

Angular momentum and energy-momentum densities as gauge currents

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If we replace the general spacetime group of diffeomorphisms by transformations taking place in the tangent space, general relativity can be interpreted as a gauge theory, and in particular as a gauge theory for the Lorentz group. In this context, it is shown that the angular momentum and the energy-momentum tensors of a general matter field can be obtained from the invariance of the corresponding action integral under transformations taking place, not in spacetime, but in the tangent space, in which case they can be considered as gauge currents.

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I. INTRODUCTION

According to the Noether theorems [1], energy-momentum conservation is related to the invariance of the action integral under translations of the spacetime coordinates, and angular-momentum conservation is related to the invariance of the action integral under Lorentz transformations. As both translation and Lorentz transformations are perfectly well defined in the Minkowski spacetime, the Noether theorems can be applied with no restrictions in this spacetime. However, on a curved spacetime, neither translations nor Lorentz transformations can be defined in a natural way [2]. The problem then arises on how to define energy momentum and angular momentum in the presence of gravitation, as in this case spacetime is represented by a curved (pseudo) Riemannian manifold.

In general relativity, the conservation of the energy-momentum tensor of any matter field is usually obtained as a consequence of the invariance of the action integral in relation to the spacetime group of diffeomorphisms (general coordinate transformations). Although this is usually considered as acceptable for the energy-momentum tensor [3], it leads to problems when considering the angular momentum conservation, mainly in the case of spinor fields. In fact, as the angular momentum conservation is related to the invariance of the action integral under Lorentz transformations, and as there is no natural action of the full group of diffeomorphisms on spinor fields [4], the spin character of these fields has necessarily to be taken into account by considering the action of the Lorentz group on the Minkowski tangent spacetime, where its action is well defined. As a consequence, the Dirac equation in general relativity must necessarily be written in terms of the spin connection, a connection assuming values in the Lie algebra of the Lorentz group, and can never be written in terms of the spacetime Levi-Civita (or Christoffel) connection.

In order to circumvent the above problems, let us then consider the following structure. At each point of spacetime—which in the presence of gravitation is a curved (pseudo) Riemannian manifold—there is always a Minkowski tangent spacetime attached to it. Now, instead of

considering the spacetime group of diffeomorphism as the fundamental group behind gravitation, we consider the *local* symmetry group of general relativity to be the Lorentz group [5], whose action takes place in the tangent space. According to this construction, general relativity can be reinterpreted as a gauge theory [6] for the Lorentz group [7], with the indices related to the Minkowski space considered as *local* Lorentz indices in relation to spacetime. As a consequence, the spin connection is to be considered as the fundamental field representing gravitation. This means to use, instead of the Levi-Civita covariant derivative, the Fock–Ivanenko operator, a covariant derivative that takes into account the spin content of the field as defined in the Minkowski tangent space. This approach, mandatory for the case of spinors [8], can actually be used for any field, being in this sense more general than the usual spacetime approach of general relativity.

By adopting the above described point of view, which means to reinterpret general relativity as a gauge theory for the Lorentz group, the aim of the present paper will be to show that the angular momentum and the energy-momentum tensors of a general matter field can be defined as the Noether currents associated to the invariance of the action integral under local transformations taking place not in spacetime, but in the tangent space. We begin in Sec. II with a review of the Lorentz transformation properties. In Sec. III we introduce the gauge potentials, define the Lorentz covariant derivative, and show how a very special tetrad field naturally shows up in this formalism. This tetrad, as we are going to see, depends on the spin connection, and this dependency will be crucial for obtaining the covariant conservation laws. In Sec. IV we obtain the gauge transformations of both the spin connection and the tetrad field. The roles played by the spin and the orbital parts of the Lorentz generators in these transformations will also be analyzed. In Sec. V we show how the angular momentum conservation can be obtained as the Noether current associated to a transformation generated by the spin generator of the Lorentz group. Then, by considering the Lorentz transformation of the tetrad field, we show in Sec. VI how the energy-momentum conservation can be obtained as the Noether current associated to a transformation generated by the orbital generator of the Lorentz group. Finally, in Sec. VII, we comment on the results obtained.

