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On some topics based on the concept of coherent pairs of measures of the second kind

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Tese apresentada como parte dos requisitos para obtenção do título de Doutor em Matemática, junto ao Programa de Pós-Graduação em Matemática, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista "Júlio de Mesquita Filho", Câmpus de São José do Rio Preto.

Orientador: Prof. Dr. Alagacone Sri Ranga

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To my family.

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Everything that is incomprehensible does not cease to exist.
Blaise Pascal

RESUMO

O principal objetivo desta tese é estudar alguns tópicos de pesquisa que podem ser classificados como estudos baseados em um conceito conhecido como "pares coerentes de medidas de segundo tipo". Um par de medidas é considerado um par coerente de medidas de segundo tipo se a derivada do polinômio ortogonal de grau $(n+1)$ associado a uma das medidas pode ser dada como combinação linear dos polinômios ortogonais de graus n e $(n-1)$ associados à outra medida. Nosso estudo inicial sobre par coerente de medidas de segundo tipo começou com medidas definidas no círculo unitário. Um dos tópicos de pesquisa que consideramos é estender a ideia de coerência no círculo unitário substituindo o operador derivada na fórmula que define o conceito por um operador q-diferença. Propriedades de polinômios ortogonais do tipo Sobolev relacionados também são exaustivamente exploradas. Outro tópico de pesquisa nesta tese é considerar uma análise minuciosa de pares de medidas na reta real que satisfaçam a propriedade de coerência de segundo tipo. Foi encontrada uma caracterização completa das medidas que satisfazem este conceito. Como tópico final de pesquisa, é também considerado um estudo sobre uma extensão do conceito de pares coerentes de medidas de segundo tipo na reta real onde as medidas são assumidas como simétricas. Polinômios ortogonais de Sobolev associados também são analisados.

Palavras-chave: Polinômios ortogonais na reta real. Polinômios ortogonais no círculo unitário. Polinômios ortogonais de Sobolev. Pares coerentes de medidas de segundo tipo. Sequências encadeadas positivas.

ABSTRACT

The main objective in this thesis is to consider some topics of research which can be classified as studies based on a concept known as "coherent pairs of measures of the second kind". A pair of measures is said to be a coherent pair of measures of the second kind if the derivative of the $(n+1)$*-th degree orthogonal polynomial associated with one of the measures can be given as a linear combination of the* n*-th degree and* $(n-1)$*-th degree orthogonal polynomials associated with the other measure. The initial studies concerning coherent pairs of measures of the second kind started with measures defined on the unit circle. One of the topics of research considered here is to extend the idea of coherence on the unit circle by replacing the derivative operator in the formula that defines the concept by a q-difference operator. Properties of related Sobolev type orthogonal polynomials are also thoroughly explored. Another topic of research in this thesis is to consider a thorough analysis of pairs of measures on the real line that satisfy the coherence property of the second kind. A complete characterization of measures that satisfy this concept has been found. As a final topic of research, a study on an extension to the concept of coherent pairs of measures of the second kind on the real line where the measures are assumed to be symmetric is also considered. Associated Sobolev orthogonal polynomials are also analyzed.*

Keywords: Orthogonal polynomials on the real line. Orthogonal polynomials on the unit circle. Sobolev orthogonal polynomials. Coherent pairs of measures of the second kind. Positive chain sequences.

Contents

Introduction and Historical Remarks

Orthogonal polynomials and associated Sobolev orthogonal polynomials that follow from pairs of positive measures which satisfy a property or concept known as coherence property of the second kind has turned out to be an important topic of study in recent years. For example, we say that the pair of positive measures $\{\nu_0, \nu_1\}$ supported on the real line is a *coherent pair of positive measures of the second kind on the real line* if the respective sequences of monic orthogonal polynomials $\{\mathcal{P}_n(\nu_0;\cdot)\}_{n\geq 0}$ and $\{\mathcal{P}_n(\nu_1;\cdot)\}_{n\geq 0}$ satisfy

$$\mathcal{P}_n(\nu_1;x) - \tau_n \mathcal{P}_{n-1}(\nu_1;x) = \frac{1}{n+1}\mathcal{P}'_{n+1}(\nu_0;x), \quad n \geq 1, \tag{1}$$

where $\tau_n \neq 0$ for $n \geq 1$.

$\{\mathcal{P}_n(\nu;\cdot)\}_{n\geq 0}$ is a sequence of monic orthogonal polynomials with respect to the positive measure ν, if

(i) $\mathcal{P}_n(\nu;x)$ is a monic polynomial of exact degree n;

(ii) $\displaystyle\int \mathcal{P}_n(\nu;x)\mathcal{P}_m(\nu;x)d\nu(x) = \begin{cases} 0 & \text{if} \quad m \neq n, \\ \hbar_n > 0 & \text{if} \quad m = n. \end{cases}$

We remark that integration here is along the support of the measure.

Our studies regarding pairs of positive measures satisfying the concept of coherence of the second kind were motivated by two reasons. The first one is that this study provide a nice and complete analysis of special pairs of positive measures on the real line and of the corresponding sequences of orthogonal polynomials with many interesting properties.

The second one is related to some problems in approximation theory. Precisely, the analysis of the Fourier expansions in terms of the sequences of polynomials orthogonal with respect to the Sobolev inner product:

$$\langle f, g \rangle_{\mathfrak{S}} = \int f(x)g(x)d\nu_0(x) + s\int f'(x)g'(x)d\nu_1(x), \tag{2}$$

with $s > 0$. It turns out that the monic Sobolev orthogonal polynomials $\mathcal{S}_n(\nu_0,\nu_1;x)$ with respect to the inner product $\langle\, .\, ,\, .\, \rangle_{\mathfrak{S}}$ satisfy the connection formulas

$$\left.\begin{aligned} &\mathcal{S}_{n+1}(\nu_0,\nu_1;x) - \gamma_n \mathcal{S}_n(\nu_0,\nu_1;x) = \mathcal{P}_{n+1}(\nu_0;x), \\ &\mathcal{S}'_{n+1}(\nu_0,\nu_1;x) - \gamma_n \mathcal{S}'_n(\nu_0,\nu_1;x) = (n+1)\left[\mathcal{P}_n(\nu_1;x) - \tau_n \mathcal{P}_{n-1}(\nu_1;x)\right], \end{aligned}\right\} \quad n \geq 1,$$

with $\mathcal{S}_1(\nu_0,\nu_1;x) = \mathcal{P}_1(\nu_0;x)$. These simple connection formulas between $\{\mathcal{S}_n(\nu_0,\nu_1;\cdot)\}_{n\geq 0}$ and the sequences of monic orthogonal polynomials $\{\mathcal{P}_n(\nu_0;\cdot)\}_{n\geq 0}$ and $\{\mathcal{P}_n(\nu_1;\cdot)\}_{n\geq 0}$ permit one to easily study the properties of the polynomials $\mathcal{S}_n(\nu_0,\nu_1;x)$. We have found that (see, for example, [40]) there exist specific examples of pairs of measures $\{\nu_0, \nu_1\}$ satisfying the property (1) for which the properties of the polynomials $\mathcal{S}_n(\nu_0, \nu_1; x)$ can be analyzed in more detail.

Topics considered in this thesis include a study on the characterization of pairs of positive measures which satisfy the concept of coherence of the second kind on the real line (this particular work, which has appeared in [29], has also been mentioned in the PhD thesis at UNESP of Gustavo Andreto Marcato) and, more importantly, include also studies concerning the following two extensions to this concept:

• In the first case, results are analyzed with respect to a family of pairs of measures supported on the unit circle, where in the formula that defines the concept of coherence the derivative operator is replaced by a q-difference operator. These results, presented in Chapter 2, have now appeared in [30];

• In the second case, results are obtained under a notion of coherence suitable for symmetric pairs of measures on the real line. These results, which are given in Chapters 4 and 5, are under preparation to be submitted for publication.

We now give some historical details which will also clarify the nomenclature adopted here. The concept of coherence (of the first kind) pairs of positive measures on the real line was introduced in 1991 by Iserles, Koch, Nørsett and Sanz-Serna [31]. As stated in [31], a pair of positive measures $\{\nu_0, \nu_1\}$ is a coherent pair of positive measures on the real line if and only if the corresponding sequences of monic orthogonal polynomials satisfy

$$\mathcal{P}_n(\nu_1; x) = \frac{1}{n+1}\left[\mathcal{P}'_{n+1}(\nu_0; x) - \rho_n \mathcal{P}'_n(\nu_0; x)\right], \quad \rho_n \neq 0, \quad n \geq 1. \tag{3}$$

It was shown in this case that the sequence of monic orthogonal polynomials with respect to the inner product $\langle\, .\, ,\, .\, \rangle_{\mathfrak{S}}$ satisfies the connection formulas

$$\begin{aligned} \mathcal{S}_{n+1}(\nu_0, \nu_1; x) - \gamma_n \mathcal{S}_n(\nu_0, \nu_1; x) &= \mathcal{P}_{n+1}(\nu_0; x) - \rho_n \mathcal{P}_n(\nu_0; x), \\ \mathcal{S}'_{n+1}(\nu_0, \nu_1; x) - \gamma_n \mathcal{S}'_n(\nu_0, \nu_1; x) &= (n+1)\mathcal{P}_n(\nu_1; x), \end{aligned} \qquad n \geq 1. \tag{4}$$

The above connection formulas proved to be very useful in studies concerning the analytic properties of the respective Sobolev orthogonal polynomials. In particular, with a novel use of these connection formulas, H. G. Meijer and M. de Bruin [62] found information about the location of zeros of these Sobolev orthogonal polynomials. Moreover, asymptotics properties have been deeply analyzed in the literature (see, for example, [43], [58], [59], [60] as well as the recent survey [50], where an updated list of references concerning this topic is presented).

The motivation in [31] for introducing such pairs of measures was their applications in connection with the Fourier expansions of functions with respect to the Sobolev inner product $\langle\, .\, ,\, .\, \rangle_{\mathfrak{S}}$. A particular case of such Fourier series expansions based on Legendre-Sobolev orthogonal polynomials had already been considered in [32], where some numerical tests comparing these Legendre-Sobolev Fourier series expansions and the ordinary Legendre-Fourier series expansions are presented. In the framework of coherent pairs of measures it is possible to obtain the associated Sobolev-Fourier coefficients with low computational cost (see [24]). The convergence of the corresponding Sobolev-Fourier expansions for the Jacobi weights is analyzed in [47], [14], [15], [16]. For other weights belonging to the so called Kufner-Opic class, see [46].

All pairs of positive measures supported on the real line that satisfy the coherence property (3) were completely determined in 1997 by H. G. Meijer [61]. He showed that if $\{\nu_0, \nu_1\}$ is a coherent pair of measures on the real line, then one of the measures must be classical (either Jacobi or Laguerre) and the other one is a rational perturbation of it. The starting point of the work in [61] are certain functional relations established in [44] with respect to pairs of quasi-definite moment functionals such that the corresponding sequences of monic orthogonal polynomials satisfy a relation as (3). Thus, what was proved by Meijer [61] is more general than what is stated above.

The idea of coherence property was carried over to measures on the unit circle in [10]. The authors of [10] introduced the concept of coherent pair for Hermitian quasi-definite linear functionals (which can be represented by signed measures supported on the unit circle). In the positive definite case (see [69]), the linear functionals are associated with nontrivial positive measures supported on the unit circle.

Following [10], a pair $\{\mu_0, \mu_1\}$ of positive measures supported on the unit circle is said to be a coherent pair of positive measures on the unit circle if the corresponding sequences of monic orthogonal polynomials $\{\Phi_n(\mu_0; \cdot)\}_{n\geq 0}$ and $\{\Phi_n(\mu_1; \cdot)\}_{n\geq 0}$ satisfy the algebraic relation

$$\Phi_n(\mu_1; z) = \frac{1}{n+1}\left[\Phi'_{n+1}(\mu_0; z) - \rho_n \Phi'_n(\mu_0; z)\right], \quad \rho_n \neq 0, \quad n \geq 1.$$

As established in [10], if $\{\mu_0, \mu_1\}$ is a coherent pair of positive measures on the unit circle then the following can be stated:

- If μ_0 is the Lebesgue measure ($d\mu_0(z) = \frac{dz}{2\pi i z}$), then the measure μ_1 is such that

$$d\mu_1(z) = \frac{d\mu_0(z)}{|z-\alpha|^2},$$

  with $|\alpha| < 1$. This means μ_1 belongs to the Bernstein-Szegő class.
- If μ_1 is the Lebesgue measure, then the measure μ_0 is such that

$$d\mu_0(z) = |z-\alpha|^2 d\mu_1(z).$$

They also prove that the only Bernstein-Szegő measure μ_0 for which $\{\mu_0, \mu_1\}$ is a coherent pair is the Lebesgue measure. A full description of all coherent pairs of measures supported on the unit circle is still not known and this remains as an open problem.

More recently, in [71], an example of a family of pairs of measures $\{\mu_0, \mu_1\}$ such that there hold

$$\Phi_n(\mu_1; z) - \tau_n \Phi_{n-1}(\mu_1; z) = \frac{1}{n+1}\Phi'_{n+1}(\mu_0; z), \quad \tau_n \neq 0, \qquad n \geq 1, \tag{5}$$

has been introduced and the corresponding sequence of monic Sobolev orthogonal polynomials has also been studied. Motivated by these results, in [71] pairs of measures on the unit circle with the property (5) were further explored in [49]. In [49] these pairs of measures have been referred to as coherent pairs of measures of the second kind on the unit circle. Thus, we refer to the pair of measures $\{\nu_0, \nu_1\}$ on the real line satisfying the property (1) as a coherent pair of measures of the second kind on the real line.

In the results considered in Chapter 2 of this thesis, we look at a special pair of measures on the unit circle $\{\mu_0, \mu_1\}$ for which the corresponding sequences of orthogonal polynomials $\{\Phi_n(\mu_0; \cdot)\}_{n\geq 0}$ and $\{\Phi_n(\mu_1; \cdot)\}_{n\geq 0}$ satisfy

$$\Phi_n(\mu_1; z) - \tau_n \Phi_{n-1}(\mu_1; z) = \frac{1}{\{n+1\}_q} D_q\Big[\Phi_{n+1}(\mu_0; z)\Big], \quad \tau_n \neq 0, \qquad n \geq 1, \tag{6}$$

where $D_q[F(z)] = \frac{F(q^{-1/2}z)-F(q^{1/2}z)}{q^{-1/2}z-q^{1/2}z}$ and $\{n\}_q$ is such that $D_q[z^n] = \{n\}_q\, z^{n-1}$. In view of (6) we say that $\{\mu_0, \mu_1\}$ satisfies a coherence type property of the second kind on the unit circle with respect to the q-difference operator D_q in which $0 < q < 1$.

Coming back to measures and orthogonal polynomials on the real line, let us recall that the connection formulas in (4) hold when the pair of measures $\{\nu_0, \nu_1\}$ satisfies the coherence property (3). This means, from results shown in [61], the formulas in (4) hold when one of the measures in $\{\nu_0, \nu_1\}$ is classical. The extension of the concept of coherence, in which

$$\mathcal{P}_n(\nu_1; x) - \tau_n \mathcal{P}_{n-1}(\nu_1; x) = \frac{1}{n+1}\left[\mathcal{P}'_{n+1}(\nu_0; x) - \rho_n \mathcal{P}'_n(\nu_0; x)\right], \quad \rho_n \neq 0, \tag{7}$$

for $n \geq 1$, where $\{\nu_0, \nu_1\}$ is known as a $(1,1)$-coherent pair when $\tau_n \neq 0$ for $n \geq 1$, is explored in [23] by including semiclassical measures. In this case the Sobolev orthogonal polynomials associated with the inner product $\langle\, .\, , .\, \rangle_{\mathfrak{S}}$ in (2) satisfy the connection formulas

$$\begin{aligned} \mathcal{S}_{n+1}(\nu_0, \nu_1; x) - \gamma_n \mathcal{S}_n(\nu_0, \nu_1; x) &= \mathcal{P}_{n+1}(\nu_0; x) - \rho_n \mathcal{P}_n(\nu_0; x), \\ \mathcal{S}'_{n+1}(\nu_0, \nu_1; x) - \gamma_n \mathcal{S}'_n(\nu_0, \nu_1; x) &= (n+1)\left[\mathcal{P}_n(\nu_1; x) - \tau_n \mathcal{P}_{n-1}(\nu_1; x)\right], \end{aligned} \qquad n \geq 1.$$

We mention that a $(1,1)$-coherent pair is a particular case of the (M, N)-coherent pair considered, for example, in [34].

We would like to emphasize that the characterization of measures that satisfy the coherence property (7) is performed in [23] assuming the restriction $\rho_n \neq 0$ for $n \geq 1$. Observe that the results associated with the special case in which $\tau_n = 0$ and $\rho_n \neq 0$ for $n \geq 1$ was already considered in [61]. Surprisingly, a complete study of the situation in which $\tau_n \neq 0$ and $\rho_n = 0$ for $n \geq 1$ has not been done previously and this study is covered in the results given in Chapter 3 of this thesis.

In Chapter 4 we consider the following extension to the study of measures which are coherent pair of measures of the second kind on the real line. That is we consider the characterization of measures $\{\nu_0, \nu_1\}$ which are now symmetric and that the corresponding orthogonal polynomials $\{\mathcal{P}_n(\nu_0; \cdot)\}_{n\geq 0}$ and $\{\mathcal{P}_n(\nu_1; \cdot)\}_{n\geq 0}$ satisfy

$$\mathcal{P}_n(\nu_1; x) - \tau_{n-1} \mathcal{P}_{n-2}(\nu_1; x) = \frac{1}{n+1} \mathcal{P}'_{n+1}(\nu_0; x), \quad n \geq 2,$$

where $\tau_n \neq 0$ for $n \geq 1$. Such a study in the case of coherent pairs of measures of the first kind on the real line is also well known and this turned out to be a non-trivial and very important extension. As one can observe from results given in Chapter 3, the same can be said about our extension to coherent pairs of measures of the second kind. We have referred to such pairs of measures as *symmetric coherent pairs of measures on the real line.*

Finally in Chapter 5 we look at the orthogonal polynomials that follow from the Sobolev inner product (2), when $\{\nu_0, \nu_1\}$ is a symmetric coherent pair of measures of the second kind on the real line.

1 Basic Background

In this chapter we collect the basic results required to the development of this work. We also introduce the definitions and terminologies used. This material is essential for what follows.

