We investigate an alternative compactification of extra dimensions using local cosmic string in the Brans-Dicke theory. In the course of development we shall use several approximations and by using a dynamical system argument we arrive at a model similar to the one in [8,10]. Nevertheless, we would like to stress that the main difference between the previous works and ours is that we use local cosmic string in the framework of a low energy string theory while they use a global string in Einstein’s gravity.

Our effort here is to present a model of alternative compactification using a local string in the Brans-Dicke theory. In the course of development we shall use several approximations and by using a dynamical system argument we arrive at a model similar to the one in [8,10]. Nevertheless, we would like to stress that the main difference between the previous works and ours is that we use local cosmic string in the framework of a low energy string theory while they use a global string in Einstein’s theory. Besides, the space is topologically different since we end up with a de Sitter behavior, and the additional scalar field plays an important role in the warp factor.

We start with the equation

$$R^p_{\mu} = 2 \delta^p_{\mu} \sigma \partial_\nu \sigma + \epsilon \left( T^p_{\mu} - \frac{\delta^p_{\mu}}{p + 1} T \right) - \frac{2 \delta^p_{\mu} \Lambda}{p + 1},$$

(1)

which comes from the Brans-Dicke action in $p + 3$ dimensions in the Einstein frame. In (1), $\Lambda$ is the cosmological constant, $\sigma$ is the scalar field, $T^p_{\mu}$ is the stress tensor of a cosmic string, $T$ is the trace of the stress tensor and $\epsilon = -\frac{1}{M_{p+3}^2}$, where $M_{p+3}$ is the analogue of the Planck mass in $p + 3$ dimensions. The equations that make the relationship between the two frames (the Jordan-Brans-Dicke and the Einstein frames) are $w = \frac{1}{4} \delta^p_{\mu} \frac{1}{2}$ (the Brans-Dicke parameter), $\delta = \frac{1}{2} e^{-2\beta_\sigma}$ and $g_{\mu\nu} = e^{2\beta_\sigma} g_{\mu\nu}$. The physical quantities (i.e., the quantities defined in the JBD frame) are the tilde ones. Note that in the Einstein’s frame the stress tensor is not conserved.

In order to calculate the metric we expect that the spacetime has the same symmetry of the source, and since our source is a cosmic string, we write down the ansatz

$$ds^2 = e^A (dt^2 + dz_i^2) + dr^2 + e^C d\theta^2,$$

(2)

where $i = 1 \ldots p$ and $A$ and $C$ are functions of $r$ only. So,
all the bulk structure has a cylindrical symmetry. Now we
turn to tell a little bit more about the source.

The standard model for a gauge cosmic string is the U(1)
invariant Lagrangean

\[
L = \frac{1}{2} (D_{\mu} \Phi)(D^{\mu} \Phi)^* + \lambda(\Phi^* \Phi - \eta^2)^2 + \frac{F_{\mu\nu} F^{\mu\nu}}{16\pi} \tag{3}
\]

where \( D_{\mu} = \partial_{\mu} + ieA_{\mu} \) and \( F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \). The ansatz to
generate a stringlike solution is \( \Phi = \eta X e^{i\theta} \) and
\( A_{\mu} = \frac{1}{2}(p-1)\partial_{\mu}\theta \). Here \( X \) and \( P \) are functions of \( r \)
only. Now, two important remarks: first, when the masses
of gauge and scalar fields are equal, the fields have a
behavior, far from the string in Einstein theory, given by [14]

\[
e^C \rightarrow r^2(1 - 4G\eta^2)^2,
\]

\[
P \rightarrow 2\sqrt{2}(1 - 4G\eta^2)\gamma r^{1/2} e^{-2\sqrt{2}r},
\]

\[
X \rightarrow \gamma r^{-1/2} e^{-2\sqrt{2}r},
\]

where \( \gamma \) is a constant that can be determined by some
boundary condition. Besides, it is possible to show that in
the Brans-Dicke theory the stress tensor associated to (3)
in Einstein frame is [13]

\[
T^r_r = \frac{n^2}{2}(X^2 e^{2\beta\sigma} + e^{-C}X^2 P^2 e^{2\beta\sigma} + 2(X^2 - 1)^2 e^{4\beta\sigma} + \frac{1}{2} e^{-C} P^2),
\]