II. LORENTZ TRANSFORMATIONS

We use the Greek alphabet $\mu, \nu, \rho, \dots = 1, 2, 3, 4$ to denote indices related to spacetime, and the Latin alphabet

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$a, b, c, \dots = 1, 2, 3, 4$ to denote indices related to each one of the Minkowski tangent spaces. The Cartesian Minkowski coordinates, therefore, is denoted by $\{x^a\}$, and its metric tensor is chosen to be

$$\eta_{ab} = \text{diag}(1, -1, -1, -1). \quad (1)$$

As is well known, the most general form of the generators of infinitesimal Lorentz transformations is [9]

$$J_{ab} = L_{ab} + S_{ab}, \quad (2)$$

where

$$L_{ab} = i(x_a \partial_b - x_b \partial_a) \quad (3)$$

is the *orbital* part of the generators, and S_{ab} is the *spin* part of the generators, whose explicit form depends on the field under consideration. The generators J_{ab} satisfy the commutation relation

$$[J_{ab}, J_{cd}] = i(\eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}), \quad (4)$$

which is to be identified with the Lie algebra of the Lorentz group. Each one of the generators L_{ab} and S_{ab} satisfies the same commutation relation as J_{ab} , and commute with each other.

A position dependent—that is, local—infinitesimal Lorentz transformation is defined as

$$\delta_L x^a = -\frac{i}{2} \epsilon^{cd} L_{cd} x^a, \quad (5)$$

where $\epsilon^{cd} \equiv \epsilon^{cd}(x^\mu)$ are the transformation parameters. By using the explicit form of L_{cd} , it becomes

$$\delta_L x^a = -\epsilon^a{}_d x^d. \quad (6)$$

An interesting property of the Lorentz transformation of the Minkowski space coordinates is that it is formally equivalent to a translation [10]. In fact, by using the explicit form of L_{cd} , the transformation (5) can be rewritten in the form

$$\delta_L x^a = -i \xi^c P_c x^a, \quad (7)$$

which is a translation with

$$\xi^c = \epsilon^c{}_d x^d \quad (8)$$

as the transformation parameters, and

$$P_c = -i \partial_c \quad (9)$$

as generators. In other words, an infinitesimal Lorentz transformation of the Minkowski coordinates is equivalent to a translation with $\xi^c \equiv \epsilon^c{}_d x^d$ as the parameters. Actually, this is a property of the Lorentz generators L_{ab} , whose action can always be reinterpreted as a translation. The reason for such equivalence is that, because the Minkowski spacetime is transitive under translations, every two points related by a Lorentz transformation can also be related by a translation. Notice that the inverse is not true.

Let us consider now a general matter field $\Psi(x^\mu)$, which is function of the spacetime coordinates $\{x^\mu\}$. Under an infinitesimal *local* Lorentz transformation of the tangent-space coordinates, the field Ψ will change according to [9]

$$\delta_J \Psi \equiv \Psi'(x) - \Psi(x) = -\frac{i}{2} \epsilon^{ab} J_{ab} \Psi(x). \quad (10)$$

The explicit form of the orbital generators L_{ab} , given by Eq. (3), is the same for all fields, whereas the explicit form of the spin generators S_{ab} depends on the spin of the field Ψ . Notice furthermore that the orbital generators L_{ab} are able to act in the spacetime argument of $\Psi(x^\mu)$ due to the relation

$$\partial_a = (\partial_a x^\mu) \partial_\mu.$$

By using the explicit form of L_{ab} , the Lorentz transformation (10) can be rewritten as

$$\delta_J \Psi = -\epsilon^{ab} x_b \partial_a \Psi - \frac{i}{2} \epsilon^{ab} S_{ab} \Psi, \quad (11)$$

or equivalently,

$$\delta_J \Psi = -i \xi^c P_c \Psi - \frac{i}{2} \epsilon^{ab} S_{ab} \Psi, \quad (12)$$

where use has been made of Eq. (8). In other words, the *orbital* part of the transformation can be reduced to a translation, and consequently the Lorentz transformation of a general field Ψ can be rewritten as a *translation* plus a strictly *spin* Lorentz transformation. Notice however that, as

$$[P_c, S_{ab}] = 0, \quad (13)$$

the transformation (12) is not a Poincaré, but a Lorentz transformation.