1.1 Positive Chain Sequences

An important theory used in the development of the present work is the theory of positive chain sequences. Following the definition adopted by Chihara [20], we say that a sequence of real numbers $\{d_n\}_{n\geq 1}$ is a *positive chain sequence* if there exists a second sequence $\{g_n\}_{n\geq 0}$ such that

(i) $0 \leq g_0 < 1, \quad 0 < g_n < 1, \quad \text{for} \quad n \geq 1;$

(ii) $d_n = (1 - g_{n-1})g_n \quad \text{for} \quad n \geq 1.$

This concept was introduced by Wall [78] in his monograph on continued fractions and has been thoroughly explored by Chihara [20] and many others in studies regarding orthogonal polynomials defined on bounded intervals on the real line. The sequence $\{g_n\}_{n\geq 0}$ is called a parameter sequence of the positive chain sequence $\{d_n\}_{n\geq 1}$. The parameter g_0 is called initial parameter. In general, a parameter sequence of a positive chain sequence needs not to be unique.

Example 1.1. *For $\alpha > -1$ the sequence $\{d_n^{(\alpha)}\}_{n\geq 1}$ defined by*

$$d_n^{(\alpha)} = \frac{n(\alpha + n)}{(\alpha + 2n - 1)(\alpha + 2n + 1)}, \quad n \geq 1,$$

is a positive chain sequence in which one of its parameter sequence $\{g_n^{(\alpha)}\}_{n\geq 0}$ is such that

$$g_n^{(\alpha)} = \frac{n}{\alpha + 2n + 1}, \quad n \geq 0.$$

Definition 1.2. Let $\{d_n\}_{n\geq 1}$ be a positive chain sequence. A parameter sequence $\{m_n\}_{n\geq 0}$ is called its *minimal parameter sequence* if $m_0 = 0$.

Notice that every positive chain sequence $\{d_n\}_{n\geq 1}$ has a minimal parameter sequence $\{m_n\}_{n\geq 0}$ which can be obtained by setting $m_0 = 0$ and

$$m_n = \frac{d_n}{1 - m_{n-1}}, \quad n \geq 1.$$

If the minimal parameter sequence $\{m_n\}_{n\geq 0}$ is the only parameter sequence of $\{d_n\}_{n\geq 1}$, we say that $\{d_n\}_{n\geq 1}$ is uniquely determined.

When the positive chain sequence $\{d_n\}_{n\geq 1}$ is not uniquely determined we can also talk about its maximal parameter sequence $\{M_n\}_{n\geq 0}$.

Definition 1.3. Let $\{d_n\}_{n\geq 1}$ be a positive chain sequence. A parameter sequence $\{M_n\}_{n\geq 0}$ is called its *maximal parameter sequence* if $M_k > g_k$, $k \geq 0$, for every other parameter sequence $\{g_n\}_{n\geq 0}$.

The constant sequence $d_n = 1/4$, $n \geq 1$, is one of the simplest examples of positive chain sequences with $m_n = n/2(n+1)$ and $M_n = 1/2$, $n \geq 0$, as the minimal parameter sequence and maximal parameter sequence, respectively.

The following results are found in [20]. In the next theorem, we state the Wall's criterion for a parameter sequence to be the maximal parameter sequence.

Theorem 1.4. *Let $\{d_n\}_{n\geq 1}$ be a positive chain sequence. A parameter sequence $\{M_n\}_{n\geq 0}$ is the maximal parameter sequence of $\{d_n\}_{n\geq 1}$ if and only if*

$$\sum_{k=1}^{\infty} \frac{M_1 M_2 \cdots M_k}{(1-M_1)(1-M_2)\cdots(1-M_k)} = \infty.$$

Theorem 1.5. *Let $\{d_n\}_{n\geq 1}$ be a positive chain sequence and let $\{m_n\}_{n\geq 0}$ and $\{M_n\}_{n\geq 0}$ be, respectively, its minimal and maximal parameter sequences. Let $\{\tilde{d}_n\}_{n\geq 1}$ be a positive chain sequence with a parameter sequence $\{h_n\}_{n\geq 0}$. If $d_n \leq \tilde{d}_n$ for $n \geq 1$, then*

$$m_n \leq h_n \leq M_n, \quad n \geq 0.$$

Moreover, if we have in addition $d_{n_0} < \tilde{d}_{n_0}$ for some $n_0 \geq 1$, then

$$m_n < h_n \quad \text{for} \quad n \geq n_0 \qquad \text{and} \qquad h_j < M_j \quad \text{for} \quad j = 0, 1, \ldots, n_0 - 1.$$

Now we state an useful *comparison test* for chain sequences.

Theorem 1.6 (Comparison Test)**.** *Let $\{d_n\}_{n\geq 1}$ be a positive chain sequence and let $\{c_n\}_{n\geq 1}$ be a sequence. If $0 < c_n \leq d_n$ for $n \geq 1$, then $\{c_n\}_{n\geq 1}$ is also a positive chain sequence.*

Theorem 1.7. *Let $\{d_n\}_{n\geq 1}$ be a positive chain sequence such that $\lim_{n\to\infty} d_n = d$, then*

$$0 \leq d \leq 1/4.$$

Moreover, if $\{d_n\}_{n\geq 1}$ is uniquely determined, then

$$\lim_{n\to\infty} m_n = \frac{1}{2}[1 + \sqrt{1-4d}].$$

Otherwise, if $\{d_n\}_{n\geq 1}$ has multiple parameters sequences, then

$$\lim_{n\to\infty} m_n = \frac{1}{2}[1 - \sqrt{1-4d}] \qquad \text{and} \qquad \lim_{n\to\infty} M_n = \frac{1}{2}[1 + \sqrt{1-4d}].$$

To denote other positive chain sequences, we will use the notation

$$b_{k,n} = b_{n+k}, \quad k \geq 1,$$

where $\{b_n\}_{n\geq 0}$ is any sequence.

Theorem 1.8. *Let $\{d_n\}_{n\geq 1}$ be a positive chain sequence with parameter sequence $\{g_n\}_{n\geq 0}$ and let $\{m_n\}_{n\geq 0}$ and $\{M_n\}_{n\geq 0}$ be, respectively, its minimal and maximal parameter sequences. Then*

(i) $\{d_{1,n}\}_{n\geq 1}$ *is a positive chain sequence with parameter sequence* $\{g_{1,n}\}_{n\geq 0}$.

(ii) *If* $\{\widetilde{m}_n\}_{n\geq 0}$ *denotes the minimal parameter sequence of* $\{d_{1,n}\}_{n\geq 1}$*, then* $\widetilde{m}_n < m_{1,n}$ *for* $n \geq 0$.

(iii) $\{M_{1,n}\}_{n\geq 0}$ *is the maximal parameter sequence of* $\{d_{1,n}\}_{n\geq 1}$.

Remark 1.9. Note that the positive chain sequence $\{d_n\}_{n\geq 1}$ can be such that $M_0 = m_0 = 0$. But, it is important to note that always $0 < m_{1,0} \leq M_{1,0} < 1$. The equality $m_{1,0} = M_{1,0}$ holds when the positive chain sequence $\{d_n\}_{n\geq 1}$ has a unique parameter sequence.

1.2 Special Functions

In this section we give a short overview on special functions, q-special functions, hypergeometric series and basic hypergeometric series. We refer to, for example, Andrews, Askey and Roy [1], Ismail [33], Gasper and Rahman [26], Koekoek, Lesky and Swarttouw [37] and Slater [70].

For $a \in \mathbb{C}$, the *shifted factorial* or *Pochhammer symbol* is defined by

$$(a)_0 = 1 \quad \text{and} \quad (a)_n = a(a+1)(a+2)\cdots(a+n-1), \quad n \geq 1. \tag{1.1}$$

Since $(1)_n = n!$ for $n \geq 0$, the above definition can be seen as a generalization of the factorial. Also, we denote by $\Gamma(z)$ the Gamma function which is defined by the gamma integral

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \quad \mathrm{Re}(z) > 0.$$

The Gamma function satisfies the well-known property $\Gamma(z+1) = z\Gamma(z)$ with $\Gamma(1) = 1$, which shows that $\Gamma(n+1) = n!$ and $\Gamma(z+n) = (z)_n\Gamma(z)$ for $n \geq 0$.

The hypergeometric function ${}_rF_s$ is defined by the series

$$\begin{aligned} {}_rF_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \middle| z\right) &= {}_rF_s(a_1, \ldots, a_r; a_1, \ldots, a_s; z) \\ &= \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!}. \end{aligned} \tag{1.2}$$

Here $b_i \neq 0, -1, -2, \ldots$, for all i. If $a_i = -n$, $n = 0, 1, 2, \ldots$, for some i, then this hypergeometric function is a polynomial in z. The radius of convergence ρ of the hypergeometric series is given by

$$\rho = \begin{cases} \infty, & \text{if} \quad r < s+1, \\ 1, & \text{if} \quad r = s+1, \\ 0, & \text{if} \quad r > s+1. \end{cases}$$

One of the most important summation formulas for hypergeometric series is given by the Binomial Theorem

$${}_1F_0\left(\begin{matrix} a \\ - \end{matrix} \middle| z\right) = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k = (1-z)^{-a}, \quad |z| < 1,$$

which is a generalization of Newton binomial

$$ {}_1F_0\left(\begin{array}{c} -n \\ - \end{array}\middle| z\right) = \sum_{k=0}^{n} \frac{(-n)_k}{k!} z^k = \sum_{k=0}^{n} \binom{n}{k} (-z)^k = (1-z)^n, \quad n = 0, 1, 2, \ldots. $$

When $p = 2$ and $q = 1$ the hypergeometric series defined on (1.2), usually called the *Gauss hypergeometric series*, is given by

$$ {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1. $$

Now let us introduce the notion of *q-analogues* (or *q-extensions*) for some special functions and classical formulas which will be necessary in the development of our work. Unless otherwise stated, we shall always assume $0 < q < 1$. Notice that

$$ \lim_{q \to 1} \frac{1-q^a}{1-q} = a, \quad a \in \mathbb{C}. $$

The number $(1-q^a)/(1-q)$ is sometimes called the *basic number* (or *q-number*).

The q-analogue of the Pochhammer symbol (1.1) is defined by

$$ (a;q)_0 = 1, \quad (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \; n = 1, 2, \ldots. $$

Clearly,

$$ \lim_{q \to 1} \frac{(q^a;q)_n}{(1-q)^n} = (a)_n. $$

The symbols $(a;q)_n$ are called *q-Pochhammer symbols* (or *q-shifted factorials*). For negative subscripts we define

$$ (a;q)_{-n} = \frac{1}{\prod_{k=1}^{n}(1-aq^{-k})} = \frac{1}{(aq^{-n};q)_n}, \quad a \neq q^n, \, n = 1, 2, \ldots. $$

We can also define

$$ (a;q)_\infty = \prod_{n=0}^{\infty} (1-aq^n), $$

which implies that

$$ (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}. $$

The multiple q-Pochhammer symbols are defined by

$$ (a_1, a_2, \ldots, a_k; q)_n = \prod_{i=1}^{k} (a_i;q)_n. $$

The function $\Gamma_q(z)$ defined by

$$ \Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, $$

is called the *q-Gamma function*. This is a q-analogue of the Gamma function $\Gamma(z)$. In fact, we have

$$ \lim_{q \to 1} \Gamma_q(z) = \Gamma(z). $$

It is easily seen that $\Gamma_q(z)$ satisfies the functional equation

$$\Gamma_q(z+1) = \frac{1-q^z}{1-q}\Gamma_q(z), \quad \text{with} \quad \Gamma_q(1) = 1.$$

A basic hypergeometric or q-hypergeometric series is given by

$$\begin{aligned} {}_r\phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q, z \right) &= {}_r\phi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, z) \\ &= \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_k}{(q, b_1, \dots, b_s; q)_k} \left((-1)^k q^{k(k-1)/2}\right)^{s+1-r} z^k. \end{aligned} \tag{1.3}$$

Here the parameters b_i are such that the denominator factors in the terms of the series are non-zero. Since $(q^{-n}; q)_k = 0, k = n+1, n+2, \dots$, if a_i for some i is of the form q^{-n}, where n is a nonnegative integer, this basic hypergeometric series is a polynomial of degree n in z. The radius of convergence of this series is $1, 0$ or ∞ accordingly $r = s+1$, $r > s+1$ or $r < s+1$, as can be seen from the ratio test.

The q-hypergeometric series is a q-analogue of the hypergeometric series defined by (1.2) since

$$\lim_{q\to 1} {}_r\phi_s\left(\begin{matrix} q^{a_1}, \dots, q^{a_r} \\ q^{b_1}, \dots, q^{b_s} \end{matrix} \middle| q, (q-1)^{s+1-r} z \right) = {}_rF_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| z \right).$$

If we set $r = 2$, $s = 1$, $a_1 = q^a$, $a_2 = q^b$ and $b_1 = q^c$ in (1.3), we get

$${}_2\phi_1\left(q^a,\, q^b;\, q^c;\, q,\, z\right) = \sum_{k=0}^{\infty} \frac{(q^a; q)_k\, (q^b; q)_k}{(q^c; q)_k\, (q; q)_k} z^k. \tag{1.4}$$

This is the q-analogue of the Gauss hypergeometric series. The ${}_2\phi_1$ series was studied by Heine around the mid 19th-century and is usually called *Heine's series* or, in view of the base q, the *basic hypergeometric series* or *q-hypergeometric series*, or simply a *q-series*.

1.3 q-Difference Operators

In this section we give a short overview on some q-difference operators. For further information, we refer to, for example, Ismail [33] and Koekoek, Lesky and Swarttouw [37].

In [28] W. Hahn introduced the linear operator $\mathcal{A}_{q,\omega}$ defined by

$$\mathcal{A}_{q,\omega}[F(z)] = \frac{F(qz+\omega) - F(z)}{(q-1)z + \omega}, \quad 0 < q < 1,$$

where w is a complex number. This operator is called *Hahn's q-operator*. As a special case of $\mathcal{A}_{q,\omega}$ we have the q-difference operator

$$D_q^H[F(z)] = \mathcal{A}_{q,0}[F(z)] = \frac{F(z) - F(qz)}{(1-q)z}. \tag{1.5}$$

Clearly,

$$D_q^H[z^n] = \frac{1-q^n}{1-q} z^{n-1}.$$

In particular for differentiable function $F(z)$ we have $\lim_{q\to 1} D_q^H[F(z)] = F'(z)$.

Another q-difference operator is the Askey-Wilson divided difference operator $D_q^{A\text{-}W}$ (see [33]) defined by

$$D_q^{A\text{-}W}[f(x)] = \frac{F(q^{1/2}e^{i\theta}) - F(q^{-1/2}e^{-i\theta})}{(q^{1/2} - q^{-1/2})(z - 1/z)/2}, \quad x = (z + 1/z)/2, \tag{1.6}$$

where $f(x) = F(z)$, $z = e^{\pm i\theta}$. Here θ is not necessarily real.

The q-difference operator D_q of our interest is defined by

$$D_q[F(z)] = \frac{F(q^{-1/2}z) - F(q^{1/2}z)}{q^{-1/2}z - q^{1/2}z}, \tag{1.7}$$

where we assumed $0 < q < 1$. It is not difficult to see that

$$\lim_{q\to 1} D_q[F(z)] = F'(z),$$

if the function F is differentiable at z.

The action of the operator D_q on the monomial z^n is given by

$$D_q[z^n] = \{n\}_q \, z^{n-1}, \quad n \geq 1,$$

where

$$\{n\}_q = \frac{1 - q^n}{q^{(n-1)/2}(1 - q)} = \frac{q^{-n/2} - q^{n/2}}{q^{-1/2} - q^{1/2}}.$$

Note that $\{0\}_q = 0$ and $\lim_{q\to 1}\{n\}_q = n$, $n \geq 1$.

Moreover, if we apply the operator D_q on the q-hypergeometric series ${}_2\phi_1$ as in (1.4), then

$$D_q\left[{}_2\phi_1\Big(q^a,\, q^b;\, q^c;\, q,\, \tilde{r}z\Big)\right] = \tilde{r}\frac{(1 - q^a)(1 - q^b)}{(1 - q^c)(1 - q)}\,{}_2\phi_1\Big(q^{a+1},\, q^{b+1};\, q^{c+1};\, q,\, q^{-1/2}\tilde{r}z\Big). \tag{1.8}$$

Remark 1.10. The q-difference operator D_q and the numbers $\{n\}_q$ can also be referred to as the $(q^{-1/2}, q^{1/2})$ - derivative and the $(q^{-1/2}, q^{1/2})$ - integers, respectively. These terminologies come from the so-called (p, q) - calculus or post-quantum calculus introduced in [19] (see also [67] and the references therein).

Remark 1.11. Note that

$$D_q^H[F(z)] = D_q[F(q^{1/2}z)] \quad \text{and} \quad (z^2 - 1)D_q^{A\text{-}W}[F(z)] = 2z^2 D_q[F(z)],$$

where D_q^H and $D_q^{A\text{-}W}$ are the operators given by (1.5) and (1.6), respectively.

1.4 Orthogonal Polynomials on the Real Line

This section summarizes the basic concepts about moment functionals and orthogonal polynomials on the real line to be used in the sequel. These concepts can be found in Chihara [20], Ismail [33] and Szegő [74].

In [51] appears an algebraic approach to the study of linear functionals defined on the space of polynomials. This work by P. Maroni has shown to be an attractive alternative approach to the study of orthogonal polynomials since it provides a general perspective to the study of the topic (see also [53, 55, 56]). Since we use here this approach, we give below some of the required basic concepts.

Let $\mathbf{v}$ be a linear functional defined on the linear space $\mathbb{P}$ of polynomials with complex coefficients and consider $\mathbb{P}'$ its algebraic dual space, i.e., the linear space of all linear functionals defined on $\mathbb{P}$. If $\mathbf{v} \in \mathbb{P}'$ and p is a polynomial, $\langle \mathbf{v}, p \rangle$ will denote the image of p by $\mathbf{v}$. We begin by presenting some definitions of some basic operations in the space $\mathbb{P}^*$.

Definition 1.12. Let $\mathbf{v} \in \mathbb{P}^*$, $\pi \in \mathbb{P}$ and $q \in \mathbb{C}$.

