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Noncommutative fluid dynamics in the Snyder space-time

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In this paper, we construct for the first time the noncommutative fluid with the deformed Poincaré invariance. To this end, the realization formalism of the noncommutative spaces is employed and the results are particularized to the Snyder space. The noncommutative fluid generalizes the fluid model in the action functional formulation to the noncommutative space. The fluid equations of motion and the conserved energy-momentum tensor are obtained.

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I. INTRODUCTION

Recent studies have shown that physical systems from a variety of fields have physical properties that can be cast simultaneously in terms of concepts from two distinct areas: the noncommutative gauge theories and the fluid mechanics [1–6]. This leads to the natural question of whether there is a well defined noncommutative fluid theory. Probably the best known example of a system in which the two fields are closely related is the quantum Hall liquid whose granular structure can be described in terms of noncommutative gauge fields. In particular, the quantum Hall effect for fraction $1/n$, the Abelian noncommutative Chern-Simons theory at level n and the Laughlin theory at $1/n$ are related by a mapping among noncommutative spaces. [1,2]. The comparison of the transformations of the fluid phase space and of symmetries of the noncommutative field theories suggests that there should be a deeper analogy between the volume preserving diffeomorphisms in the commutative phase space and the symplectic preserving diffeomorphism in the noncommutative space that might lead to the noncommutative analogue of the Bernoulli equation [7–11]. More arguments in favor of the noncommutative fluids can be found in Ref. [12] where it was shown that the lowest Landau levels of the charged particles are related to the noncommutative curvilinear coordinate operators, and in Ref. [13] where the linear cosmological perturbations of a quantum fluid were shown to exhibit noncommutative properties. In Ref. [14], a generalization of the symplectic structure of the irrotational and rotational nonrelativistic fluids to the noncommutative space was proposed.

The generalization of the fluid equations to define noncommutative fluids is not obvious since many extensive long range degrees of freedom of the commutative systems do not have a simple interpretation in terms of quantities defined on the noncommutative spaces. Therefore, finding the noncommutative correspondents of the statistical mechanics or thermodynamical concepts is a nontrivial open problem which has not been fully undertaken in the literature (see for some tentative approaches [15,16]). However, there are canonical formulations of the ideal fluids in terms of the Lagrangian functionals over the set of fluid potentials [17] that can be generalized to noncommutative functionals. This procedure needs to be supplemented by a correspondence principle needed to fix the constraints which are imposed on the noncommutative fluid fields in order to obtain the known fluid equations in the commutative limit. By pursuing this line of reasoning, some of us have proposed a noncommutative fluid action in terms of the Moyal deformed algebra of functions over the Minkowski space-time $\mathcal{M}$ [18] that generalizes the commutative relativistic ideal fluid in the Kähler parametrization [19] in which the fluid is parametrized in terms of one real $\theta(x)$ and two complex potentials $z(x)$ and $\bar{z}(x)$, respectively. As shown in Ref. [17], the description of the fluid degrees of freedom in terms of fluid potentials allows one to lift the obstruction to inverting the symplectic form in the canonical phase space of the fluid variables. (For other applications of the Kähler parametrization of the fluid potentials see Refs. [19–25].)

The action functional from Ref. [18] describes the noncommutative fluid model on *canonical noncommutative space-times* [26], i.e., spaces with the coordinate algebra characterized by a constant antisymmetric matrix $\theta_{\mu\nu}$. However, the canonical coordinate algebra and the Lorentz algebra are inconsistent with each other. Therefore, a *Lie-algebra noncommutative space* structure is needed [26] in order to properly generalize the

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relativistic fluid to a noncommutative Lorentz-covariant model. One interesting alternative is the Snyder space $\mathcal{S}$ [27] in which the noncommutative coordinates are interpreted as the Lie generators of $so(1,4)/so(1,3)$. The algebra of functions over the Snyder $\mathcal{F}(\mathcal{S})$ space can be endowed with the star product and the coproduct constructed recently in Refs. [28–30] and is isomorphic to the deformed algebra over the Minkowski space-time $(C^{\infty}(\mathcal{M}), \star)$. However, the formulation of the field theory in the Snyder space is not trivial, since the star product is nonassociative and the momenta associated to the coordinates do not form a Lie group. A particularly important problem for these systems is to define and calculate the relevant physical quantities such as the energy and the linear momenta.

