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Classical integrable $N = 1$ and $N = 2$ super sinh-Gordon models with jump defects

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Abstract. The structure of integrable field theories in the presence of jump defects is discussed in terms of boundary functions under the Lagrangian formalism. Explicit examples of bosonic and fermionic theories are considered. In particular, the boundary functions for the $N = 1$ and $N = 2$ super sinh-Gordon models are constructed and shown to generate the Backlund transformations for its soliton solutions. As a new and interesting example, a solution with an incoming boson and an outgoing fermion for the $N = 1$ case is presented. The resulting integrable models are shown to be invariant under supersymmetric transformation.

1. Introduction

The classical Lagrangian formulation of a class of relativistic integrable field theories admiting certain discontinuities (defects) has been studied recently [1] - [3]. In particular, in ref. [1] the authors have considered a field theory in which different soliton solutions of the sine-Gordon model are linked in such a way that the integrability is preserved. The integrability of the total system imposes severe constraints specifying the possible types of defects. These are characterized by Backlund transformations which are known to connect two different soliton solutions.

The presence of the defect indicates the breakdown of space isotropy and henceforth of momentum conservation. The key ingredient to classify integrable defects is to impose certain first order differential relations between the different solutions (Backlund transformation). This introduces certain boundary functions (BF) which are specific of each integrable model and leads to the conservation of the total momentum.

Here, we analize the structure of the possible boundary terms for various cases. We first consider, for pedagogical purposes, the pure bosonic case studied in refs. [1] - [3] and derive, according to them, the border functions by imposing conservation of the total momentum. Next, we consider a pure fermionic theory and propose Backlund transformation in terms of an auxiliary fermionic non local field. Such structure is then generalized to include both bosonic and fermionic fields. In particular we construct boundary functions for the $N = 1$ supersymmetric sinh-Gordon model and show that it leads to the Backlund transformation proposed in [4].

We extend the formalism to the $N = 2$ supersymmetric sinh-Gordon model [5]. This system was originally proposed in [6] and [7] (see also [8]). Later in [9] and [10] a systematic algebraic aproach was developed.

2. General formalism: bosonic case

In this section we introduce the Lagrangian approach proposed in [1]. Consider a system described by

$$\mathcal{L} = \theta(-x)\mathcal{L}_{p=1} + \theta(x)\mathcal{L}_{p=2} + \delta(x)\mathcal{L}_D, \tag{1}$$

where $\mathcal{L}_p(\phi_p, \partial_\mu \phi_p) = \frac{1}{2}(\partial_x \phi_p)^2 - \frac{1}{2}(\partial_t \phi_p)^2 - V(\phi_p)$ describes a set of fields denoted by ϕ_1 for $x < 0$ and ϕ_2 for $x > 0$. A defect placed at $x = 0$, is described by

$$\mathcal{L}_D = \frac{1}{2}(\phi_2 \partial_t \phi_1 - \phi_1 \partial_t \phi_2) + B_0 \tag{2}$$

where B_0 is the border function. The equations of motion are therefore given by

$$\begin{aligned} \partial_x^2 \phi_1 - \partial_t^2 \phi_1 &= \partial_{\phi_1} V(\phi_1), \qquad x < 0 \\ \partial_x^2 \phi_2 - \partial_t^2 \phi_2 &= \partial_{\phi_2} V(\phi_2), \qquad x > 0 \end{aligned} \tag{3}$$

and $x = 0$,

$$\begin{aligned} \partial_x \phi_1 - \partial_t \phi_2 &= -\partial_{\phi_1} B_0, \\ \partial_x \phi_2 - \partial_t \phi_1 &= \partial_{\phi_2} B_0, \end{aligned} \tag{4}$$

The momentum is

$$P = \int_{-\infty}^{0} \partial_x \phi_1 \partial_t \phi_1 + \int_0^\infty \partial_x \phi_2 \partial_t \phi_2. \tag{5}$$

Acting with time derivative and inserting eqns. of motion (3) we find

$$\begin{aligned} \frac{dP}{dt} &= \int_{-\infty}^{0} (\frac{1}{2}\partial_x(\partial_t \phi_1)^2 + \frac{1}{2}\partial_x(\partial_x \phi_1)^2 - \partial_x \phi_1 \frac{\delta V_1}{\delta \phi_1})dx \\ &+ \int_0^\infty (\frac{1}{2}\partial_x(\partial_t \phi_2)^2 + \frac{1}{2}\partial_x(\partial_x \phi_2)^2 - \partial_x \phi_2 \frac{\delta V_2}{\delta \phi_2})dx \end{aligned} \tag{6}$$

