization.

While this work was being completed, a related preprint [60] appeared which analyzed the effects of light-meson exchange on a bound state of heavy mesons near a three-meson threshold. This work used a scalar-meson model and calculated the entire line shape of the resonance to second order in the heavy-light meson coupling. Our work is complementary to that of Ref. [60] in that we do not use a model but rather a Lagrangian that is directly relevant to the $X(3872)$ and we go to higher order in the heavy-light meson coupling, where renormalization requires the introduction of higher-derivative contact operators. On the other hand, we do not calculate the full line shape but work at the resonance peak where a Breit-Wigner is a suitable approximation.

II. LAGRANGIAN AND POWER COUNTING

The mass of the $X(3872)$ in Eq. (1) is extremely close to the $D^0\bar{D}^{*0}$ threshold. Assuming that the $X(3872)$ is a hadronic molecule whose constituents are a superposition of the $D^0\bar{D}^{*0}$ and $D^{*0}\bar{D}^0$, the $X(3872)$ binding energy is given by Eq. (4). For reasons stated earlier, our calculations assume positive binding energy and a molecular interpretation of the $X(3872)$. The upper bound on the typical momentum of the D and $\bar{D}^*$ in the bound state is then $\gamma \equiv (2M_{DD*}E_X)^{1/2} \leq 48$ MeV, where M_{DD*} is the reduced mass of the D^0 and $\bar{D}^{*0}$. For this binding momentum the typical velocity of the D and D^* is approximately $v_D \simeq (E_X/2M_{DD*})^{1/2} \lesssim 0.02$, and both the D and D^* are clearly nonrelativistic. We will use nonrelativistic fields for the D and D^*.

The pion degrees of freedom are also treated nonrelativistically. The maximum energy of the pion emitted in the decay $X \to D^0\bar{D}^0\pi^0$ is

$$E_\pi = \frac{m_X^2 - 4m_D^2 + m_\pi^2}{2m_X} = 142 \text{ MeV}, \tag{8}$$

which is just 7 MeV above the π^0 mass at 134.98 MeV. The maximum pion momentum is approximately 44 MeV, which is comparable to both the typical D-meson momentum, $p_D \sim \gamma \lesssim 48$ MeV, and the momentum scale appearing in the pion-exchange graph, $\mu \simeq 45$ MeV. Since the velocity of the pions is $v_\pi = p_\pi/m_\pi \lesssim 0.34$, a nonrelativistic treatment of the pion fields is valid. In this respect the treatment of pions differs from ordinary chiral perturbation theory or the NN theory of Refs. [51,52].

The effective Lagrangian includes the charm mesons, the anticharm mesons, and the pion fields. We denote the fields that annihilate the D^{*0}, $\bar{D}^{*0}$, D^0, $\bar{D}^0$, and π^0 as $\boldsymbol{D}$, $\bar{\boldsymbol{D}}$, D, $\bar{D}$, and π, respectively. To the order we are working we will not need diagrams with charged pions and charged D mesons so these are neglected in what follows. We construct an effective Lagrangian that is relevant for low-energy S-wave DD^* scattering, where the initial and the final states are the $C = +$ superposition of $D^0\bar{D}^{*0}$ and $D^{*0}\bar{D}^0$:

$$|DD^*\rangle \equiv \frac{1}{\sqrt{2}}[|D^0\bar{D}^{*0}\rangle + |D^{*0}\bar{D}^0\rangle]. \tag{9}$$

An interpolating field with these quantum numbers will be used to calculate the properties of the $X(3872)$. We integrate out all momentum scales much larger than the momentum scale set by $p_D \sim p_\pi \sim \mu$. For D mesons this corresponds to kinetic energy $\lesssim 1$ MeV; for pions the kinetic energy is $\lesssim 7$ MeV. The hyperfine splitting Δ and m_π should be treated as large compared to the typical energy scale in the theory. We start from the Lagrangian of HHχPT [57–59], which describes the interactions of D and D^* mesons with Goldstone bosons, and integrate out the scales m_π and Δ by rephasing fields to eliminate the large components of their energy. The method is similar to the rephasing used to remove the large mass from the energies of the fields in heavy-quark effective theory [61]. Details are given in Appendix A. The effective

[1] A pionless effective theory of shallow nuclear bound states in which the leading nonderivative contact interaction is resummed to all orders was first proposed in Ref. [55]. For a similar theory of the $X(3872)$ see Ref. [56].

Lagrangian is

$$\mathcal{L} = \boldsymbol{D}^\dagger\Big(i\partial_0 + \frac{\vec{\nabla}^2}{2m_{D^*}}\Big)\boldsymbol{D} + D^\dagger\Big(i\partial_0 + \frac{\vec{\nabla}^2}{2m_D}\Big)D + \bar{\boldsymbol{D}}^\dagger\Big(i\partial_0 + \frac{\vec{\nabla}^2}{2m_{D^*}}\Big)\bar{\boldsymbol{D}} + \bar{D}^\dagger\Big(i\partial_0 + \frac{\vec{\nabla}^2}{2m_D}\Big)\bar{D} + \pi^\dagger\Big(i\partial_0 + \frac{\vec{\nabla}^2}{2m_\pi} + \delta\Big)\pi$$
$$+ \frac{g}{\sqrt{2}f_\pi}\frac{1}{\sqrt{2m_\pi}}(D\boldsymbol{D}^\dagger\cdot\vec{\nabla}\pi + \bar{D}^\dagger\bar{\boldsymbol{D}}\cdot\vec{\nabla}\pi^\dagger) + \text{H.c.} - \frac{C_0}{2}(\bar{\boldsymbol{D}}D + \boldsymbol{D}\bar{D})^\dagger\cdot(\bar{\boldsymbol{D}}D + \boldsymbol{D}\bar{D})$$
$$+ \frac{C_2}{16}(\bar{\boldsymbol{D}}D + \boldsymbol{D}\bar{D})^\dagger\cdot(\bar{\boldsymbol{D}}(\overleftrightarrow{\nabla})^2D + \boldsymbol{D}(\overleftrightarrow{\nabla})^2\bar{D}) + \text{H.c.} + \frac{B_1}{\sqrt{2}}\frac{1}{\sqrt{2m_\pi}}(\bar{\boldsymbol{D}}D + \boldsymbol{D}\bar{D})^\dagger\cdot D\bar{D}\vec{\nabla}\pi + \text{H.c.} + \cdots, \tag{10}$$