In this paper, we are going to construct the noncommutative fluid in the Snyder space-time by generalizing the Lagrangian functional approach from Ref. [18]. To this end, we have found it convenient to formulate the geometry of $\mathcal{S}$ in the *realization formalism* developed in Refs. [31–38] (see for similar ideas [39,40]) that has been used recently in Ref. [41] in an attempt to formulate the scalar field theory. The realization method has at least two nice features: it allows one to circumvent the problems related to the nonassociativity in the interacting field theories, and it represents a unified framework for handling simultaneously different types of noncommutative spaces such as the Snyder, the Maggiore and the Weyl spaces [41]. Also, it can be used to interpolate between the k-deformed Minkowski and the Snyder space-times [42]. In the realization formalism, the coordinates belong to the noncommutative space in which the algebra of the coordinate operators closes over the generators of the Lorentz symmetry. The corresponding momenta are defined as being the duals to the coordinates and they belong to a coset space. In general, the algebra of coordinates does not fix the commutation relations either among the momenta or among the coordinates and momenta. In order to obtain a noncommutative fluid with the largest symmetry group, we require that the symmetries of the noncommutative space-time be described by the undeformed Poincaré algebra. Also, we require that the commutative limit of the noncommutative fluid be the relativistic ideal fluid in the Clebsch parametrization in which the fluid potentials are given in terms of three real fields $\theta(x)$, $\alpha(x)$ and $\beta(x)$, and that they parametrize the velocity of the fluid elements as $v_{\mu} = \partial_{\mu}\theta + \alpha\partial_{\mu}\beta$ [17]. The present construction can be easily applied to the relativistic fluid in the Kähler parametrization.

The paper is organized as follows. In Sec. II we review the geometry of the Snyder space-time in the realization formalism and establish our notations. In Sec. III we construct the noncommutative Lagrangian that generalizes the relativistic ideal fluid in the Clebsch parametrization. We discuss the transformation of the action under the symmetries of the noncommutative space. The energy-momentum tensor is defined from the variation of the action under the deformed translations and we show that it satisfies a conservation equation. The last section is devoted to conclusions.

II. GEOMETRY OF THE SNYDER SPACE-TIME

In this section we are going to review the geometry of the Snyder space-time in the framework of the realization formalism following [34,41]. The Snyder space-time is a lattice space characterized by a length scale l_s and compatible with the Lorentz symmetry. These two properties are obtained by associating noncommutative position operators $\tilde{x}_{\mu}$ to the sites of the lattice. The algebra of $\tilde{x}_{\mu}$'s is closed over the generators of $so(1,3)$. The Snyder algebra was originally obtained by descending from five dimensions and can be interpreted as a deformed algebra of the $so(1,3)$ with the deformation parameter $s = l_s^2$ [27].

Let us start with the deformed algebra generated by the operators $\{\tilde{x}_{\mu}, p_{\mu}, M_{\mu\nu}\}$ that satisfy the following commutation relations:

$$[\tilde{x}_{\mu}, \tilde{x}_{\nu}] = sM_{\mu\nu}, \tag{1}$$

$$[p_{\mu}, p_{\nu}] = 0, \tag{2}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\sigma}M_{\mu\rho}, \tag{3}$$

$$[M_{\mu\nu}, \tilde{x}_{\rho}] = \eta_{\nu\rho}\tilde{x}_{\mu} - \eta_{\mu\rho}\tilde{x}_{\nu}, \tag{4}$$

$$[M_{\mu\nu}, p_{\rho}] = \eta_{\nu\rho}p_{\mu} - \eta_{\mu\rho}p_{\nu}, \tag{5}$$

where $\mu, \nu = \overline{0,3}$ and the deformation parameter is $s > 0$. The generators $M_{\mu\nu}$ satisfy the commutation relations of the Lorentz group and can be written in terms of the commutative coordinates of the underlying Minkowski space-time $\mathcal{M}$ in the usual way, $M_{\mu\nu} = i(x_{\mu}p_{\nu} - x_{\nu}p_{\mu})$. Thus, the Snyder algebra defined by the commutators (1)–(5) can be interpreted as a deformation of the commutative Poincaré algebra of $\mathcal{M}$. In fact, the relation (1) shows that the noncommutative coordinates $\tilde{x}_{\mu}$ are functions of the commutative phase space variables x_{μ} and p_{μ}. However, the Snyder algebra leaves the functions $\tilde{x}_{\mu}(x,p)$ and the commutators $[\tilde{x}_{\mu}(x,p), p_{\nu}]$ undetermined.[1] The *realizations* of the noncommutative Snyder geometry are defined by the simplest choice possible for the coordinate operators $\tilde{x}_{\mu}(x,p)$ as momentum-dependent rescalings of the coordinates x_{μ}:

