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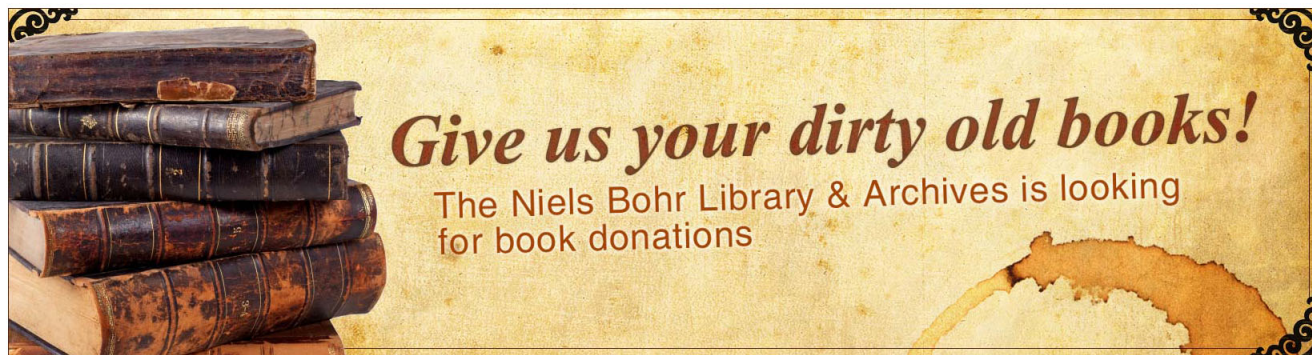
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The canonical structure of Podolsky's generalized electrodynamics on the null-plane

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In this work, we analyze the canonical structure of Podolsky's generalized electrodynamics on the null-plane. We show that the constraint structure presents a set of second-class constraints, which are exclusive of the analysis on the null-plane, and an expected set of first-class ones. An inspection on the field equations leads to the generalized radiation gauge on the null-plane. Dirac Brackets are then introduced by considering the problem of uniqueness under the chosen initial-boundary conditions of the theory. © 2011 American Institute of Physics. [doi:10.1063/1.3653510]

I. INTRODUCTION

Most physical systems, including fundamental fields in quantum field theory, are described by Lagrangians that depend at most on first-order derivatives. However, there is a continuous interest on theories with higher-order derivatives, either do accomplish generalizations or to get rid of some undesirable properties of first-order theories. This interest had begun in the half of the 19th century, when Ostrogradski¹ developed the generalized Hamiltonian formalism in classical mechanics.

As examples of systems treated by higher-order Lagrangians, we mention the attempts to solve the problem of renormalization of the gravitational field by inserting quadratic terms of the Riemann tensor and its contractions² on the Einstein-Hilbert action. Recent developments in this direction have been made by Cuzinatto *et al.*^{3,4} We may also highlight the recently discovered new massive gravity,⁵ which describes a unitary, parity-preserving second-order gravitational theory with massive gravitons in $2 + 1$ dimensions.

Higher-order Lagrangians have also emerged as effective theories on the infrared sector of the QCD,⁶ where it enforces a good asymptotic behavior of the gluon propagator. It is important to remark that the inclusion of higher-order derivatives in field theory of supersymmetric fields has shown to be a powerful regularization mechanism.⁷ A very attractive property of quantum field theories with higher-order terms is that of the improvement of the convergence of the corresponding Feynman diagrams.⁸

The first model of a higher-order derivative field theory is a generalization of the electromagnetic field proposed in the works of Podolsky, Schwed, and Bopp,⁹ which culminated in the Podolsky's generalized electrodynamics. It is suggested to modify the Maxwell-Lorentz theory in order to avoid divergences such as the electron self-energy and the vacuum polarization current. These difficulties can be traced to the fact that the classical electrodynamics involve an r^{-1} singularity that results in an infinite value of the electron self-energy. The Lagrangian density is, therefore, modified by a

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second-order derivative term:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}a^2\partial_\lambda F^{\mu\lambda}\partial^\nu F_{\mu\nu}. \quad (1)$$

Podolsky's theory already has many interesting features at the classical level. It solves the problem of infinite energy in the electrostatic case and also gives the correct expression for the self-force of charged particles at short distances, as showed by Frenkel,¹⁰ solving the problem of the singularity at $r \rightarrow 0$. It has been shown by Cuzinato *et al.*⁴ that the above Lagrangian density is the only possible generalization of the electromagnetic field that preserves invariance under $U(1)$. Besides, the theory yields field equations that are still linear in the fields.

Another important prediction of the model is the existence of massive photons, whose mass is proportional to the inverse of the Podolsky's parameter a . This feature allows experiments that may test the generalized electrodynamics as a viable effective theory. The determination of an upper bound value for the mass of the photon is actually a current concern both in the theoretical¹¹ and the experimental framework.¹²

The canonical quantization of the field was tried in the work of Podolsky and Schwed.⁹ However, Podolsky's theory inherits the same difficulties from the standard electromagnetic field, the presence of a degenerate variable, which forced them to use a Fermi-like Lagrangian. The chosen gauge fixing condition, the usual Lorenz condition, does not fulfill the requirements for a good choice of gauge in the context of Podolsky's theory. The first consistent approach to the quantization of the field was given by Galvão and Pimentel,¹³ where Dirac's canonical formalism¹⁴ is used with the correct choice of gauge.