where $\delta = \Delta - m_\pi \simeq 7$ MeV. Note that $\mu^2 = \Delta^2 - m_\pi^2 \approx 2m_\pi\delta$. We use the notation $\overleftrightarrow{\nabla} = \overleftarrow{\nabla} - \vec{\nabla}$, and "$\cdots$" in Eq. (10) denotes higher-order interactions. The pion decay constant is $f_\pi = 132$ MeV with our choice of normalization. Notice that, since we are only interested in a $C = +$ superposition of the $D^0\bar{D}^{*0}$ and $D^{*0}\bar{D}^0$ defined in Eq. (9), contact interactions are written in terms of the combination of fields $(\bar{\boldsymbol{D}}D + \boldsymbol{D}\bar{D})/\sqrt{2}$. Because π is a nonrelativistic field, π annihilates and $\pi^\dagger$ creates π^0 quanta, so that the Lagrangian in Eq. (10) allows $D^{0*} \to D^0 + \pi^0$ and $D^0 + \pi^0 \to D^{*0}$ transitions and forbids $D^{*0} + \pi^0 \to D^0$ and $D^0 \to D^{*0} + \pi^0$. Therefore in this effective field theory the only channels that appear are $\bar{D}^{*0}D^0 + \bar{D}^0D^{*0}$ and $D^0\bar{D}^0\pi^0$. In amplitudes with external pions, we must multiply by $\sqrt{2m_\pi}$ because of the normalization of the nonrelativistic pion fields. In the $X(3872) \to D^0\bar{D}^0\pi^0$ decay diagrams, this will cancel the factors of $1/\sqrt{2m_\pi}$ in the axial coupling and in the term proportional to B_1 in Eq. (10).

Other channels can of course couple to the $X(3872)$. The three-body channels $D^\pm\bar{D}^0\pi^\mp$ and $D^+D^-\pi^0$ are above the $X(3872)$ by only 2.8 ± 0.6 MeV and 3.0 ± 0.6 MeV, respectively. These channels can only appear as virtual intermediate states in $X(3872)$ decay and self-energy graphs that contain at least two-pion exchanges. These graphs are NNLO and therefore do not appear at the order we are working.[2] The $D^{*+}D^-$ threshold lies 8.7 MeV above the $X(3872)$, and we may integrate out these states because they lie outside the range of the effective theory. If kept in the theory, this intermediate state would also only appear at NNLO. One may worry about other nearby thresholds, especially $J/\psi\rho$ and $J/\psi\omega$ which are only 1.4 ± 1.1 MeV and 8.2 ± 1.0 MeV, respectively, above the $X(3872)$. The $J/\psi\rho$ channel has a much smaller energy gap than the others. However, one should take into account that the magnitude of the complex energy gap includes the width of the ρ, $\Gamma_\rho/2 = 73$ MeV, so the $J/\psi\rho$ channel can be safely integrated out [46]. A higher-precision analysis of the $X(3872)$ may need to include these thresholds explicitly, especially if one wishes to describe the decays $X \to J/\psi\pi^+\pi^-$ and $X \to J/\psi\pi^+\pi^-\pi^0$. We leave this to future work.

Matching onto HHχPT yields the D^0, D^{*0}, and π^0 kinetic terms as well as the axial D^{*0}-D^0-π^0 coupling. The coupling constant, g, is determined from data on the decays of D^* mesons. The CLEO measurements of the D^{*+} width yield $g = 0.59 \pm 0.07$ at tree level [48,49]. A NLO analysis of D^* decays in Ref. [62] yields $g = 0.27^{+0.06}_{-0.03}$. A more recent analysis [50] obtains $g = 0.61$ at tree level and $g = 0.66(0.53)$ at NLO, where the number outside parentheses refers to the result when virtual low-lying even-parity charmed mesons are included in the loop calculations and the number in parentheses refers to the result obtained when these states are integrated out. The uncertainty in the NLO extraction of g is estimated to be 20%. We will use $g = 0.6 \pm 0.1$ in this paper.

The remaining terms in Eq. (10) with coefficients C_0, C_2, and B_1 are contact interactions that are not obtained from matching HHχPT but must also be included. They incorporate effects that come from shorter distance scales than the scale coming from π^0 exchange. We have only included operators needed to the order we are working. C_0 and C_2 mediate $D^0\bar{D}^{*0} + \bar{D}^0D^{*0}$ scattering in the $C = +$, S-wave channel and have zero and two derivatives, respectively. B_1 mediates a transition between $D^0\bar{D}^{*0} + \bar{D}^0D^{*0}$ in the $C = +$, S-wave channel to a state with a D^0, $\bar{D}^0$, and π^0.

In our power counting, $p_D \sim p_{D^*} \sim p_\pi \sim \mu \sim \gamma \sim Q$, and we calculate amplitudes in an expansion in powers of Q. Since the D^0, D^{*0}, and π^0 are all nonrelativistic, $E_D \sim E_{D^*} \sim E_\pi \sim Q^2$, so the propagators of all particles are order Q^{-2}. Loop integrations are order Q^5. The D^{*0}-D^0-π^0 axial coupling is order Q. In the exchange diagram of Fig. 1, one can drop the energy dependence in the pion propagator. The factors of $\sqrt{2m_\pi}$ from the vertices cancel the factors of $1/(2m_\pi)$ in the momentum-dependent term in the pion propagator and combine with δ to give $2m_\pi\delta = \mu^2$, reproducing the expression in Eq. (5). The pion-exchange amplitude is order Q^0 as is easily seen from Eq. (5).

Only counting powers of momentum, the Feynman rules for the terms in the Lagrangian with coefficients C_0, C_2, and B_1 are naively of order Q^0, Q^2, and Q^1, respectively. However, with this power counting the theory is perturbative and cannot produce a bound state. Instead we will treat

[2]This assumes that the interpolating field for the $X(3872)$ is $\propto \bar{\boldsymbol{D}}D + \boldsymbol{D}\bar{D}$, i.e. is constructed from neutral D-meson fields only. Since physical results should not depend on the choice of interpolating field, we are free to make this choice.

C_0 nonperturbatively, along the lines of Refs. [51,55], and sum diagrams with C_0 to all orders. At LO, using the power divergence subtraction (PDS) scheme one then finds [51]

$$C_0 = \frac{2\pi}{M_{DD^*}} \frac{1}{\gamma - \Lambda_{\rm PDS}}, \tag{11}$$

where $\Lambda_{\rm PDS}$ is the dimensional-regularization parameter.[3] Taking $\Lambda_{\rm PDS}$ of order Q we find C_0 is order Q^{-1} which justifies its resummation. In PDS, the coefficient C_2 is order Q^{-2} as is B_1, as we shall see below. No other short-distance operators are needed for our NLO calculation of $X \to D^0\bar{D}^0\pi^0$. Feynman diagrams with C_2 and B_1 first contribute to $X \to D^0\bar{D}^0\pi^0$ at NLO.