(i) The *left multiplication of* $\mathbf{v}$ *by* π, denoted by $\pi\mathbf{v}$, is the linear functional on $\mathbb{P}^*$ defined by

$$\langle \pi\mathbf{v}, p \rangle = \langle \mathbf{v}, \pi p \rangle, \quad p \in \mathbb{P}.$$

(ii) The *distributional derivative of* $\mathbf{v}$, denoted by $\mathcal{D}\mathbf{v}$, is the linear functional on $\mathbb{P}^*$ defined by

$$\langle \mathcal{D}\mathbf{v}, p \rangle = -\langle \mathbf{v}, p' \rangle, \quad p \in \mathbb{P}.$$

It satisfies

$$\mathcal{D}(\pi\mathbf{v}) = \pi'\mathbf{v} + \pi\mathcal{D}\mathbf{v}.$$

(iii) We define the *division of* $\mathbf{v}$ *by* $(x-q)$ as

$$\left\langle \frac{1}{x-q}\mathbf{v}, p \right\rangle = \left\langle \mathbf{v}, \frac{p(x)-p(q)}{x-q} \right\rangle, \quad p \in \mathbb{P}.$$

(iv) The linear functional $\boldsymbol{\delta}_q$ given by

$$\langle \boldsymbol{\delta}_q, p \rangle = p(q), \quad p \in \mathbb{P},$$

is said to be the *Dirac delta* linear functional supported at q.

It is straightforward verified that

$$(x-q)\left[\frac{1}{x-q}\mathbf{v}\right] = \mathbf{v} \qquad \text{and} \qquad \frac{1}{x-q}\left[(x-q)\,\mathbf{v}\right] = q\mathbf{v} - (\mathbf{v})_0\,\boldsymbol{\delta}_q.$$

Every linear functional $\mathbf{v} \in \mathbb{P}^*$ can be associated with a sequence of complex numbers $\{(\mathbf{v})_n\}_{n\geq 0}$ where

$$(\mathbf{v})_n = \langle \mathbf{v}, x^n \rangle, \quad n = 0, 1, 2, \ldots,$$

which is called the *sequence of moments* of $\mathbf{v}$. Each $(\mathbf{v})_n$ is said to be the moment of order n of $\mathbf{v}$, $n \geq 0$. Since the moments $(\mathbf{v})_n$ play a central role in the study of these linear functionals, it is also customary to call such linear functionals as *moment functionals*.

The Gram matrix associated with the moment functional $\mathbf{v}$ in terms of the canonical basis $\{x^n\}_{n\geq 0}$ of $\mathbb{P}$ is given by

$$\mathbf{H} = \left[(\mathbf{v})_{i+j}\right]_{i,j=0}^{\infty} = \begin{bmatrix} (\mathbf{v})_0 & (\mathbf{v})_1 & \cdots & (\mathbf{v})_n & \cdots \\ (\mathbf{v})_1 & (\mathbf{v})_2 & \cdots & (\mathbf{v})_{n+1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ (\mathbf{v})_n & (\mathbf{v})_{n+1} & \cdots & (\mathbf{v})_{2n} & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{bmatrix}, \tag{1.9}$$

and it is known in the literature as *Hankel matrix*. We denote the determinant of the $(n+1)$-th principal leading submatrix of $\mathbf{H}$ as Δ_n.

Throughout this thesis $\delta_{m,n}$ will denote the Kronecker's delta symbol

$$\delta_{m,n} = \begin{cases} 1 & \text{if} \quad m = n, \\ 0 & \text{if} \quad m \neq n. \end{cases}$$

Definition 1.13. A sequence of monic polynomials $\{P_n\}_{n\geq 0}$ such that

$$deg(P_n) = n \quad \text{and} \quad \langle \mathbf{v}, P_m P_n \rangle = \hbar_n \delta_{m,n}, \quad \hbar_n \neq 0, \quad m, n = 0, 1, 2, \ldots,$$

is said to be a sequence of *monic orthogonal polynomials* (MOP) with respect to $\mathbf{v}$.

A direct consequence of the above definition is that a sequence of MOP $\{P_n\}_{n\geq 0}$ with respect to the moment functional $\mathbf{v}$ is a basis of $\mathbb{P}$. Then, there exists a unique sequence of linear functionals $\{\mathbf{v}_n\}_{n\geq 0}$, called the dual basis, such that

$$\langle \mathbf{v}_n, P_m \rangle = \delta_{n,m}, \quad n, m \geq 0, \tag{1.10}$$

where $\delta_{n,m}$ denotes the Kronecker delta. As a consequence, the linear functional $\mathbf{v}$ can be expressed as

$$\mathbf{v} = \sum_{n=0}^{\infty} \gamma_n \mathbf{v}_n, \quad \gamma_n = \langle \mathbf{v}, P_n \rangle .$$

Next, we introduce some preliminary results (see [41], [55]).

Lemma 1.14. *Let $\{P_n\}_{n\geq 0}$ be a sequence of MOP with respect to the moment functional $\mathbf{v}$ and $\{\mathbf{v}_n\}_{n\geq 0}$ the corresponding dual basis, then*

$$\mathbf{v}_n = \frac{P_n(x)}{\hbar_n} \mathbf{v}, \quad n \geq 0.$$

Lemma 1.15. *If $\{P_n\}_{n\geq 0}$, $\{\tilde{P}_n\}_{n\geq 0}$ are sequences of monic polynomials and $\{\mathbf{v}_n\}_{n\geq 0}$, $\{\tilde{\mathbf{v}}_n\}_{n\geq 0}$ are their corresponding dual bases and $\tilde{P}_n = P'_{n+1}/(n+1)$, then*

$$\mathcal{D}(\tilde{\mathbf{v}}_n) = -(n+1)\mathbf{v}_{n+1}.$$

Moreover, if $\{P_n\}_{n\geq 0}$ is a sequence of MOP with respect to the moment functional $\mathbf{v}$

$$\mathcal{D}(\tilde{\mathbf{v}}_n) = -(n+1) \frac{P_{n+1}(x)}{\hbar_{n+1}} \mathbf{v}, \quad n \geq 1.$$

The existence of a sequence of MOP with respect to moment functional $\mathbf{v}$ can be characterized by the next theorem.

Theorem 1.16. *Let $\mathbf{v}$ be a moment functional with its corresponding Hankel matrix given by* (1.9)*. A necessary and sufficient condition for the existence of a sequence of MOP with respect to $\mathbf{v}$ is*

$$\Delta_n \neq 0, \quad n = 0, 1, 2, \ldots .$$

In this situation, $\mathbf{v}$ is said to be quasi-definite.

The most important occurrence of orthogonal polynomials emerges when $\Delta_n > 0$ for all $n \geq 0$. In this case $\mathbf{v}$ is said to be a *positive definite* moment functional (see [20]) and it has the integral representation

$$\langle \mathbf{v}, x^n \rangle = \int_E x^n \, d\nu(x), \quad n = 0, 1, 2, \ldots,$$

where ν is a nontrivial positive Borel measure supported on some infinite subset $E \subseteq \mathbb{R}$. The above integral is also known in the literature as *Stieltjes integral.*

If ν is absolutely continuous, then we have

$$d\nu(x) = \omega(x)dx,$$

where $\omega : [a, b] \to \mathbb{R}$ is a non-negative function supported on some interval $[a, b] \subseteq \mathbb{R}$ (where $-\infty \leq a < b \leq \infty$). The function ω is known as a *weight function.* In this case one can write

$$\langle \mathbf{v}, x^n \rangle = \int_a^b x^n \, \omega(x)dx, \quad n = 0, 1, 2, \ldots. \tag{1.11}$$

Remark 1.17. Since a positive definite moment functional $\mathbf{v}$ can be defined by a positive measure ν, we can also refer to $\{P_n\}_{n\geq 0}$ as the sequence of MOP with respect to the positive measure ν.

One of the most important properties of orthogonal polynomials is that they satisfy a very simple relation known as *three-term recurrence relation* (TTRR, for short) that we state in the next theorem.

Theorem 1.18. *Let* $\mathbf{v}$ *be a quasi-definite moment functional and let* $\{P_n\}_{n\geq 0}$ *be its corresponding sequence of MOP. Then the polynomials* $P_n(x)$ *satisfy the three-term recurrence relation*

$$P_{n+1}(x) = (x - \beta_{n+1})P_n(x) - \alpha_{n+1}P_{n-1}(x), \quad n \geq 1, \tag{1.12}$$

with $P_0(x) = 1$ *and* $P_1(x) = x - \beta_1$*, where the coefficients* β_n *and* α_{n+1} *are given by*

$$\beta_n = \frac{1}{\hbar_{n-1}} \left\langle \mathbf{v}, \, xP_{n-1}^2 \right\rangle \quad \textit{and} \quad \alpha_{n+1} = \frac{\hbar_n}{\hbar_{n-1}} \neq 0, \quad n \geq 1.$$

Moreover, if $\mathbf{v}$ *is positive definite, then* β_n *is real and* $\alpha_{n+1} > 0$ *for* $n \geq 1$.

The converse of the previous theorem is valid and it is an important characterization of orthogonal polynomials and quasi-definite moment functionals, it is known in the literature as Favard's Theorem.

Theorem 1.19 (Favard)**.** *Let* $\{\beta_n\}_{n\geq 1}$ *and* $\{\alpha_n\}_{n\geq 1}$ *be two arbitrary sequences of complex numbers and let* $\{P_n\}_{n\geq 0}$ *be a sequence of monic polynomials defined by the TTRR* (1.12)*. Then there exists a unique moment functional* $\mathbf{v}$ *such that*

$$\langle \mathbf{v}, 1 \rangle = \alpha_1 \quad \textit{and} \quad \langle \mathbf{v}, P_m P_n \rangle = 0 \quad \textit{if} \quad n \neq m \quad m, n = 0, 1, 2, \ldots.$$

Moreover, $\mathbf{v}$ *is quasi-definite and* $\{P_n\}_{n\geq 0}$ *is its corresponding sequence of MOP if and only if* $\alpha_n \neq 0$ *for* $n \geq 1$*, while* $\mathbf{v}$ *is positive definite with* $\{P_n\}_{n\geq 0}$ *as its corresponding sequence of MOP if and only if* $\beta_n \in \mathbb{R}$ *and* $\alpha_n > 0$ *for each* $n \geq 1$.

Next we introduce the so-called *semiclassical* moment functionals. Let ϕ and ψ be two nonzero polynomials such that

$$\phi(x) = \lambda_j^{\phi} x^j + \dots, \ \ \lambda_j^{\phi} \neq 0, \ \ j \geq 0, \quad \text{and} \quad \psi(x) = \lambda_k^{\psi} x^k + \dots, \ \ \lambda_k^{\psi} \neq 0, \ \ k \geq 1.$$

Then, (ϕ, ψ) is said to be an *admissible pair* of polynomials if either $k \neq j-1$ or if $k = j-1$, then $n\lambda_{k+1}^{\phi} + \lambda_k^{\psi} \neq 0$, $n \geq 0$.

Definition 1.20. A moment functional $\mathbf{v}$ is said to be *semiclassical* if $\mathbf{v}$ is quasi-definite and there exist two polynomials ϕ and ψ such that

$$\mathcal{D}(\phi \mathbf{v}) = \psi \mathbf{v}, \tag{1.13}$$

where (ϕ, ψ) is an admissible pair of polynomials.

Notice that, the pair of polynomials satisfying the Pearson equation given in Definition 1.20 is not unique since, for instance, if the admissible pair of polynomials (ϕ_1, ψ_1) satisfies (1.13), then for every nonzero polynomial π, the pair $(\pi\phi_1, \pi\psi_1 + \pi'\phi_1)$ also satisfies (1.13) and it is an admissible pair of polynomials.

To a semiclassical moment functional $\mathbf{v}$ one can associate the *class* of $\mathbf{v}$ as the non-negative integer number s given by

$$\mathsf{s} = \min \Big\{ \max\{deg(\phi) - 2, deg(\psi) - 1\} : \mathcal{D}(\phi \mathbf{v}) = \psi \mathbf{v} \text{ and } \ (\phi, \psi) \ \text{ is admissible} \Big\}.$$

Observe that, s is defined as the minimum value among all pairs of admissible polynomials satisfying (1.13). For more details regarding the class of a semiclassical moment functional see for example [55].

The equation (1.13) is known in the literature as *Pearson equation* (see [52], [55], [68]).

One of the most important families of polynomials on the real line are the so-called *classical orthogonal polynomials*. They are the Hermite, Laguerre, Besel and Jacobi polynomials (some special cases are the Gegenbauer, Chebyshev and Legendre polynomials). These families are a special case of the semiclassical orthogonal polynomials since their corresponding moment functional $\mathbf{v}$ is of class $\mathsf{s} = 0$.

In the next table, we describe the main parameters of the classical families of MOP.

P_n	**Hermite**	**Laguerre**	**Jacobi**	**Bessel**
$\mathbf{v}$	$\mathcal{H}$	$\mathcal{L}_\alpha$	$\mathcal{J}_{\alpha,\beta}$	$\mathcal{B}_\alpha$
ϕ	1	x	$1-x^2$	x^2
ψ	$-2x$	$-x+\alpha+1$	$-(\alpha+\beta+2)x+\beta-\alpha$	$(\alpha+2)x+2$
$\omega(x)$	e^{-x^2}	$x^\alpha e^{-x}$	$(1-x)^\alpha(1-x)^\beta$	$x^\alpha e^{-2/x}$
E	$\mathbb{R}$	$(0,+\infty)$	$[-1,1]$	$\{z \in \mathbb{C} : \lvert z\rvert = 1\}$
β_n	0	$\alpha+2n-1$	$\frac{\beta^2-\alpha^2}{(\alpha+\beta+2n-2)(\alpha+\beta+2n)}$	$-\frac{2\alpha}{(\alpha+2n-2)(\alpha+2n)}$
α_{n+1}	$\frac{n}{2}$	$n(\alpha+n)$	$\frac{4n(\alpha+n)(\beta+n)(\alpha+\beta+n)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+2n+1)}$	$\frac{-4n(\alpha+n)}{(\alpha+2n-1)(\alpha+2n)^2(\alpha+\beta+2n+1)}$
		$-\alpha \notin \mathbb{N}$	$-\alpha, -\beta, -(\alpha+\beta) \notin \mathbb{N}$	$\alpha \notin \{0,-1,-2,\dots\}$

Table 1.1: The Classical families of MOP

The classical MOP are the only monic orthogonal polynomials such that their corresponding moment functional $\mathbf{v}$ satisfies the Pearson equation (1.13) with $deg(\phi) \leq 2$ and $deg(\psi) = 1$. In this case, we say that $\mathbf{v}$ is *classical*. In addition, a classical moment functional $\mathbf{v}$ has an integral representation as (1.11).

Remark 1.21. The moment functionals $\mathcal{H}$, $\mathcal{L}_\alpha$, $\mathcal{J}_{\alpha,\beta}$ and $\mathcal{B}_\alpha$ of the Table 1.1 are quasi-definite for all ranges of their parameters. Moreover, $\mathcal{H}$ is positive definite, $\mathcal{L}_\alpha$ is positive definite if $\alpha > -1$ and $\mathcal{J}_{\alpha,\beta}$ is positive definite if $\alpha > -1$ and $\beta > -1$.

Another important fact is that sequences of MOP satisfy the so-called Christoffel-Darboux identity.

Theorem 1.22 (Christoffel-Darboux Formula)**.** *Let $\{P_n\}_{n\geq 0}$ be a sequence of MOP satisfying the TTRR* (1.12) *with $\alpha_{n+1} \neq 0$ for $n \geq 1$. Then*

$$\sum_{j=0}^{n} \frac{P_j(x)P_j(y)}{\hbar_j} = \frac{1}{\hbar_n} \frac{P_{n+1}(x)P_n(y) - P_{n+1}(y)P_n(x)}{x-y}, \quad n \geq 0.$$

Moreover, if $\mathbf{v}$ is positive definite then

$$\sum_{j=0}^{n} \frac{P_j^2(x)}{\hbar_j} = \frac{1}{\hbar_n}[P'_{n+1}(x)P_n(x) - P_{n+1}(x)P'_n(x)] > 0, \quad n \geq 0.$$

Let $\mathbf{v}$ be a quasi-definite moment functional and let $\{P_n\}_{n\geq 0}$ be its corresponding sequence of MOP. Given a real or complex number κ and let the moment functional $(x-\kappa)\mathbf{v}$ be defined by (see [20])

$$\langle (x-\kappa)\mathbf{v}, x^n \rangle = (\mathbf{v})_{n+1} - \kappa(\mathbf{v})_n, \quad n = 0, 1, 2, \ldots.$$

It follows immediately that for every polynomial $p \in \mathbb{P}$

$$\langle (x-\kappa)\mathbf{v}, p \rangle = \langle \mathbf{v}, (x-\kappa)p \rangle .$$

We define the polynomials $P_n^*(\kappa, x)$ by

$$P_n^*(\kappa, x) = (x-\kappa)^{-1} \left[P_{n+1}(x) - \frac{P_{n+1}(\kappa)}{P_n(\kappa)} P_n(x) \right], \quad n \geq 0,$$

where κ is assumed not to be a zero of $P_n(x)$.

Theorem 1.23. *Let $\mathbf{v}$ be a quasi-definite moment functional and let $\{P_n\}_{n\geq 0}$ be its corresponding sequence of MOP. If κ is not a zero of $P_n(x)$ for any n, then $(x-\kappa)\mathbf{v}$ is quasi-definite and $\{P_n^*(\kappa, \cdot)\}_{n\geq 0}$ its the corresponding sequence of MOP.*

Moreover, if $\mathbf{v}$ is positive-definite on $[a, b]$, then $(x-\kappa)\mathbf{v}$ is also positive-definite on $[a, b]$ if and only if $a \geq \kappa$.

We will refer to $P_n^*(\kappa, x)$ as the *monic kernel polynomials* corresponding to $\mathbf{v}$ (or corresponding to the sequence $\{P_n\}_{n\geq 0}$) with K-parameter κ.