The first attempts of quantization of Podolsky's field were made in instant-form, where the "laboratory time" $t = x^0$ is the evolution parameter of the theory. It is actually possible to define five different forms of Hamiltonian dynamics, each one related to different sub-groups of the Poincaré group.¹⁵ Other of the simplest form of dynamics is the front-form dynamics, which is the Hamiltonian dynamics of fields over a null-plane $x^0 + x^3 = cte$. The "time" or evolution parameter is chosen to be the coordinate $x^+ \equiv 1/\sqrt{2}(x^0 + x^3)$. This parameter choice was mistaken, for some time, to the so-called infinite-momentum frame,¹⁶ which is a limit process to analyze field theories in a frame near to the speed of light. The front-form dynamics implies a choice of coordinate system, not a physical reference frame, where the classical (quantum) evolution of the system is given by the definition of appropriate fundamental "equal-time" brackets (commutators), defined on a null-plane of constant x^+ , plus a special set of initial-boundary data.

In this paper, we study the Hamiltonian dynamics of the Podolsky's generalized electrodynamics on the null-plane. Since this theory is constrained, we may build a consistent canonical approach using the Dirac's Hamiltonian method. Our focus is to find the complete set of constraints of the theory, to build a consistent gauge fixing procedure and Dirac brackets, as well as to establish the complete equivalence between the Lagrangian and Hamiltonian approaches.

There are some good reasons, both physical and mathematical, to analyze relativistic fields on the null-plane. One of them is the fact that this kind of dynamics usually presents a set of second-class constraints not present in instant-form, which reduces the number of independent degrees of freedom necessary to describe such theories. This is closely related to the fact that the stability group of the Poincaré group in front-form, which is the sub-group of transformations that relates field configurations in a single surface $x^+ = cte$, has seven generators, one more than the six kinematic generators in instant-form. Besides, the algebra of these kinematic generators takes its simplest form in front-form dynamics. This is so because a boost in a fixed spatial direction is simply a diagonal scale transformation, which does not mix the coordinates x^+ and x^- . For some important systems, this feature is responsible for a complete separation of physical degrees of freedom, resulting in a clean and excitation-free quantum vacuum. This is actually verified, for example, in Yang-Mills,¹⁷ QCD,¹⁸ and spontaneous symmetry breaking models.¹⁹

The paper is organized as follows. In Sec. II, we discuss the null-plane coordinates, which are the natural coordinate system for the front-form dynamics, and we review the initial-boundary value problem for the fields, establishing appropriate conditions to achieve a unique solution of the dynamic equations on the null-plane. Section III will be devoted to a review on the Hamiltonian

formalism for higher-order Lagrangians. In Sec. IV, the canonical approach is applied for the generalized electromagnetic field in null-plane coordinates. In Sec. V, we establish a set of consistent gauge conditions and corresponding Dirac brackets to describe the physical dynamics of the theory. Section VI will be devoted for the final remarks.

II. THE NULL-PLANE COORDINATES

As the start point for the analysis of a second-order field theory, we have the action

$$S[\phi] \equiv \int_{\Omega} d\omega \mathcal{L}(\phi, \partial\phi, \partial^2\phi),$$

where \mathcal{L} is a Lagrangian density and $d\omega$ is a four-volume element of a four-volume Ω of the space-time. For relativistic theories, the Lagrangian density must be chosen to be invariant under any particular parameter choice. However, although the Lagrangian formalism preserves this invariance, the same does not occur in the Hamiltonian formalism, which requires a parameterization in order to be fully carried out.

Dirac has shown²⁰ that the usual dynamics, the instant-form, where the Galilean time $x^0 = t$ is the parameter that defines the evolution of the system from a given initial three-surface $\Gamma_{t=t_0}$ to a later surface $\Gamma_{t=t_1}$, is not the only possible choice of parameterization. He calls attention for two other forms of Hamiltonian dynamics: the punctual-form and the front-form. Later, two other forms were discovered.²¹

An important advantage pointed out by Dirac is the fact that seven of the ten Poincaré generators are kinematical on the null-plane, while the conventional theory constructed in instant-form has only six of these generators. Therefore, the structure of the phase space is distinct in both cases. As such, a description of the physical systems on the null-plane could give additional information from those provided by the conventional formalism.²² Another remarkable feature is that regular theories become constrained when analyzed on the null-plane. In general, it leads to a reduction in the number of independent field operators in the respective phase space due to the presence of second-class constraints.

The natural coordinate system of instant-form dynamics is the rectangular system $x^\mu \equiv (x^0, x^1, x^2, x^3)$. We can pass to null-plane coordinates with the linear transformation $x' = \Sigma x$ where the transformation matrix and its inverse are given by

$$\Sigma \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} \cdot \mathbf{I} & 0 \end{pmatrix}, \quad \Sigma^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \cdot \mathbf{I} \\ 1 & -1 & 0 \end{pmatrix},$$

where \mathbf{I} is the 2×2 identity matrix and $x'^\mu \equiv (x^+, x^-, x^1, x^2)$.

Lorentz tensors are covariant under this transformation, but the transformation itself is not of Lorentz type:²² if in usual coordinates, we define the Minkowski metric as $\eta \equiv \text{diag}(1, -1, -1, -1)$, the metric in null-plane coordinates will be given by

$$\eta' = \Sigma \eta \Sigma^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\mathbf{I} \end{pmatrix}.$$

Using this metric (from now on, we will ignore the comma), we can see that the norm of a vector is not a quadratic form, but will be linear in the longitudinal components.