In addition to expanding in Q, we will make one more approximation in the NLO calculation of $X \to D^0\bar{D}^0\pi^0$. In many cases it greatly simplifies calculations to expand in $m_\pi/m_D \sim 0.07$. This is an approximation we will perform when evaluating loop diagrams. It is not systematized in our power counting scheme.

As emphasized earlier, the perturbative character of pion exchange depends on the smallness of the parameter appearing in Eq. (6). Our effective theory can be used even if dimensionless parameters conspire to render pion exchange nonperturbative, but in this case one-pion exchange would have to be resummed as done in the NN system [63].

III. DECAY RATE FOR $X(3872) \to D^0\bar{D}^0\pi^0$

Here we describe our method for calculating the width of the $X(3872)$ resonance. We consider the following two-point function of interpolating fields $X^i = (D^0\bar{D}^{0*i} + \bar{D}^0 D^{0*i})/\sqrt{2}$ for the $X(3872)$ with the spin index i:

$$G(E)\delta^{ij} = \int d^4x e^{-iEt}\langle 0|T[X^i(x)X^j(0)]|0\rangle = i\delta^{ij}\frac{Z(-E_X)}{E + E_X + i\Gamma/2} + \ldots, \tag{12}$$

where E_X is the binding energy of the $X(3872)$ and the ellipsis represents terms that are less important in the resonance region, $E + E_X \sim \Gamma$. We can define a function $\Sigma(E)$, where $-i\Sigma(-E_x)$ represents the C_0-irreducible graphs contributing to $G(E)$. Our definition of $\Sigma(E)$ is similar to the function Σ defined in Appendix A of Ref. [64]. In terms of $\Sigma(E)$, $G(E)$ is

$$G(E) = \frac{-i\Sigma(E)}{1 + C_0\Sigma(E)} = \frac{-i\,{\rm Re}\Sigma(E) + {\rm Im}\Sigma(E)}{1 + C_0\,{\rm Re}\Sigma(E) + iC_0\,{\rm Im}\Sigma(E)}. \tag{13}$$

Since the real part of the denominator must vanish at $E = -E_X$, we have $1 + C_0\,{\rm Re}\Sigma(-E_X) = 0$, and expanding about $E = -E_X$ we obtain for $G(E)$

$$G(E) = \frac{i(1/C_0 - (E + E_X)\,{\rm Re}\Sigma'(-E_X)) + {\rm Im}\Sigma(-E_X)}{C_0(E + E_X)\,{\rm Re}\Sigma'(-E_X) + iC_0\,{\rm Im}\Sigma(-E_X)} = \frac{i}{C_0^2(E + E_X)\,{\rm Re}\Sigma'(-E_X) + iC_0^2\,{\rm Im}\Sigma(-E_X)} - \frac{i}{C_0}, \tag{14}$$

where $\Sigma' = d\Sigma/dE$. From Eq. (14), we immediately see that

$$Z(E) = \frac{1}{C_0^2\,{\rm Re}\Sigma'(-E_X)}, \qquad \Gamma = \frac{2\,{\rm Im}\Sigma(-E_X)}{{\rm Re}\Sigma'(-E_X)}. \tag{15}$$

The function $2\,{\rm Im}\Sigma$ corresponds to the square of the decay diagrams. The LO decay and wave function renormalization diagrams are shown in Figs. 2 and 3, respectively. The NLO decay diagrams are shown in Figs. 4 and 5, and diagrams for the NLO wave function renormalization are shown in Fig. 6.

It is interesting to compare the result of evaluating the loop diagrams and taking the imaginary part with direct evaluation of the decay diagrams. Consider, for example, the evaluation of the two-loop diagram in Fig. 6(a) in Appendix B. The result of evaluating the graph is

$$\text{Fig. 6a} = -i\frac{g^2}{2f_\pi^2}\frac{1}{2m_\pi}\left(\frac{\Lambda_{\rm PDS}}{2}\right)^{8-2D}\int\frac{d^Dq}{(2\pi)^D}\int\frac{d^Dl}{(2\pi)^D}\frac{1}{q_0 + E_X/2 - q^2/(2m_{D^*}) + i\epsilon}\frac{1}{-q_0 + E_X/2 - q^2/(2m_D) + i\epsilon} \times \frac{1}{l_0 + E_X/2 - l^2/(2m_{D^*}) + i\epsilon}\frac{1}{-l_0 + E_X/2 - l^2/(2m_D) + i\epsilon}\frac{(q + l)_i(q + l)_j}{q_0 + l_0 - (q + l)^2/(2m_\pi) + \delta + i\epsilon}. \tag{16}$$

We perform the energy integrals by contour integration, taking the poles of the D-meson propagators. This yields

[3] The dimensional-regularization parameter is usually denoted μ but we use a different symbol here to avoid confusion with the scale appearing in pion exchange.

$$\begin{aligned}\text{Fig. 6a} &= i\frac{g^2}{2f_\pi^2}\left(\frac{\Lambda_{\rm PDS}}{2}\right)^{8-2D}\int\frac{d^{D-1}q}{(2\pi)^{D-1}}\int\frac{d^{D-1}l}{(2\pi)^{D-1}}\frac{1}{E_X-q^2/(2M_{DD^*})+i\epsilon}\frac{1}{E_X-l^2/(2M_{DD^*})+i\epsilon}\\ &\quad\times\frac{(q+l)_i(q+l)_j}{2m_\pi(E_X-q^2/(2m_D)-l^2/(2m_D))-(q+l)^2+\mu^2+i\epsilon}\\ &= i\frac{g^2}{2f_\pi^2}(2M_{DD^*})^2\left(\frac{\Lambda_{\rm PDS}}{2}\right)^{8-2D}\int\frac{d^{D-1}q}{(2\pi)^{D-1}}\int\frac{d^{D-1}l}{(2\pi)^{D-1}}\frac{1}{q^2+\gamma^2-i\epsilon}\frac{1}{l^2+\gamma^2-i\epsilon}\\ &\quad\times\frac{(q+l)_i(q+l)_j}{-m_\pi(\gamma^2/M_{DD^*}+q^2/m_D+l^2/m_D)-(q+l)^2+\mu^2+i\epsilon}.\end{aligned}\tag{17}$$