Using eqns. (4) after integration, we find

$$\frac{dP}{dt} = [-\frac{\partial B_0}{\partial \phi_+}\dot{\phi}_+ + \frac{\partial B_0}{\partial \phi_-}\dot{\phi}_- + \frac{1}{2}(\frac{\partial B_0}{\partial \phi_1})^2 - \frac{1}{2}(\frac{\partial B_0}{\partial \phi_2})^2 - V_1 + V_2]|_{x=0} \tag{7}$$

where $\phi_\pm = \phi_1 \pm \phi_2$ and $\dot{\phi}_\pm = \partial_t \phi_\pm$. If the border function factorizes into $B_0 = B_0^+(\phi_+) + B_0^-(\phi_-)$, the modified momentum $\mathcal{P} = P + \left(B_0^+ - B_0^-\right)|_{x=0}$ is conserved provided B_0 satisfies its defining condition, i.e.,

$$[\frac{1}{2}(\frac{\partial B_0}{\partial \phi_1})^2 - \frac{1}{2}(\frac{\partial B_0}{\partial \phi_2})^2 - V_1 + V_2]|_{x=0} = 0 \tag{8}$$

Let us illustrate the above structure by first considering the free massive bosonic theory for which $V_p = \frac{1}{2}m^2\phi_p^2$.

$$\frac{1}{2}(\frac{\partial B_0}{\partial \phi_1})^2 - \frac{1}{2}(\frac{\partial B_0}{\partial \phi_2})^2 = \frac{m^2}{2}(\phi_1^2 - \phi_2^2) = \frac{m^2}{2}\phi_+\phi_- \tag{9}$$

The solution is easely found to be

$$B_0 = -\frac{m\beta^2}{4}\phi_-^2 - \frac{m}{4\beta^2}\phi_+^2 \tag{10}$$

and β^2 denotes a free (spectral) parameter.

As second example, consider the sinh-Gordon model for which $V_p = 4m^2\cosh(2\phi_p)$. The defining eqn. (8) indicates the natural decomposition

$$\begin{aligned}\frac{1}{2}(\frac{\partial B_0}{\partial\phi_1})^2 - \frac{1}{2}(\frac{\partial B_0}{\partial\phi_2})^2 &= 4m^2(\cosh(2\phi_1) - \cosh(2\phi_2)) \\ &= 8m^2\sinh(\phi_+)\sinh(\phi_-)\end{aligned} \tag{11}$$

yielding

$$B_0 = -m\beta^2\cosh(\phi_-) - \frac{4m}{\beta^2}\cosh(\phi_+) \tag{12}$$

and hence we rederive from (4) the Backlund transformation for the sinh-Gordon model

$$\begin{aligned}\partial_x\phi_1 - \partial_t\phi_2 &= m\beta^2\sinh(\phi_-) + \frac{4m}{\beta^2}\sinh(\phi_+), \\ \partial_x\phi_2 - \partial_t\phi_1 &= m\beta^2\sinh(\phi_-) - \frac{4m}{\beta^2}\sinh(\phi_+),\end{aligned} \tag{13}$$

3. Fermions and the $N = 1$ super sinh-Gordon model

Before discussing the Super sinh-Gordon Model let us consider the pure fermionic prototype described by the Lagrangian density

$$\mathcal{L}_p = \bar{\psi}_p\partial_t\bar{\psi}_p - \bar{\psi}_p\partial_x\bar{\psi}_p + \psi_p\partial_t\psi_p + \psi_p\partial_x\psi_p + W(\psi_p, \bar{\psi}_p) \tag{14}$$

where, for the free fermionic theory, $W(\psi_p, \bar{\psi}_p) = 2m\bar{\psi}_p\psi_p$. For the half line, $x < 0$ or $x > 0$ the equations of motion are given by

$$\partial_x\psi_p + \partial_t\psi_p = -\frac{1}{2}\partial_{\psi_p}W_p, \qquad \partial_x\bar{\psi}_p - \partial_t\bar{\psi}_p = \frac{1}{2}\partial_{\bar{\psi}_p}W_p \tag{15}$$

according to $p = 1$ or 2 respectively.