Of special interest is the D'Alembertian operator

$$\square \equiv \partial_\mu \partial^\mu = 2\partial_- \partial_+ + \partial_i \partial^i. \quad (2)$$

Since the evolution parameter is x^+ , a field equation like $(\square + m^2)\phi = 0$ will be linear on the velocity $\partial_+ \phi$, which does not occur in instant-form. Therefore, the analysis of initial-boundary value problem is changed from a Cauchy to a characteristics initial-boundary value problem. This

is due to the fact that a quadratic Lagrangian on $\partial_0\phi$ is actually of first order on $\partial_+\phi$. In the case of the scalar field on the null-plane, it is sufficient to fix the values of the fields on both characteristics surfaces to solve the field equations.²³

This can be seen in Podolsky's case by the Euler-Lagrange (EL) equations of the Lagrangian (1):

$$(1 + a^2\Box)\Box A_\mu - \partial_\mu(1 + a^2\Box)\partial^\nu A_\nu = 0. \quad (3)$$

This equation is of fourth-order in $\partial_0 A_\mu$ but only second-order in $\partial_+ A_\mu$. Therefore, in instant-form, it is necessary to specify four conditions, the values of the field and its derivatives until third-order on an initial surface $x^0 = 0$ to uniquely write a solution.

On the null-plane, the equation is just of second-order, but the existence of two characteristics surfaces demands the knowledge of four initial-boundary conditions as well. The normal vector of a null-plane lies in the same plane; therefore, the knowledge of the value A_μ on a null-plane implies in its normal derivative $\partial_+ A_\mu$. Thus, the solution of the field equations is uniquely determined if A is specified on the null-plane $x^+ = cte$ and three boundary conditions are imposed on $x^- = cte$, which, in our case, consists on the value of the derivatives of the field up to third-order.

In the canonical framework, it was Steinhart²⁴ who showed that for linear Lagrangians, the initial condition on $x^+ = cte$ plus a Hamiltonian function are insufficient to predict uniquely all physical processes. Boundary conditions along the $x^- = cte$ plane must also be determined. He also observed that the matrix formed by the Poisson brackets (PBs) of the second-class constraints does not have a unique inverse and that the presence of arbitrary functions is associated with the insufficiency of the initial value data. It is also responsible for the existence of a hidden subset of first-class constraints, which is associated with improper gauge transformations.²⁵ By imposing appropriate initial-boundary conditions on the fields, the hidden first-class constraints can be eliminated in order to the total Hamiltonian be a true generator of the physical evolution. It will also determine a unique inverse of the second class constraint matrix that allows to obtain the correct Dirac brackets (DBs) among the fundamental variables. Thus, in the study of the Podolsky's theory, we follow the same tune outlined in Ref. 26.

III. SECOND-ORDER DERIVATIVES ON THE NULL-PLANE

Let us consider a generic Lagrangian density $\mathcal{L}(\phi, \partial\phi, \partial^2\phi)$ dependent on a number n of fields $\phi^a(x)$ and its first and second derivatives. The application of Hamilton's principle yields the following EL equations:

$$\frac{\delta\mathcal{L}}{\delta\phi^a} - \partial_\mu \left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)} \right] + \partial_\mu\partial_\nu \left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\partial_\nu\phi^a)} \right] = 0,$$

which is the equation that originates (3) from (1). Because the system is Poincaré invariant, the EL equations imply conservation of the symmetric energy-momentum tensor

$$\begin{aligned} T_{\mu\nu} \equiv & \partial_\mu\phi^a \frac{\delta\mathcal{L}}{\delta(\partial^\nu\phi^a)} - \mathcal{L}\eta_{\mu\nu} - 2\partial_\mu\phi^a\partial_\lambda \left[\frac{\delta\mathcal{L}}{\delta(\partial^\nu\partial_\lambda\phi^a)} \right] \\ & + \partial_\lambda \left[\partial_\mu\phi^a \frac{\delta\mathcal{L}}{\delta(\partial^\nu\partial_\lambda\phi^a)} \right] - \partial^\lambda (\Xi_{\mu\lambda\nu} + \Pi_{\mu\lambda\nu}), \end{aligned} \quad (4)$$

where

$$\begin{aligned} \Xi_{\mu\lambda\nu} \equiv & \frac{1}{2} \left[\frac{\delta\mathcal{L}}{\delta(\partial^\mu\phi^a)} - \partial_\nu \left(\frac{\delta\mathcal{L}}{\delta(\partial^\mu\partial_\nu\phi^a)} \right) \right] (\mathbf{I}_{\lambda\nu})^a_b \phi^b, \\ \Pi_{\mu\lambda\nu} \equiv & \frac{1}{2} \frac{\delta\mathcal{L}}{\delta(\partial^\mu\partial^\alpha\phi^a)} (\mathbf{I}_{\lambda\nu})^a_b \partial^\alpha\phi^b. \end{aligned}$$

$(\mathbf{I}_{\lambda\nu})^a_b$ are the infinitesimal generators of the Poincaré group.

The conserved charge is given by the expression

$$G \equiv -a^\mu P_\mu - \frac{1}{2} \omega^{\mu\nu} M_{\mu\nu}$$

with generators

$$P_\mu \equiv \int_\sigma d\sigma^\nu T_{\mu\nu},$$

$$M_{\mu\nu} \equiv \int_\sigma d\sigma^\alpha (T_{\alpha\mu} x_\nu - T_{\alpha\nu} x_\mu).$$

In the above expressions, σ is a three-surface orthogonal to the parameterization axis.