The first two propagators that come from the D^* mesons clearly scale as Q^{-2}. The last two terms in the pion propagator denominator, $-(q+l)^2+\mu^2$, scale as Q^2 while the other terms scale as $(m_\pi/m_D)Q^2$. Since $m_\pi/m_D\sim 0.07$ is comparable to our expansion parameter in Eq. (6), these terms can be systematically dropped. The neglected terms come from the pion kinetic energy, and in dropping them we are treating the pions in the potential approximation [65]. The final answer is then

$$\begin{aligned}\text{Fig. 6a} &= -i\frac{g^2}{2f_\pi^2}(2M_{DD^*})^2\left(\frac{\Lambda_{\rm PDS}}{2}\right)^{8-2D}\int\frac{d^{D-1}q}{(2\pi)^{D-1}}\int\frac{d^{D-1}l}{(2\pi)^{D-1}}\frac{1}{q^2+\gamma^2-i\epsilon}\frac{1}{l^2+\gamma^2-i\epsilon}\frac{(q+l)_i(q+l)_j}{(q+l)^2-\mu^2-i\epsilon}\\ &= -i\frac{g^2}{2f_\pi^2}\frac{\delta_{ij}}{3}\left(\frac{M_{DD^*}}{2\pi}\right)^2\left[(\Lambda_{\rm PDS}-\gamma)^2+\mu^2\left(\frac{1}{4\hat\epsilon}+\frac{1}{2}+\log\left(\frac{\Lambda_{\rm PDS}}{2\gamma-i\mu}\right)\right)\right],\end{aligned}\tag{18}$$

where $1/(4\hat\epsilon)=1/(4\epsilon)+(\ln\pi-\gamma_{\rm E})/2$. We have used three-dimensional rotational invariance to replace $(q+l)_i(q+l)_j$ with $(q+l)^2\delta_{ij}/3$ and use the PDS scheme to evaluate the remaining scalar integrals.

There is one instance when dropping m_π/m_D corrections is not appropriate. To see this consider evaluating the imaginary part of Fig. 6(a) by evaluating the cut diagram. The cut runs through the D-meson and pion propagators. In the cut diagrams, these propagators are replaced with δ functions. For the D-meson propagators integrating over the δ functions is equivalent to taking the pole using contour integration. So the cut diagram is obtained from Eq. (17) simply by replacing the pion propagator with the corresponding δ function. Doing this and making the substitutions $q\to p_{\bar D}$ and $l\to p_D$ so that $q+l=p_D+p_{\bar D}=-p_\pi$, one obtains

$$\frac{g^2}{2f_\pi^2}(2M_{DD^*})^2\int\frac{d^3p_Dd^3p_{\bar D}}{(2\pi)^5}|\vec\epsilon\cdot\vec p_\pi|^2\frac{1}{p_D^2+\gamma^2}\frac{1}{p_{\bar D}^2+\gamma^2}\times\delta\left(\mu^2-p_\pi^2-\frac{m_\pi}{M_{DD^*}}\gamma^2-\frac{m_\pi}{m_D}p_D^2-\frac{m_\pi}{m_D}p_{\bar D}^2\right).\tag{19}$$

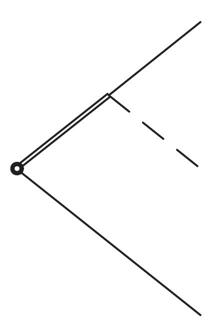


FIG. 2. LO diagram for decay rate.

This clearly reproduces the interference term in Voloshin's effective-range calculation of $X\to D^0\bar D^0\pi^0$ [39]. The δ function in Eq. (19) imposes the constraint on the phase space due to energy conservation. Dropping m_π/m_D-suppressed terms in the δ function corresponds to neglecting the final-state D-meson's kinetic energy and would leave the integrals over their momentum unconstrained. Clearly this is not a good approximation. Physically, it is also clear that the on-shell propagating pion in the final state cannot be treated in the potential approximation.

Therefore, in evaluating $\text{Im}\Sigma(-E_X)$ we will calculate the decay amplitudes for the diagrams and integrate over the physical three-body phase. In diagrams with virtual pions, we drop the kinetic energy so the pions are potential.[4] In the virtual diagrams this approximation is valid up to $O(m_\pi/m_D)$ corrections. Since our expansion parameter is expected to be 0.05–0.1, making this approximation in the virtual NLO graphs induces an error of the same size as the NNLO correction.

The LO decay diagram is shown in Fig. 2. The D^* propagator scales as $1/Q^2$ and the axial coupling scales as Q so the LO diagram is order Q^{-1}. We show only one diagram, but there are two channels related by C-conjugation that are implied. It is straightforward to evaluate these diagrams and obtain

$$i\frac{g}{f_\pi}\frac{M_{DD^*}}{p_D^2+\gamma^2}\vec p_\pi\cdot\vec\epsilon_X+(p_D\to p_{\bar D}).\tag{20}$$

The LO contribution to the wave function diagram is shown in Fig. 3. The graph is $O(Q)$ and therefore

[4] For further discussion on the role of recoil corrections, see Ref. [66].

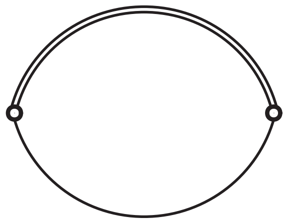


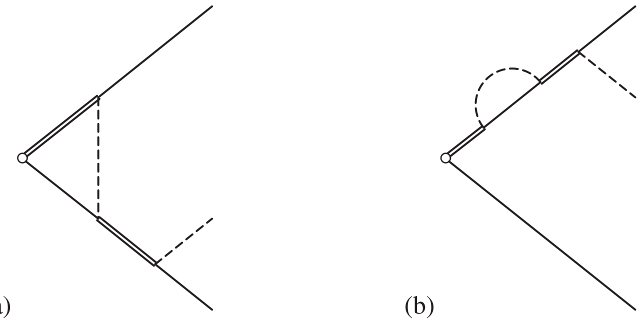
FIG. 3. LO diagram for calculating wave function renormalization.

$\mathrm{Re}\Sigma'(-E_X)$ is $O(Q^{-1})$. The result of evaluating this graph and taking the derivative is

$$\mathrm{Re}\,\Sigma'_{\mathrm{LO}} = \frac{M_{DD^*}^2}{2\pi\gamma}. \tag{21}$$

The LO decay diagram in Fig. 2 is $O(Q^{-1})$ so the leading contribution to $\mathrm{Im}\Sigma(-E_X)$ from Fig. 2 is $O(Q^{-2})$. Dividing by the LO wave function renormalization which is $O(Q^{-1})$ one sees that the leading contribution to the decay rate is $O(Q^{-1})$. The result reproduces Voloshin's calculation of $X \to D^0\bar{D}^0\pi^0$ [39]:

$$\frac{d\Gamma_{\mathrm{LO}}}{dp_D^2 dp_{\bar{D}}^2} = \frac{g^2}{32\pi^3 f_\pi^2} 2\pi\gamma(\vec{p}_\pi\cdot\vec{\epsilon}_X)^2\left[\frac{1}{p_D^2+\gamma^2}+\frac{1}{p_{\bar{D}}^2+\gamma^2}\right]^2. \tag{22}$$



(a) (b)

FIG. 4. NLO diagrams for the decay rate involving pion exchange.