As a final remark, it is important to notice that, instead of four scalar functions, the coordinates x^a of the Minkowski spacetime can also be interpreted as a vector field $x^a(x^\mu)$. In this case, however, the Lorentz generators must be written in the vector representation

$$(S_{cd})^a{}_b = i(\delta_c^a \eta_{db} - \delta_d^a \eta_{cb}). \quad (14)$$

Consequently, its Lorentz transformation will be written as

$$\delta_S x^a = -\frac{i}{2} \epsilon^{cd} (S_{cd})^a{}_b x^b, \quad (15)$$

which yields

$$\delta_S x^a = \epsilon^a{}_d x^d. \quad (16)$$

Therefore, from Eqs. (6) and (16) we see that a Lorentz transformation of the Minkowski coordinates written with the complete generator J_{cd} vanishes identically:

$$\delta_J x^a \equiv -\frac{i}{2} \epsilon^{cd} J_{cd} x^a = 0. \quad (17)$$

The interpretation of this result is that, under a Lorentz transformation generated by J_{cd} , all vector fields $V^a(x)$ undergo a transformation at the same point:

$$\delta_J V^a \equiv V^{a'}(x) - V^a(x) = -\frac{i}{2} \epsilon^{cd} J_{cd} V^a.$$

In the specific case of the coordinate itself, which is also a Lorentz vector field, the transformations generated by S_{cd} and L_{cd} cancel each other, yielding a vanishing net result.

III. LORENTZ COVARIANT DERIVATIVE

In a gauge theory for the Lorentz group, the fundamental field representing gravitation is the spin connection $\overset{\circ}{A}_\mu$, a field assuming values in the Lie algebra of the Lorentz group,

$$\overset{\circ}{A}_\mu = \frac{1}{2} \overset{\circ}{A}^{ab} J_{ab}. \quad (18)$$

Equivalently, we can write

$$\overset{\circ}{A}_\mu = c^{-2} B^a{}_\mu P_a + \frac{1}{2} \overset{\circ}{A}^{ab} S_{ab}, \quad (19)$$

where a new gauge potential $B^a{}_\mu$ assuming values in the Lie algebra of the translation group, has been defined [5]

$$c^{-2} B^a{}_\mu = \overset{\circ}{A}^a{}_{b\mu} x^b, \quad (20)$$

with the velocity of light c introduced for dimensional reasons. It is important to remark once more that, despite the existence of a gauge field related to translations, and another one related to the Lorentz group, the structure group underlying this construction is not the Poincaré, but the Lorentz group.

We consider now the Lorentz covariant derivative of the matter field Ψ , whose general form is [11]

$$\overset{\circ}{D}_c \Psi = \partial_c \Psi + \frac{1}{2} \overset{\circ}{A}^{ab}{}_c \frac{\delta_J \Psi}{\delta \epsilon^{ab}}. \quad (21)$$

Substituting the transformation (11), it becomes [5]

$$\overset{\circ}{D}_c \Psi = h^\mu{}_c \overset{\circ}{D}_\mu \Psi, \quad (22)$$

where $h^\mu{}_c$ is the inverse of the tetrad field

$$h^c{}_\mu = \partial_\mu x^c + \overset{\circ}{A}^c{}_{d\mu} x^d \equiv \partial_\mu x^c + c^{-2} B^c{}_\mu, \quad (23)$$

and

$$\overset{\circ}{D}_\mu = \partial_\mu - \frac{i}{2} \overset{\circ}{A}^{ab}{}_\mu S_{ab} \quad (24)$$

is the Fock-Ivanenko covariant derivative operator [12]. According to this construction, therefore, the *orbital* part of the Lorentz generators is reduced to a translation, which gives then rise to a tetrad that depends on the spin connection.