If we choose the null-plane, we will be interested in the dynamical generator P_+ , which is given by

$$P_+ \equiv \int d^3x T_{+-},$$

where we adopt $d^3x \equiv dx^- dx^1 dx^2$. Here, we have the canonical Hamiltonian density

$$\mathcal{H}_c \equiv T_{+-} = \left[\frac{\delta\mathcal{L}}{\delta(\partial_+\phi^a)} - \partial_+ \frac{\delta\mathcal{L}}{\delta(\partial_+\partial_+\phi^a)} - 2\partial_- \frac{\delta\mathcal{L}}{\delta(\partial_-\partial_+\phi^a)} - 2\partial_i \frac{\delta\mathcal{L}}{\delta(\partial_i\partial_+\phi^a)} \right] \partial_+\phi^a$$

$$+ \partial_+\partial_+\phi^a \frac{\delta\mathcal{L}}{\delta(\partial_+\partial_+\phi^a)} - \mathcal{L}. \quad (5)$$

This result suggests the following definition for the canonical momenta:

$$p_a \equiv \int d^3x \left[\frac{\delta\mathcal{L}}{\delta(\partial_+\phi^a)} - \partial_+ \frac{\delta\mathcal{L}}{\delta(\partial_+\partial_+\phi^a)} - 2\partial_- \frac{\delta\mathcal{L}}{\delta(\partial_-\partial_+\phi^a)} - 2\partial_i \frac{\delta\mathcal{L}}{\delta(\partial_i\partial_+\phi^a)} \right], \quad (6a)$$

$$\pi_a \equiv \int d^3x \frac{\delta\mathcal{L}}{\delta(\partial_+\partial_+\phi^a)}, \quad (6b)$$

where the fields ϕ and $\partial_+\phi$ are treated as independent canonical fields.

It is straightforward to show that the EL equations can be written by

$$W_{ab} \partial_+^4 \phi^b = F_a(\phi, \partial\phi, \partial^2\phi, \partial^3\phi)$$

where the generalized Hessian matrix is

$$W_{ab} \equiv \frac{\delta\pi_a}{\delta(\partial_+^2\phi^b)} = \int d^3x \frac{\delta\mathcal{L}}{\delta(\partial_+^2\phi^a)\delta(\partial_+^2\phi^b)}. \quad (7)$$

It is the regularity or the singularity of this matrix that determines the regularity or the singularity of the system.

In this analysis, we have ignored the boundary conditions of the fields, which is a quite misleading attitude, since the null-plane dynamics requires a different analysis of initial-boundary conditions than the instant-form dynamics. The discussion about the initial-boundary value problem in this case will be made properly during the canonical procedure, so at this point, we just make sure that the conditions of the fields are equivalent of those in instant-form, in other words, the fields and all required derivatives go to zero at the boundary of the three-surface.

IV. THE HAMILTONIAN ANALYSIS

From the Lagrangian density (1) and the definitions (6), follow the canonical momenta for the Podolsky's field

$$p^\mu = F^{\mu+} - a^2 [\eta^{\mu-} \partial_- \partial_\lambda F^{+\lambda} + \eta^{\mu i} \partial_i \partial_\lambda F^{+\lambda} - 2\partial_- \partial_\lambda F^{\mu\lambda}], \quad (8a)$$

$$\pi^\mu = a^2 \eta^{\mu+} \partial_\lambda F^{+\lambda}. \quad (8b)$$

The Hessian matrix of this system is

$$W^{\mu\nu} = \frac{\delta\pi^\mu}{\delta(\partial_+^2 A_\nu)} = -a^2\eta^{\mu+}\delta_\nu^+\eta^{++} = 0.$$

As we saw in the earlier section, the fields A_μ and $\partial_+ A_\mu$ should be treated as independent variables. Therefore, we will use the notation $\bar{A}_\mu \equiv \partial_+ A_\mu$, being A_μ and \bar{A}_μ independent fields. Then, we are able to define the primary constraints

$$\begin{aligned} \phi_1 &= \pi^+ \approx 0, \\ \phi_2^i &= \pi^i \approx 0, \\ \phi_3 &= p^+ - \partial_- \pi^- \approx 0, \\ \phi_4^i &= p^i - \partial_i \pi^- + F_{i-} + 2a^2 \partial_- [\partial_i \bar{A}_- - 2\partial_- \bar{A}_i + \partial_i \partial_- A_+ - \partial_j F_{ij}] \approx 0. \end{aligned}$$

The canonical Hamiltonian density can be expressed by

$$\begin{aligned} \mathcal{H}_c &= p^\mu \bar{A}_\mu + \pi^- (\partial_- \bar{A}_+ - \partial^i \bar{A}_i + \partial^i \partial_i A_+) + \frac{1}{4} F_{ij} F^{ij} - \frac{1}{2} (\bar{A}_- - \partial_- A_+)^2 \\ &\quad - (\bar{A}_i - \partial_i A_+) F_{-i} + \frac{1}{2} a^2 (\partial_i \bar{A}_- - 2\partial_- \bar{A}_i + \partial_i \partial_- A_+ - \partial_j F_{ij})^2. \end{aligned} \tag{9}$$

With the canonical Hamiltonian $H_c = \int d^3x \mathcal{H}_c(x)$ and the primary constraints, we build the primary Hamiltonian

$$H_P \equiv H_c + \int d^3x u^a(x) \phi_a(x), \quad \{a\} = \{1, 2, 3, 4\}. \tag{10}$$