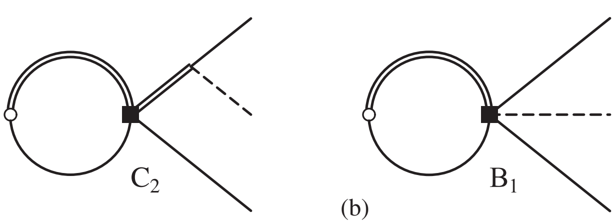
The NLO corrections to the decay rate are suppressed by one power of Q. They come from graphs, shown in Figs. 4 and 5, with one additional pion exchange or one insertion of C_2 and B_1. These coefficients scale as Q^{-2}. NLO contributions to the wave function renormalization are down by one power of Q as well. These contributions are given by the two-loop self-energy diagrams involving pion exchange or an insertion of C_2 shown in Fig. 6.

The results for individual diagrams are given in Appendix B. The final expression for the NLO differential rate is

$$\begin{aligned}\frac{d\Gamma_{\mathrm{NLO}}}{dp_D^2 dp_{\bar{D}}^2} &= \frac{d\Gamma_{\mathrm{LO}}}{dp_D^2 dp_{\bar{D}}^2}\left(1+\frac{g^2 M_{DD^*}\gamma}{6\pi f_\pi^2}\left(\frac{4\gamma^2-\mu^2}{4\gamma^2+\mu^2}\right)+C_2(\Lambda_{\mathrm{PDS}})\frac{M_{DD^*}\gamma(\gamma-\Lambda_{\mathrm{PDS}})^2}{\pi}\right)\\ &\quad -\frac{g\gamma}{16\pi^3 f_\pi}\left(\frac{gM_{DD^*}}{f_\pi}C_2(\Lambda_{\mathrm{PDS}})+B_1(\Lambda_{\mathrm{PDS}})\right)(\Lambda_{\mathrm{PDS}}-\gamma)(\vec{p}_\pi\cdot\vec{\epsilon}_X)^2\left[\frac{1}{p_D^2+\gamma^2}+\frac{1}{p_{\bar{D}}^2+\gamma^2}\right]\\ &\quad -\frac{g^4 M_{DD^*}\gamma}{64\pi^3 f_\pi^4}(\vec{p}_\pi\cdot\vec{\epsilon}_X)^2\left[\frac{\mathrm{Re}h_1(p_D)}{p_D^2+\gamma^2}+\frac{\mathrm{Re}h_1(p_{\bar{D}})}{p_{\bar{D}}^2+\gamma^2}\right]\left[\frac{1}{p_D^2+\gamma^2}+\frac{1}{p_{\bar{D}}^2+\gamma^2}\right]\\ &\quad +\frac{g^4 M_{DD^*}\gamma}{64\pi^3 f_\pi^4}\left[\frac{\mathrm{Re}h_2(p_D)}{p_D^2+\gamma^2}\vec{p}_\pi\cdot\vec{\epsilon}_X\vec{p}_D\cdot\vec{\epsilon}_X\vec{p}_\pi\cdot\vec{p}_D+(p_D\to p_{\bar{D}})\right]\left[\frac{1}{p_D^2+\gamma^2}+\frac{1}{p_{\bar{D}}^2+\gamma^2}\right].\end{aligned} \tag{23}$$

The functions $h_1(p)$ and $h_2(p)$ are given in Appendix B. The first line in Eq. (23) is a multiplicative correction to the LO decay rate. Note that in the absence of pions

$$C_2(\Lambda_{\mathrm{PDS}}) = \frac{2\pi}{M_{DD^*}}\frac{r_0}{2}\frac{1}{(\Lambda_{\mathrm{PDS}}-\gamma)^2}, \tag{24}$$



(a) (b)

FIG. 5. NLO diagrams for the decay rate which involve contact interaction.

where r_0 is the effective range. The term proportional to C_2 in the first line of Eq. (23) reproduces the expected correction from the effective-range theory, in which the leading correction involving r_0 comes from the modification of the normalization of the wave function:

$$\psi^{\mathrm{ER}}(r) = \sqrt{\frac{\gamma}{4\pi(1-\gamma r_0)}}\frac{e^{-\gamma r}}{r}. \tag{25}$$

The second line in Eq. (23) is the interference between a short-distance local coupling of the X to the $D^0\bar{D}^0\pi^0$ state and the LO amplitude. Note that the coefficient of this term scales as $1/(\Lambda_{\mathrm{PDS}}-\gamma)$ and disappears if one takes $\Lambda_{\mathrm{PDS}}\to\infty$, confirming the short-distance nature of the contribution. The final terms are nonanalytic corrections due to pion exchange. These contributions turn out to give a very small ($\sim 1\%$) contribution to the decay rate, so the

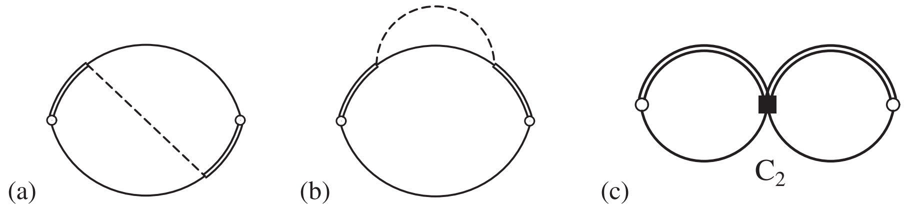


FIG. 6. NLO diagrams for calculating wave function renormalization.

NLO correction is entirely dominated by the contact interaction contributions.