An important family of moment functionals is constituted by the symmetric moment functionals, i.e., $\langle \mathbf{v}, x^{2n+1} \rangle = 0$ for every $n \geq 0$. If $\mathbf{v}$ is a quasi-definite moment functional and $\{P_n\}_{n\geq 0}$ is its corresponding sequence of MOP, then we can define the moment functional $\mathbf{u}$ by

$$\langle \mathbf{u}, x^n \rangle = \left\langle \mathbf{v}, x^{2n} \right\rangle, \quad n \geq 0. \tag{1.14}$$

We can also define the sequences of monic polynomials $\{\mathcal{Q}_n\}_{n\geq 0}$ and $\{\widetilde{\mathcal{Q}}_n\}_{n\geq 0}$ by

$$P_{2n}(x) = \mathcal{Q}_n(x^2) \quad \text{and} \quad P_{2n+1}(x) = x\widetilde{\mathcal{Q}}_n(x^2). \tag{1.15}$$

Thus $\mathbf{u}$ is a quasi-definite moment functional and $\{\mathcal{Q}_n\}_{n\geq 0}$ and $\{\widetilde{\mathcal{Q}}_n\}_{n\geq 0}$ are the sequences of MOP with respect to $\mathbf{u}$ and $x\mathbf{u}$, respectively (see [20]).

Conversely, if $\mathbf{u}$ is a quasi-definite moment functional, it is possible to define the symmetric moment functional $\mathbf{v}$ by

$$\left\langle \mathbf{v}, x^{2n} \right\rangle = \langle \mathbf{u}, x^n \rangle \quad \text{and} \quad \left\langle \mathbf{v}, x^{2n+1} \right\rangle = 0, \quad n \geq 0. \tag{1.16}$$

If $\mathbf{u}$ and $x\mathbf{u}$ are quasi-definite moment functionals and $\{\mathcal{Q}_n\}_{n\geq 0}$ and $\{\widetilde{\mathcal{Q}}_n\}_{n\geq 0}$ are, respectively, the corresponding sequence of MOP. Then the symmetric moment functional $\mathbf{v}$ defined by (1.16) is quasi-definite and its sequence of MOP $\{P_n\}_{n\geq 0}$ is given by (1.15) (see [20]).

Remark 1.24. Note that $\{\widetilde{\mathcal{Q}}_n\}_{n\geq 0}$ are the kernel polynomials corresponding to $\mathbf{u}$ with K-parameter 0, i.e., $\widetilde{\mathcal{Q}}_n(x) = \mathcal{Q}_n^*(0, x)$. Besides, $\mathbf{v}$ is called the symmetrized linear functional of $\mathbf{u}$.

Given a semiclassical quasi-definite moment functional $\mathbf{u}$, the semiclassical character of the symmetrized linear functional of $\mathbf{u}$, its class, and the respective Pearson equation are described in the next theorem and were proved in [3].

Theorem 1.25. *Let $\mathbf{u}$ be semiclassical moment functional of class $\tilde{\mathsf{s}}$ satisfying the Pearson equation*

$$\mathcal{D}(\Phi\mathbf{u}) = \Psi\mathbf{u}.$$

Let $\mathbf{v}$ denote the symmetrization of $\mathbf{u}$ and let $x\mathbf{u}$ be a quasi-definite moment functional and $\Pi(x) = -\Phi'(x) + 2\Psi(x)$. Then, $\mathbf{v}$ is semiclassical of class s satisfying the Pearson equation given by (1.13)*, where the number s and the polynomials ϕ and ψ are defined according to the next cases:*

(i) *If $\Phi(0) = 0$ and $\Pi(0) = 0$, then*

$$\phi(x) = (\theta_0\Phi)(x^2), \quad \psi(x) = x[-(\theta_0^2\Phi)(x^2) + 2(\theta_0\Psi)(x^2)],$$

and $\mathsf{s} = 2\tilde{\mathsf{s}}$.

(ii) *If $\Phi(0) = 0$ and $\Pi(0) \neq 0$, then*

$$\phi(x) = x(\theta_0\Phi)(x^2), \quad \psi(x) = 2\Psi(x^2),$$

and $\mathsf{s} = 2\tilde{\mathsf{s}} + 1$.

(iii) *If $\Phi(0) \neq 0$, then*

$$\phi(x) = x\Phi(x^2), \quad \psi(x) = 2[x^2\Psi(x^2) + \Phi(x^2)],$$

and $\mathsf{s} = 2\tilde{\mathsf{s}} + 3$.

Here, $(\theta_q p)(x) = \dfrac{p(x) - p(q)}{x - q}, \quad p \in \mathbb{P}.$

Corollary 1.26. *If* s *is odd, then the polynomials* ϕ *and* ψ *in* (1.13) *are, respectively, odd and even functions. If* s *is even, then the polynomials* ϕ *and* ψ *in* (1.13) *are, respectively, even and odd functions.*

Finally, we present a well known relation between a sequence of MOP and positive chain sequence (see [20, Chapter IV, Theorem 2.4]). This result will be required in our study.

Theorem 1.27. *Let* $\{P_n\}_{n\geq 0}$ *be the sequence of MOP with respect to the positive definite moment functional* $\mathbf{v}$ *with support on* $(a,b)\subseteq\mathbb{R}$ *and let the sequence* $\{a_n(t)\}_{n\geq 1}$ *be given by*

$$a_n(t)=\frac{\alpha_{n+1}}{(t-\beta_n)(t-\beta_{n+1})},\quad n\geq 1,$$

where $\{\beta_n\}_{n\geq 1}$ *and* $\{\alpha_{n+1}\}_{n\geq 1}$ *are the coefficients of the corresponding TTRR* (1.12)*. Then for* $t\in\mathbb{R}\setminus(a,b)$*, the sequence* $\{a_n(t)\}_{n\geq 1}$ *is a positive chain sequence and its minimal parameter sequence* $m_n(t)_{n\geq 0}$ *is such that*

$$m_0(t)=0\quad\text{and}\quad m_n(t)=1-\frac{P_{n+1}(t)}{(t-\beta_{n+1})P_n(t)}=\frac{\alpha_{n+1}P_{n-1}(t)}{(t-\beta_{n+1})P_n(t)},\quad n\geq 1.$$

Example 1.28. *For* $\lambda>-1$ *and* $0<q<1$*, let us denote by* $\{\widehat{C}_n(x;q^{\lambda+1}|q)\}_{n\geq 0}$ *the sequence of monic continuous q-ultraspherical polynomials given by*

$$\widehat{C}_n(x;q^{\lambda+1}|q)=\frac{(q;q)_n}{2^n\,(q^{\lambda+1};q)_n}\,C_n(x;q^{\lambda+1}|q),\quad n\geq 0,$$

where

$$C_n(x;q^{\lambda+1}|q)=\frac{(q^{\lambda+1};q)_n}{(q;q)_n}e^{in\theta}\,{}_2\phi_1\left(\begin{matrix}q^{-n},\,q^{\lambda+1}\\ q^{-\lambda-n}\end{matrix}\,\middle|\,q,q^{-\lambda}e^{-2i\theta}\right)$$

is the continuous q-ultraspherical (or Rogers) polynomials the n-th degree usually defined (see, for example, [5] *and* [37, p. 469]*).*

It is known that, with the positive measure $\nu(x;q^{\lambda+1}|q)$ *on* $[-1,1]$ *given by*

$$d\nu(x;q^{\lambda+1}|q)=\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(q^{\lambda+1}e^{2i\theta},q^{\lambda+1}e^{-2i\theta};q)_\infty}\,(\sin\theta)^{-1}dx,\quad x=\cos\theta,$$

there holds the orthogonality relation

$$\int_{-1}^{1}\widehat{C}_n(x;q^{\lambda+1}|q)\,\widehat{C}_m(x;q^{\lambda+1}|q)\,d\nu(x;q^{\lambda+1}|q)=\hbar_n(q^{\lambda+1}|q)\delta_{m,n},\quad\text{for}\quad m,n=0,1,2,\ldots,$$

with

$$\hbar_n(q^{\lambda+1}|q)=\frac{\pi(q^{\lambda+1},q^{\lambda+2};q)_\infty}{(q,q^{2\lambda+2};q)_\infty}\frac{(1-q^{\lambda+1})(q^{2\lambda+2};q)_n(q;q)_n}{2^{2n-1}(1-q^{\lambda+n+1})(q^{\lambda+1};q)_n^2},\quad n\geq 0.$$

The monic continuous q-ultraspherical polynomials $\{\widehat{C}_n(x;q^{\lambda+1}|q)\}_{n\geq 0}$ *satisfy the TTRR*

$$\widehat{C}_{n+1}(x;q^{\lambda+1}|q)=x\widehat{C}_n(x;q^{\lambda+1}|q)-d^{(\lambda)}_{n+1}(q)\widehat{C}_{n-1}(x;q^{\lambda+1}|q),\quad n\geq 1,$$

with $\widehat{C}_0(x;q^{\lambda+1}|q)=1$ *and* $\widehat{C}_1(x;q^{\lambda+1}|q)=x$*, where the coefficients* $d^{(\lambda)}_{n+1}(q)$ *are given by*

$$d^{(\lambda)}_{n+1}(q)=\frac{(1-q^n)\,(1-q^{2\lambda+n+1})}{4\Big(1-q^{\lambda+n}\Big)\Big(1-q^{\lambda+n+1}\Big)},\quad n\geq 1.\tag{1.17}$$

Since the coefficients of the TTRR for the monic q-ultraspherical orthogonal polynomials are

$$\beta_n^{(\lambda)}(q) = 0 \quad \text{and} \quad \alpha_{n+1}^{(\lambda)}(q) = d_{n+1}^{(\lambda)}(q), \quad n \geq 1,$$

by Theorem 1.27, the sequence

$$a_{n,q}^{\lambda}(t) = \frac{\alpha_{n+1}^{(\lambda)}(q)}{(t - \beta_n^{(\lambda)}(q))(t - \beta_{n+1}^{(\lambda)}(q))} = \frac{d_{n+1}^{(\lambda)}(q)}{t^2}, \quad n \geq 1,$$

is a positive chain sequence for $t \in \mathbb{R} \setminus (-1, 1)$. In particular, if $t = 1$, the sequence $\{d_{n+1}^{(\lambda)}(q)\}_{n\geq 1}$ is a positive chain sequence.

1.5 Orthogonal Polynomials on the Unit Circle

In this section we give some of the basic results on the theory of orthogonal polynomials on the unit circle to be used in the next chapter. The basic sources for this section are Ismail [33] and Simon [69].

We consider a linear functional $\mathcal{L}$ in the linear space $\Lambda = \text{span}\{z^n : n \in \mathbb{Z}\}$ of the Laurent polynomials with complex coefficients such that

$$\mu_n = \langle \mathcal{L}, z^n \rangle = \overline{\langle \mathcal{L}, z^{-n} \rangle} = \overline{\mu}_{-n}, \quad n \in \mathbb{Z}.$$

The complex numbers $\{\mu_n\}_{n\in\mathbb{Z}}$ are said to be the moments associated with $\mathcal{L}$. The linear functional $\mathcal{L}$ can be referred to as a moment functional. Under these conditions, we can introduce a bilinear form associated with $\mathcal{L}$ in the space $\mathbb{P}$ of polynomials with complex coefficients as follows

$$\langle p(z), q(z) \rangle_{\mathcal{L}} = \left\langle \mathcal{L}, p(z)\overline{q(1/\overline{z})} \right\rangle, \quad p, q \in \mathbb{P}.$$

The Gram matrix associated with this bilinear form in terms of the monomial basis $\{z^n\}_{n\geq 0}$ of $\mathbb{P}$ is

$$\mathbf{T} = \left[\langle z^i, z^j \rangle_{\mathcal{L}}\right]_{i,j=0}^{\infty} = \begin{bmatrix} \mu_0 & \mu_{-1} & \cdots & \mu_{-n} & \cdots \\ \mu_1 & \mu_0 & \cdots & \mu_{-n+1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ \mu_n & \mu_{n+1} & \cdots & \mu_0 & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{bmatrix},$$

known in the literature as *Toeplitz matrix*. Notice that $\mathbf{T}$ is an Hermitian matrix. We denote the determinant of the $(n+1) \times (n+1)$ principal leading submatrix of $\mathbf{T}$ as ∇_n, with the convention $\nabla_{-1} = 1$.

The moment functional $\mathcal{L}$ is said to be *quasi-definite* (respectively, *positive definite*) if $\nabla_n \neq 0$ (respectively, $\nabla_n > 0$) for $n \geq 0$, and there exists a sequence of monic polynomials $\{\Phi_n\}_{n\geq 0}$ such that

$$\langle \Phi_n(z), \Phi_m(z) \rangle_{\mathcal{L}} = \mathbf{k}_n \delta_{n,m},$$

where $\mathbf{k}_n \neq 0$, $n \geq 0$. This sequence is said to be the monic orthogonal polynomial sequence corresponding to $\mathcal{L}$.

In the positive definite case, there exists a nontrivial positive measure $\mu(z) = \mu(e^{i\theta})$ supported on the unit circle $\mathbb{T} = \{z = e^{i\theta} : 0 \le \theta \le 2\pi\}$ such that

$$\langle f, g\rangle_\mu = \langle f(z), g(z)\rangle_{\mathcal{L}} = \int_{\mathbb{T}} f(z)\overline{g(z)}d\mu(z)$$

and the associated sequence of monic orthogonal polynomials $\{\Phi_n\}_{n\ge 0}$ is usually defined by

$$\langle \Phi_n, \Phi_m\rangle_\mu = \int_{\mathbb{T}} \Phi_n(z)\overline{\Phi_m(z)}d\mu(z) = \int_0^{2\pi} \overline{\Phi_n(e^{i\theta})}\Phi_m(e^{i\theta})d\mu(e^{i\theta}) = \kappa_n^{-2}\delta_{n,m}, \quad n, m \ge 0,$$

where $\kappa_n^{-2} = ||\Phi_n||^2 = \int_{\mathbb{T}} |\Phi_n(z)|^2 d\mu(z)$. This sequence is said to be the sequence of monic *orthogonal polynomials on the unit circle* (OPUC, in short) that are also known in the literature as *Szegő polynomials*. Here, μ is a nontrivial measure if its support is infinite. In addition, μ is said to be a probability measure if $\int_{\mathbb{T}} d\mu(z) = 1$.

The monic OPUC satisfy the so-called forward and backward recurrence relations, respectively,

$$\begin{aligned} \Phi_n(z) &= z\Phi_{n-1}(z) - \overline{\alpha}_{n-1}\Phi_{n-1}^*(z), \\ \Phi_n(z) &= (1 - |\alpha_{n-1}|^2)z\Phi_{n-1}(z) - \overline{\alpha}_{n-1}\Phi_n^*(z), \end{aligned} \qquad n \ge 1, \tag{1.18}$$

where $\overline{\alpha}_{n-1} = -\Phi_n(0)$ and $\Phi_n^*(z) = z^n\overline{\Phi_n(1/\overline{z})}$ denotes the reversed (reciprocal) polynomial of $\Phi_n(z)$. Following Simon [69] we refer to the numbers α_n as *Verblunsky coefficients*. It is known that these coefficients are such that $|\alpha_n| < 1$, $n \ge 0$. It is also well known that OPUC are completely characterized by the coefficients $\{\alpha_n\}_{n\ge 0}$ as given by the following theorem.

Theorem 1.29. *Given an arbitrary sequence of complex numbers $\{\alpha_n\}_{n\ge 0}$, where $|\alpha_n| < 1$, $n \ge 0$, then associated with this sequence there exists a unique nontrivial probability measure on the unit circle such that the polynomials generated by* (1.18) *are the corresponding OPUC.*

In the next examples we introduce the "general circular Jacobi monic polynomials" and "general Pastro orthogonal monic polynomials". These polynomials, which are a generalization of the polynomials analyzed in [4] and [66], were extensively studied in [72] and [18]. Neither of these polynomials are in the list of examples discussed in the Section 1.6 of [69].

Example 1.30 (General circular Jacobi monic polynomials)**.** *Let $b = \lambda + i\eta$ be such that $\lambda > -1/2$. The polynomials*

$$\Phi_n^{(b)}(z) \; = \frac{(\overline{b})_n}{(b+1)_n}\, {}_2F_1(-n,\, b+1;\, -\overline{b} - n + 1;\, z), \quad n \ge 0, \tag{1.19}$$

are the monic OPUC defined by

$$\left\langle \Phi_m^{(b)}, \Phi_n^{(b)} \right\rangle_{\mu^{(b)}} \; = \int_{\mathbb{T}} \overline{\Phi_m^{(b)}(\zeta)}\, \Phi_n^{(b)}(\zeta)\, d\mu^{(b)}(\zeta) = A_n^{(b)}\, \delta_{n,m},$$

with respect to the probability measure $\mu^{(b)}$ supported on the unit circle. The probability measure $\mu^{(b)}$ is such that

$$i2\pi\zeta\frac{d\mu^{(b)}(\zeta)}{d\zeta} = \tau^{(b)}w^{(b)}(\zeta), \quad \textit{with} \quad w^{(b)}(e^{i\theta}) = e^{(\pi-\theta)\,\mathrm{Im}(b)}(4\sin^2(\theta/2))^{\mathrm{Re}(b)}, \tag{1.20}$$

where

$$\tau^{(b)} = \frac{|\Gamma(b+1)|^2}{\Gamma(b+\bar{b}+1)}.$$

Moreover, the coefficients $A_n^{(b)}$ *satisfy*

$$A_n^{(b)} = \frac{(b+\bar{b}+1)_n \, n!}{|(b+1)_n|^2}, \quad n \geq 0.$$

There is also an alternative expression for $\Phi_n^{(b)}$ (see [72])

$$\Phi_n^{(b)}(z) = \frac{(b+\bar{b}+1)_n}{(b+1)_n} \; {}_2F_1(-n, b+1; b+\bar{b}+1; 1-z), \quad n \geq 0.$$

Example 1.31 (General Pastro orthogonal monic polynomials)**.** *Let* $b = \lambda + i\eta$ *be such that* $\lambda > -1/2$. *The polynomials*

$$\Phi_n^{(b)}(q;z) \;=\; \frac{(q^{\bar{b}};\, q)_n}{(q^{b+1};\, q)_n}\, q^{n/2} \, {}_2\phi_1(q^{-n},\, q^{b+1};\, q^{-\bar{b}-n+1};\, q,\, q^{-\bar{b}+1/2}z), \quad n \geq 0, \tag{1.21}$$

are q-analogues of the polynomials given by (1.19) *since* $\lim_{q\to 1} \Phi_n^{(b)}(q;z) = \Phi_n^{(b)}(z)$. *These monic polynomials are OPUC satisfying*

$$\left\langle \Phi_m^{(b)}(q;\,.\,), \Phi_n^{(b)}(q;\,.\,)\right\rangle_{\mu_q^{(b)}} \;=\; \int_{\mathbb{T}} \overline{\Phi_m^{(b)}(q;\zeta)}\, \Phi_n^{(b)}(q;\zeta)\, d\mu_q^{(b)}(\zeta) = A_n^{(b)}(q)\, \delta_{n,m}, \tag{1.22}$$

with respect to the probability measure $\mu_q^{(b)}$ *supported on the unit circle. The probability measure* $\mu_q^{(b)}$ *is such that*

$$i2\pi\zeta \frac{d\mu_q^{(b)}(\zeta)}{d\zeta} = \tau_q^{(b)} w_q^{(b)}(\zeta), \quad \textit{with} \quad w_q^{(b)}(\zeta) = \frac{|(q^{1/2}\zeta;\, q)_\infty|^2}{|(q^{b+1/2}\zeta;\, q)_\infty|^2}, \tag{1.23}$$

where

$$\tau_q^{(b)} = \frac{(q;\, q)_\infty (q^{b+\bar{b}+1};\, q)_\infty}{(q^{b+1};\, q)_\infty (q^{\bar{b}+1};\, q)_\infty} = \frac{|\Gamma_q(b+1)|^2}{\Gamma_q(b+\bar{b}+1)}. \tag{1.24}$$

Also, the coefficients $A_n^{(b)}(q)$ *satisfy*

$$A_n^{(b)}(q) = \frac{(q^{b+\bar{b}+1};\, q)_n\, (q;\, q)_n}{|(q^{b+1};\, q)_n|^2}, \quad n \geq 0.$$

1.6 Para-orthogonal Polynomials on the Unit Circle and Positive Chain Sequences

Given a sequence $\{\Phi_n\}_{n\geq 0}$ of monic OPUC with respect to the positive measure μ on the unit circle, the associated sequence $\{\Psi_n\}_{n\geq 0}$ of monic *para-orthogonal polynomials on the unit circle* (POPUC for short) is such that

$$\Psi_n(z) = z\Phi_{n-1}(z) - \rho_n \Phi_{n-1}^*(z), \quad n \geq 1,$$

where ρ_n is a any sequence of complex numbers such that $|\rho_n| = 1$. These polynomials are interesting since their zeros are all simple and lie on the unit circle $\mathbb{T}$. As an application, the zeros of these polynomials are used in the quadrature rules on the unit circle. These facts were first observed in [35].