To proceed with the calculus of the consistency conditions, we use the primary Hamiltonian as the generator of the x^+ evolution and define the fundamental equal x^+ Poisson Brackets with the expressions

$$\{A_\mu(x), p^\nu(y)\} = \{\bar{A}_\mu(x), \pi^\nu(y)\} = \delta_\mu^\nu \delta^3(x - y), \tag{11}$$

where $\delta^3(x - y) \equiv \delta(x^+ - y^+) \delta^2(\mathbf{x} - \mathbf{y})$. We verify that the condition $\dot{\phi}_1 \approx 0$ gives just the constraint $\phi_3 \approx 0$, which is already satisfied. The consistency for the remaining constraints gives equations for some Lagrange multipliers. Notice that the conditions for ϕ_2^i and ϕ_3 ,

$$\begin{aligned} \dot{\phi}_2^i &= -\phi_4^i + 4a^2 \partial_- \partial_- u_i^4 \approx 0, \\ \dot{\phi}_3 &= \partial_- p^- + \partial_i p^i + 4a^2 \partial_i \partial_- \partial_- u_i^4 \approx 0, \end{aligned}$$

give equations for the same parameters u_i^4 . These equations must be consistent to each other. From the first, we have $\partial_- \partial_- u_i^4 \approx 0$ and applying this result on the second condition, a secondary constraints appears:

$$\chi \equiv \partial_- p^- + \partial_i p^i \approx 0.$$

For this secondary constraint, $\dot{\chi} = 0$ and no more constraints can be found. The analysis leaves us with the following set:

$$\begin{aligned} \chi &= \partial_- p^- + \partial_i p^i \approx 0, \\ \phi_1 &= \pi^+ \approx 0, \\ \phi_2^i &= \pi^i \approx 0, \\ \phi_3 &= p^+ - \partial_- \pi^- \approx 0, \\ \phi_4^i &= p^i - \partial_i \pi^- + F_{i-} + 2a^2 \partial_- [\partial_i \bar{A}_- - 2\partial_- \bar{A}_i + \partial_i \partial_- A_+ - \partial_j F_{ij}] \approx 0. \end{aligned}$$

It happens that χ and ϕ_1 are first-class constraints, while ϕ_2^i , ϕ_3 , and ϕ_4^i are second-class ones. However, by constructing the matrix of the second-class constraints, we found that it is singular of rank 4, which indicates that there must exist a first-class constraint, associated with the zero mode of this matrix, and its construction is made from the corresponding eigenvector that gives a linear

combination of second-class constraints. The combination happens to be just $\Sigma_2 \equiv \phi_3 - \partial_i \phi_2^i$, and it is independent of χ and ϕ_1 . Therefore, we have the renamed set of first-class constraints

$$\Sigma_1 \equiv \pi^+ \approx 0, \quad (12a)$$

$$\Sigma_2 \equiv p^+ - \partial_- \pi^- - \partial_k \pi^k \approx 0, \quad (12b)$$

$$\Sigma_3 \equiv \partial_- p^- + \partial_i p^i \approx 0, \quad (12c)$$

and a set of irreducible second-class constraints

$$\Phi_1^i \equiv \pi^i \approx 0, \quad (13a)$$

$$\Phi_2^i \equiv p^i - \partial_i \pi^- + F_{i-} + 2a^2 \partial_- [\partial_i \bar{A}_- - 2\partial_- \bar{A}_i + \partial_i \partial_- A_+ - \partial_j F_{ij}] \approx 0. \quad (13b)$$

The second-class constraints do not appear in the instant-form dynamics for this theory: they are a common effect of the null-plane dynamics.

Here, we are in position to write the total Hamiltonian

$$H_T \equiv H_c + \int d^3x u^a(x) \Sigma_a(x) + \int d^3x \lambda_i^l(x) \Phi_l^i(x), \quad (14)$$

with which we are able to calculate the canonical equations of the system for the variables A_μ , \bar{A}_μ , p^μ , and π^μ .

For A_μ , we have the equations

$$\partial_+ A_\mu = \bar{A}_\mu + \delta_\mu^+ u^2 - \delta_\mu^- \partial_- u^3 - \delta_\mu^i [\partial_i u^3 - \lambda_i^2], \quad (15)$$

which just means that the canonical variable \bar{A}_μ is defined as $\partial_+ A_\mu$ plus a linear combination of the still arbitrary Lagrange multipliers. The equations for \bar{A}_μ give

$$\partial_+ \bar{A}_\mu \approx \delta_\mu^+ u^1 + \delta_\mu^- [\partial_- \bar{A}_+ + \partial_i \bar{A}_i - \partial_i \partial_i A_+ + \partial_- u^2 + \partial_i \lambda_i^2] + \delta_\mu^i [\partial_i u^2 + \lambda_i^1]. \quad (16)$$

The equation for \bar{A}_+ is just $\partial_+ \bar{A}_+ \approx u^1$, which is expected since \bar{A}_+ is a degenerate variable. The expression for \bar{A}_- can be written, using (15), as

$$\partial_\mu F^{-\mu} \approx -[\partial^i \partial_i + \partial^+ \partial_+] u^3. \quad (17)$$

The Hamiltonian equations for the momenta p^μ are given, with (15) and $\pi^- = a^2 \partial_\lambda F^{+\lambda}$, by

$$\partial_+ p^+ \approx \partial_\lambda F^{\lambda+} - a^2 \partial_i \partial_- \partial_\lambda F^{\lambda i} - a^2 \partial_i \partial_i \partial_\lambda F^{\lambda+} + (1 + a^2 \partial_i \partial_i) \partial_- \partial_- u^3,$$

$$\partial_+ p^- \approx \partial_i F^{i-} + \partial_i \partial_i u^3,$$

$$\partial_+ p^i \approx \partial_- F^{-i} + \partial_j F^{ji} - a^2 \partial_\mu \partial^\mu \partial_j F^{ij} - \partial_- \partial_i u^3.$$

The equations for π^μ are, using the fact that π^+ and π^i are weakly zero,

$$p^+ \approx a^2 \partial_- \partial_\lambda F^{+\lambda},$$

$$p^- \approx F^{-+} + a^2 \partial_- \partial_\lambda F^{-\lambda} + \partial_- u^3 - a^2 \partial_- \partial_i \partial_i u^3,$$

$$p^i \approx F^{i+} - a^2 (\partial^i \partial_\lambda F^{+\lambda} - 2\partial_- \partial_\lambda F^{i\lambda}) + 2a^2 \partial_- \partial_- \partial_i u^3.$$