We will parametrize C_2 according to Eq. (24), where r_0 is to be interpreted as the short-distance contribution to the effective range. Since we have integrated out the scales m_π and Δ, it is reasonable to take $r_0 \sim (100\text{ MeV})^{-1}$. We will parametrize

$$\left(\frac{gM_{DD^*}}{f_\pi}C_2(\Lambda_{\text{PDS}}) + B_1(\Lambda_{\text{PDS}})\right)(\Lambda_{\text{PDS}} - \gamma) = \frac{\eta}{(100\text{ MeV})^3}, \tag{26}$$

where η is a dimensionless parameter we expect to be of order unity. Figure 7 shows the partial width $\Gamma[X \to D^0\bar{D}^0\pi^0]$ as a function of the binding energy. The central solid line is the LO result. We use the central value for the tree-level extraction of the D-meson axial coupling, $g = 0.6$. The band in Fig. 7 shows the NLO rate with the parameters r_0 and η varied between

$$0 \le r_0 \le \frac{1}{100\text{ MeV}}, \qquad -1 \le \eta \le 1. \tag{27}$$

As stated earlier the nonanalytic calculable corrections from pion exchange in Eq. (23) give negligible corrections. The band is dominated entirely by the contact interaction contributions. Measurements of the X mass and partial decay width into $D^0\bar{D}^0\pi^0$ can naturally be explained within a molecular picture if the corresponding point in Fig. 7 falls within, or—due to higher orders—close to, this band. Values far outside the band can be accommodated only if short-range parameters or higher-order effects are anomalously large. In either case the appeal of our framework would be strongly diminished.

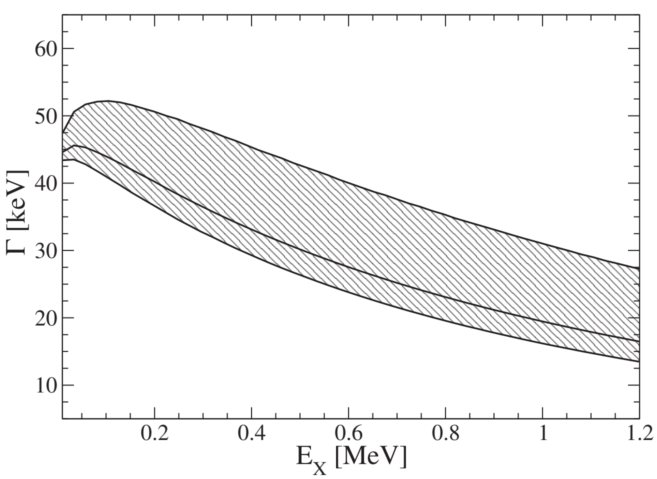


FIG. 7. Decay rate for $X \to D^0\bar{D}^0\pi^0$ as a function of E_X. We use $g = 0.6$. The central solid line corresponds to the LO prediction. The band is the result of the NLO calculation when the parameters r_0 and η are varied in the ranges $0 \le r_0 \le (100\text{ MeV})^{-1}$ and $-1 \le \eta \le 1$.

IV. SUMMARY

In this paper we have developed an effective field theory of nonrelativistic pions and D mesons that can be used to describe the properties of the $X(3872)$, assuming it is a weakly bound state of $D^0\bar{D}^{*0}$ and $D^{*0}\bar{D}^0$ with anomalously small binding energy. Because of an accidental cancellation between the D-meson hyperfine splitting and the mass of the π^0, pion exchange is characterized by a smaller scale than is typically the case in nuclear physics. This relatively small scale and the small axial coupling in the D-meson system (compared to the nucleon's axial coupling) combine to make the corrections from π^0-meson exchange amenable to perturbation theory. This justifies the application of a theory similar to that proposed by Kaplan, Savage, and Wise for low-energy NN interactions [51,52], in which a leading-order contact interaction is resummed to all orders to produce the bound state, and pion exchange and higher-derivative contact interactions are treated within perturbation theory.

This theory reproduces at leading order the calculation of $\Gamma[X \to D^0\bar{D}^0\pi^0]$ by Voloshin [39] which exploits the universal behavior of the DD^* wave function in the limit of small binding energy. Effective-range corrections as well as other corrections from short-distance scales are encoded in higher-dimension contact operators in the theory. These corrections turn out to completely dominate nonanalytic calculable corrections from π^0 exchange. Varying these coefficients within ranges determined by naturalness allows us to estimate the size of corrections to the leading-order calculations of Voloshin. While it is somewhat disappointing that the nonanalytic calculable corrections from π^0 exchange are so small that an experimental test of this aspect of the theory seems unlikely in the foreseeable future, the smallness of these corrections confirms one of the main points of this work, namely, that pion exchange can be dealt with using perturbation theory.

A naive estimate of the size of the NNLO corrections based on the expansion parameter in Eq. (6) is 1% or smaller. It is important to remember that in conventional nuclear physics, large corrections come from graphs with two or more pion exchanges in the $^3S_1$ channel, which first arise at NNLO. The two-pion-exchange graphs at NNLO come with large coefficients, $\sim$5, which ruin the perturbative expansion of KSW for two-nucleon systems [54]. In our case similar size coefficients in two-pion-exchange graphs should not ruin perturbation theory since even with a large coefficient $\sim$5, they would only be expected to be 5% or smaller. It would be interesting to perform the NNLO calculation to check this. A NNLO correction of 5% would dominate the nonanalytic NLO contribution but would be smaller than the uncertainty in the contact interaction contribution, indicating convergence of the expansion. In the unlikely case that pion exchange is nonperturbative, it can be resummed as done in nuclear physics [63].

It is straightforward to extend the analysis of this paper to other $X(3872)$ decay and production processes, such as $X \to D^0\bar{D}^0\gamma$ or $X \to J/\psi\rho^* \to J/\psi\pi^+\pi^-$. Coupling to $J/\psi\rho$ and $J/\psi\omega$ channels can be incorporated by including these degrees of freedom explicitly in the theory and coupling them to $D^0\bar{D}^{*0} + D^{*0}\bar{D}^0$ via contact interactions. It would be interesting to calculate π^0 exchange to other decays of the $X(3872)$ to see if these corrections lead to any interesting observable effects. It would also be interesting to use data or theoretical calculations to fix some of the counterterms appearing in the theory so as render calculations in this paper more predictive.