Let us propose the following Backlund transformation

$$\psi_1 + \psi_2 = -i\beta\sqrt{m}f_1 = \partial_{\psi_1}B_1, \qquad \bar{\psi}_1 - \bar{\psi}_2 = -\frac{2i\sqrt{m}}{\beta}f_1 = -\partial_{\bar{\psi}_1}B_1, \tag{16}$$

where f_1 satisfies

$$\begin{aligned}\dot{f}_1 &= -\frac{i\beta\sqrt{m}}{2}(\psi_1 - \psi_2) + \frac{i}{2\beta}\sqrt{m}(\bar{\psi}_1 + \bar{\psi}_2) = -\frac{1}{4}\partial_{f_1}B_1, \\ \partial_x f_1 &= \frac{i\beta\sqrt{m}}{2}(\psi_1 - \psi_2) + \frac{i}{2\beta}\sqrt{m}(\bar{\psi}_1 + \bar{\psi}_2)\end{aligned} \tag{17}$$

written in terms of a border function $B_1 = B_1(\bar{\psi}_1, \bar{\psi}_2, \psi_1, \psi_2, f_1)$, which now, due to the Grassmanian character of the fermions, depends upon the non local fermionic field f_1. By considering the Lagrangian system (1) with $\mathcal{L}_p$ given by (14) and

$$\mathcal{L}_D = -\psi_1\psi_2 - \bar{\psi}_1\bar{\psi}_2 + 2f_1\partial_t f_1 + B_1(\bar{\psi}_1, \bar{\psi}_2, \psi_1, \psi_2, f_1) \tag{18}$$

construct the momentum to be

$$P = \int_{-\infty}^{0}(-\bar{\psi}_1\partial_x\bar{\psi}_1 - \psi_1\partial_x\psi_1)dx + \int_{0}^{\infty}(-\bar{\psi}_2\partial_x\bar{\psi}_2 - \psi_2\partial_x\psi_2)dx \tag{19}$$

In analising its conservation, we find

$$\frac{dP}{dt} = [-W_1 + W_2 - \bar{\psi}_1\partial_t\bar{\psi}_1 + \bar{\psi}_2\partial_t\bar{\psi}_2 - \psi_1\partial_t\psi_1 + \psi_2\partial_t\psi_2]|_{x=0} \tag{20}$$

after using equations of motion (15) to eliminate time derivatives and integrating over x. Using the Backlund transformation (16) and (17), eqn. (20) becomes

$$\begin{aligned}\frac{dP}{dt} &= -[W_1 - W_2 - \partial_{\bar{\psi}_1}B_1\dot{\bar{\psi}}_1 + \partial_{\psi_1}B_1\dot{\psi}_1 \\ &- \partial_{\bar{\psi}_2}B_1\dot{\bar{\psi}}_2 + \partial_{\psi_2}B_1\dot{\psi}_2 + \partial_t(\bar{\psi}_2\bar{\psi}_1) - \partial_t(\psi_2\psi_1)]|_{x=0}\end{aligned} \tag{21}$$

If we assume the border function to decompose as $B_1 = B_1^+(\bar{\psi}_+, f_1) + B_1^-(\psi_-, f_1)$, where $\bar{\psi}_\pm = \bar{\psi}_1 \pm \bar{\psi}_2, \quad \psi_\pm = \psi_1 \pm \psi_2$, the modified momentum

$$\mathcal{P} = P + [\bar{\psi}_2\bar{\psi}_1 - \psi_2\psi_1 + B_1^+ - B_1^-]_{x=0} \tag{22}$$

is conserved provided

$$[W_2 - W_1 - \partial_{f_1}B_1^-\dot{f}_1 + \partial_{f_1}B_1^+\dot{f}_1]|_{x=0} = 0 \tag{23}$$

For the free fermi fields system (14)-(15) eqn. (23) becomes

$$\frac{1}{2}(\partial_{f_1}B_1^+)(\partial_{f_1}B_1^-) = 2m(\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2) \tag{24}$$

The solution is

$$B_1 = -\frac{2i}{\beta}\sqrt{m}f_1\bar{\psi}_+ + i\beta\sqrt{m}f_1\psi_- \tag{25}$$