The last equations reproduce the definition of the canonical momenta p with some combination of the Lagrange multipliers. If we use these equation on the earlier equations for $\partial_+ p^\mu$, and also using (17), we have

$$(1 + a^2 \square) \partial_\lambda F^{\lambda+} + (1 + a^2 \partial_i \partial_i) \partial_- \partial_- u^3 \approx 0, \quad (18a)$$

$$(1 + a^2 \square) \partial_\lambda F^{\lambda-} + a^2 \partial_+ \partial_- \partial_i \partial_i u^3 \approx 0, \quad (18b)$$

$$(1 + a^2 \square) \partial_\lambda F^{\lambda i} - (1 + 2a^2 \partial_+ \partial_-) \partial_- \partial_i u^3 \approx 0. \quad (18c)$$

These equations are compatible with the Lagrangian field equations (3) only if suitable gauge conditions are chosen in order to eliminate the Lagrange multiplier u^3 .

V. GAUGE FIXING AND DIRAC BRACKETS

At this stage, we consider the set of first-class constraints (12) that must be considered as generators of gauge transformations. The problem of choosing proper gauge conditions has to be solved to fully eliminate the redundant variables of the theory at the classical level, and therefore, to proceed with a consistent quantization of the Podolsky's field.

As it has been already stated in Sec. I, the first attempt to find gauge conditions in the instant-form of the theory was made by using the Lorenz gauge $\partial_\mu A^\mu = 0$. However, as shown in Ref. 13, the Lorenz condition is not a good gauge choice for the Podolsky's field, since it does not fulfill the necessary requirements for a consistent gauge: it does not fix the gauge, it is not preserved by the equations of motion, and it is not attainable. Moreover, it is also clear that the solutions of the field equations (3) cannot consist only by transverse fields.

The analysis of the correct gauge fixing on the null-plane can be made by closely inspecting the EL equations of the system. If we look for the $\mu = +$ equation, it produces the explicit solution for A_+

$$A_+ = - (U^{-1}) \partial_+ (1 + a^2 \square) (\partial^- A_- + \partial^i A_i), \quad (19)$$

where $U \equiv (1 + a^2 \square) \nabla^2$ and $\nabla^2 \equiv \partial_i \partial_i$. The remaining equations of motion can be written, eliminating the A_+ variable, by

$$(1 + a^2 \square) \square \mathbf{A}_- = 0, \quad (1 + a^2 \square) \square \mathbf{A}_i = 0,$$

with

$$\begin{aligned} \mathbf{A}_- &\equiv A_- + \partial_- (U^{-1}) (1 + a^2 \square) (\partial^- A_- + \partial^i A_i), \\ \mathbf{A}_i &\equiv A_i + \partial_i (U^{-1}) (1 + a^2 \square) (\partial^- A_- + \partial^j A_j). \end{aligned}$$

Therefore, we can achieve the variables \mathbf{A} through a gauge transformation such that the gauge function is

$$\Delta = (U^{-1}) (1 + a^2 \square) (\partial^- A_- + \partial^i A_i).$$

In addition, these fields satisfy the condition

$$(1 + a^2 \square) (\partial^- \mathbf{A}_- + \partial^i \mathbf{A}_i) = 0, \quad (20)$$

which is the generalized Coulomb condition on the null-plane.

For that reason, the most natural gauge choice that is compatible with the field equations is given by

$$(1 + a^2 \square) (\bar{A}_- + \partial^i A_i) \approx 0. \quad (21)$$

Back to (19), we see that the time preservation of this relation is guaranteed if we set $A_+ \approx 0$. Whereas consistency requires $\bar{A}_+ \approx 0$ as well.

In this gauge, the field equations are written by

$$(1 + a^2 \square) \square A_B = 0,$$

which is a generalized wave equation on the null-plane for the variables $A_B \equiv (A_-, A_i)$.

Back to the Hamiltonian framework, this analysis leads to the gauge conditions

$$\Omega_1 \equiv \bar{A}_+ \approx 0, \quad (22a)$$

$$\Omega_2 \equiv A_+ \approx 0, \quad (22b)$$

$$\Omega_3 \equiv (1 + a^2 \square) (\bar{A}_- + \partial^i A_i) \approx 0, \quad (22c)$$

which is the generalized radiation gauge on the null-plane. The next step is to calculate the DB for the set of ten constraints of the theory, but due to the present of the second-class constraints (13), it

is more convenient to evaluate the reduced dynamics for these constraints first. Taking the matrix of the Poisson Brackets of the second-class constraints, we have

$$M^{ij} \equiv 2\eta^{ij} \partial_-^x \begin{pmatrix} 0 & -2a^2 \partial_-^x \\ 2a^2 \partial_-^x & 1 - 2a^2 \nabla_x^2 \end{pmatrix} \delta^3(x - y). \tag{23}$$

The explicit evaluation of the inverse involves the knowledge of the inverse of the operators $(\partial_-^x)^{-1}$, $(\partial_-^x)^{-2}$, and $(\partial_-^x)^{-3}$, which are Green's functions of the operators ∂_-^x , $(\partial_-^x)^2$, and $(\partial_-^x)^3$. To achieve a unique solution, it is necessary and sufficient to impose $\partial_-^x A_\mu = 0$, $\partial_-^x \partial_-^x A_\mu = 0$, and $\partial_-^x \partial_-^x \partial_-^x A_\mu = 0$ on $x^- \rightarrow -\infty$ as the appropriate initial conditions of the theory. This choice is also consistent with the definition of momenta (8), since their definitions are also dependent on initial-boundary conditions. Therefore, we write the unique inverse

$$N_{ij} \equiv \frac{1}{2} \eta_{ij} \begin{pmatrix} \alpha(x, y) & \beta(x, y) \\ \gamma(x, y) & 0 \end{pmatrix}, \tag{24}$$

with the coefficients

$$\alpha = \frac{1}{4a^4} (x^- - y^-)^2 \epsilon(x^- - y^-) (1 - 2a^2 \nabla_x^2) \delta^2(\mathbf{x} - \mathbf{y}), \tag{25}$$

$$\beta = -\gamma = \frac{1}{a^2} |x^- - y^-| \delta^2(\mathbf{x} - \mathbf{y}). \tag{26}$$