Because its action reduces ultimately to a translation, the *orbital* generator L_{ab} is responsible for the universality of gravitation. In fact, as L_{ab} acts in the fields through their arguments, all fields will respond equally to its action. Notice also that, whereas the tangent space indices are raised and lowered with the metric η_{ab} , spacetime indices are raised and lowered with the Riemannian metric

$$g_{\mu\nu} = h^a{}_\mu h^b{}_\nu \eta_{ab}. \quad (25)$$

It is important to remark that the Fock-Ivanenko derivative has the definition

$$\overset{\circ}{D}_c \Psi = \partial_c \Psi + \frac{1}{2} \overset{\circ}{A}^{ab}{}_c \frac{\delta_S \Psi}{\delta \epsilon^{ab}},$$

where [9]

$$\delta_S \Psi \equiv \Psi'(x') - \Psi(x) = -\frac{i}{2} \epsilon^{ab} S_{ab} \Psi. \quad (26)$$

Accordingly, the Fock-Ivanenko covariant derivative of the vector field $x^c(x^\mu)$ is

$$\overset{\circ}{D}_\mu x^c = \partial_\mu x^c + \frac{1}{2} \overset{\circ}{A}^{ab}{}_\mu \frac{\delta_S x^c}{\delta \epsilon^{ab}}. \quad (27)$$

Using the transformation (16), it becomes

$$\overset{\circ}{D}_\mu x^c = \partial_\mu x^c + \overset{\circ}{A}^c{}_{b\mu} x^b \equiv h^c{}_\mu, \quad (28)$$

which shows that the tetrad coincides with the Fock-Ivanenko covariant derivative of the vector field $x^c(x^\mu)$.

IV. GAUGE TRANSFORMATIONS

Under a local Lorentz transformation generated by

$$U = \exp \left[-\frac{i}{2} \epsilon^{ab} S_{ab} \right], \quad (29)$$

the covariant derivative $\overset{\circ}{D}_a \Psi$ will change according to

$$\overset{\circ}{D}'_a \Psi'(x') = U \overset{\circ}{D}_a \Psi(x). \quad (30)$$

As

$$\overset{\circ}{D}_a \Psi(x) = h^\mu{}_a \overset{\circ}{D}_\mu \Psi(x), \quad (31)$$

and taking into account that $h'^\mu{}_a(x') = U_a{}^b h^\mu{}_b(x)$, with $U_a{}^b$ the usual element of the Lorentz group in the vector representation, we can rewrite Eq. (30) in the form

$$h'^\mu{}_a(x') \overset{\circ}{D}'_\mu \Psi'(x') = U_a{}^b h^\mu{}_b(x) U \overset{\circ}{D}_\mu \Psi(x). \quad (32)$$

It then follows that

$$\overset{\circ}{D}'_\mu \Psi'(x') = U \overset{\circ}{D}_\mu \Psi(x), \quad (33)$$

or equivalently,

$$\mathring{D}'_\mu = U \mathring{D}_\mu U^{-1}. \quad (34)$$

Using the Fock-Ivanenko derivative (24), we obtain the usual gauge transformation

$$\mathring{A}'_\mu = U \mathring{A}_\mu U^{-1} + iU \partial_\mu U^{-1}. \quad (35)$$

The infinitesimal form of U is

$$U \approx 1 - \frac{i}{2} \epsilon^{cd} S_{cd}. \quad (36)$$

By using the commutation relation (4) for S_{ab} , we get from Eq. (35)

$$\delta_S \mathring{A}^{cd}{}_\mu = -(\partial_\mu \epsilon^{cd} + \mathring{A}^c{}_{a\mu} \epsilon^{ad} + \mathring{A}^d{}_{a\mu} \epsilon^{ca}) \equiv -\mathring{D}_\mu \epsilon^{cd}. \quad (37)$$

Notice that, as $\mathring{A}^{cd}{}_\mu$ does not respond to the orbital generators, we have that $\delta_S \mathring{A}^{cd}{}_\mu = \delta_J \mathring{A}^{cd}{}_\mu$.

