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Introduction to Focus Issue: Statistical mechanics and billiard-type dynamical systems

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Dynamical systems of the billiard type are of fundamental importance for the description of numerous phenomena observed in many different fields of research, including statistical mechanics, Hamiltonian dynamics, nonlinear physics, and many others. This Focus Issue presents the recent progress in this area with contributions from the mathematical as well as physical stand point. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4730155]

Billiard-type dynamical systems are at the heart of the foundations of statistical mechanics and the theory of dynamical systems, thanks to their clear meaning, very rich dynamics and relevance to many problems in physics. This Focus Issue presents some recent developments in the theory and applications of these systems. The contributions treat both fundamental and advanced questions in billiard systems from both mathematical and physical points of view.

The statistical theory of dynamical systems was created by Boltzmann and Gibbs as a natural foundation of statistical mechanics. Later on, this theory acquired the name ergodic theory thanks to Boltzmann's celebrated ergodic hypothesis, which asserts that a gas of elastically colliding particles of hard spheres in a box is an ergodic dynamical system. Although the Boltzmann hypothesis is still not proved in full generality, it was one of the major inspirations behind the development of a modern theory of dynamical systems. An elastically colliding gas of hard spheres is indeed a billiard dynamical system. These impressive developments again started with statistical physics. A brilliant Russian physicist Krylov[1] observed that dynamics of hard spheres is very unstable and, moreover, this intrinsic instability reminds one very much of the instability of other mechanical systems, namely geodesic flows on surfaces of negative curvature which were (at that time recently) rigorously investigated by Hedlund[2] and Hopf,[3] who proved that these systems are hyperbolic (a term which is often substituted by "chaotic" in the physics literature) and ergodic. Rigorous studies of the chaotic properties of billiards started with a remarkable groundbreaking 1970 paper by Sinai.[4] Since that time the theory and applications of billiards have experienced enormous development. Billiards acquired a leading role in the theory of dynamical systems as well as in applications. A new fundamental mechanism of hyperbolicity (the mechanism of "chaos"), called defocusing, was discovered in billiards[5] and subsequently was proved to work also in other classes of dynamical systems (including of course geodesic flows). Moreover, billiards became one of the most favorite systems for theoretical investigations and real experiments. There are many physical labs over the world where billiard experimental devices were constructed, which allowed to obtain new insights into various branches of physics. All these advances are not surprising because billiards are simultaneously very visual (arguably, billiards form the most visual class of dynamical systems) and billiards naturally appear as relevant models in numerous branches in physics. Unfortunately, there is not much interaction (and sometimes even knowledge of recent developments) between the physics and mathematics communities. The present Focus Issue is intended to make a contribution to increasing such interaction and collaborations. The issue contains both physical and mathematical papers, as well as papers that combine rigorous, numerical and phenomenological results demonstrating the breadth and depth of the modern theory of billiard-type systems and its growing influence in such classical areas of research as time-dependent (non-autonomous) dynamics and general theory of Hamiltonian systems.

It is well known (although not proved rigorously in a general case) that a generic Hamiltonian system has a "divided" phase space, i.e., in its phase space there coexist chaotic and ergodic components of positive measure and KAM tori with regular (even integrable) dynamics. Such structure strongly influences the dynamics on chaotic ergodic components: the dynamics becomes intermittent, i.e., quasi-regular dynamics generated by the "stickiness" in vicinity of KAM tori coexist with strongly chaotic dynamics far from them. One of the fundamental problems in this respect is an appearance of KAM tori under perturbations of chaotic dynamical systems. The paper by Rom-Kedar and Turaev[6] addresses and analyzes a process of creation of KAM tori under perturbation of the billiard reflection law, i.e., by considering steep potentials instead of a hard core (billiard) potential. Bunimovich and Vela-Arevalo[7] discuss a notion of stickiness and demonstrate that KAM islands can be of two types (elliptic and parabolic) and that they can in general be sticky as well as non-sticky.

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Studies of billiards inspire the investigation of various non-conservative systems which can be naturally considered as billiards in some fields (electrical, gravitational, etc) or just models with modified laws of reflection. The paper by Bálint, Borbély, and Varga[8] presents an analysis of statistical properties of the system of two point masses moving along a vertical line in a gravitational field and elastically colliding with each other and with a "floor." Chernov, Korepanov, and Simanyi[9] investigate a gas of N hard disks in a box with various modifications to the elastic law of reflection from the boundary. A major result is that all the perturbations, however small they are, tend to induce a collapse to stable regimes of motion, whereas unperturbed billiard dynamics is chaotic. A very much related in spirit paper by Del Magno *et al.*[10] considers a billiard in a square with modified reflection law. A study of non-elastic triangular billiards is conducted by Arroyo, Markarian, and Sanders.[11] A major goal is to investigate a topological structure of strange attractors which appear in this system and can be very complicated. An elegant example of a dissipative dynamical system is considered in the paper by Gallavotti, Gentile, and Giuliani.[12] They rigorously study dynamics of a chaotic system subject to a weak periodic perturbation. The resulting dynamics is characterized by periodic visits to a strange attractor, the shape of which is exactly established.