Let us consider some results established in [13] and [9]. Let the sequences of polynomials $\{R_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ be such that

$$\begin{aligned} R_{n+1}(z) &= [(1+ic_{n+1})z + (1-ic_{n+1})]R_n - 4d_{n+1}zR_{n-1}(z), \\ Q_{n+1}(z) &= [(1+ic_{n+1})z + (1-ic_{n+1})]Q_n - 4d_{n+1}zQ_{n-1}(z), \end{aligned} \quad n \geq 1. \tag{1.25}$$

with $R_0(z) = 1$, $Q_0(z) = 0$, $R_1(z) = (1+ic_1)z + (1-ic_1)$ and $Q_1(z) = 2d_1$, where $\{c_n\}_{n\geq 1}$ is a sequence of real numbers and $\{d_n\}_{n\geq 1}$ is a positive chain sequence.

Remark 1.32. Note that the first element d_1 of the positive chain sequence $\{d_n\}_{n\geq 1}$ doest not affect the sequence of polynomials $\{R_n\}_{n\geq 0}$, however has an influence in the sequence of polynomials $\{Q_n\}_{n\geq 0}$ and consequently, in the results of this section (see [13] for more details).

From the recurrence formula (1.25) we have

$$R_n^*(z) = z^n\overline{R_n(1/\overline{z})} = R_n(z) \quad \text{and} \quad Q_n^*(z) = z^{n-1}\overline{Q_n(1/\overline{z})} = Q_n(z), \quad n \geq 1.$$

With this property the polynomials $\{R_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ can be called self-inversive polynomials. Moreover, if $R_n(z) = \sum_{k=0}^{n} r_{n,k}z^k$ and $Q_n(z) = \sum_{k=0}^{n-1} q_{n,k}z^k$, then

$$r_{n,n} = \overline{r_{n,0}} = \prod_{k=1}^{n}(1+ic_k), \;\; n \geq 1, \quad \text{and} \quad q_{n,n-1} = \overline{q_{n,0}} = 2d_1\prod_{k=2}^{n}(1+ic_k), \;\; n \geq 2.$$

Lemma 1.33. *Let $\{R_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ be the sequences of polynomials obtained from* (1.25)*. Then there exist two series expansions*

$$E_0(z) = -\sum_{n=0}^{\infty}\nu_{n+1}z^n \quad \text{and} \quad E_\infty(z) = \sum_{n=1}^{\infty}\nu_{-n+1}z^{-n},$$

where $\nu_n = -\overline{\nu}_{-n+1}$, $n \geq 1$, such that there hold the correspondence properties

$$E_0(z) - \frac{Q_n(z)}{R_n(z)} = \frac{\overline{\gamma}_n}{\overline{r}_{n,n}}z^n + O(z^{n+1}) \quad \text{and} \quad E_\infty(z) - \frac{Q_n(z)}{R_n(z)} = \frac{\gamma_n}{r_{n,n}}\frac{1}{z^{n+1}} + O((1/z)^{n+2}),$$

for $n \geq 0$. Moreover, if the moment functional $\mathcal{L}$ on the space of Laurent polynomials is defined by

$$\left\langle \mathcal{L}, z^{-n} \right\rangle = \nu_n, \quad n = 0, \pm 1, \pm 2, \ldots,$$

then the polynomials R_n satisfy the orthogonality property

$$\left\langle \mathcal{L}, z^{-n+j}R_n \right\rangle = \begin{cases} -\overline{\gamma}_n, & j = -1, \\ 0, & j = 0, 1, \ldots, n-1, \\ \gamma_n, & j = n, \end{cases} \quad n \geq 1.$$

Here, $\gamma_0 = \nu_0 = \dfrac{2d_1}{1+ic_1}$ and $\gamma_n = \dfrac{4d_{n+1}}{1+ic_{n+1}}\gamma_{n-1}$, $n \geq 1$.

Theorem 1.34. *Taking into account the real sequence* $\{c_n\}_{n\geq 1}$ *and the positive chain sequence* $\{d_n\}_{n\geq 1}$ *there exists a unique nontrivial probability measure* μ *on the unit circle. If* $M_0 > 0$*, where* $\{M_n\}_{n\geq 0}$ *is the maximal parameter sequence of* $\{d_n\}_{n\geq 1}$*, then* μ *has a pure point of mass* M_0 *at* $z = 1$*. Let* $\mathcal{L}$ *be the moment functional associated with* $\{c_n\}_{n\geq 1}$ *and* $\{d_n\}_{n\geq 1}$ *as given by Lemma 1.33. Then*

$$\left\langle \mathcal{L}, z^{-n} \right\rangle = \int_{\mathbb{T}} z^{-n}(1-z)d\mu(z), \quad n = 0, \pm 1, \pm 2, \ldots.$$

Thus, from Lemma 1.33 and Theorem 1.34, it follows that

$$\nu_n = \left\langle \mathcal{L}, z^{-n} \right\rangle = \int_{\mathbb{T}} z^{-n}(1-z)d\mu(z), \quad n = 0, \pm 1, \pm 2, \ldots,$$

and for $n \geq 1$

$$\int_{\mathbb{T}} z^{-n} R_n(z)(1-z)d\mu(z) = 0, \quad 0 \leq k \leq n-1.$$

Moreover, the authors of [13] proved that, if $\hat{\gamma}_n = \int_{\mathbb{T}} R_n(z)d\mu(z)$, $n \geq 0$, then

$$\hat{\gamma}_0 = 1 \quad \text{and} \quad \hat{\gamma}_n = 2(1-m_n)\hat{\gamma}_{n-1}, \quad n \geq 1,$$

where $\{m_n\}_{n\geq 0}$ is the minimal parameter sequence of the positive chain sequence $\{d_n\}_{n\geq 1}$.

The following theorem is an immediate consequence of the above results.

Theorem 1.35. *If the sequence of polynomials* $\{\Phi_n\}_{n\geq 0}$ *is such that*

$$\Phi_0(z) = 1 \quad \textit{and} \quad \Phi_n(z)\prod_{k=1}^{n}(1+ic_k) = R_n(z) - 2(1-m_n)R_{n-1}(z), \quad n \geq 1,$$

then $\{\Phi_n\}_{n\geq 0}$ *is the sequence of monic OPUC with respect to* μ*.*

Example 1.36. *Let the sequences* $\{c_n\}_{n\geq 1}$ *and* $\{d_n\}_{n\geq 1}$ *be given by*

$$\begin{aligned} c_n &= \frac{\eta}{\lambda+n}, \quad n \geq 1, \\ d_1 &= d_1^{(\lambda,\epsilon)} = \frac{1}{2}\frac{2\lambda+1}{\lambda+1}(1-\epsilon), \quad d_{n+1} = \frac{1}{4}\frac{n(2\lambda+n+1)}{(\lambda+n)(\lambda+n+1)}, \quad n \geq 1 \end{aligned} \tag{1.26}$$

where $\lambda > -1/2$, $\eta \in \mathbb{R}$ *and* $0 \leq \epsilon < 1$.

Notice that, as verified in [17]*, the sequence* $\{d_n\}_{n\geq 1}$ *is a positive chain sequence with maximal parameter sequence* $\{M_n^{(\epsilon)}\}_{n\geq 0}$ *given by*

$$M_0^{(\epsilon)} = \epsilon, \quad M_n^{(\epsilon)} = \frac{1}{2}\frac{2\lambda+n}{\lambda+n}, \quad n \geq 1.$$

The polynomials $R_n^{(b)}(z)$ *generated by the sequences* $\{c_n\}_{n\geq 1}$ *and* $\{d_{n+1}\}_{n\geq 1}$ *given by* (1.26) *together with the first recurrence formula* (1.25) *are*

$$\begin{aligned} R_n^{(b)}(z) &= \frac{(\bar{b}+1)_n}{(\lambda+1)_n}\, {}_2F_1(-n,\, b+1;\, -\bar{b}-n;\, z) \\ &= \frac{(2\lambda+2)_n}{(\lambda+1)_n}\, {}_2F_1(-n,\, b+1;\, b+\bar{b}+2;\, 1-z), \quad n \geq 0, \end{aligned} \tag{1.27}$$

where $b = \lambda + i\eta$. *Note that* d_1 *doest not affect the polynomials* $R_n^{(b)}(z)$.

From Theorem 1.35, if

$$\Phi_n^{(b,\epsilon)}(z) = \frac{(\lambda+1)_n}{(b+1)_n}\Big[R_n^{(b)}(z) - 2[1 - m_n^{(\lambda,\epsilon)}]R_{n-1}^{(b)}(z)\Big], \tag{1.28}$$

where $\{m_n^{(\lambda,\epsilon)}\}_{n\geq 0}$ *is the minimal parameter sequence of the positive chain sequence* $\{d_n\}_{n\geq 1}$, *the sequence of polynomials* $\{\Phi_n^{(b,\epsilon)}\}_{n\geq 0}$ *is the sequence of monic OPUC with respect the probability measure* $\mu^{(b,\epsilon)}$ *given by*

$$\int_{\mathbb{T}} f(\zeta)d\mu^{(b,\epsilon)}(\zeta) = (1-\epsilon)\int_{\mathbb{T}} f(\zeta)d\mu^{(b)}(\zeta) \ + \ \epsilon f(1).$$

Observe that $\{\Phi_n^{(b,0)}\}_{n\geq 0}$ *is the sequence of general circular Jacobi polynomials* $\{\Phi_n^{(b)}\}_{n\geq 0}$ *and* $\mu^{(b)}$ *is the probability measure given in* (1.20).

2 Pastro Polynomials and Sobolev-Type OPUC Based on the D_q Operator

The aim in this chapter is to consider the monic orthogonal polynomials $\Psi_n^{(b,\epsilon,s)}(q;z)$ with respect to the Sobolev-type inner product

$$\left\langle f,g\right\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}} = \left\langle f,g\right\rangle_{\widetilde{\mu}_q^{(b,\epsilon)}} + s\left\langle D_q[f],D_q[g]\right\rangle_{\mu_q^{(b+1)}}, \tag{2.1}$$

where the pair of measures $\{\widetilde{\mu}_q^{(b,\epsilon)},\mu_q^{(b+1)}\}$ satisfy a coherence type property of the second kind on the unit circle with respect to the q-difference operator D_q given by (1.7). That is, the associated sequences of monic OPUC $\{\widetilde{\Phi}_n^{(b,\epsilon)}(q;\cdot)\}_{n\geq 0}$ and $\{\Phi_n^{(b+1)}(q;\cdot)\}_{n\geq 0}$ of $\widetilde{\mu}_q^{(b,\epsilon)}$ and $\mu_q^{(b+1)}$, respectively, satisfy

$$D_q[\widetilde{\Phi}_n^{(b,\epsilon)}(q;z)] = \{n\}_q\left[\Phi_{n-1}^{(b+1)}(q;z) - \chi_n^{(b,\epsilon)}(q)\,\Phi_{n-2}^{(b+1)}(q;z)\right], \quad n\geq 2.$$

with $D_q[\widetilde{\Phi}_1^{(b,\epsilon)}(q;z)] = 1 = \Phi_0^{(b+1)}(q;z)$. This property is proved in Theorem 2.9.

The measures $\mu_q^{(b+1)}$ and $\widetilde{\mu}_q^{(b,\epsilon)}$, which we consider to be probability measures on the unit circle, are defined as follows:

(i) $\mu_q^{(b+1)}$ is the probability measure given in (1.23). We write

$$d\mu_q^{(b+1)}(\zeta) = \frac{1}{i2\pi\zeta}\tau_q^{(b+1)}\frac{|(q^{1/2}\zeta;\,q)_\infty|^2}{|(q^{b+3/2}\zeta;\,q)_\infty|^2}d\zeta,$$

with

$$\tau_q^{(b+1)} = \frac{(q;\,q)_\infty(q^{b+\bar{b}+3};\,q)_\infty}{(q^{b+2};\,q)_\infty(q^{\bar{b}+2};\,q)_\infty} = \frac{|\Gamma_q(b+2)|^2}{\Gamma_q(b+\bar{b}+3)}.$$

Hence, its corresponding sequence of monic OPUC are the general Pastro orthogonal monic polynomials $\{\Phi_n^{(b+1)}(q;\cdot)\}_{n\geq 0}$ given in the Example 1.31.

(ii) $\widetilde{\mu}_q^{(b,\epsilon)}$ is the probability measure such that

$$\left\langle f,g\right\rangle_{\widetilde{\mu}_q^{(b,\epsilon)}} = (1-\epsilon)\tilde{\tau}_q^{(b)}\int_{\mathbb{T}} f(\zeta)\overline{g(\zeta)}\,\frac{|(q\zeta;q)_\infty|^2}{|(q^{b+1}\zeta;q)_\infty|^2}\frac{1}{i2\pi\zeta}\,d\zeta \;+\; \epsilon\, f(1)\overline{g(1)}, \tag{2.2}$$

where

$$\tilde{\tau}_q^{(b)} = \frac{(1-q^{\bar{b}+1})(1-q^{\lambda+1}\cos\eta_q)(q;q)_\infty(q^{2\lambda+2};q)_\infty}{(1-q^{\lambda+1}\cos\eta_q)\,{}_2\phi_1(q,\,q^{-b};\,q^{\bar{b}+2};\,q,\,q^{b+1})}\,\frac{1}{|(q^{b+1};q)_\infty|^2}, \quad \eta_q = -\eta\ln(q).$$

More information on this probability measure will be presented in Theorems 2.6 and 2.7.

We study some properties of $\Psi_n^{(b,\epsilon,s)}(q;z)$ and establish the connection formulas that they satisfy with the monic OPUC corresponding to the measure $\widetilde{\mu}_q^{(b,\epsilon)}$. Bounds for the connection coefficients as well as outer relative asymptotics are also provided. The results given in this chapter have appeared in the paper [30].

2.1 Introduction

The sequence of polynomials $\{p_n(z;\alpha,\beta)\}_{n\geq 0}$ given by

$$p_n(z;\alpha,\beta) = {}_2\phi_1\Big(q^{-n}, q^{\alpha};\, q^{-\beta-n+2};\, q,\, q^{-\beta+3/2}z\Big), \quad n \geq 0, \quad \alpha,\beta\in\mathbb{R}, \tag{2.3}$$

was introduced by P. I. Pastro [66], where it was showed that these polynomials satisfy the biorthogonal relation on the unit circle

$$\int_{\mathbb{T}} p_n(\zeta;\alpha,\beta)p_m(\overline{\zeta};\beta,\alpha)\, w(\zeta;\alpha,\beta)\frac{1}{i2\pi\zeta}d\zeta = \mathfrak{h}_n\, \delta_{m,n}.$$

Here, $w(\zeta;\alpha,\beta)$ is the q-beta weight function given by

$$w(\zeta;\alpha,\beta) = \frac{(q^{1/2}\zeta;q)_\infty\, (q^{1/2}\overline{\zeta};q)_\infty}{(q^{\alpha-1/2}\zeta;q)_\infty\, (q^{\beta-1/2}\overline{\zeta};q)_\infty}.$$

We remind that $\mathbb{T}\equiv\{\zeta=e^{i\theta}\colon 0\leq\theta<2\pi\}$ and it is also assumed that $0<q<1$.

Remark 2.1. When $\beta=\alpha$, the sequence of polynomials $\{p_n(z;\alpha,\alpha)\}_{n\geq 0}$ is a sequence of OPUC with respect to the positive weight function $w(\zeta,\alpha,\alpha)$.

The results obtained in Pastro [66] were inspired by the following results given by R. Askey in [4, p. 304]. Let the sequence of hypergeometric polynomials $\{P_n(z;\alpha,\beta)\}_{n\geq 0}$ be such that

$$P_n(z;\alpha,\beta) = {}_2F_1\Big(-n,\, \alpha;\, -\beta-n+2;\, z\Big), \quad n\geq 0, \quad \alpha,\beta\in\mathbb{R}. \tag{2.4}$$

Then these polynomials satisfy the biorthogonal relation

$$\int_{\mathbb{T}} P_n(\zeta;\alpha,\beta)P_m(\overline{\zeta};\beta,\alpha)\, W(\zeta;\alpha,\beta)\frac{1}{i2\pi\zeta}d\zeta = \mathfrak{H}_n\delta_{m,n},$$

with $W(e^{i\theta};\alpha,\beta) = e^{i(\alpha-\beta)\theta/2}|\sin(\theta/2)|^{\alpha+\beta-2}$.