Let us obtain now the infinitesimal Lorentz transformations of the tetrad field. First of all, we have the transformation generated by S_{ab} , which yields the *total* change in the tetrad, that is, $\delta_S h^a{}_\mu \equiv h'^a{}_\mu(x') - h^a{}_\mu(x)$. From Eq. (23) we see that

$$\delta_S h^a{}_\mu = \partial_\mu (\delta_S x^a) + (\delta_S \mathring{A}^a{}_{d\mu}) x^d + \mathring{A}^a{}_{d\mu} (\delta_S x^d). \quad (38)$$

Using the transformations (16) and (37), we get

$$\delta_S h^a{}_\mu = \epsilon^a{}_c h^c{}_\mu \equiv -\frac{i}{2} \epsilon^{cd} (S_{cd})^a{}_b h^b{}_\mu, \quad (39)$$

as it should be since $h^a{}_\mu$ is a Lorentz vector field in the tangent-space index. On the other hand, the tetrad transformation generated by J_{ab} corresponds to a transformation at the same x^a , that is, $\delta_J h^a{}_\mu \equiv h'^a{}_\mu(x) - h^a{}_\mu(x)$. Such a transformation can be obtained from Eq. (23) by keeping x^a fixed, and substituting $\delta_J \mathring{A}^a{}_{b\mu} \equiv \delta_S \mathring{A}^a{}_{b\mu}$ as given by Eq. (37). The result is

$$\delta_J h^a{}_\mu = -x_b \mathring{D}_\mu \epsilon^{ab}. \quad (40)$$

Finally, there is also the transformation generated by the orbital generator L_{ab} . As the spin connection transformation is generated by the spin generators S_{ab} only—see Eqs. (35) and (36)—this corresponds to a transformation due to the variation of the coordinate x^a only, with $\mathring{A}^a{}_{d\mu}$ fixed: $\delta_L h^a{}_\mu \equiv h'^a{}_\mu(x') - h^a{}_\mu(x)$. From Eq. (23), we see that such a transformation is given by

$$\delta_L h^a{}_\mu = \partial_\mu (\delta_L x^a) + \mathring{A}^a{}_{d\mu} (\delta_L x^d). \quad (41)$$

Substituting Eq. (6), and making use of the definition (8), we get

$$\delta_L h^a{}_\mu = -\mathring{D}_\mu \xi^a. \quad (42)$$

This transformation shows that the tetrad behaves as a translational gauge potential under a Lorentz transformation of the tangent space coordinates, in which only the change due to the variation of the coordinates is considered. In other words, the tetrad behaves like a translational gauge potential under a Lorentz transformation generated by the orbital generator L_{ab} , whose action, as we have already seen, can always be reinterpreted as a translation.

Notice finally that, by using the above results, the transformation (39) can be rewritten in the form

$$\delta_S h^a{}_\mu \equiv \delta_J h^a{}_\mu - \delta_L h^a{}_\mu = -x^b \mathring{D}_\mu \epsilon^{ab} + \mathring{D}_\mu \xi^a. \quad (43)$$

We remark that this result is easily seen to be equivalent to Eq. (39) by using the fact that the tetrad is the covariant derivative of the tangent space coordinate x^a .

V. ANGULAR MOMENTUM CONSERVATION

Let us consider now a general matter field Ψ with the action integral

$$S = \frac{1}{c} \int \mathcal{L} d^4x \equiv \frac{1}{c} \int L h d^4x, \quad (44)$$

where $h = \det(h^a{}_\mu) = \sqrt{-g}$, with $g = \det(g_{\mu\nu})$. We assume a first-order formalism, according to which the Lagrangian depends only on the fields and on their first derivatives. Under a local Lorentz transformation of the tangent-space coordinates, both $\mathring{A}^a{}_{b\mu}$ and $h^a{}_\mu$ will change. The transformation of the spin connection $\mathring{A}^a{}_{b\mu}$ is generated by the spin part of the Lorentz generators, whereas the transformation of the tetrad $h^a{}_\mu$ is generated by both the spin and the orbital parts.