A classical area of study in billiards is the analysis of the set of their periodic orbits. Although this set is believed (although not proved in the general case) to have measure zero, it plays an important role for many aspects of dynamics of billiards. The paper by Pinto-de-Carvalho and Ramirez-Ros[13] deals with billiards inside planar, strictly convex tables with smooth boundaries. It is shown, in particular, that Birkhoff's classical result provides an optimal lower bound for a number of periodic points with a fixed structure in such billiards. A classification of symmetric periodic orbits in ellipsoidal billiards is presented in the paper by Casas and Ramirez-Ros.[14]

Regarding the dynamical regime of stickiness, Oliveira, Emidio, and Beims[15] show the appearance of stickiness in a soft triangular billiard whose walls are appropriate to study the soft to hard wall transition by changing the softness parameter. The emergence of chaotic motion inside the quasi-regular region is shown by Manchein and Beims[16] to occur inside Arnold stripes, which are intervals of initial conditions leading to the chaotic motion and which increase with the nonlinear parameter. For higher-dimensional systems such stripes define the chaotic channels inside which Arnold diffusion starts to occur. The effect of stickiness is quantified by Dettmann and Georgiou[17] in the open "drive-belt" billiard from the point of view of the escape through the hole. The survival probability decays algebraically due to the presence of multiple families of marginal unstable periodic orbits and decays exponentially in the limit of small hole size. When noise is considered in the dynamics of a generic and mixed-phase space, the survival probability has a total of five different decay regimes that prevail for different intermediate times, as observed for the annular billiard by Altmann, Leitão, and Lopes.[18]

One of the most famous billiard-type system in statistical mechanics is a Lorentz gas generated by a motion of a point particle in an array of immovable scatterers. Nandori and Szasz[19] consider a Lorentz gas in a half strip with a modified (time-dependent) reflection law in a wall located at the origin. They demonstrate that in a certain scaling limit, a new type of a Brownian motion arises which is a generalization of the Brownian motion and reflected Brownian motion.

The dynamics of billiards in billiard tables with the boundary of zero curvature cannot be chaotic (hyperbolic). In particular, it has a zero Kolmogorov entropy. However, such billiards in polygons and polyhedra may demonstrate high complexity of their orbits. Mathematical tools to study such billiards are quite different from the ones developed for billiards in tables with non-everywhere-flat boundaries. A review by Gutkin[20] discusses the current state and some open problems in this area.

As mentioned above, billiard systems are suitable models for attempting to understand non-equilibrium statistical mechanics. This is done by Gaspard and Gilbert,[21] who analyze the energy transfer processes of interacting hard spheres. Energy spreading is discussed in the work of Roy and Pikovsky[22] in the regular lattice of the Ding-Dong model.

It is natural to ask if billiard particles can acquire unlimited energy when colliding with a moving boundary. If this does occur, a phenomena called "Fermi acceleration" is taking place. In such approach, Itin and Neishtadt[23] discuss the particle dynamics inside a rectangular billiard with moving boundaries. For small perturbations from the integrable problem, adiabatic invariants are destroyed, and this can lead to Fermi acceleration. The standard description of Fermi acceleration, developed in a class of time-dependent billiards, is generally given in terms of a diffusion process which takes place in momentum space. Therefore, the evolution of the probability density function for the magnitude of particle velocities as a function of the number of collisions is determined by the Fokker-Planck equation, as explained by Karlis, Diakonos, and Constantoudis.[24] However, the phenomenon is not robust. Therefore, the introduction of dissipation leads the dynamics to contract the area in the phase space and to present attractors which lead to the suppression of the phenomenon. Considering these attractors are far away from infinity, the suppression of Fermi acceleration results. Indeed, the unlimited energy growth in time-dependent billiards can be suppressed by considering different kinds of dissipation. This is confirmed in the work of Ryabov and Loskutov[25] for a time-dependent non-integrable focusing billiard and by Livorati, Caldas, and Leonel[26] for the stadium like-billiard with inelastic collisions. Additionally, this suppression can be carefully described by the use of scaling arguments leading to the characterization of scaling exponents describing a phase transition from limited to unlimited energy growth, as shown in the breathing Lorentz gas by Oliveira and Leonel[27] considering both in-flight dissipation and inelastic collisions. A rigorous study of various piecewise smooth Fermi-Ulam models was conducted by DeSimoi and Dolgopyat.[28] They succeeded to prove that for all possible motions of the wall which have just one discontinuity, there is a single parameter which determines the dynamics of the system for large energies.

Many authors of the papers published in this Focus issue had the opportunity to meet each other in an International School-Conference Mathematics and Physics of Billiard-Type dynamical systems: Billiards'11, held in Ubatuba-Brazil in February 2011 when discussions of a Focus issue in this subject arose. At that time, our friend Alexander Loskutov was among us. However, it is our sad duty to acknowledge a crucial contribution to this Focus Issue of Sasha Loskutov and say that he started work enthusiastically in this direction. His sudden and untimely death happened in 05 November 2011 and was a shock for everybody who knew Sasha. This issue is a tribute to his memory.

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