Remark 2.2. When $\alpha=\beta$, the weight function $W(\zeta;\alpha,\alpha)$ is positive on the unit circle and the polynomials $P_n(z;\alpha,\alpha)$ are OPUC with respect to this weight function.

Following [4] and [66], the parameters α and β in the polynomials (2.3) and (2.4) are assumed to be real. However, it is now known that (see [72, 18]) the parameters α and β can be extended to complex values.

In this chapter our goal is to consider some properties of the following three sequences of monic polynomials

$$\{\widehat{R}_n^{(b)}(q;\cdot)\}_{n\geq 0}, \quad \{\widetilde{\Phi}_n^{(b,\epsilon)}(q;\cdot)\}_{n\geq 0} \;\text{ and }\; \{\Psi_n^{(b,\epsilon,s)}(q;\cdot)\}_{n\geq 0},$$

where $\widehat{R}_n^{(b)}(q;z)$ is defined in (2.8) below, $\widetilde{\Phi}_n^{(b,\epsilon)}(q;z)$ is the monic OPUC with respect to inner product (2.2), and $\Psi_n^{(b,\epsilon,s)}(q;z)$ are the monic orthogonal polynomials with respect to Sobolev-type inner product (2.1).

Using the q-difference operator (1.7), in the next sections we present the proof the following properties:

A) The polynomials $\widehat{R}_n^{(b)}(q;z)$ are such that

$$D_q[\widehat{R}_n^{(b)}(q;z)] = \{n\}_q\, \Phi_{n-1}^{(b+1)}(q;z), \quad n \geq 1.$$

B) The polynomials $\widetilde{\Phi}_n^{(b,\epsilon)}(q;z)$ satisfy $D_q[\widetilde{\Phi}_1^{(b,\epsilon)}(q;z)] = 1 = \Phi_0^{(b+1)}(q;z)$ and

$$D_q[\widetilde{\Phi}_n^{(b,\epsilon)}(q;z)] = \{n\}_q \left[\Phi_{n-1}^{(b+1)}(q;z) - \chi_n^{(b,\epsilon)}(q)\, \Phi_{n-2}^{(b+1)}(q;z)\right], \quad n \geq 2.$$

C) The polynomials $\Psi_n^{(b,\epsilon,s)}(q;z)$ satisfy the connection formula

$$\Psi_n^{(b,\epsilon,s)}(q;z) - \beta_n^{(b,\epsilon,s)}(q)\, \Psi_{n-1}^{(b,\epsilon,s)}(q;z) = \widetilde{\Phi}_n^{(b,\epsilon)}(q;z), \quad n \geq 1.$$

The polynomials $\Phi_n^{(b)}(q;z)$ are the general Pastro orthogonal polynomials, given in (1.21).

2.2 Preliminary Results on q-Hypergeometric Polynomials

In this section we mention some results of certain families of q-hypergeometric polynomials, given in [18] and [2], that we need for what follows.

First we consider the family of polynomials given by

$$Q_n^{(b,c,d)}(q;z) = \frac{(q^{c-b+1};q)_n}{(q^{b+1};q)_n} q^{n(b-d+1)}\, {}_2\phi_1\Big(q^{-n},\, q^{b+1};\, q^{-c+b-n};\, q,\, q^{-c+d-1}z\Big), \quad n \geq 0,$$

where $0 < q < 1$ and b, c and d are complex parameters with $b \neq -1, -2, \ldots$ and $c - b + 1 \neq -1, -2, \ldots$. These polynomials satisfy the three term recurrence formula (see [18])

$$Q_{n+1}^{(b,c,d)}(q;z) = \Big(z + \beta_{n+1}^{(b,c,d)}\Big) Q_n^{(b,c,d)}(q;z) - \alpha_{n+1}^{(b,c,d)}\, z Q_{n-1}^{(b,c,d)}(q;z), \quad n \geq 1, \tag{2.5}$$

with $Q_0^{(b,c,d)}(q;z) = 1$ and $Q_1^{(b,c,d)}(q;z) = z + \beta_1^{(b,c,d)}$, where

$$\beta_n^{(b,c,d)} = \frac{1 - q^{c-b+n}}{1 - q^{b+n}}\, q^{b-d+1}, \quad \alpha_{n+1}^{(b,c,d)} = \frac{(1-q^n)\,(1-q^{c+n+1})}{(1-q^{b+n})\,(1-q^{b+n+1})}\, q^{b-d+1}, \quad n \geq 1.$$

In addition, it was also shown in [18] that if $\mathrm{Re}(c+2) > \mathrm{Re}(d) > 0$, then

$$\int_{\mathbb{T}} \zeta^{-j} \tau^{(b,c)} v_q^{(b,c,d)}(\zeta)\, \frac{1}{i2\pi\zeta}\, d\zeta = \frac{(q^{-b};q)_j}{(q^{c-b+2};q)_j} q^{jd}, \quad j = 0, \pm1, \pm2, \ldots \tag{2.6}$$

and

$$\int_{\mathbb{T}} \zeta^{-k} Q_n^{(b,c,d)}(\zeta)\, \tau^{(b,c)} v_q^{(b,c,d)}(\zeta)\, \frac{1}{i2\pi\zeta}\, d\zeta = \rho_n^{(b,c)}\, \delta_{n,k}, \quad 0 \leq k \leq n, \quad n \geq 0, \tag{2.7}$$

where $\rho_n^{(b,c)} = \dfrac{(q;q)_n\,(q^{c+2};q)_n}{(q^{b+1};q)_n\,(q^{c-b+2};q)_n}$, $n \geq 0$,

$$v_q^{(b,c,d)}(\zeta) = \frac{(q^{-b+d}\zeta;q)_\infty\,(q^{b-d+1}/\zeta;q)_\infty}{(q^d\zeta;q)_\infty\,(q^{c-d+2}/\zeta;q)_\infty} \quad \text{and} \quad \tau^{(b,c)} = \frac{(q;q)_\infty\,(q^{c+2};q)_\infty}{(q^{c-b+2};q)_\infty\,(q^{b+1};q)_\infty}.$$

Now let us consider the monic polynomials $\{\widehat{R}_n^{(b)}(q;\cdot)\}_{n\geq 0}$ given by

$$\widehat{R}_n^{(b)}(q;z) = \frac{(q^{\bar{b}+1};q)_n}{(q^{b+1};q)_n}\ {}_2\phi_1\Big(q^{-n},\ q^{b+1};\ q^{-\bar{b}-n};\ q,\ q^{-\bar{b}}z\Big), \quad n\geq 0, \tag{2.8}$$

where $b=\lambda+i\eta$ with $\lambda=\mathrm{Re}(b)>-1$. These polynomials, which were extensively studied in [2], have only simple zeros, all of them lying on the unit circle.

The polynomials $\widehat{R}_n^{(b)}(q;z)$ are a particular case of the polynomials $Q_n^{(b,c,d)}(q;z)$ with the choice of the parameters $d=b+1$ and $c=b+\bar{b}$, i.e.,

$$Q_n^{(b,b+\bar{b},b+1)}(q;z) = \widehat{R}_n^{(b)}(q;z), \quad n\geq 0.$$

Another particular case of the polynomials $Q_n^{(b,c,d)}(q;z)$ are the general Pastro orthogonal monic polynomials $\Phi_n^{(b)}(q;z)$ stated in the Example 1.31. Indeed, with the choice $d=b+1/2$ and $c=b+\bar{b}-1$, we have

$$Q_n^{(b,b+\bar{b}-1,b+1/2)}(q;z) = \Phi_n^{(b)}(q;z), \quad n\geq 0.$$

Remark 2.3. The general Pastro orthogonal monic polynomials $\Phi_n^{(b)}(q;z)$ are different from the polynomials $\widehat{R}_n^{(b)}(q;z)$ since they have their zeros inside the open unit disk.

Multiplying both sides of the expression (2.8) by $(q^{b+1};q)_n/(q^{\lambda+1}\cos\eta_q;q)_n$, we obtain the modified polynomials

$$\begin{aligned} R_n^{(b)}(q;z) &= \frac{(q^{b+1};q)_n}{(q^{\lambda+1}\cos\eta_q;q)_n}\widehat{R}_n^{(b)}(q;z) \\ &= \frac{(q^{\bar{b}+1};q)_n}{(q^{\lambda+1}\cos\eta_q;q)_n}\ {}_2\phi_1\Big(q^{-n},\ q^{b+1};\ q^{-\bar{b}-n};\ q,\ q^{-\bar{b}}z\Big), \quad n\geq 0, \end{aligned} \tag{2.9}$$

where $\eta_q=-\eta\ln(q)$. From (2.5) it is possible to verify that these polynomials satisfy

$$R_{n+1}^{(b)}(q;z) = \Big[\Big(1+ic_{n+1}^{(b)}(q)\Big)z + \Big(1-ic_{n+1}^{(b)}(q)\Big)\Big]R_n^{(b)}(q;z) - 4d_{n+1}^{(b)}(q)zR_{n-1}^{(b)}(q;z), \tag{2.10}$$

for $n\geq 1$, with $R_0^{(b)}(q;z)=1$ and $R_1^{(b)}(q;z)=\Big(1+ic_1^{(b)}(q)\Big)z+\Big(1-ic_1^{(b)}(q)\Big)$. Here,

$$c_n^{(b)}(q) = \frac{q^{\lambda+n}\sin\eta_q}{1-q^{\lambda+n}\cos\eta_q} \quad \text{and} \quad d_{n+1}^{(b)}(q) = \frac{(1-q^n)\,(1-q^{2\lambda+n+1})}{4(1-q^{\lambda+n}\cos\eta_q)\,(1-q^{\lambda+n+1}\cos\eta_q)}. \tag{2.11}$$

Notice that the sequence $\{d_{n+1}^{(b)}(q)\}_{n\geq 1}$ satisfy

$$d_{n+1}^{(b)}(q) \leq d_{n+1}^{(\lambda)}(q), \quad n\geq 1,$$

where $d_{n+1}^{(\lambda)}(q)$ are the coefficients given in (1.17). Moreover, in the Example 1.28 we observed that the sequence $\{d_{n+1}^{(\lambda)}(q)\}_{n\geq 1}$ is a positive chain sequence. Therefore, by Theorem 1.6 wich is the comparison test for positive chain sequences, the sequence $\{d_{n+1}^{(b)}(q)\}_{n\geq 1}$ is a positive chain sequence. Moreover,

$$\lim_{n\to\infty} d_{n+1}^{(b)}(q) = \lim_{n\to\infty}\frac{(1-q^n)\,(1-q^{2\lambda+n+1})}{4(1-q^{\lambda+n}\cos\eta_q)\,(1-q^{\lambda+n+1}\cos\eta_q)} = \frac{1}{4}. \tag{2.12}$$

Remark 2.4. The polynomials $\{R_n^{(b)}(q;\cdot)\}_{n\geq 0}$ and the sequences $\{c_n^{(b)}(q)\}$ and $\{d_{n+1}^{(b)}(q)\}$ are, respectively, the q-analogues of the polynomials $\{R_n^{(b)}\}_{n\geq 0}$ and the sequences $\{c_n^{(b)}\}$ and $\{d_{n+1}^{(b)}\}$ given in the Example 1.36. Indeed

$$\lim_{q\to 1} R_n^{(b)}(q;z) = R_n^{(b)}(z), \quad \lim_{q\to 1} c_n^{(b)}(q) = c_n^{(b)} \quad \text{and} \quad \lim_{q\to 1} d_{n+1}^{(b)}(q) = d_{n+1}^{(\lambda)}. \tag{2.13}$$

2.3 Some Further Properties

From [71], it is known that the monic polynomials $\widehat{R}_n^{(b)}(z)$ given by

$$\widehat{R}_n^{(b)}(z) = \frac{(\lambda+1)_n}{(b+1)_n} R_n^{(b)}(z), \quad n \geq 0,$$

where $R_n^{(b)}(z)$ are as in (1.27), satisfy the differential property

$$\frac{d}{dz}\widehat{R}_n^{(b)}(z) = n\,\Phi_{n-1}^{(b+1)}(z), \quad n \geq 1. \tag{2.14}$$

Here, $\Phi_n^{(b)}(z)$ are the general circular Jacobi polynomials (1.19).

Similar to the differential property (2.14) between polynomials $\widehat{R}_n^{(b)}(z)$ and $\Phi_n^{(b+1)}(z)$, we state the following q-differential property between polynomials $\widehat{R}_n^{(b)}(q;z)$ and $\Phi_n^{(b+1)}(q;z)$, where we substitute the derivative operator by q-difference operator D_q. This corresponds to the Property **A** stated in Section 2.1.

Theorem 2.5. *With respect to the q-difference operator D_q given by* (1.7)*, the polynomial $\widehat{R}_n^{(b)}(q;z)$ given in* (2.8) *and the general Pastro monic OPUC $\Phi_{n-1}^{(b+1)}(q;z)$ given in* (1.21)*, satisfy*

$$D_q[\widehat{R}_n^{(b)}(q;z)] = \{n\}_q\,\Phi_{n-1}^{(b+1)}(q;z), \quad n \geq 1. \tag{2.15}$$

Proof. By applying the q-difference operator D_q to both sides of the q-hypergeometric expression (2.8) and using the equality (1.8), we get

$$\begin{aligned} D_q[\widehat{R}_n^{(b)}(q;z)] &= \frac{(q^{\bar{b}+1};q)_n}{(q^{b+1};q)_n}\, D_q\Big[\,{}_2\phi_1\Big(q^{-n},\, q^{b+1};\, q^{-\bar{b}-n};\, q,\, q^{-\bar{b}}z\Big)\Big] \\ &= \frac{(q^{\bar{b}+1};q)_n}{(q^{b+1};q)_n}\, q^{-\bar{b}}\frac{(1-q^{-n})(1-q^{b+1})}{(1-q^{-\bar{b}-n})(1-q)}\, {}_2\phi_1\Big(q^{-n+1},\, q^{b+2};\, q^{-\bar{b}-n+1}\, q,\, q^{-1/2}q^{-\bar{b}}z\Big), \end{aligned}$$

for $n \geq 1$, and by adequately manipulating the q-Pochhammer symbols, we have

$$D_q[\widehat{R}_n^{(b)}(q;z)] = \frac{1-q^n}{q^{(n-1)/2}(1-q)}\,\frac{(q^{\bar{b}+1};q)_{n-1}}{(q^{b+2};q)_{n-1}}\, q^{(n-1)/2}\; {}_2\phi_1\Big(q^{-n+1},\, q^{b+2};\, q^{-\bar{b}-n+1}\, q,\, q^{-\bar{b}-1/2}z\Big),$$

for $n \geq 1$. Here, the right hand side to the above equality is equal to the q-hypergeometric expressions for $\{n\}_q\Phi_{n-1}^{(b+1)}(q;z)$. This confirms the result of the theorem. ■

The results given below, which follows from Lemma 1.33, Theorem 1.34 and Theorem 1.35, are based on the double sequence $\{(c_n^{(b)}(q), d_{n+1}^{(b)}(q))\}_{n\geq 1}$ given in (2.11).

Let $\mathrm{Re}(b) > -1$ and let $\{M_{n+1}^{(b)}(q)\}_{n\geq 0}$ be the maximal parameter sequence of the positive chain sequence $\{d_{n+1}^{(b)}(q)\}_{n\geq 1}$. That is, $\{M_{n+1}^{(b)}(q)\}_{n\geq 0}$ is the largest sequence such that $0 < M_n^{(b)}(q) < 1$ for $n \geq 1$ and

$$[1 - M_n^{(b)}(q)]M_{n+1}^{(b)}(q) = d_{n+1}^{(b)}(q), \quad n \geq 1.$$

An explicit expression for $M_1^{(b)}(q)$ is in (2.22) below.

Using $M_1^{(b)}(q)$ and $0 \le \epsilon < 1$, we now consider the augmented positive chain sequence $\{\tilde{d}_n^{(b,\epsilon)}(q)\}_{n\ge1}$, where

$$\tilde{d}_1^{(b,\epsilon)}(q) = (1-\epsilon)M_1^{(b)}(q) \quad \text{and} \quad \tilde{d}_{n+1}^{(b,\epsilon)}(q) = d_{n+1}^{(b)}(q), \quad n \ge 1,$$

and let $\{\tilde{m}_n^{(b,\epsilon)}(q)\}_{n\ge0}$ be its minimal parameter sequence. That is,

$$\tilde{m}_0^{(b,\epsilon)}(q) = 0 \quad \text{and} \quad [1-\tilde{m}_{n-1}^{(b,\epsilon)}(q)]\tilde{m}_n^{(b,\epsilon)}(q) = \tilde{d}_n^{(b,\epsilon)}(q), \quad n \ge 1. \tag{2.16}$$

Observe that the sequence $\{\epsilon, M_1^{(b)}(q), M_2^{(b)}(q), M_3^{(b)}(q), \ldots\}$ is the maximal parameter sequence of $\{\tilde{d}_n^{(b,\epsilon)}(q)\}_{n\ge1}$. Observe also that $\tilde{m}_1^{(b,\epsilon)}(q) = (1-\epsilon)M_1^{(b)}(q)$ and $\tilde{m}_n^{(b,0)}(q) = M_n^{(b)}(q)$ for $n \ge 1$.