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APPENDIX A: DERIVING THE EFFECTIVE LAGRANGIAN FROM HHχPT

Heavy-hadron chiral perturbation theory (HHχPT) [57–59] can be used to derive the low-energy effective Lagrangian for D mesons, D^* mesons, and pions relevant to the $X(3872)$. We begin with the two-component HHχPT Lagrangian introduced in Ref. [67],

$$\mathcal{L} = \mathrm{Tr}[H^\dagger(iD_0)H] - g\,\mathrm{Tr}[H^\dagger H\boldsymbol{\sigma}\cdot\mathbf{A}] + \frac{\Delta}{4}\mathrm{Tr}[H^\dagger\boldsymbol{\sigma}H\boldsymbol{\sigma}], \tag{A1}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices, $\Delta = m_{H^*} - m_H$, $H = \mathbf{D}\cdot\boldsymbol{\sigma} + D$, and $\mathbf{A} = -\vec{\nabla}\pi/f_\pi + \mathcal{O}(\pi^3)$. Here $\mathbf{D}$ is a heavy vector field, D is a heavy pseudoscalar field, and π is the pion field,

$$\pi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 & \pi^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 \end{pmatrix}. \tag{A2}$$

Evaluating the traces in Eq. (A1) we obtain

$$\mathcal{L} = 2\mathbf{D}^\dagger\left(iD_0 - \frac{\Delta}{4}\right)\mathbf{D} + 2D^\dagger\left(iD_0 + \frac{3\Delta}{4}\right)D - 2g(\mathbf{D}^\dagger\cdot\mathbf{A}D + D^\dagger\mathbf{D}\cdot\mathbf{A}) - 2ig\mathbf{D}^\dagger\cdot\mathbf{D}\times\mathbf{A}. \tag{A3}$$

Since we wish to describe a bound state of two heavy mesons the power counting of HHχPT in powers of $1/m_H$ is inappropriate. Instead we need to power count in the relative velocity $v \ll 1$ of the heavy mesons. The kinetic energy which is subleading in $1/m_H$ is leading in v, and as a consequence we must include the kinetic term in our Lagrangian,

$$\mathcal{L} = 2\mathbf{D}^\dagger\left(iD_0 + \frac{\vec{\nabla}^2}{2m_{D^*}} - \frac{\Delta}{4}\right)\mathbf{D} + 2D^\dagger\left(iD_0 + \frac{\vec{\nabla}^2}{2m_D} + \frac{3\Delta}{4}\right)D - 2g(\mathbf{D}^\dagger\cdot\mathbf{A}D + D^\dagger\mathbf{D}\cdot\mathbf{A}) - 2ig\mathbf{D}^\dagger\cdot\mathbf{D}\times\mathbf{A}. \tag{A4}$$

We now rescale the heavy-meson fields

$$\{D, \mathbf{D}\} \to \frac{1}{\sqrt{2}}e^{i3\Delta t/4}\{D, \mathbf{D}\}, \tag{A5}$$

which gives

$$\mathcal{L} = \mathbf{D}^\dagger\left(iD_0 + \frac{\vec{\nabla}^2}{2m_{D^*}} - \Delta\right)\mathbf{D} + D^\dagger\left(iD_0 + \frac{\vec{\nabla}^2}{2m_D}\right)D - g(\mathbf{D}^\dagger\cdot\mathbf{A}D + D^\dagger\mathbf{D}\cdot\mathbf{A}) - ig\mathbf{D}^\dagger\cdot\mathbf{D}\times\mathbf{A}. \tag{A6}$$

Since we are only interested in those terms involving D^{*0}, D^0, and π^0 we keep only these fields in the Lagrangian:

$$\mathcal{L} = \mathbf{D}^\dagger\left(iD_0 + \frac{\vec{\nabla}^2}{2m_{D^*}} - \Delta\right)\mathbf{D} + D^\dagger\left(iD_0 + \frac{\vec{\nabla}^2}{2m_D}\right)D + \frac{g}{\sqrt{2}f_\pi}(\mathbf{D}^\dagger\cdot\vec{\nabla}\pi^0 D + D^\dagger\mathbf{D}\cdot\vec{\nabla}\pi^0) - i\frac{g}{\sqrt{2}f_\pi}\mathbf{D}^\dagger\cdot\boldsymbol{D}\times\vec{\nabla}\pi^0. \tag{A7}$$

Next the kinetic term for the pion is derived from the chiral Lagrangian

$$\begin{aligned}\mathcal{L}_\pi &= \frac{f_\pi^2}{8}\,\mathrm{Tr}[\partial_\mu\Sigma\partial^\mu\Sigma^\dagger] + \frac{f_\pi^2}{4}B_0\,\mathrm{Tr}[\mathcal{M}(\Sigma+\Sigma^\dagger)]\\ &= \frac{1}{2}\partial_\mu\pi^0\partial^\mu\pi^0 - \frac{1}{2}m_\pi^2(\pi^0)^2 + \text{self-interactions}\\ &= \frac{1}{2}\pi^0(-\partial^2 - m_\pi^2)\pi^0 + \cdots,\end{aligned} \tag{A8}$$

where $\Sigma = \exp(2i\pi/f_\pi)$, $\mathcal{M} = \mathrm{diag}(m_u, m_d)$ is the quark-mass matrix, and B_0 is a constant. The pion self-interaction terms are not needed at the order we are working so they are dropped, and we add the pion kinetic term to Eq. (A7) to obtain our Lagrangian.

However, this Lagrangian still includes the large scales m_π and Δ, which must be integrated out of the theory. Since we are interested in a nonrelativistic theory of pions we are justified in splitting the pion fields into creation and annihilation operators $\pi^0 = \hat{\pi} + \hat{\pi}^\dagger$. In addition we rescale the meson fields to make the large scales explicit:

$$\hat{\pi} = \frac{1}{\sqrt{2m_\pi}}e^{-im_\pi t}\pi, \qquad \hat{\pi}^\dagger = \frac{1}{\sqrt{2m_\pi}}e^{im_\pi t}\pi^\dagger, \qquad \mathbf{D} \to e^{-im_\pi t}\mathbf{D}. \tag{A9}$$

The pion kinetic term can then be expanded:

$$\begin{aligned}\mathcal{L}_\pi &= \frac{1}{2}\pi^0(-\partial_0^2 + \vec{\nabla}^2 - m_\pi^2)\pi^0\\ &= \frac{1}{4m_\pi}\{\pi^\dagger(2im_\pi\partial_0 - \partial_0^2 + \vec{\nabla}^2)\pi + \pi(-2im_\pi\partial_0\\ &\quad - \partial_0^2 + \vec{\nabla}^2)\pi^\dagger + e^{-2im_\pi t}\pi(2im_\pi\partial_0 - \partial_0^2 + \vec{\nabla}^2)\pi\\ &\quad + e^{2im_\pi t}\pi^\dagger(-2im_\pi\partial_0 - \partial_0^2 + \vec{\nabla}^2)\pi^\dagger\}\\ &= \pi^\dagger\left(i\partial_0 + \frac{\vec{\nabla}^2}{2m_\pi}\right)\pi\\ &\quad + \text{higher-order relativistic corrections.}\end{aligned} \tag{A10}$$

The terms above that include a large phase factor can be integrated out. In addition to modifying the pion propagator the field redefinition in Eq. (A9) modifies the kinetic term for $\mathbf{D}$,

$$\mathbf{D}^\dagger\left(i\partial_0 + \frac{\vec{\nabla}^2}{2m_{D^*}} - \Delta\right)\mathbf{D} \to \mathbf{D}^\dagger\left(i\partial_0 + \frac{\vec{\nabla}^2}{2m_{D^*}} - \delta\right)\mathbf{D}, \tag{A11}$$

where $\delta = \Delta - m_\pi \simeq 7$ MeV. Note that after the field rescaling the last term in Eq. (A7) contains a phase factor $e^{-im_\pi t}$, and can be dropped.