[1]It has been shown in Ref. [36] that there are infinitely many commutation relations among $\tilde{x}_{\mu}$ and p_{ν} that are compatible with the Snyder algebra.

$$\tilde{x}_{\mu}(x, p) = \Phi_{\mu\nu}(s; p)x_{\nu}. \quad (6)$$

The smooth functions $\Phi_{\mu\nu}(s; p)$ can be reduced to a set of two dependent functions φ_1 and φ_2:

$$\tilde{x}_{\mu}(x, p) = x_{\mu}\varphi_1(A) + s\langle xp\rangle p_{\mu}\varphi_2(A), \quad (7)$$

$$\varphi_2(A) = \left[1 + 2\frac{d\varphi_1(A)}{dA}\right]\left[\varphi_1(A) - 2A\frac{d\varphi_1(A)}{dA}\right]^{-1}, \quad (8)$$

and $A = s\eta^{\mu\nu}p_{\mu}p_{\nu}$. The commutative scalar product is denoted by $\langle ab\rangle = \eta^{\mu\nu}a_{\mu}b_{\nu}$. The realizations defined by the relations (6)–(8) show that the Snyder geometry defined by the algebra (1)–(5) can be viewed as a noncanonical deformation of the commutative phase space. Different realizations can be obtained by choosing different functions $\varphi_1(A)$. For example, the Weyl, the Maggiore and the Snyder noncommutative space-times can be obtaining by choosing $\varphi_1(A) = \sqrt{A}\cot(A)$, $\varphi_1(A) = \sqrt{1 - sp^2}$ and $\varphi_1(A) = 1$, respectively [41]. The physical momenta depend on the specific realization since

$$\tilde{p}_{\mu} = f(A)p_{\mu}, \qquad f(A) = [\varphi_1(A)^2 + A]^{-(1/2)}. \quad (9)$$

An important property is that from the point of view of the realizations, the algebras generated by $\{\tilde{x}_{\mu}, p_{\mu}, M_{\mu\nu}\}$ are deformed Heisenberg algebras

$$[\tilde{x}_{\mu}, p_{\nu}] = i(\eta_{\mu\nu}\varphi_1(A) + sp_{\mu}p_{\nu}\varphi_2(A)). \quad (10)$$

The symmetries of the Snyder space-time are described by the algebra of the Lorentz generators and the co-algebra of the translation generators acting on the noncommutative coordinates $\tilde{x}_{\mu}$ according to the relations (4) and (10), respectively. The translation algebra acts covariantly from the left on the space of commutative functions as follows. If $\tilde{\phi}(\tilde{x})$ is a noncommutative function and $\mathbf{1}$ is the identity element of the algebra of commutative functions over x_{μ} then

$$\tilde{\phi}(\tilde{x})\triangleright\mathbf{1} = \psi(x), \quad (11)$$

where $\psi(x)$, in general, differs from $\phi(x)$. Since the noncommutative functions can be expanded formally in terms of the noncommutative wave functions $e^{i\langle k\tilde{x}\rangle}$, the deformed momentum $K_{\mu} = K_{\mu}(k)$ is defined by the following relation:

$$e^{i\langle k\tilde{x}\rangle}\triangleright\mathbf{1} = e^{i\langle K\tilde{x}\rangle}, \quad (12)$$

with its inverse given by

$$e^{i\langle K^{-1}(k)\tilde{x}\rangle}\triangleright\mathbf{1} = e^{i\langle kx\rangle}.$$