Let us now consider the super sinh-Gordon model described by

$$\begin{aligned}\mathcal{L}_p &= \frac{1}{2}(\partial_x\phi_p)^2 - \frac{1}{2}(\partial_t\phi_p)^2 + \bar{\psi}_p\partial_t\bar{\psi}_p - \bar{\psi}_p\partial_x\bar{\psi}_p + \psi_p\partial_t\psi_p \\ &+ \psi_p\partial_x\psi_p + V(\phi_p) + W(\phi_p, \psi_p, \bar{\psi}_p)\end{aligned} \tag{26}$$

and

$$\begin{aligned}\mathcal{L}_D &= \frac{1}{2}(\phi_2\partial_t\phi_1 - \phi_1\partial_t\phi_2) - \psi_1\psi_2 - \bar{\psi}_1\bar{\psi}_2 + 2f_1\partial_t f_1 \\ &+ B_0(\phi_1, \phi_2) + B_1(\phi_1, \phi_2, \bar{\psi}_1, \bar{\psi}_2, \psi_1, \psi_2, f_1)\end{aligned} \tag{27}$$

where $V_p = 4m^2\cosh(2\phi_p)$ and $W_p = 8m\bar{\psi}_p\psi_p\cosh(\phi_p)$.

Propose the following Backlund transformation [11],

$$\begin{aligned}&\partial_x\phi_1 - \partial_t\phi_2 = -\partial_{\phi_1}(B_0 + B_1), \qquad \partial_x\phi_2 - \partial_t\phi_1 = \partial_{\phi_2}(B_0 + B_1) \\ &\psi_1 + \psi_2 = \partial_{\psi_1}B_1, \quad \bar{\psi}_1 - \bar{\psi}_2 = -\partial_{\bar{\psi}_2}B_1, \quad \dot{f}_1 = -\frac{1}{4}\partial_{f_1}B_1\end{aligned} \tag{28}$$

Assuming the decomposition

$$B_0 = B_0^+(\phi_+) + B_0^-(\phi_-), \qquad B_1 = B_1^+(\phi_+, \bar{\psi}_+, f_1) + B_1^-(\phi_-, \psi_-, f_1) \tag{29}$$

we find that the modified momentum

$$\mathcal{P} = P + [B_0^+(\phi_+) - B_0^-(\phi_-) + B_1^+(\phi_+, \bar{\psi}_+, f_1) - B_1^-(\phi_-, \psi_-, f_1) - \bar{\psi}_1\bar{\psi}_2 + \psi_1\psi_2]_{x=0} \tag{30}$$

is conserved provided the border functions B_0 and B_1 satisfy

$$\begin{aligned} W_1 - W_2 &= \frac{1}{2}(\partial_{f_1} B_1^+)(\partial_{f_1} B_1^-) + 2(\partial_{\phi_+} B_0^+)(\partial_{\phi_-} B_1^-) + 2(\partial_{\phi_-} B_0^-)(\partial_{\phi_+} B_1^+) \\ V_1 - V_2 &= \frac{1}{2}(\frac{\partial B_0}{\partial \phi_1})^2 - \frac{1}{2}(\frac{\partial B_0}{\partial \phi_2})^2 \end{aligned} \tag{31}$$

The solution for (31) is found to be

$$\begin{aligned} B_0 &= -m\beta^2 \cosh(\phi_-) - \frac{4m}{\beta^2}\cosh(\phi_+), \\ B_1 &= -\frac{4i}{\beta}\sqrt{m}\cosh(\frac{\phi_+}{2}) f_1 \bar{\psi}_+ + 2i\beta\sqrt{m}\cosh(\frac{\phi_-}{2}) f_1 \psi_- \end{aligned} \tag{32}$$

The boundary functions B_0 and B_1 in (32) generate the Backlund transformation which agrees with the one proposed in [4] for the $N = 1$ super sinh-Gordon model. The equations of motion obtained from (26) and (27) are verified to be invariant under the supersymmetry transformation

$$\delta\bar{\psi}_p = \epsilon\partial_z\phi_p, \quad \delta\phi_p = \epsilon\bar{\psi}_p, \quad \delta\psi_p = 2\epsilon m \sinh\phi_p \tag{33}$$

where $\partial_z = 1/2(\partial_x + \partial_t)$.