With this inverse, we are able to define the first DB for two observables $A(x)$ and $B(y)$,

$$\{A(x), B(y)\}^* = \{A(x), B(y)\} - \iint d^3z d^3w \{A(x), \Phi_I^i(z)\} N_{ij}^{IJ}(z, w) \{\Phi_J^j(w), B(y)\}, \tag{27}$$

where $\{I, J\} = \{1, 2\}$. This definition implies the elimination of the second-class constraints and the definition of an extended Hamiltonian where Φ_I are strongly zero. Thus, we are left with the first-class constraints Σ and the gauge conditions Ω . To proceed with the evaluation of the complete DB, we should calculate the matrix of the first DB of these constraints. It is given by

$$C(x, y) \equiv \{\chi_A(x), \chi_B(y)\}^* = \begin{pmatrix} 0 & \mathcal{O}(x, y) \\ -\mathcal{O}^T(x, y) & 0 \end{pmatrix},$$

with $\chi_A \equiv (\Sigma_a, \Omega_a)$. If we write $\mathcal{D}_x \equiv (1 - a^2 \nabla_x^2)$, the matrix \mathcal{O} follows:

$$\mathcal{O}(x, y) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & \mathcal{D}_x \partial_-^x \\ 0 & 0 & -\mathcal{D}_x \nabla_x^2 \end{pmatrix} \delta^3(x - y).$$

The inverse is given by

$$C^{-1}(x, y) = \begin{pmatrix} 0 & -(\mathcal{O}^{-1})^T(x, y) \\ \mathcal{O}^{-1}(x, y) & 0 \end{pmatrix}, \tag{28}$$

in which

$$\mathcal{O}^{-1}(x, y) = \begin{pmatrix} -\delta^3(x - y) & 0 & 0 \\ 0 & -\delta^3(x - y) & \gamma(x, y) \\ 0 & 0 & \rho(x, y) \end{pmatrix}. \tag{29}$$

Under the considered boundary conditions, the coefficients are given by

$$\gamma(x, y) = -\partial_-^x (\nabla_x^2)^{-1} \delta^3(x - y), \tag{30}$$

$$\rho(x, y) = -\mathcal{D}_x^{-1} (\nabla_x^2)^{-1} \delta^3(x - y), \quad (31)$$

where

$$(\nabla_x^2)^{-1} \delta^3(x - y) = \frac{1}{4\pi} \ln(\mathbf{x} - \mathbf{y})^2.$$

Then, we are able to define the complete Dirac Brackets of the generalized radiation gauge:

$$\begin{aligned} \{A(x), B(y)\}^{**} &\equiv \{A(x), B(y)\}^* \\ &+ \int d^3z d^3w \{A(x), \Sigma_a(z)\}^* \left[(\mathcal{O}^{-1})^T \right]^{ab}(z, w) \{\Omega_b(w), B(y)\}^* \\ &- \int d^3z d^3w \{A(x), \Omega_a(z)\}^* [\mathcal{O}^{-1}]^{ab}(z, w) \{\Sigma_b(w), B(y)\}^*. \end{aligned} \quad (32)$$

With these brackets, we can deduce the fundamental ones that will lead, through the correspondence principle, to a consistent quantization of the field:

$$\begin{aligned} \{A_\mu, \bar{A}_\nu\}^{**} &= -\frac{1}{8a^2} \delta_\mu^i \delta_\nu^j \eta_{ij} |x^- - y^-| \delta^2(\mathbf{x} - \mathbf{y}), \\ \{A_\mu, p^\nu\}^{**} &= \left[\delta_\mu^\nu - \delta_\mu^+ \delta_\nu^+ + (\delta_\mu^- \partial_- + \delta_\mu^i \partial_i) (\delta_+^\nu \partial_- + \delta_j^\nu \partial^j) \right] \frac{1}{\nabla^2} \delta^3(x - y), \\ \{A_\mu, \pi^\nu\}^{**} &= \delta_-^\nu [\delta_\mu^- \partial_- + \delta_\mu^i \partial_i] \frac{1}{\nabla^2} \delta^3(x - y), \\ \{\bar{A}_\mu, \bar{A}_\nu\}^{**} &= \frac{1}{64a^4} \delta_\mu^i \delta_\nu^j \eta_{ij} (x^- - y^-)^2 \epsilon(x^- - y^-) [1 - 2a^2 \nabla^2] \delta^2(\mathbf{x} - \mathbf{y}) \\ &- \frac{1}{8a^2} \eta_{ij} |x^- - y^-| [\delta_\mu^i \delta_\nu^- + \delta_\mu^- \delta_\nu^i] \partial_j \delta^2(\mathbf{x} - \mathbf{y}), \\ \{\bar{A}_\mu, p^\nu\}^{**} &= \frac{1}{8a^2} \delta_\mu^i \delta_\nu^+ |x^- - y^-| \partial_i \delta^2(\mathbf{x} - \mathbf{y}) \\ &- \frac{1}{4} \delta_\mu^i \epsilon(x^- - y^-) \left[\delta_k^\nu \partial_k \partial_i + \frac{1}{2a^2} \delta_i^\nu (1 - 2a^2 \nabla^2) \right] \delta^2(\mathbf{x} - \mathbf{y}) \\ &- \delta_+^\nu \left[[\delta_\mu^- \partial_- + \delta_\mu^i \partial_i] - \frac{1}{2} \delta_\mu^i \partial_i \right] \delta^3(x - y), \\ \{\bar{A}_\mu, \pi^\nu\}^{**} &= [\delta_\mu^\nu - \delta_\mu^+ \delta_\nu^+ + \eta_{\mu j} \delta_j^\nu] \delta^3(x - y) - \frac{1}{4} \eta_{\mu j} \delta_-^\nu \epsilon(x^- - y^-) \partial_j \delta^2(\mathbf{x} - \mathbf{y}). \end{aligned}$$