The left-action can be extended to products of a finite number of noncommutative wave functions:

$$e^{i\langle K_1^{-1}(k_1)\tilde{x}\rangle}e^{i\langle K_2^{-1}(k_2)\tilde{x}\rangle}\cdots e^{i\langle K_3^{-1}(k_3)\tilde{x}\rangle}\triangleright\mathbf{1} = e^{i\langle D^{(m)}(k_m,k_{m-1},\ldots,k_1)x\rangle}, \quad (13)$$

where the functions $D^{(m)}(k_1, k_2, \ldots, k_m)$ are defined recursively as

$$D_{\mu}^{(m)}(k_m, k_{m-1}, \ldots, k_1) = D_{\mu}^{(2)}(k_m, D^{(m-1)}(k_{m-1}, \ldots, k_1)). \quad (14)$$

In particular, the product of two wave functions determines the $\star$-product, the coproduct and the antipode S of the Poincaré coalgebra as follows:

$$e^{i\langle K_1^{-1}(k_1)\tilde{x}\rangle}\star e^{i\langle K_2^{-1}(k_2)\tilde{x}\rangle} = e^{i\langle D^{(2)}(k_2,k_1)x\rangle}, \quad (15)$$

$$\triangle p_{\mu} = D_{\mu}^{(2)}(p\otimes\mathbf{1}, \mathbf{1}\otimes p), \quad (16)$$

$$D_{\mu}^{(2)}(g, S(g)) = 0, \quad (17)$$

for any element of the deformed Poincaré group. Thus, the whole structure of the coalgebra is encoded in the two-functions $D^{(2)}(k_2, k_1)$. These functions depend on the realization of the Snyder geometry. The $\star$-product and the coproduct are nonassociative and noncommutative, and that makes it difficult to construct the field theories. The coproduct of the Lorentz generators takes the following form:

$$\triangle M_{\mu\nu} = M_{\mu\nu}\otimes\mathbf{1} + \mathbf{1}\otimes M_{\mu\nu}. \quad (18)$$

The $\star$-product can also be given a representation in terms of differential operators by taking $p_{\mu} = -i\partial_{\mu}$ [34]. Then one can write

$$(f\star g)(x) = \lim_{y\to x}\lim_{z\to x}\exp[i\langle(D^{(2)}(p_y, p_z) - p_y - p_z)x\rangle]. \quad (19)$$

The deformed Poincaré group can be obtained from the coproduct of the translation generators which is compatible with the Lorentz subgroup of the deformed Poincaré group according to the relation (18). In the realization formalism, any realization represents a deformation of the Poincaré algebra that is a generalized Hopf algebra, and that describes the symmetries of the Snyder geometry with the translation space given by a deformation of the de Sitter space $SO(1,4)/SO(1,3)$.

III. NONCOMMUTATIVE FLUID IN THE SNYDER SPACE-TIME

In this section, we are going to use the realization formalism to derive the action functional of the noncommutative relativistic fluid in the Snyder space-time $\mathcal{S}$. The functions over $\mathcal{S}$ can be mapped into the deformed algebra of the Minkowski space-time $(C^{\infty}(\mathcal{M}), \star)$. Thus, the action can be represented by a functional over $(C^{\infty}(\mathcal{M}), \star)$. In the same way, the deformed Poincaré group over $\mathcal{S}$ can be mapped bijectively into the deformed Poincaré group over $\mathcal{M}$.

A. Action of the noncommutative fluid

The dynamics of the relativistic ideal fluid in the Minkowski space-time $\mathcal{M}$ in the Clebsch parametrization can be obtained from an action functional that depends on the density current and three real fluid potentials $\phi(x) = \{j^\mu(x), \theta(x), \alpha(x), \beta(x)\}$ [17]. The first step to be taken in order to construct the action of the noncommutative fluid is to generalize the potentials to functions $\tilde{\phi}(\tilde{x}) = \{\tilde{j}^\mu(x), \tilde{\theta}(\tilde{x}), \tilde{\alpha}(\tilde{x}), \tilde{\beta}(\tilde{x})\}$ over the Snyder space-time that should be identified with the degrees of freedom of the noncommutative fluid. The correspondence principle in this case is

$$\lim_{s\to 0} S_s[\tilde{\phi}(\tilde{x})] = S[\phi(x)], \tag{20}$$

where $S_s[\tilde{\phi}(\tilde{x})]$ is the action functional of the noncommutative fluid and $S[\phi(x)]$ is the action of the perfect relativistic fluid in the Clebsch parametrization. Guided by this principle, we propose the following Lagrangian for the noncommutative fluid in the Snyder space-time:

$$\tilde{\mathcal{L}}[\tilde{\theta}(\tilde{x}), \tilde{\alpha}(\tilde{x}), \tilde{\beta}(\tilde{x})] = -\tilde{j}^\mu(\tilde{x})[\partial_\mu \tilde{\theta}(\tilde{x}) + \tilde{\alpha}(\tilde{x})\partial_\mu \tilde{\beta}(\tilde{x})] - \tilde{f}\Big(\sqrt{-\tilde{j}^\mu(\tilde{x})\tilde{j}_\mu(\tilde{x})}\Big), \tag{21}$$

where $\tilde{j}^\mu(\tilde{x})$ is an arbitrary smooth function of the noncommutative coordinates that generalizes the fluid current and $\tilde{f}$ is an arbitrary smooth function that characterizes the equation of state of a specific model. According to the realization method discussed in the previous section, the Lagrangian functional $\tilde{\mathcal{L}}[\tilde{j}^\mu(x), \tilde{\theta}(\tilde{x}), \tilde{\alpha}(\tilde{x}), \tilde{\beta}(\tilde{x})]$ is mapped to a functional $\mathcal{L}_s[j^\mu(x), \theta(x), \alpha(x), \beta(x)]$ that depends on functions from the algebra $(C^\infty(\mathcal{M}), \star)$ where the $\star$-product is given by the relation (19). In order to determine the form of $\mathcal{L}_s[j^\mu(x), \theta(x), \alpha(x), \beta(x)]$ we perform the Fourier transformation of the noncommutative potentials,

$$\tilde{\phi}(\tilde{x}) = \int [dk]_s \hat{\phi}(k) \exp(i\langle K^{-1}(k)\tilde{x}\rangle). \tag{22}$$

The integration-invariant measure depends on the antipode $S(k_\mu) = -k_\mu$ which is a realization-dependent quantity. However, since the momenta in different realizations are related by the relations (9) the antipode is exactly trivial in all realizations [28,41] and the measure takes the following form:

$$[dk]_s = \frac{d^4k}{(2\pi)^4}. \tag{23}$$

From the Fourier transform (22) and the definition of the $\star$-product (19) we can derive the first term of the Lagrangian $\mathcal{L}_s[j^\mu(x), \theta(x), \alpha(x), \beta(x)]$ as follows:

$$\begin{aligned}(\tilde{j}^\mu(\tilde{x})\partial_\mu\tilde{\theta}(\tilde{x}))\triangleright\mathbf{1} &= i\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\hat{j}^\mu(k_1)k_{2,\mu}\hat{\theta}(k_2) \\ &\quad\times(\exp(i\langle K^{-1}(k_1)\tilde{x}\rangle)\exp(i\langle k_2\tilde{x}\rangle)) \\ &= i\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\hat{j}^\mu(k_1)k_{2,\mu}\hat{\theta}(k_2) \\ &\quad\times\exp(i\langle D^{(2)}(k_1,k_2)x\rangle) \\ &= j^\mu(x)\star\partial_\mu\theta(x).\end{aligned} \tag{24}$$

The last product is defined on the algebra $(C^\infty(\mathcal{M}), \star)$. The second term of the Lagrangian $\tilde{\mathcal{L}}[\tilde{\theta}(\tilde{x}), \tilde{\alpha}(\tilde{x}), \tilde{\beta}(\tilde{x})]$ can be mapped into $(C^\infty(\mathcal{M}), \star)$ in exactly the same way. The third term involves a triple $\star$-product. Its Fourier transform can be calculated by using the relation (14). After some algebraic manipulations, the following result is obtained:

$$\begin{aligned}&(\tilde{j}^\mu(\tilde{x})\tilde{\alpha}(\tilde{x})\partial_\mu\tilde{\beta}(\tilde{x}))\triangleright\mathbf{1} \\ &\quad= i\int\Big(\prod_{m=1}^{3}\frac{d^4k_m}{(2\pi)^4}\Big)\hat{j}^\mu(k_1)\hat{\alpha}(k_2)k_{3,\mu}\hat{\beta}(k_3) \\ &\qquad\times\exp(i\langle D^{(3)}(k_3,k_2,k_1)x\rangle) \\ &\quad= j^\mu(x)\star(\alpha(x)\star\partial_\mu\beta(x)).\end{aligned} \tag{25}$$