In ref. [12] the general soliton solutions for the $N = 1$ super sinh-Gordon model were constructed using vertex functions techniques and in [11] different solutions were analysed in the context of the Backlund transformation. Besides those examples discussed in [11], a new and interesting case containing *an incoming boson and an outcoming fermion* is given when $\phi_1 \neq 0, \psi_1 = \bar{\psi}_1 = 0$ and $\phi_2 = 0, \psi_2, \bar{\psi}_2 \neq 0$. Under such conditions the Backlund transformation (28) reads,

$$\begin{aligned} \partial_x\phi_1 &= -2(\sigma + \frac{1}{\sigma})\sinh\phi_1, \qquad \partial_t\phi_1 = -2(\sigma - \frac{1}{\sigma})\sinh\phi_1, \\ \psi_2 &= 2\sqrt{\frac{2}{\sigma}}\cosh\left(\frac{\phi_1}{2}\right) f_1, \qquad \bar{\psi}_2 = 2\sqrt{2\sigma}\cosh\left(\frac{\phi_1}{2}\right) f_1, \\ \partial_t f_1 &= 2(\sigma - \frac{1}{\sigma})\cosh^2\left(\frac{\phi_1}{2}\right) f_1, \qquad \partial_x f_1 = 2(\sigma + \frac{1}{\sigma})\cosh^2\left(\frac{\phi_1}{2}\right) f_1 \end{aligned} \tag{34}$$

where we took $m = 1$ and $\sigma = -\frac{2}{\beta^2}$. As solution of eqns. (34) we find

$$\phi_1 = ln\left(\frac{1 + \frac{b_1}{2}\rho}{1 - \frac{b_1}{2}\rho}\right), \qquad f_1 = \epsilon\rho^{-1}\sqrt{1 - \frac{b_1^2}{4}\rho^2} \tag{35}$$

where $\rho = \exp\left(-2\sigma(x+t) - \frac{2}{\sigma}(x-t)\right)$, ϵ is a grassmaniann constant and ψ_2 and $\bar{\psi}_2$ are obtained from the second eqn. (34) after inserting the above ϕ_1 and f_1.

The integrability of the model is verified by construction of the Lax pair representation of the equations of motion. This is achieved by splitting the space into two overlapping regions, namely, $x \le b$ and $x \ge a$ with $a < b$ and defining a corresponding Lax pair within each of them. The integrability is ensured by the existence of a gauge transformation relating the two sets of Lax pairs within the overlapping region. In ref. [11] we have explicitly constructed such gauge transformation for the $N = 1$ super sinh-Gordon model in terms of the $SL(2,1)$ affine Lie algebra.

4. $N = 2$ super sinh-Gordon model

The Lagrangian density for the $N = 2$ Super sinh-Gordon Model is given by (see for instance [8])

$$\begin{aligned}
\mathcal{L}_p &= \frac{1}{2}(\partial_x\phi_p)^2 - \frac{1}{2}(\partial_t\phi_p)^2 + 2\bar{\psi}_p\partial_t\bar{\psi}_p + 2\bar{\psi}_p\partial_x\bar{\psi}_p + 2\psi_p\partial_t\psi_p - 2\psi_p\partial_x\psi_p \\
&- \frac{1}{2}(\partial_x\varphi_p)^2 + \frac{1}{2}(\partial_t\varphi_p)^2 - 2\bar{\chi}_p\partial_t\bar{\chi}_p - 2\bar{\chi}_p\partial_x\bar{\chi}_p - 2\chi_p\partial_t\chi_p + 2\chi_p\partial_x\chi_p \\
&- 16(\psi_p\bar{\psi}_p + \chi_p\bar{\chi}_p)\cosh\varphi_p\cosh\phi_p - 4\cosh(2\varphi_p) \\
&+ 16(\psi_p\bar{\chi}_p + \chi_p\bar{\psi}_p)\sinh\varphi_p\sinh\phi_p + 4\cosh(2\phi_p)
\end{aligned} \tag{36}$$

whose equations of motion are invariant under the supersymmetry transformations

$$\delta(\phi_p \pm \varphi_p) = 2(\psi_p \mp \chi_p)\epsilon_\pm, \quad \delta(\psi_p \pm \chi_p) = -\partial_z(\phi_p \mp \varphi_p)\epsilon_\pm \tag{37}$$

and

$$\delta(\bar{\psi}_p \pm \bar{\chi}_p) = 2\sinh(\phi_p \pm \varphi_p)\epsilon_\mp \tag{38}$$