Let us consider first the response of the action integral due to the change of the Lorentz gauge potential $\mathring{A}^{ab}{}_\mu$. As a Lorentz scalar, the action integral is invariant under a local Lorentz transformation generated by S_{ab} . Under such a transformation, it changes according to

$$\delta S = \frac{1}{2c} \int \left[\frac{\partial \mathcal{L}}{\partial \mathring{A}^{ab}{}_\mu} - \partial_\rho \frac{\partial \mathcal{L}}{\partial \partial_\rho \mathring{A}^{ab}{}_\mu} \right] \delta_S \mathring{A}^{ab}{}_\mu d^4x, \quad (45)$$

where we have not written the variation in relation to the field Ψ because it gives the associated field equation [13]. Introducing the notation

$$\left[\frac{\partial \mathcal{L}}{\partial \mathring{A}^{ab}{}_\mu} - \partial_\rho \frac{\partial \mathcal{L}}{\partial \partial_\rho \mathring{A}^{ab}{}_\mu} \right] \equiv \frac{\delta \mathcal{L}}{\delta \mathring{A}^{ab}{}_\mu} = h \mathcal{J}^{\mu ab}, \quad (46)$$

where $\mathcal{J}^{\mu ab}$ is the angular momentum tensor, it follows that

$$\delta S = \frac{1}{2c} \int \mathcal{J}^{\mu ab} \delta_S \mathring{A}^{ab}{}_\mu h d^4x. \quad (47)$$

Substituting the transformation (37), integrating by parts, and neglecting the surface term, we obtain

$$\delta S = -\frac{1}{2c} \int \overset{\circ}{D}_\mu (h \mathcal{J}^\mu_{ab}) \epsilon^{ab} d^4x. \quad (48)$$

Due to the arbitrariness of ϵ^{ab} , it follows from the invariance of the action integral under local Lorentz transformations that

$$\overset{\circ}{D}_\mu (h \mathcal{J}^\mu_{ab}) = 0. \quad (49)$$

Using the identity

$$\partial_\mu h = h \overset{\circ}{\Gamma}^\lambda_{\lambda\mu}, \quad (50)$$

with $\overset{\circ}{\Gamma}^\lambda_{\lambda\mu} = \overset{\circ}{\Gamma}^\lambda_{\mu\lambda}$ the Levi-Civita connection of the metric (25), we get

$$\partial_\mu \mathcal{J}^\mu_{ab} + \overset{\circ}{\Gamma}^\mu_{\lambda\mu} \mathcal{J}^\lambda_{ab} - \overset{\circ}{A}^c_{a\mu} \mathcal{J}^\mu_{cb} - \overset{\circ}{A}^c_{b\mu} \mathcal{J}^\mu_{ac} = 0, \quad (51)$$

which is the usual covariant conservation law of the angular momentum tensor in general relativity. According to this construction, therefore, we see that the angular momentum conservation is related to the response of the action integral under a Lorentz transformation of the spin connection, which is a transformation generated by the spin generator S_{ab} .

VI. ENERGY-MOMENTUM CONSERVATION

The angular momentum tensor can be rewritten in the form

$$\mathcal{J}^\mu_{ab} = -T^p_c \frac{\delta h^c_\rho}{\delta \overset{\circ}{A}^{ab}_\mu}, \quad (52)$$

where

$$T^p_c = -\frac{1}{h} \frac{\delta \mathcal{L}}{\delta h^c_\rho} \equiv \frac{\partial \mathcal{L}}{\partial h^c_\rho} - \partial_\lambda \frac{\partial \mathcal{L}}{\partial \partial_\lambda h^c_\rho} \quad (53)$$

is the energy-momentum tensor. From Eq. (23), we see that

$$\frac{\delta h^c_\rho}{\delta \overset{\circ}{A}^{ab}_\mu} = \delta^\mu_\rho (\delta^c_{ax} x_b - \delta^c_{bx} x_a).$$

Therefore, Eq. (52) becomes

$$\mathcal{J}^\mu_{ab} = x_a T^\mu_b - x_b T^\mu_a, \quad (54)$$

which is the usual expression of the *total* angular momentum tensor in terms of the symmetric energy-momentum tensor [14]. Reversing the argument, we can say that the usual relation between \mathcal{J}^μ_{ab} and T^μ_a requires a tetrad of the form (23).