Theorem 2.6. *Associated with the sequence of polynomials $\{R_n^{(b)}(q;\cdot)\}_{n\ge0}$, (i.e., associated with $\{c_n^{(b)}(q)\}_{n\ge1}$ and $\{d_{n+1}^{(b)}(q)\}_{n\ge1}$), there exists a unique absolutely continuous probability measure on $\mathbb{T}$, which we denote by $\tilde{\mu}_q^{(b,0)}$. Moreover, for $0 \le \epsilon < 1$, if $\{\tilde{m}_n^{(b,\epsilon)}(q)\}_{n\ge0}$ is as in (2.16) and if the probability measure $\tilde{\mu}_q^{(b,\epsilon)}$ is such that*

$$\int_{\mathbb{T}} f(\zeta)\, d\tilde{\mu}_q^{(b,\epsilon)}(\zeta) = (1-\epsilon)\int_{\mathbb{T}} f(\zeta)\, d\tilde{\mu}_q^{(b,0)}(\zeta) \ + \ \epsilon f(1), \tag{2.17}$$

then the following hold:

- *The polynomials $R_n^{(b)}(q;z)$ satisfy*

$$\int_{\mathbb{T}} \zeta^{-n+k} R_n^{(b)}(q;\zeta)\,(1-\zeta) d\tilde{\mu}_q^{(b,\epsilon)}(\zeta) = \delta_{n,k}\, \frac{1}{2} \prod_{j=0}^{n} \frac{4\tilde{d}_{j+1}^{(b,\epsilon)}(q)}{1+ic_{j+1}^{(b)}(q)}, \quad 0 \le k \le n, \quad n \ge 0. \tag{2.18}$$

- *The sequence of monic polynomials $\{\widetilde{\Phi}_n^{(b,\epsilon)}(q;z)\}_{n\ge0}$ given by*

$$\begin{aligned} \widetilde{\Phi}_n^{(b,\epsilon)}(q;z) &= \frac{(q^{\lambda+1}\cos\eta_q;q)_n}{(q^{b+1};q)_n}\Big[R_n^{(b)}(q;z) - 2[1-\tilde{m}_n^{(b,\epsilon)}(q)]R_{n-1}^{(b)}(q;z)\Big] \\ &= \widehat{R}_n^{(b)}(q;z) - 2[1-\tilde{m}_n^{(b,\epsilon)}(q)]\frac{(1-q^{\lambda+n}\cos\eta_q)}{(1-q^{b+n})}\widehat{R}_{n-1}^{(b)}(q;z), \quad n \ge 1, \end{aligned} \tag{2.19}$$

is the sequence of monic OPUC with respect to $\tilde{\mu}_q^{(b,\epsilon)}$. That is,

$$\left\langle \widetilde{\Phi}_m^{(b,\epsilon)}(q;\,.\,), \widetilde{\Phi}_n^{(b,\epsilon)}(q;\,.\,)\right\rangle_{\tilde{\mu}_q^{(b,\epsilon)}} = \int_{\mathbb{T}} \overline{\widetilde{\Phi}_m^{(b,\epsilon)}(q;\zeta)}\, \widetilde{\Phi}_n^{(b,\epsilon)}(q;\zeta)\, d\tilde{\mu}_q^{(b,\epsilon)}(\zeta) = 0, \quad n \ne m.$$

From (2.13) and (2.19) one can verify that

$$\lim_{q\to1} \widetilde{\Phi}_n^{(b,\epsilon)}(q;z) = \Phi_n^{(b,\epsilon)}(z), \quad n \ge 1,$$

where the orthogonal polynomials $\Phi_n^{(b,\epsilon)}(z)$ are given in (1.28).

From (2.19), for the Verblunsky coefficients $\{-\overline{\widetilde{\Phi}_{n+1}^{(b,\epsilon)}(q;0)}\}_{n\ge0}$ associated with the measure $\tilde{\mu}_q^{(b,\epsilon)}$ we have

$$-\overline{\widetilde{\Phi}_n^{(b,\epsilon)}(q;0)} = -\frac{(q^{b+1};q)_n}{(q^{\bar{b}+1};q)_n}\left[1 - 2[1-\tilde{m}_n^{(b,\epsilon)}(q)]\frac{1-q^{\lambda+n}\cos\eta_q}{1-q^{b+n}}\right], \quad n \ge 1,$$

and

$$\lim_{n\to\infty} \Big| - \overline{\widetilde{\Phi}_n^{(b,\epsilon)}(q;0)}\Big| = 0.$$

Thus, the measure $\widetilde{\mu}_q^{(b,\epsilon)}$ belongs to the Nevai class (see [69]) and hence, there holds the outer ratio asymptotics

$$\lim_{n\to\infty} \frac{\widetilde{\Phi}_n^{(b,\epsilon)}(q;z)}{\widetilde{\Phi}_{n-1}^{(b,\epsilon)}(q;z)} = z, \tag{2.20}$$

in every compact subset of $\mathbb{C}\setminus\overline{\mathbb{D}}$.

Theorem 2.6 states the existence of the probability measures $\widetilde{\mu}_q^{(b,0)}$ associated with $\{c_n^{(b)}(q)\}_{n\geq1}$ and $\{d_{n+1}^{(b)}(q)\}_{n\geq1}$. The next theorem gives information about the explicit form of $\widetilde{\mu}_q^{(b,0)}$ and also information regarding the values of $\widetilde{A}_n^{(b,\epsilon)}(q)$.

Theorem 2.7. *The probability measure $\widetilde{\mu}_q^{(b,\epsilon)}$ given in Theorem 2.6 is such that*

$$d\widetilde{\mu}_q^{(b,0)}(\zeta) = \tilde{\tau}_q^{(b)}\, \widetilde{w}_q^{(b)}(\zeta)\frac{1}{i2\pi\zeta}\, d\zeta, \tag{2.21}$$

where

$$\widetilde{w}_q^{(b)}(\zeta) = \frac{|(q\zeta;q)_\infty|^2}{|(q^{b+1}\zeta;q)_\infty|^2} \quad \text{and} \quad \tilde{\tau}_q^{(b)} = 2M_1^{(b)}(q)\,(1-q^{\lambda+1}\cos\eta_q)\frac{(q;q)_\infty(q^{2\lambda+2};q)_\infty}{|(q^{b+1};q)_\infty|^2}.$$

Moreover, if $\widetilde{A}_n^{(b,\epsilon)}(q) = \left\langle \widetilde{\Phi}_n^{(b,\epsilon)}(q;\,.\,), \widetilde{\Phi}_n^{(b,\epsilon)}(q;\,.\,)\right\rangle_{\widetilde{\mu}_q^{(b,\epsilon)}}$*, then*

$$\widetilde{A}_n^{(b,\epsilon)}(q) = \frac{\tilde{m}_1^{(b,\epsilon)}(q)}{\tilde{m}_{n+1}^{(b,\epsilon)}(q)}\,\frac{1-q^{\lambda+1}\cos\eta_q}{1-q^{\lambda+n+1}\cos\eta_q}\,\frac{(q;q)_n(q^{2\lambda+2};q)_n}{(q^{b+1};q)_n(q^{\bar{b}+1};q)_n}, \quad n\geq 1.$$

Proof. From (2.7) and (2.9), with $v_q^{(b,b+\bar{b},b+1)}(\zeta) = (1-\zeta^{-1})\widetilde{w}_q^{(b)}(\zeta)$, we have

$$\int_{\mathbb{T}} \zeta^{-k} R_n^{(b)}(q;\zeta)\,\tau^{(b,b+\bar{b})}(1-\zeta^{-1})\widetilde{w}_q^{(b)}(\zeta)\,\frac{1}{i2\pi\zeta}\,d\zeta = \delta_{n,k}\,\frac{(q^{b+1};q)_n\,\rho_n^{(b,b+\bar{b})}}{(q^{\lambda+1}\cos\eta_q;q)_n},$$

for $0\leq k\leq n$ and $n\geq 0$. Hence, by conjugation of this expression and then using the conjugate reciprocal (reverse) property

$$R_n^{(b)*}(q;z) = z^n\overline{R_n^{(b)}(q;1/\overline{z})} = R_n^{(b)}(q;z), \quad n\geq 0,$$

which can be easily verified from (2.10), we can also write

$$\int_{\mathbb{T}} \zeta^{-n+k} R_n^{(b)}(q;\zeta)\,(1-\zeta)\tilde{\tau}_q^{(b)}\widetilde{w}_q^{(b)}(\zeta)\,\frac{1}{i2\pi\zeta}\,d\zeta = \delta_{n,k}\,\frac{\tilde{\tau}_q^{(b)}}{\tau^{(\bar{b},b+\bar{b})}}\,\frac{(q^{\bar{b}+1};q)_n\,\rho_n^{(\bar{b},b+\bar{b})}}{(q^{\lambda+1}\cos\eta_q;q)_n},$$

for $0\leq k\leq n$ and $n\geq 0$. Comparing this with (2.18) we then have

$$\frac{\tilde{\tau}_q^{(b)}}{\tau^{(\bar{b},b+\bar{b})}}\,\frac{(q^{\bar{b}+1};q)_n\,\rho_n^{(\bar{b},b+\bar{b})}}{(q^{\lambda+1}\cos\eta_q;q)_n} = \frac{1}{2}\prod_{j=0}^{n}\frac{4\tilde{d}_{j+1}^{(b,0)}(q)}{1+ic_{j+1}^{(b)}(q)}, \quad n\geq 0.$$

Thus, by using (2.11) we confirm the value of $\tilde{\tau}_q^{(b)}$.

Now to find the value of $\widetilde{A}_n^{(b,\epsilon)}(q)$, we first observe that

$$\widetilde{A}_n^{(b,\epsilon)}(q) = \int_{\mathbb{T}} \overline{\widetilde{\Phi}_n^{(b,\epsilon)}(q;\zeta)}\, \widetilde{\Phi}_n^{(b,\epsilon)}(q;\zeta)\, d\widetilde{\mu}_q^{(b,\epsilon)}(\zeta) = \int_{\mathbb{T}} (\overline{\zeta}^n - 1)\, \widetilde{\Phi}_n^{(b,\epsilon)}(q;\zeta)\, d\widetilde{\mu}_q^{(b,\epsilon)}(\zeta), \quad n \geq 1.$$

Hence, from $\overline{\zeta} - 1 = (1-\zeta)/\zeta$ and

$$\frac{\overline{\zeta}^n - 1}{\overline{\zeta} - 1} = \sum_{j=0}^{n-1} \zeta^{-j},$$

we find

$$\widetilde{A}_n^{(b,\epsilon)}(q) = \int_{\mathbb{T}} \Big[\sum_{j=0}^{n-1} \zeta^{-j-1}\Big] \widetilde{\Phi}_n^{(b,\epsilon)}(q;\zeta)\, (1-\zeta) d\widetilde{\mu}_q^{(b,\epsilon)}(\zeta), \quad n \geq 1.$$

Thus, from (2.19) we can write

$$\begin{aligned}&\widetilde{A}_n^{(b,\epsilon)}(q) \\ &= \frac{(q^{\lambda+1}\cos\eta_q;q)_n}{(q^{b+1};q)_n} \int_{\mathbb{T}} \Big[\sum_{j=0}^{n-1} \zeta^{-j-1}\Big] \Big[R_n^{(b)}(q;z) - 2[1-\tilde{m}_n^{(b,\epsilon)}(q)] R_{n-1}^{(b)}(q;z)\Big]\, (1-\zeta) d\widetilde{\mu}_q^{(b,\epsilon)}(\zeta),\end{aligned}$$

and, hence, we obtain from (2.18) that

$$\widetilde{A}_n^{(b,\epsilon)}(q) = -\frac{(q^{\lambda+1}\cos\eta_q;q)_n}{(q^{b+1};q)_n} 2[1-\tilde{m}_n^{(b,\epsilon)}(q)] \int_{\mathbb{T}} \zeta^{-n} R_{n-1}^{(b)}(q;\zeta)\, (1-\zeta) d\widetilde{\mu}_q^{(b,\epsilon)}(\zeta), \quad n \geq 1.$$

By conjugating this expression and using the conjugate reciprocal property of $R_{n-1}^{(b)}(q;\zeta)$ we then have

$$\overline{\widetilde{A}_n^{(b,\epsilon)}(q)} = \frac{(q^{\lambda+1}\cos\eta_q;q)_n}{(q^{\bar{b}+1};q)_n} 2[1-\tilde{m}_n^{(b,\epsilon)}(q)] \int_{\mathbb{T}} R_{n-1}^{(b)}(q;\zeta)\, (1-\zeta) d\widetilde{\mu}_q^{(b,\epsilon)}(\zeta), \quad n \geq 1.$$

Thus, again from (2.18)

$$\begin{aligned}\overline{\widetilde{A}_n^{(b,\epsilon)}(q)} &= \frac{(q^{\lambda+1}\cos\eta_q;q)_n}{(q^{\bar{b}+1};q)_n} 2[1-\tilde{m}_n^{(b,\epsilon)}(q)] \frac{1}{2} \prod_{j=0}^{n-1} \frac{4\tilde{d}_{j+1}^{(b,\epsilon)}(q)}{1+ic_{j+1}^{(b)}(q)} \\ &= \frac{\tilde{m}_1^{(b,\epsilon)}(q)}{\tilde{m}_{n+1}^{(b,\epsilon)}(q)} \prod_{j=1}^{n} \frac{4\, d_{j+1}^{(b)}(q)}{\big|1+ic_j^{(b)}(q)\big|^2}, \quad n \geq 1,\end{aligned}$$

and the required result of the theorem follows. ■

Remark 2.8. In the results given in this section the augmented positive chain sequence $\{\tilde{d}_n^{(b,\epsilon)}(q)\}_{n\geq 1}$ plays an important role. However, this chain sequence is based on the value of $M_1^{(b)}(q)$ which one can derive from the evaluation of the integral

$$\int_{\mathbb{T}} \widetilde{w}_q^{(b)}(\zeta) \frac{1}{i2\pi\zeta}\, d\zeta = \frac{1}{\tilde{\tau}_q^{(b)}} = \frac{|(q^{b+1};q)_\infty|^2}{2M_1^{(b)}(q)\,(1-q^{\lambda+1}\cos\eta_q)(q;q)_\infty(q^{2\lambda+2};q)_\infty}.$$

Hence, using $\widetilde{w}_q^{(b)}(\zeta) = \zeta v_q^{(b,b+\bar{b},b+1)}(\zeta)/(\zeta-1)$ and (2.6) one can also show that

$$M_1^{(b)}(q) = \frac{1}{2} \frac{1-q^{\bar{b}+1}}{1-q^{\lambda+1}\cos\eta_q} \frac{1}{{}_2\phi_1(q,\, q^{-b};\, q^{\bar{b}+2};\, q,\, q^{b+1})}. \tag{2.22}$$

The next theorem states that $\{\widetilde{\mu}_q^{(b,\epsilon)}, \mu_q^{(b+1)}\}$ satisfies a coherence type property of the second kind on the unit circle with respect to the q-difference operator D_q. This is the Property **B** stated in Section 2.1.

Theorem 2.9. *The pair of measures $\{\widetilde{\mu}_q^{(b,\epsilon)}, \mu_q^{(b+1)}\}$, where $\widetilde{\mu}_q^{(b,\epsilon)}$ is given by Theorem 2.6 and $\mu_q^{(b+1)}$ is given by* (1.23)*, satisfies the coherence type property of the second kind on the unit circle with respect to the q-difference operator D_q given by* (1.7)*. That is, the respectives associated monic OPUC satisfy*

$$D_q[\widetilde{\Phi}_n^{(b,\epsilon)}(q;z)] = \{n\}_q \Big[\Phi_{n-1}^{(b+1)}(q;z) - \chi_n^{(b,\epsilon)}(q)\, \Phi_{n-2}^{(b+1)}(q;z)\Big], \quad n \geq 2, \tag{2.23}$$

where $\chi_n^{(b,\epsilon)}(q) = 2[1-\tilde{m}_n^{(b,\epsilon)}(q)]\dfrac{(1-q^{\lambda+n}\cos\eta_q)}{(1-q^{b+n})}\dfrac{\{n-1\}_q}{\{n\}_q}$, $n \geq 2$.
Moreover, $\lim_{n\to\infty} \chi_n^{(b,\epsilon)}(q) = q^{1/2}$.

Proof. By applying the q-difference operator D_q to (2.19) we have

$$D_q[\widetilde{\Phi}_n^{(b,\epsilon)}(q;z)] = D_q[\widehat{R}_n^{(b)}(q;z)] - 2[1-\tilde{m}_n^{(b,\epsilon)}(q)]\frac{(1-q^{\lambda+n}\cos\eta_q)}{(1-q^{b+n})} D_q[\widehat{R}_{n-1}^{(b)}(q;z)],$$

for $n \geq 1$, and using Theorem 2.5 leads to the required coherence formula.

On the other hand, notice that

$$\lim_{n\to\infty} \frac{\{n-1\}_q}{\{n\}_q} = q^{1/2},$$

and from (2.12) it follows that $\lim_{n\to\infty} \tilde{d}_n^{(b,\epsilon)}(q) = \lim_{n\to\infty} d_n^{(b,\epsilon)}(q) = 1/4$, then using Theorem 1.7 we obtain

$$\lim_{n\to\infty} \tilde{m}_n^{(b,\epsilon)}(q) = \frac{1}{2}.$$

The above asymptotic results confirm the asymptotic result of $\chi_n^{(b,\epsilon)}(q)$. ∎

2.4 The Sobolev-Type OPUC

As we have pointed out in the introduction of this work, coherent pairs of measures constitute an useful tool to study sequences of orthogonal polynomials with respect to the Sobolev inner product associated with such a pair of measures.

Following the results obtained in [8, 71] and taking into account (2.23), we can now deal with the sequence $\{\Psi_n^{(b,\epsilon,s)}(q;\cdot)\}_{n\geq 0}$ of monic orthogonal polynomials associated with the Sobolev-type inner product

$$\Big\langle f, g \Big\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}} = \Big\langle f, g \Big\rangle_{\widetilde{\mu}_q^{(b,\epsilon)}} + s\,\Big\langle D_q[f], D_q[g] \Big\rangle_{\mu_q^{(b+1)}}, \tag{2.24}$$

where $\mu_q^{(b+1)}$ is the probability measure given in (1.23) and $\widetilde{\mu}_q^{(b,\epsilon)}$ is the probability measure given by (2.17) and (2.21). Moreover, we assume b to be such that $\mathrm{Re}(b) > -1$.

We define the sequences $\{\mathfrak{p}_n^{(b,\epsilon,s)}(q)\}_{n\geq 1}$ and $\{\mathfrak{q}_n^{(b,\epsilon,s)}(q)\}_{n\geq 1}$, which will play an important role in the sequel, by

$$\begin{aligned} \mathfrak{q}_{n+1}^{(b,\epsilon,s)}(q) &= s\,\{n\}_q\{n+1\}_q A_{n-1}^{(b+1)}(q)\, \chi_{n+1}^{(b,\epsilon)}(q), \\ \mathfrak{p}_n^{(b,\epsilon,s)}(q) &= \widetilde{A}_n^{(b,\epsilon)}(q) + s\,\{n\}_q^2\Big[A_{n-1}^{(b+1)}(q) + A_{n-2}^{(b+1)}(q)\,|\chi_n^{(b,\epsilon)}(q)|^2\Big], \end{aligned} \quad n \geq 1, \tag{2.25}$$

where $\widetilde{A}_n^{(b,\epsilon)}(q)$ is given in Theorem 2.7 and $A_n^{(b+1)}(q)$ is given by (1.22). As we are dealing with probability measures, we also remind that $\widetilde{A}_0^{(b,\epsilon)}(q) = A_0^{(b+1)}(q) = 1$. Moreover, $A_{-1}^{(b+1)}(q) = 0$.