\[
T^r_\theta = \frac{n^2}{2}(-X^2 e^{2\beta\sigma} - e^{-C}X^2 P^2 e^{2\beta\sigma} - 2(X^2 - 1)^2 e^{4\beta\sigma} + \frac{1}{2} e^{-C} P^2),
\]

\[
T^\theta_\theta = \frac{n^2}{2}(-X^2 e^{2\beta\sigma} + e^{-C}X^2 P^2 e^{2\beta\sigma} - 2(X^2 - 1)^2 e^{4\beta\sigma} + \frac{1}{2} e^{-C} P^2),
\]

where prime means derivative with respect to \( r \).

A remarkable characteristic of the solution found by
Gundlach and Ortiz [13] is that the string solution in the
Brans-Dicke theory can be obtained using an amazing
approximation far enough from the core, so that (4) is
valid, in such way that \( e^{\beta\sigma} \approx 1 \) is still a good approxima-
tion. We shall use it here. Note that in the components of
stress tensor it means \( \sigma = \text{cte} \), i.e. the contribution of a
scalar Brans-Dicke field, as a source, is too small compared
with that one coming from the cosmic string, even in the
\( r \gg \) region. Since we are interested in an exotic compac-
tification model, our system will be analyzed in the \( \gamma \rightarrow \) region.

Taking everything into account, i.e., substituting the
expressions (4) into (5) and using \( e^{\beta\sigma} \approx 1 \) we have, at
\( 1/r \) order,

\[
T^r_r - \frac{T}{p + 1} \approx 16\left(1 - \frac{p}{p + 1}\right) \eta^2 \gamma^2 e^{-4\sqrt{2}r},
\]

\[
T^r_\theta - \frac{T}{p + 1} = T^\theta_\theta - \frac{T}{p + 1} = \frac{32\eta^2 \gamma^2}{p + 1} e^{-4\sqrt{2}r}.
\]

So, the Einstein-Brans-Dicke equation. (1), after some
calculation and calling \( f(r) = \frac{32\eta^2 \gamma^2}{p + 1} e^{-4\sqrt{2}r} \), gives

\[
(p + 1)\left(\frac{A'' + \frac{A'^2}{2}}{2} + \frac{C'' + \frac{C'^2}{2}}{2}\right) \approx -4\sigma^2 - 2f(r) + \frac{4\Lambda}{p + 1},
\]

\[
C'' + \frac{C'^2}{2} + \frac{(p + 1)}{2} A'C' \approx -2f(r) + \frac{4\Lambda}{p + 1},
\]

\[
A'' + \frac{(p + 1)}{2} A'^2 + \frac{A'C'}{2} = f(r)(p - 1) + \frac{4\Lambda}{p + 1}.
\]

It is easy to note that even in such approximation the
equations above are not simple. We shall perform, as in
[8], a dynamical system formulation and then an analysis in
the phase plane. We start with the variables

\[
x = pA' + C', \quad y = C'.
\]

Equations (7) and (8), together, give \( (f(r) = f) \)

\[
-4\sigma^2 = (p + 1)\left(\frac{A'' + \frac{A'^2}{2}}{2} - \frac{A'C'}{2}\right),
\]

and substituting \( A'' \) from (9) we have

\[
-4\sigma^2 = (p + 1)((p - 1)f + \frac{4\Lambda}{p + 1} - \frac{pA'^2}{2} - A'C').
\]

Then, taking into account that \( \frac{x^2 - y^2}{2p} = \frac{pA'^2}{2} + A'C' \), we
arrive at

\[
4\sigma^2 = -(p^2 - 1)f - 4\Lambda + \frac{(p + 1)}{2p}(x^2 - y^2).
\]

The dynamical system (10) can be studied in the \( x,y \)
plane and the relation (13) is a consistency equation that
plays an important role later on. From (8) it is easy to see
that

\[
y' = -2f + \frac{4\Lambda}{p + 1} - \frac{1}{2p}[xy(p + 1) - y^2],
\]

and (7), (9), and (13) led to

\[
x' = (p + 1)(p - 2)f + 4\Lambda - \frac{1}{2p}[(p + 1)x^2 - xy].
\]