Finally, we obtain the kinetic terms and axial coupling appearing in the Lagrangian of Eq. (10) by another rephasing of the $\mathbf{D}$ and π fields,

$$\mathbf{D} \to e^{-i\delta t}\mathbf{D}, \qquad \pi \to e^{-i\delta t}\pi. \tag{A12}$$

This just shifts the residual mass from the $\mathbf{D}$ kinetic term to the π kinetic term. This last step is not essential, however it is convenient. The remaining terms in Eq. (10) are short-distance interactions allowed by power counting and the symmetries of the theory.

APPENDIX B: NLO DIAGRAMS

The NLO decay diagrams involving pion exchange are shown in Fig. 4. The result for Fig. 4(a) is

$$\begin{aligned} i\frac{g^3M_{DD^*}^2}{8\pi f_\pi^3(p_D^2+\gamma^2)}\bigg[&\left(\frac{2}{3}\Lambda_{\mathrm{PDS}} - h_1(p_D)\right)\vec{p}_\pi\cdot\vec{\epsilon}_X\\ &+ h_2(p_D)\vec{p}_D\cdot\vec{\epsilon}_X\vec{p}_D\cdot\vec{p}_\pi\bigg] + (p_D \to p_{\bar{D}}),\end{aligned} \tag{B1}$$

where the functions $h_1(p)$ and $h_2(p)$ are given by

$$\begin{aligned} h_1(p) &= \int_0^1 dx\sqrt{-p^2x^2 + (p^2+\gamma^2+\mu^2)x - \mu^2 - i\epsilon}\\ &= \frac{p^2-\gamma^2-\mu^2}{4p^2}\gamma + \frac{(p^2+\gamma^2+\mu^2)^2 - 4p^2\mu^2}{8p^3}\tan^{-1}\frac{2p\gamma}{-p^2+\gamma^2+\mu^2}\\ &\quad - i\frac{\mu}{4p^2}(p^2+\gamma^2+\mu^2) - i\frac{(p^2+\gamma^2+\mu^2)^2 - 4p^2\mu^2}{16p^3}\ln\frac{\gamma^2+(\mu-p)^2}{\gamma^2+(\mu+p)^2}\end{aligned} \tag{B2}$$

$$\begin{aligned} h_2(p) &= \int_0^1 dx\frac{x^2}{\sqrt{-p^2x^2 + (p^2+\gamma^2+\mu^2)x - \mu^2 - i\epsilon}}\\ &= -\frac{5p^2+3\gamma^2+3\mu^2}{4p^4}\gamma + \frac{3(p^2+\gamma^2+\mu^2)^2 - 4p^2\mu^2}{8p^5}\tan^{-1}\frac{2p\gamma}{-p^2+\gamma^2+\mu^2} - i\frac{3\mu}{4p^4}(p^2+\gamma^2+\mu^2)\\ &\quad - i\frac{3(p^2+\gamma^2+\mu^2)^2 - 4p^2\mu^2}{16p^5}\ln\frac{\gamma^2+(\mu-p)^2}{\gamma^2+(\mu+p)^2},\end{aligned} \tag{B3}$$

and the result for Fig. 4(b) is

$$-\frac{g^3 M_{DD^*}^2}{12\pi f_\pi^3}\frac{\mu^3}{(p_D^2+\gamma^2)^2}\vec{p}_\pi\cdot\vec{\epsilon}_X+(p_D\rightarrow p_{\bar{D}}). \quad \text{(B4)}$$

Note that Fig. 4(b) includes, as a subgraph, the one-loop D^* self-energy contribution. In the PDS scheme the self-energy graph has a linear divergence which gives an additive renormalization to the residual mass term of the D^*. At tree level, we performed a field redefinition which moved the residual mass term from the kinetic term of the D^* to the kinetic term of the pion through a field redefinition. In order that loop corrections do not reintroduce a residual mass for the D^*, we introduce a counterterm $-\delta_{\rm ct}(\boldsymbol{D}^\dagger\boldsymbol{D}+\bar{\boldsymbol{D}}^\dagger\bar{\boldsymbol{D}})$, which is defined to cancel the residual mass term at each order in perturbation theory. At one-loop order, $\delta_{\rm ct}=g^2\mu^2\Lambda_{\rm PDS}/24\pi f_\pi^2$. This linearly divergent contribution to the self-energy also appears in Fig. 6(b) and is canceled by an insertion of the residual mass counterterm in a DD^* bubble (not shown in the figure).

The NLO diagrams with the counterterms C_2 and B_1 are shown in Fig. 5. The result for Fig. 5(a) is

$$-iC_2(\Lambda_{\rm PDS})\frac{gM_{DD^*}^2(\Lambda_{\rm PDS}-\gamma)}{4\pi f_\pi}\frac{p_D^2-\gamma^2}{p_D^2+\gamma^2}\vec{p}_\pi\cdot\vec{\epsilon}_X + (p_D\rightarrow p_{\bar{D}}), \quad \text{(B5)}$$

and the result for Fig. 5(b) is

$$-iB_1(\Lambda_{\rm PDS})\frac{M_{DD^*}(\Lambda_{\rm PDS}-\gamma)}{2\pi}\vec{p}_\pi\cdot\vec{\epsilon}_X. \quad \text{(B6)}$$

Finally, we show the NLO wave function renormalization diagrams in Fig. 6. The NLO contribution to ${\rm Re}\Sigma'(-E_X)$ from the graphs in Fig. 6 is

$$\frac{g^2M_{DD^*}^3}{12\pi^2 f_\pi^2}\left[\frac{\Lambda_{\rm PDS}-\gamma}{\gamma}+\frac{2\mu^2}{4\gamma^2+\mu^2}\right] - C_2(\Lambda_{\rm PDS})\frac{M_{DD^*}^3(\gamma-\Lambda_{\rm PDS})(2\gamma-\Lambda_{\rm PDS})}{2\pi^2}. \quad \text{(B7)}$$

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