Inspired from the $N = 1$ case (see eqn. (27)) we propose the following Lagrangian description for the defect

$$\begin{aligned}
\mathcal{L}_D &= \frac{1}{2}(\phi_2\partial_t\phi_1 - \phi_1\partial_t\phi_2) - 2\psi_1\psi_2 - 2\bar{\psi}_1\bar{\psi}_2 + f_1\partial_t f_2 \\
&- \frac{1}{2}(\varphi_2\partial_t\varphi_1 - \varphi_1\partial_t\varphi_2) + 2\chi_1\chi_2 + 2\bar{\chi}_1\bar{\chi}_2 + f_2\partial_t f_1 \\
&+ B_0(\phi_p, \varphi_p) + B_1(\phi_p, \varphi_p, \psi_p, \chi_p, \bar{\psi}_p, \bar{\chi}_p, f_1, f_2)
\end{aligned} \tag{39}$$

leading to the Backlund transformation

$$\begin{aligned}
\partial_x\phi_1 - \partial_t\phi_2 &= -\partial_{\phi_1}B, \\
\partial_x\phi_2 - \partial_t\phi_1 &= \partial_{\phi_2}B, \\
\partial_x\varphi_1 - \partial_t\varphi_2 &= \partial_{\varphi_1}B, \\
\partial_x\varphi_2 - \partial_t\varphi_1 &= -\partial_{\varphi_2}B, \\
\psi_1 - \psi_2 &= -\frac{1}{2}\partial_{\psi_1}B = -\frac{1}{2}\partial_{\psi_2}B \\
\chi_1 - \chi_2 &= \frac{1}{2}\partial_{\chi_1}B = \frac{1}{2}\partial_{\chi_2}B \\
\bar{\psi}_1 + \bar{\psi}_2 &= \frac{1}{2}\partial_{\bar{\psi}_1}B = -\frac{1}{2}\partial_{\bar{\psi}_2}B, \\
\bar{\chi}_1 + \bar{\chi}_2 &= -\frac{1}{2}\partial_{\bar{\chi}_1}B = \frac{1}{2}\partial_{\bar{\chi}_2}B, \\
\dot{f}_1 &= -\frac{1}{2}\partial_{f_2}B, \\
\dot{f}_2 &= -\frac{1}{2}\partial_{f_1}B
\end{aligned} \tag{40}$$

where $B = B_0 + B_1$. The canonical momentum

$$\begin{aligned} P &= \int_{-\infty}^{0} (\partial_x\phi_1\partial_t\phi_1 - 2\bar{\psi}_1\partial_x\bar{\psi}_1 - 2\psi_1\partial_x\psi_1 \\ &- \partial_x\varphi_1\partial_t\varphi_1 + 2\bar{\chi}_1\partial_x\bar{\chi}_1 + 2\chi_1\partial_x\chi_1)\,dx \\ &+ \int_{0}^{\infty} (\partial_x\phi_2\partial_t\phi_2 - 2\bar{\psi}_2\partial_x\bar{\psi}_2 - 2\psi_2\partial_x\psi_2 \\ &- \partial_x\varphi_2\partial_t\varphi_2 + 2\bar{\chi}_2\partial_x\bar{\chi}_2 + 2\chi_2\partial_x\chi_2)\,dx \end{aligned} \tag{41}$$

is no longer conserved. Proposing the modified momentum to be

$$\mathcal{P} = P + [B_0^{(+)} - B_0^{(-)} + B_1^{(+)} - B_1^{(-)} + 2\bar{\psi}_1\bar{\psi}_2 - 2\psi_1\psi_2 - 2\bar{\chi}_1\bar{\chi}_2 + 2\chi_1\chi_2]_{x=0}$$

which is conserved provided the border function satisfies

$$\partial_{\phi_+}B_0^{(+)}\partial_{\phi_-}B_0^{(-)} - \partial_{\varphi_+}B_0^{(+)}\partial_{\varphi_-}B_0^{(-)} = 4\sinh\phi_+\sinh\phi_- - 4\sinh\varphi_+\sinh\varphi_-, \tag{42}$$