Substituting now Eq. (52) in the transformation (47), it follows that

$$\delta S = -\frac{1}{2c} \int T^p_c \frac{\delta h^c_\rho}{\delta \overset{\circ}{A}^{ab}_\mu} \delta_S \overset{\circ}{A}^{ab}_\mu h d^4x, \quad (55)$$

or equivalently,

$$\delta S = -\frac{1}{c} \int T^p_c \delta_S h^c_\rho h d^4x. \quad (56)$$

Substituting $\delta_S h^c_\rho$ as given by Eq. (43), integrating both terms by parts and neglecting the corresponding surface terms, we obtain

$$\delta S = \frac{1}{c} \int [\overset{\circ}{D}_\mu (h \mathcal{J}^\mu_{ab}) \epsilon^{ab} - \overset{\circ}{D}_\mu (h T^\mu_a) \xi^a] d^4x. \quad (57)$$

Using the fact that the angular momentum is covariantly conserved, we get

$$\delta S = -\frac{1}{c} \int \overset{\circ}{D}_\mu (h T^\mu_a) \xi^a d^4x. \quad (58)$$

Due to the arbitrariness of ξ^a , it follows from the invariance of the action integral under a local Lorentz transformation that

$$\overset{\circ}{D}_\mu (h T^\mu_a) = 0. \quad (59)$$

Using the identity (50), this expression can be rewritten in the form

$$\partial_\mu T^\mu_a + \overset{\circ}{\Gamma}^\mu_{\lambda\mu} T^\lambda_a - \overset{\circ}{A}^c_{a\mu} T^\mu_c = 0, \quad (60)$$

which is the usual covariant conservation law of general relativity.

It is important to notice that the energy-momentum covariant conservation in this case turns out to be related to the response of the action integral under a transformation of the tetrad field generated by the orbital generators L_{ab} , which as we have already seen are transformations that can be reinterpreted as translations. In fact, after integrating (back) by parts and neglecting the surface term, Eq. (58) can be rewritten in the form

$$\delta S = \frac{1}{c} \int T^\mu_a \delta_L h^a_\mu h d^4x, \quad (61)$$

which holds provided the angular momentum \mathcal{J}^μ_{ab} is covariantly conserved. Furthermore, it is easy to see that $\delta_L h^a_\mu$, given by Eq. (42), induces in the metric tensor (25) the transformation

$$\delta_L g_{\mu\nu} = -\overset{\circ}{\nabla}_\mu \xi_\nu - \overset{\circ}{\nabla}_\nu \xi_\mu, \quad (62)$$

where $\xi_\mu = \xi_a h^a_\mu$, and $\overset{\circ}{\nabla}_\mu$ is the Levi-Civita covariant derivative. As is well known, this equation represents the response of $g_{\mu\nu}$ to a general transformation of the spacetime coordinates, and its use in the Noether theorem yields the covariant conservation law of the matter energy-momentum tensor in the usual context of general relativity [13].

VII. FINAL REMARKS

According to the Noether theorems, energy-momentum conservation is related to the invariance of the action integral under spacetime translations, and angular momentum conservation is related to the invariance of the action integral under spacetime Lorentz transformations. However, as is well known, in the presence of gravitation spacetime becomes a (pseudo) Riemannian manifold. As the above transformations cannot be defined on such spacetimes [2], it is necessary to introduce a local procedure in which the corresponding covariant conservation laws can be obtained from the invariance of the action integral under transformations taking place in the Minkowski tangent space, where they are well defined.

By considering general relativity as a gauge theory for the Lorentz group, where the spin connection—that is, the Lorentz gauge potential—is the fundamental field representing gravitation, we have shown that it is possible to obtain the angular momentum and the energy-momentum covariant conservation laws from the invariance of the action integral under transformations taking place in the tangent space. The crucial point of this formalism is the Lorentz covariant derivative (22), in which the action of the orbital Lorentz generators reduces to a translation, giving then rise to a *translational* gauge potential $c^{-2}B^a{}_{\mu} = \overset{\circ}{A}{}^a{}_{b\mu}x^b$ that appears as the nontrivial part of the tetrad field:

$$h^a{}_{\mu} = \partial_{\mu}x^a + \overset{\circ}{A}{}^a{}_{b\mu}x^b. \quad (63)$$