The relations (21), (24), and (25) lead to the following action of the noncommutative fluid defined on the algebra $(C^\infty(\mathcal{M}), \star)$:

$$\begin{aligned}&S_s[j^\mu(x), \theta(x), \alpha(x), \beta(x)] \\ &\quad= \int d^4x\,\tilde{\mathcal{L}}[\tilde{\theta}(\tilde{x}), \tilde{\alpha}(\tilde{x}), \tilde{\beta}(\tilde{x})]\triangleright\mathbf{1} \\ &\quad= \int d^4x\Big[-j^\mu(x)\star[\partial_\mu\theta(x) + \alpha(x)\star\partial_\mu\beta(x)] \\ &\qquad - f_s\Big(\sqrt{-j^\mu(x)\star j_\mu(x)}\Big)\Big],\end{aligned} \tag{26}$$

where in the second term from (26) the $\star$-product from the square bracket should be computed first. The relationship between $\tilde{f}$ and f_s is given by the map (11). Since the function $\tilde{f}$ is arbitrary, the action (26) describes a class of noncommutative fluids parametrized by α, β and f_s for any given value of s.[2] Equation (26) can be used to construct the noncommutative deformation of a commutative fluid model characterized by a particular function f by deforming f to f_s such that $\lim_{s\to 0} f_s = f$. Then it is a simple exercise to verify that the action (26) satisfies the corresponding principle (20).

[2]This is different from the noncommutative fluid in the Kähler parametrization in which the action describes a class of fluids parametrized by f_λ *and* the Kähler potential K_λ where λ is the noncommutative parameter [18].

B. Deformed Poincaré transformations

The common point of view adopted to define the physical quantities associated to a noncommutative field theory is that they should be associated to the group of transformations of the noncommutative structure underlying the theory. According to this point of view, the infinitesimal variation of the action $\delta_\varepsilon S_s$ under the deformed Poincaré algebra should define physical quantities relevant to the noncommutative fluid described by (26). Note that, in general, the variation δ_ε viewed as an operator on $(C^\infty(\mathcal{M}), \star)$ is linear but does not necessarily satisfy the Leibniz condition. The variation of the action under the deformed Poincaré transformations is defined by the usual relation

$$\delta_\varepsilon S_s = S_s(\varepsilon) - S_s, \tag{27}$$

where $S_s(\varepsilon)$ represents the action with all variables acted upon by the infinitesimal deformed Poincaré transformations

$$x_\mu \to x_\mu + \delta_\varepsilon x_\mu, \tag{28}$$

where $\delta_\varepsilon x$ is defined by (4) and (10). In the first case, the parameter ε_μ is an infinitesimal constant vector on $\mathcal{M}$ while in the second case it is an infinitesimal antisymmetric constant matrix $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$. The transformation (28) induces a map between the translated functions from $\mathcal{F}(\mathcal{S})$ and $(C^\infty(\mathcal{M}), \star)$,

$$\tilde{\phi}(\tilde{x} + \delta_\varepsilon \tilde{x}) \triangleright \mathbf{1} = \psi(x + \delta_\varepsilon x), \tag{29}$$

where ψ is the function defined by Eq. (11).

Let us consider the deformed translations

$$\delta_\varepsilon \tilde{x}_\mu = [\tilde{x}_\mu, \langle \varepsilon p \rangle]. \tag{30}$$

For any realization, the variation (30) induces a transformation in $\mathcal{M}$ which, in general, is not the commutative translation. This can be seen by inverting the relation (7) and by calculating the variation of x_μ from

$$\delta_\varepsilon x_\mu = [x_\mu, \langle \varepsilon p \rangle] = i\varepsilon_\mu + isp_\mu \langle \varepsilon p \rangle \frac{\varphi_2(A)}{\varphi_1(A)}. \tag{31}$$

As can be easily checked, the transformation (31) leaves the volume element invariant. After some algebra, we can show that the relation (27) takes the following form:

$$\delta_\varepsilon S_s = \int d^4x [\mathcal{L}_s(x + \delta_\varepsilon x) - \mathcal{L}_s(x)], \tag{32}$$

where $\mathcal{L}_s(x + \delta_\varepsilon x)$ is the Lagrangian from (26) with the $\star$-product computed at $x + \delta_\varepsilon x$.