Equations (14) and (15) form a nonautonomous dynamical
system that is characterized by two critical points

\[
(\ddot{x}, \ddot{y})_\pm = \pm \Delta \left(4\Lambda + f(p + 1)(p - 2), \frac{4\Lambda}{p + 1} - 2f\right).
\]

where \( \Delta = \frac{2(p + 1)}{4\Lambda(p + 2) + f(p - 3)(p + 1)} \).
(\vec{x}, \vec{y}) are the attractor and repellor points, respectively. Basically it means that there is at least one stable configuration for the fields (attractor). Looking at (2) one notes that if \( p = 3 \) we have a brane-world picture, with a transverse space \((r, \theta)\). Beyond this, the attractor point in the phase plane gives the integral equations to the fields in \( r \gg \), i.e.

\[
\sigma^2 = -\frac{4f(\Lambda + 2f)}{5\Lambda + 6f}, \quad (17)
\]

\[
C' = \left(\frac{2}{5\Lambda + 6f}\right)^{1/2}(\Lambda - 2f), \quad (18)
\]

\[
A' = \left(\frac{2}{5\Lambda + 6f}\right)^{1/2}(\Lambda + 2f). \quad (19)
\]

From Eq. (17) we note that, since \( f < 0 \)(because \( \varepsilon < 0 \)), the cosmological constant is positive and must obey the following inequality:

\[
\Lambda > 2|f(r)|. \quad (20)
\]

It is a quite remarkable relation. First, it tells that we are, in fact, dealing with a de Sitter space. Second, this is a new characteristic of models of this kind. Like in Ref. [8], there are two branches and the system never goes to the repellor. In a more explicit way, Eq. (20) gives

\[
\Lambda > 16|\varepsilon| \gamma^2 \frac{e^{-4\sqrt{2}\gamma r}}{r}, \quad (21)
\]

which leads to a very small cosmological constant.

The next step is to analyze how this model can help in solving the hierarchy problem. Then, after coming back to the physical frame we have

\[
ds^2 = W(r)(\eta_{\mu\nu}dx^\mu dx^\nu) + e^2\beta r dr^2 + H(r)d\theta^2,\]

where \( W(r) = e^{2\beta r + \lambda} \), \( H(r) = e^{2\beta r + \sigma} \) and \( \lambda, \sigma \) are the fields which asymptotic behavior is found in (17)–(19). Explicitly, the function \( W(r) \) intervenes directly in the masses generated by the Higgs mechanism, just like in [3]. Nevertheless, we should emphasize here that there is a new possible adjustment provided by the Brans-Dicke scalar field. However, the way it appears can not tell, in a rigorous way, its influence in the Higgs mechanism once the equations are only valid far from the brane. By all means, it is clear that inclusion of the Brans-Dicke field opens a new possibility.

It is easy to note from Eqs. (17)–(19) that if one implements the limit \( f(r) \to 0 \), the scalar field becomes constant, and the fields \( A \) and \( C \) are equal, apart of a constant. When this occurs the functions \( W(r) \) and \( H(r) \) become equal and we arrive into a new brane-world picture, given by

\[
ds^2 = e^{k\Lambda^{1/2}}(\eta_{\mu\nu}dx^\mu dx^\nu + d\theta^2) + dr^2, \quad (22)
\]

where \( k \), in general, depending on the dimension and the constants, had been absorbed in new variables \((x^\mu, \theta, r)\). Again, Eq. (22) is valid only in \( r \gg \) region (far from the string) and we stress the minuteness of the cosmological constant. Of course, the shape found in (22) can also be obtained solving the Einstein’s equations with cylindrical symmetry without any source and free of approximations, however there is no physical reason to do it.

We conclude this work stressing that we believe that this is a promising field of research, i.e. to realize brane-gravity on this model, analyzing how the extra-dimensions influence in the calculation of measurable physical amounts [15]. Special techniques used in other models cannot be used here, like among others, Israel junction conditions [16] at the brane that makes explicitly use of the \( Z_2 \) symmetry.

The combination of a local cosmic string and the low energy string theory such as Brans-Dicke seems to be a good alternative model for compactifying extra dimensions from both physical and topological points of view.

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