We remark that this constraint between $h^a{}_{\mu}$ and $\overset{\circ}{A}{}^a{}_{b\mu}$ yields naturally the usual relation, given by Eq. (54), between the energy-momentum and the angular momentum tensors, showing in this way the consistency of the tetrad (63). In this approach, the covariant conservation law of the angular momentum tensor turns out to be related to the response of the action integral under Lorentz transformations of the spin connection $\overset{\circ}{A}{}^a{}_{b\mu}$, which is a transformation generated by the spin part of the Lorentz generators. On the other hand, the

energy-momentum conservation turns out to be related to the response of the action integral under a Lorentz transformation of the tetrad field. Differently from $\overset{\circ}{A}{}^a{}_{b\mu}$, the tetrad field $h^a{}_{\mu}$ responds simultaneously to both the spin and the orbital Lorentz generators. The part related to spin generator S_{ab} yields again the conservation of the angular momentum tensor, written now in the form (54). The part related to the orbital generator L_{ab} yields the conservation of the energy-momentum tensor, a result consistent with the fact that the Lorentz transformation generated by L_{ab} can always be reduced to a translation. In fact, the tetrad transformation generated by L_{ab} , given by Eq. (42), induces in the metric tensor $g_{\mu\nu}$ the transformation (62), which is the usual transformation of $g_{\mu\nu}$ under a general transformation of the spacetime coordinates, and which yields the covariant conservation law of the matter energy-momentum tensor in the usual context of general relativity. We have in this way established a relation between spacetime diffeomorphisms and tangent space Lorentz transformations generated by the orbital generator L_{ab} . This is a crucial result in the sense that it is responsible for obtaining the covariant conservation law for the energy-momentum tensor under transformations taking place in the tangent space. We notice in passing that even in the tetrad approach to general relativity, as the tetrad is *invariant* under a *true translation* of the tangent space coordinates, no energy-momentum covariant conservation law can be obtained. Summing up, with this construction we have succeeded in obtaining an internal Noether theorem from which the covariant conservation laws for angular momentum and energy-momentum tensors are obtained from the invariance of the action integral under “internal”—that is, tangent space—transformations. Accordingly, the associated densities can be considered as “gauge” currents.

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- [1] See, for example, N.P. Konopleva and V.N. Popov, *Gauge Fields* (Harwood, Chur, 1981).
 - [2] See, for example, R.M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
 - [3] A. Trautman, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).
 - [4] M.J.G. Veltman, in *Methods in Field Theory*, Les Houches 1975, edited by R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976).
 - [5] M. Calçada and J.G. Pereira, *Int. J. Theor. Phys.* **41**, 729 (2002).
 - [6] For a general description of the gauge approach to gravitation, see F.W. Hehl, J.D. McCrea, E.W. Mielke, and Y. Ne’emann, *Phys. Rep.* **258**, 1 (1995).
 - [7] The first attempt to construct a gauge theory for the Lorentz group was made in R. Utiyama, *Phys. Rev.* **101**, 1597 (1956).
 - [8] P.A.M. Dirac, in *Planck Festschrift*, edited by W. Frank (Deutscher Verlag der Wissenschaften, Berlin, 1958).
 - [9] See, for example, P. Ramond, *Field Theory: A Modern Primer*, 2nd ed. (Addison-Wesley, Redwood City, CA, 1989).
 - [10] T.W.B. Kibble, *J. Math. Phys.* **2**, 212 (1961).
 - [11] See, for example, R. Aldrovandi and J.G. Pereira, *An Introduction to Geometrical Physics* (World Scientific, Singapore, 1995).
 - [12] V.A. Fock and D. Ivanenko, *Z. Phys.* **54**, 798 (1929); V.A. Fock, *ibid.* **57**, 261 (1929).
 - [13] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975).
 - [14] See, for example, S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), p. 46; see also K. Hayashi, *Lett. Nuovo Cimento Soc. Ital. Fis.* **5**, 529 (1972).