The deformed rotations have the following form:

$$\delta_\varepsilon \tilde{x}_\mu = [\tilde{x}_\mu, \langle \varepsilon M \rangle], \tag{33}$$

where $\langle \varepsilon M \rangle = \varepsilon^{\mu\nu} M_{\mu\nu}$. The induced transformation in $\mathcal{M}$ can be obtained from the inverse of the relation (7) and takes the following form:

$$\delta_\varepsilon x_\mu = [x_\mu, \langle \varepsilon M \rangle] = -\frac{2}{\varphi_1(A)} \varepsilon_\mu^\nu \tilde{x}_\nu. \tag{34}$$

The volume element is invariant under the transformation (28) with (34). Therefore, the variation of the action under (33) has the form (32).

In general, there is no simple expression that describes the variation $\delta_\varepsilon S_s$ due to the nonassociativity of the $\star$-product. Indeed, one can compute the $\star$-product at $x + \delta_\varepsilon x$ and compare it with the Baker-Campbell-Hausdorff formula. We consider the mapping of the additive subgroup by the exponential at right.[3] Then after some calculations one can show that under the deformed translations,

$$\begin{aligned} &\exp[i\langle (D^{(2)}(p_y, p_z) - p_y - p_z)x\rangle] \exp[i\langle (D^{(2)}(p_y, p_z) - p_y - p_z)\delta_\varepsilon x\rangle] \\ &= \exp\Big\{ i\langle (D^{(2)}(p_y, p_z) - p_y - p_z)(x + \delta_\varepsilon x)\rangle - \frac{s\varphi_2(A)}{2\varphi_1(A)}(\eta_{\mu\nu}\langle \varepsilon p_x\rangle + p_\nu^x \varepsilon_\mu)(D^{(2)\mu}(p_y, p_z) - p_y^\mu - p_z^\mu) \\ &\quad \times (D^{(2)\nu}(p_y, p_z) - p_y^\nu - p_z^\nu) - \frac{is\varphi_2(A)}{12\varphi_1(A)}(\varepsilon_\mu \eta_{\rho\sigma} + \eta_{\mu\nu}\varepsilon_\rho)(D^{(2)\rho}(p_y, p_z) - p_y^\rho - p_z^\rho) \\ &\quad \times (D^{(2)\nu}(p_y, p_z) - p_y^\nu - p_z^\nu)(D^{(2)\mu}(p_y, p_z) - p_y^\mu - p_z^\mu)\Big\}. \end{aligned} \tag{35}$$

As can be seen, the exponential that defines the $\star$-product at $x + \delta_\varepsilon x$ does factorize in the Maggiore realization but not in the Snyder's. In general, even if the exponential factorizes, the $\star$-product does not. This behavior is not restricted to the relativistic fluid. Actually, it is the result of the structure of the noncommutative algebra given by the relations (1)–(5) and it is expected to hold for any field theory. Similar conclusions can be drawn for the deformed rotations. Apparently, the difficulties generated by the nonassociativity of the $\star$-product could be circumvented by defining the variation of the action functional as generated by the operator δ_ε instead of (27) with the following action:

$$\delta_\varepsilon S_s = \left[\int d^4x \tilde{\mathcal{L}}[\tilde{j}^\mu(x), \tilde{\theta}(\tilde{x}), \tilde{\alpha}(\tilde{x}), \tilde{\beta}(\tilde{x})] \triangleright \mathbf{1}, \langle \varepsilon G \rangle \right], \tag{36}$$

[3] The additive subgroup is mapped to right and left product of exponentials which are not isomorphic to each other due to the noncommutativity of the product.

where G is either p_μ or $M_{\mu\nu}$. However, the problems related to the nonassociativity return in the form of the variation of the $\star$-products from $\mathcal{L}_s[j^\mu(x), \theta(x), \alpha(x), \beta(x)]$ as can be verified easily.