The physical degrees of freedom can be found with the analysis of the constraints as strong relations. Of course, the fields A_+ , \bar{A}_+ , π^+ , and π^i are not independent since they are strongly zero in the formalism. Thus, p^i , p^+ , and p^- can be written in function of π^- and other variables. The gauge condition (22c) also eliminates \bar{A}_- . Therefore, the only independent variables are actually given by A_- , A_i , \bar{A}_i , p^i , and π^- . They are eight independent fields, less than the dynamics in instant-form,¹³ which can be seen as a good feature, but the structure of the phase space comes out to be quite more complicate.

Considering, for example, the brackets

$$\{A_-(x), p^i(y)\}^{**} = \frac{\partial_- \partial^i}{\nabla^2} \delta^3(x - y), \quad (33)$$

we can see that the longitudinal field acquires a non-local character. This is expected for every system analyzed on the null-plane, since this component lies on the light-cone and no criterium of causality can be employed for this field. The non-locality is due to the second-class constraints, which does not appear in instant-form dynamics of the system.

For the transverse fields, the brackets

$$\{A_i(x), p^j(y)\}^{**} = \left[\delta_i^j - \frac{\partial_i \partial_j}{\nabla^2} \right] \delta^3(x - y) \quad (34)$$

indicate that a Coulomb-type interaction is present, this case in two dimensions, what justifies to call the gauge condition (21) the generalized Coulomb condition. This is also expected, since these brackets depend exclusively on the first-class constraints plus gauge conditions, just like in the instant-form.

VI. FINAL REMARKS

We have analyzed the canonical structure of Podolsky's electrodynamics on the null-plane. The theory has high-order derivatives in the Lagrangian function, so we followed the procedure outlined in Ref. 13 for the definition of the Hamiltonian density (5) and the canonical momenta (8) associated with the fields A_μ and \bar{A}_μ , which result from the definition of the conserved energy-momentum tensor.

We have observed in the study of the initial-boundary problem of Podolsky's equation that, because it is a second-order equation, the uniqueness of the solution is obtained when the field A_μ is specified on the null-plane $x^+ = cte$ and three boundary conditions are imposed on $x^- = cte$. These conditions were chosen to be $\partial_- A_\mu = 0$, $(\partial_-)^2 A_\mu = 0$, and $(\partial_-)^3 A_\mu = 0$ on $x^- \rightarrow -\infty$. Then, we have specified a single initial condition along with three boundary data situation that differs from the instant-form dynamics, where it is necessary to specify four initial conditions on an initial surface $x^0 = cte$ to uniquely determine a solution for the field equations.

In the canonical analysis of the Podolsky's theory, we found a set of three first-class constraints (12) and a set of four second-class ones (13). The first-class constraints are responsible for the $U(1)$ invariance of the action, which is expected since the gauge character of the field should not be destroyed by the choice of parameterization. The number of first-class constraints is the same of the found in instant-form,¹³ even though their functional forms are distinct.

The new feature on the null-plane is the second-class constraints, which are not present in the conventional instant-form dynamics.¹³ The appearance of second-class constraints is a common effect of the null-plane dynamics,^{17,24,26} and they are responsible to the fact that the analysis on the null-plane requires a lesser number of degrees of freedom. Because of the second-class constraints, the longitudinal components of the fields turned out to be non-local.

To evaluate the physical degrees of freedom, it was necessary to choose proper gauge conditions for the theory, which was a subject that needed closer inspection. Gauge conditions must obey a set of requirements to be consistent with the formalism: they must fix completely the gauge, they must be consistent with the field equations, they must not affect Lorentz covariance, and last but not least, they must be attainable. We found that the generalized radiation gauge (22) on the null-plane fulfills all these requirements. Of course, this gauge choice is not the only consistent possible choice. There is, for example, the so-called null-plane gauge, which will be studied in a future work concerning the Podolsky's field coupled with scalar and spinor fields.

Since the first- and second-class constraints, together with the gauge conditions, were known, we calculated the Dirac brackets that had clarified the physical fields of the system. However, these brackets are not unique unless we specify all the information about the initial-boundary value problem of the theory. By imposing the value of the field on the null-plane $x^+ = cte$, and the considered boundary conditions on $x^- = cte$, we have fixed the hidden subset of the first-class constraints^{24,25} and got a unique inverse for the second-class constraints matrix when the ambiguity on the operators $(\partial_-^x)^{-1}$, $(\partial_-^x)^{-2}$, and $(\partial_-^x)^{-3}$ was eliminated.

Finally, an analysis of the physical fields results in the true degrees of freedom, which are given by A_- , A_i , \bar{A}_i , p^i , and π^- . The complete Dirac brackets of these fields implicated the non-locality of the longitudinal component A_- and a Coulomb-type interaction in the electrostatic case, in two dimensions.

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