and

$$\begin{aligned} &\partial_{\phi_+}B_0^{(+)}\partial_{\phi_-}B_1^{(-)} + \partial_{\phi_-}B_0^{(-)}\partial_{\phi_+}B_1^{(+)} - \partial_{\varphi_+}B_0^{(+)}\partial_{\varphi_-}B_1^{(-)} - \partial_{\varphi_-}B_0^{(-)}\partial_{\varphi_+}B_1^{(+)} \\ +\; &\partial_{\phi_+}B_1^{(+)}\partial_{\phi_-}B_1^{(-)} - \partial_{\varphi_+}B_1^{(+)}\partial_{\varphi_-}B_1^{(-)} - \frac{1}{2}(\partial_{f_1}B_1^{(-)}\partial_{f_2}B_1^{(+)} + \partial_{f_2}B_1^{(-)}\partial_{f_1}B_1^{(+)}) \\ =\; &-2(\psi_+\bar{\psi}_+ + \psi_-\bar{\psi}_- + \chi_+\bar{\chi}_+ + \chi_-\bar{\chi}_-)\Lambda_- - 2(\psi_+\bar{\psi}_- + \psi_-\bar{\psi}_+ + \chi_+\bar{\chi}_- + \chi_-\bar{\chi}_+)\Lambda_+ \\ +\; &2(\psi_+\bar{\chi}_+ + \psi_-\bar{\chi}_- + \chi_+\bar{\psi}_+ + \chi_-\bar{\psi}_-)\Delta_- + 2(\psi_+\bar{\chi}_- + \psi_-\bar{\chi}_+ + \chi_+\bar{\psi}_- + \chi_-\bar{\psi}_+)\Delta_+ \end{aligned} \tag{43}$$

where

$$\chi_\pm = \chi_1 \pm \chi_2, \qquad \bar{\chi}_\pm = \bar{\chi}_1 \pm \bar{\chi}_2, \qquad \varphi_\pm = \varphi_1 \pm \varphi_2, \cdots \text{etc}$$

and

$$\begin{aligned} \Lambda_\pm &= \cosh(\frac{\phi_+ + \phi_-}{2})\cosh(\frac{\varphi_+ + \varphi_-}{2}) \pm \cosh(\frac{\phi_+ - \phi_-}{2})\cosh(\frac{\varphi_+ - \varphi_-}{2}), \\ \Delta_\pm &= \sinh(\frac{\phi_+ + \phi_-}{2})\sinh(\frac{\varphi_+ + \varphi_-}{2}) \pm \sinh(\frac{\phi_+ - \phi_-}{2})\sinh(\frac{\varphi_+ - \varphi_-}{2}), \end{aligned}$$

The solution of (42) and (43) is

$$B_0^{(+)} = B_0^{(+)}(\phi_+, \varphi_+) = \frac{2\alpha_3}{\alpha_2}(\cosh\phi_+ - \cosh\varphi_+), \tag{44}$$

$$B_0^{(-)} = B_0^{(-)}(\phi_-, \varphi_-) = \frac{2\alpha_2}{\alpha_3}(\cosh\phi_- - \cosh\varphi_-), \tag{45}$$

and

$$\begin{aligned} B_1^{(+)} &= B_1^{(+)}(\phi_+, \varphi_+, \psi_+, \chi_+, f_1, f_2) \\ &= \frac{i}{\sqrt{2}}f_1\left(-\alpha_3(\psi_+ - \chi_+)\cosh\frac{1}{2}(\phi_+ + \varphi_+)\right) \\ &+ \frac{i}{\sqrt{2}}f_2\left(\frac{8}{\alpha_2}(\psi_+ + \chi_+)\cosh\frac{1}{2}(\phi_+ - \varphi_+)\right), \end{aligned} \tag{46}$$

$$
\begin{aligned}
B_1^{(-)} &= B_1^{(-)}(\phi_-, \varphi_-, \bar{\psi}_-, \bar{\chi}_-, f_1, f_2) \\
&= \frac{i}{\sqrt{2}} f_1 \left(\alpha_2 (\bar{\psi}_- + \bar{\chi}_-) \cosh \frac{1}{2}(\phi_- - \varphi_-) \right) \\
&+ \frac{i}{\sqrt{2}} f_2 \left(-\frac{8}{\alpha_3} (\bar{\psi}_- - \bar{\chi}_-) \cosh \frac{1}{2}(\phi_- + \varphi_-) \right)
\end{aligned} \tag{47}
$$

where α_2 and α_3 are arbitrary constants.

These results are consistent with the Backlund transformation for the $N = 2$ super sinh-Gordon obtained from a superfield formalism [5]. We should mention that the Backlund transformation (40) with B_0 and B_1 given by (44)-(47) are invariant under the supersymmetry transformation (37)-(38). In [5] some examples of Backlund solutions are discussed also.

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