A very important consequence of the nonassociativity of the $\star$-product is that it makes it difficult to calculate and even to define relevant physical quantities associated with the fields, such as the energy and the momentum. Nevertheless, some important properties of the quantities associated with the variations $\delta_\varepsilon x_\mu$ can be derived in the general case. To this end, we write the deformed Poincaré transformations (31) and (34) as

$$\delta_\nu x_\mu = i\left(\eta_{\nu\mu} + s p_\nu p_\mu \frac{\varphi_2(A)}{\varphi_1(A)}\right), \tag{37}$$

$$\delta_{\rho\sigma} x_\mu = \left[\eta_{\rho\sigma}(x_\sigma + s\langle xp\rangle p_\sigma)\frac{\varphi_2(A)}{\varphi_1(A)}\right] - \left[\eta_{\sigma\mu}(x_\rho + s\langle xp\rangle p_\rho)\frac{\varphi_2(A)}{\varphi_1(A)}\right]. \tag{38}$$

Then one can show by direct calculations that the variation of the action under the deformed transformations $\delta_\varepsilon x_\mu$ produces the following equation:

$$\partial_\mu(\Theta^{\mu\nu}(\phi)\delta_\varepsilon x_\nu) = \partial^\mu(\mathcal{L}\delta_\varepsilon x_\mu), \tag{39}$$

where $\Theta^{\mu\nu}(\phi)$ is a functional of the fluid potentials and their derivatives up to the third order. Since (38) depends linearly on x_μ, it follows that the equations of motion alone are not sufficient to guarantee the conservation of the quantities described by the functions $\Theta^{\mu\nu}(\phi)$ associated to $\delta_{\rho\sigma}x_\mu$. On the other hand, the right-hand side of Eq. (37) is independent of x_μ. One can check that the following quantity associated to the deformed translations,

$$T^\mu_\nu = \Theta^\mu_\sigma(\phi) - \mathcal{L}\eta^\mu_\nu, \tag{40}$$

is conserved. Note that, in general, the functions $\Theta^{\mu\nu}(\phi)$ that correspond to the translation are different from the ones derived from the rotations. T^μ_ν represents the variation of the action S_s under the deformed translation. Therefore, it can be interpreted as being the energy-momentum tensor of the noncommutative fluid in an arbitrary realization. Note that the tensor $T_{\mu\nu} = \eta_{\mu\rho}T^\rho_\nu$ does not have a definite symmetry. Actually, a symmetric energy-momentum tensor can be obtained by coupling the fluid with a c-number metric $g_{\mu\nu}$ and by deriving the action with respect to it (see Ref. [18]). However, by this procedure information about the noncommutative properties of the fluid could be lost due to the contraction between the antisymmetric components of the $\star$-product and the metric. The invariance of the theory under the noncommutative translations has been discussed and used in the literature to define the energy-momentum tensor of different field theories [40,43–51]. Equation (31) generalizes the deformed translations to the realization formalism which treats simultaneously various noncommutative spaces, as we have seen in the previous section. From this point of view, Eq. (40) represents a generalization of the previous results within the realization formalism.

IV. CONCLUSIONS AND DISCUSSIONS

In this paper, we have constructed for the first time a model of the noncommutative fluid on the Snyder space-time. To this end, we have used the realization formalism of the noncommutative spaces and we have generalized the action functional formulation of the relativistic perfect fluid in the Minkowski space-time. This model is important for understanding the behavior of the effective (or long-wave) degrees of freedom on noncommutative spaces. It provides a new class of noncommutative field theories with deformed Poincaré symmetry that generalizes the field theories on the commutative space-time in the first order formulation. By using the realization maps from the algebra $\mathcal{F}(\mathcal{S})$ to the $(C^\infty(\mathcal{M}), \star)$, we have obtained a representation of a large class of noncommutative fluids parametrized by three arbitrary functions $\alpha(x)$, $\beta(x)$ and $f(x)$ in terms of the deformed algebra of smooth functions on the Minkowski space-time. In this formulation, the fluid dynamics is given by the equation of motion of the fluid potentials viewed as fields on the Snyder space and subjected to the conservation of the energy-momentum tensor. Establishing these equations in the general case is a difficult task due to the interactions among the fluid potentials that involve infinitely many derivatives of fields. This particular structure is the result of the action of the $\star$-product, which is neither commutative nor associative. However, it is possible to study the theory perturbatively in the noncommutative parameter s. Nevertheless, even at the first order, the equations of motion are highly nonlinear even in the simplest case of the model that reduces to the irrotational fluid in the commutative limit. It is an important and interesting problem to study these equations and to determine their integrability and possible solutions. We hope to report on these topics elsewhere.

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