For later use, we now observe that $A_n^{(b+1)}(q)$ and $\widetilde{A}_n^{(b,\epsilon)}(q)$ stay bounded as $n \to \infty$. Indeed, from Theorem 2.7 we have

$$\widetilde{A}_n^{(b,\epsilon)}(q) = \frac{\tilde{m}_1^{(b,\epsilon)}(q)}{\tilde{m}_{n+1}^{(b,\epsilon)}(q)} \frac{1-q^{\lambda+1}\cos\eta_q}{1-q^{\lambda+n+1}\cos\eta_q} \frac{(1-q^n)(1-q^{2\lambda+2})}{|1-q^{b+1}|^2} A_{n-1}^{(b+1)}(q), \quad n \geq 1. \tag{2.26}$$

Therefore, since $\lim_{n\to\infty} A_n^{(b+1)}(q) = \tau_q^{(b+1)}$, we get

$$\lim_{n\to\infty} \widetilde{A}_n^{(b,\epsilon)}(q) = \frac{2\tilde{m}_1^{(b,\epsilon)}(q)(1-q^{\lambda+1}\cos\eta_q)(1-q^{2\lambda+2})}{|1-q^{b+1}|^2}\tau_q^{(b+1)}.$$

Here, $\tau_q^{(b+1)}$ is as in (1.24).

In the next result, we consider the expression for $\Psi_n^{(b,\epsilon,s)}(q;z)$ as a linear combination of the polynomials $\{\widetilde{\Phi}_j^{(b,\epsilon)}(q;\,.\,)\}_{j=0}^n$ given by (2.19).

Theorem 2.10. *If $\{\Psi_n^{(b,\epsilon,s)}(q;\,.\,)\}_{n\geq 0}$ and $\{\widetilde{\Phi}_n^{(b,\epsilon)}(q;\,.\,)\}_{n\geq 0}$ are the sequences of monic orthogonal polynomials with respect to the inner products $\big\langle f,g\big\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}}$ and $\big\langle f,g\big\rangle_{\widetilde{\mu}_q^{(b,\epsilon)}}$, respectively, then*

$$\Psi_1^{(b,\epsilon,s)}(q;z) = \widetilde{\Phi}_1^{(b,\epsilon)}(q;z) \quad \textit{and}$$
$$\Psi_{n+1}^{(b,\epsilon,s)}(q;z) = \sum_{j=1}^{n} \mathfrak{a}_{n+1,j}^{(b,\epsilon,s)}(q)\,\widetilde{\Phi}_j^{(b,\epsilon)}(q;z) + \widetilde{\Phi}_{n+1}^{(b,\epsilon)}(q;z), \quad n \geq 1.$$

Here, if $\mathbf{a}_n^{(b,\epsilon,s)}(q) = [\mathfrak{a}_{n+1,1}^{(b,\epsilon,s)}(q), \mathfrak{a}_{n+1,2}^{(b,\epsilon,s)}(q), \ldots, \mathfrak{a}_{n+1,n}^{(b,\epsilon,s)}(q)]^T$, then

$$\mathfrak{a}_{2,1}^{(b,\epsilon,s)}(q) = s\{2\}_q\chi_2^{(b,\epsilon)}(q)/[\widetilde{A}_1^{(b,\epsilon)}(q)+s] \quad \textit{and}$$
$$\mathbf{T}_n^{(b,\epsilon,s)}(q)\,\mathbf{a}_n^{(b,\epsilon,s)}(q) = \mathfrak{q}_{n+1}^{(b,\epsilon,s)}(q)\,\mathbf{e}_n, \quad n \geq 2,$$

where $\mathbf{e}_n$ is the n-th column of the $n\times n$ identity matrix and $\mathbf{T}_n^{(b,\epsilon,s)}(q)$ is the $n\times n$ Hermitian tridiagonal matrix

$$\begin{bmatrix} \mathfrak{p}_1^{(b,\epsilon,s)}(q) & -\mathfrak{q}_2^{(b,\epsilon,s)}(q) & & & & \\ -\overline{\mathfrak{q}}_2^{(b,\epsilon,s)}(q) & \mathfrak{p}_2^{(b,\epsilon,s)}(q) & -\mathfrak{q}_3^{(b,\epsilon,s)}(q) & & & \\ & \ddots & \ddots & \ddots & & \\ & & & -\overline{\mathfrak{q}}_{n-1}^{(b,\epsilon,s)}(q) & \mathfrak{p}_{n-1}^{(b,\epsilon,s)}(q) & -\mathfrak{q}_n^{(b,\epsilon,s)}(q) \\ & & & & -\overline{\mathfrak{q}}_n^{(b,\epsilon,s)}(q) & \mathfrak{p}_n^{(b,\epsilon,s)}(q) \end{bmatrix},$$

with $\mathfrak{p}_k^{(b,\epsilon,s)}(q)$ and $\mathfrak{q}_k^{(b,\epsilon,s)}(q)$ as in (2.25).

Proof. If we write $\Psi_{n+1}^{(b,\epsilon,s)}(q;z) = \sum_{j=0}^{n+1} r_{n,j}\widetilde{\Phi}_j^{(b,\epsilon)}(q;z)$, with $r_{n,n+1} = 1$, by considering the orthogonality between $\Psi_{n+1}^{(b,\epsilon,s)}(q;z)$ and $\Psi_0^{(b,\epsilon,s)}(q;z)$ with respect to the inner product $\big\langle f,g\big\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}}$:

$$\begin{aligned}0 &= \left\langle \Psi_0^{(b,\epsilon,s)}(q;\,.\,), \Psi_{n+1}^{(b,\epsilon,s)}(q;\,.\,)\right\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}} \\ &= \left\langle 1, \sum_{j=0}^{n+1} r_{n,j}\widetilde{\Phi}_j^{(b,\epsilon)}(q;\,.\,)\right\rangle_{\widetilde{\mu}_q^{(b,\epsilon)}} + s\left\langle 0, \sum_{j=0}^{n+1} r_{n,j}D_q\Big[\widetilde{\Phi}_j^{(b,\epsilon)}(q;\,.\,)\Big]\right\rangle_{\mu_q^{(b+1)}} = r_{n,0}\left\langle 1,1\right\rangle_{\widetilde{\mu}_q^{(b,\epsilon)}},\end{aligned}$$

for $n \geq 0$. Thus, $r_{n,0} = 0$ and we can write $\Psi_{n+1}^{(b,\epsilon,s)}(q;z) = \sum_{j=1}^{n+1} r_{n,j}\widetilde{\Phi}_j^{(b,\epsilon)}(q;z)$.

Hence, with $n \geq 1$, from

$$\begin{aligned}0 &= \left\langle \widetilde{\Phi}_1^{(b,\epsilon)}(q,\,.\,), \Psi_{n+1}^{(b,\epsilon,s)}(q;\,.\,)\right\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}} \\ &= \left\langle \widetilde{\Phi}_1^{(b,\epsilon)}(q;\,.\,), \sum_{j=1}^{n+1} r_{n,j}\widetilde{\Phi}_j^{(b,\epsilon)}(q;\,.\,)\right\rangle_{\widetilde{\mu}_q^{(b,\epsilon)}} + s\left\langle 1, \sum_{j=1}^{n+1} r_{n,j}\, D_q\Big[\widetilde{\Phi}_j^{(b,\epsilon)}(q;\,.\,)\Big]\right\rangle_{\mu_q^{(b+1)}}\end{aligned}$$

and (2.23), we get

$$\begin{aligned}0 = r_{n,1}&\left\langle \widetilde{\Phi}_1^{(b,\epsilon)}(q;\,.\,), \widetilde{\Phi}_1^{(b,\epsilon)}(q;\,.\,)\right\rangle_{\widetilde{\mu}_q^{(b,\epsilon)}} \\ &+ s\left\langle \Phi_0^{(b+1)}(q;\,.\,), \sum_{j=1}^{n+1} r_{n,j}\{j\}_q\Big[\Phi_{j-1}^{(b+1)}(q;\,.\,) - \chi_j^{(b,\epsilon)}(q)\,\Phi_{j-2}^{(b+1)}(q;\,.\,)\Big]\right\rangle_{\mu_q^{(b+1)}},\end{aligned}$$

where $\chi_1^{(b,\epsilon)}(q)\,\Phi_{-1}^{(b+1)}(q;\,.\,) = 0$. Hence,

$$\Big[\widetilde{A}_1^{(b,\epsilon)}(q) + sA_0^{(b+1)}(q)\Big]\, r_{n,1} - \{2\}_q sA_0^{(b+1)}(q)\,\chi_2^{(b,\epsilon)}(q)\, r_{n,2} = 0. \tag{2.27}$$

Thus, when $n = 1$, with the observation $r_{1,2} = 1$ we obtain the expression for $r_{1,1} = \mathfrak{a}_{2,1}^{(b,\epsilon,s)}(q)$. With $r_{n,j} = \mathfrak{a}_{n+1,j}^{(b,\epsilon,s)}(q)$, $j = 1,2,\ldots,n$, we also have from (2.27)

$$\mathfrak{p}_1^{(b,\epsilon,s)}(q)\,\mathfrak{a}_{n+1,1}^{(b,\epsilon,s)}(q) - \mathfrak{q}_2^{(b,\epsilon,s)}(q)\,\mathfrak{a}_{n+1,2}^{(b,\epsilon,s)}(q) = 0, \quad n \geq 2.$$

Now with $n \geq 2$ and $2 \leq k \leq n$, from

$$\begin{aligned}0 &= \left\langle \widetilde{\Phi}_k^{(b,\epsilon)}(q;\,.\,), \Psi_{n+1}^{(b,\epsilon,s)}(q;\,.\,)\right\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}} \\ &= \left\langle \widetilde{\Phi}_k^{(b,\epsilon)}(q;\,.\,), \sum_{j=1}^{n+1} r_{n,j}\widetilde{\Phi}_j^{(b,\epsilon)}(q;\,.\,)\right\rangle_{\widetilde{\mu}_q^{(b,\epsilon)}} + s\left\langle D_q\Big[\widetilde{\Phi}_k^{(b,\epsilon)}(q;\,.\,)\Big], D_q\Big[\Psi_{n+1}^{(b,\epsilon,s)}(q;\,.\,)\Big]\right\rangle_{\mu_q^{(b+1)}} \\ &= r_{n,k}\left\langle \widetilde{\Phi}_k^{(b,\epsilon)}(q;\,.\,), \widetilde{\Phi}_k^{(b,\epsilon)}(q;\,.\,)\right\rangle_{\widetilde{\mu}_q^{(b,\epsilon)}} \\ &\quad + s\Big\langle \{k\}_q\Big[\Phi_{k-1}^{(b+1)}(q;\,.\,) - \chi_k^{(b,\epsilon)}(q)\,\Phi_{k-2}^{(b+1)}(q;\,.\,)\Big], \\ &\qquad\quad r_{n,1}\Phi_0^{(b+1)}(q;\,.\,) + \sum_{j=2}^{n+1} r_{n,j}\,\{j\}]_q\Big[\Phi_{j-1}^{(b+1)}(q;\,.\,) - \chi_j^{(b,\epsilon)}(q)\,\Phi_{j-2}^{(b+1)}(q;\,.\,)\Big]\Big\rangle_{\mu_q^{(b+1)}},\end{aligned}$$

we obtain

$$\begin{aligned}0 = -s\{k\}_q\,\{k-1\}_q\, &A_{k-2}^{(b+1)}(q)\,\overline{\chi}_k^{(b,\epsilon)}(q)\, r_{n,k-1} \\ &+\Big[\widetilde{A}_k^{(b,\epsilon)}(q) + s\{k\}_q^2\Big(A_{k-1}^{(b+1)}(q) + A_{k-2}^{(b+1)}(q)\,|\chi_k^{(b,\epsilon)}(q)|^2\Big)\Big]\, r_{n,k} \\ &\qquad\qquad -s\{k\}_q\,\{k+1\}_q\, A_{k-1}^{(b+1)}(q)\,\chi_{k+1}^{(b,\epsilon)}(q)\, r_{n,k+1}.\end{aligned}$$

We can write this as

$$-\overline{\mathfrak{q}}_k^{(b,\epsilon,s)}(q)\, r_{n,k-1} + \mathfrak{p}_k^{(b,\epsilon,s)}(q)\, r_{n,k} - \mathfrak{q}_{k+1}^{(b,\epsilon,s)}(q)\, r_{n,k+1} = 0, \quad 2 \leq k \leq n, \quad n \geq 2.$$

Hence, with $r_{n,n+1} = 1$ and $r_{n,j} = \mathfrak{a}_{n+1,j}^{(b,\epsilon,s)}(q)$, $j = 1,2,\ldots,n$, we obtain the linear system of equations $\mathbf{T}_n^{(b,\epsilon,s)}(q)\,\mathbf{a}_n^{(b,\epsilon,s)}(q) = \mathfrak{q}_{n+1}^{(b,\epsilon,s)}(q)\,\mathbf{e}_n$. This completes the proof of the theorem.■

Since $\mathbf{T}_n^{(b,\epsilon,s)}(q)$ is a tridiagonal matrix, it is easily seen that

$$\det(\mathbf{T}_n^{(b,\epsilon,s)}(q)) = \mathfrak{p}_n^{(b,\epsilon,s)}(q)\ \det(\mathbf{T}_{n-1}^{(b,\epsilon,s)}(q)) - |\mathfrak{q}_n^{(b,\epsilon,s)}(q)|^2\ \det(\mathbf{T}_{n-2}^{(b,\epsilon,s)}(q)), \quad n \geq 2,$$

with $\det(\mathbf{T}_0^{(b,\epsilon,s)}(q)) = 1$ and $\det(\mathbf{T}_1^{(b,\epsilon,s)}(q)) = \mathfrak{p}_1^{(b,\epsilon,s)}(q)$. Thus,

$$\frac{\det(\mathbf{T}_n^{(b,\epsilon,s)}(q))}{\mathfrak{p}_n^{(b,\epsilon,s)}(q)\ \det(\mathbf{T}_{n-1}^{(b,\epsilon,s)}(q))}\left[1 - \frac{\det(\mathbf{T}_{n+1}^{(b,\epsilon,s)}(q))}{\mathfrak{p}_{n+1}^{(b,\epsilon,s)}(q)\ \det(\mathbf{T}_n^{(b,\epsilon,s)}(q))}\right] = \frac{|\mathfrak{q}_{n+1}^{(b,\epsilon,s)}(q)|^2}{\mathfrak{p}_n^{(b,\epsilon,s)}(q)\,\mathfrak{p}_{n+1}^{(b,\epsilon,s)}(q)}, \quad n \geq 1.$$

The sequence $\{\mathfrak{d}_n^{(b,\epsilon,s)}(q)\}_{n=1}^{\infty}$, where

$$\mathfrak{d}_n^{(b,\epsilon,s)}(q) = \frac{|\mathfrak{q}_{n+1}^{(b,\epsilon,s)}(q)|^2}{\mathfrak{p}_n^{(b,\epsilon,s)}(q)\,\mathfrak{p}_{n+1}^{(b,\epsilon,s)}(q)}, \quad n \geq 1, \tag{2.28}$$

is a positive chain sequence. This will be confirmed later in the proof of Theorem 2.13. This means that the sequence $\{\mathfrak{m}_n^{(b,\epsilon,s)}(q)\}_{n=0}^{\infty}$, where

$$[1 - \mathfrak{m}_{n-1}^{(b,\epsilon,s)}(q)]\mathfrak{m}_n^{(b,\epsilon,s)}(q) = \mathfrak{d}_n^{(b,\epsilon,s)}(q) \quad \text{and} \quad \mathfrak{m}_{n-1}^{(b,\epsilon,s)}(q) = 1 - \frac{\det(\mathbf{T}_n^{(b,\epsilon,s)}(q))}{\mathfrak{p}_n^{(b,\epsilon,s)}(q)\ \det(\mathbf{T}_{n-1}^{(b,\epsilon,s)}(q))},$$

for $n \geq 1$, is the minimal parameter sequence of $\{\mathfrak{d}_n^{(b,\epsilon,s)}(q)\}_{n=1}^{\infty}$. Therefore,

$$0 < \det(\mathbf{T}_n^{(b,\epsilon,s)}(q)) < \mathfrak{p}_n^{(b,\epsilon,s)}(q)\ \det(\mathbf{T}_{n-1}^{(b,\epsilon,s)}(q)), \quad n \geq 2.$$

Remark 2.11. The positiveness of the determinant $\det(\mathbf{T}_n^{(b,\epsilon,s)}(q))$ guarantees the unique existence of the solution of the system $\mathbf{T}_n^{(b,\epsilon,s)}(q)\,\mathbf{a}_n^{(b,\epsilon,s)}(q) = \mathfrak{q}_{n+1}^{(b,\epsilon,s)}(q)\,\mathbf{e}_n$, thus, also the existence of the Sobolev-type orthogonal polynomials $\Psi_n^{(b,\epsilon,s)}(q;\,.\,)$.

The following result states that two consecutive Sobolev-type orthogonal polynomials with respect to (2.24) are connected to an orthogonal polynomial corresponding to the measure $\widetilde{\mu}_q^{(b,\epsilon)}$. This is the Property **C** stated in Section 2.1.

Theorem 2.12. *The orthogonal polynomials $\Psi_n^{(b,\epsilon,s)}(q;z)$ with respect to the Sobolev type inner product $\left\langle f, g\right\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}}$ given by* (2.24) *satisfy*

$$\Psi_n^{(b,\epsilon,s)}(q;z) - \beta_n^{(b,\epsilon,s)}(q)\,\Psi_{n-1}^{(b,\epsilon,s)}(q;z) = \widetilde{\Phi}_n^{(b,\epsilon)}(q;z), \quad n \geq 1. \tag{2.29}$$

Moreover, the connection coefficients in (2.29) *satisfy $\beta_1^{(b,\epsilon,s)}(q) = 0$ and*

$$\beta_{n+1}^{(b,\epsilon,s)}(q) = \frac{\mathfrak{q}_{n+1}^{(b,\epsilon,s)}(q)}{\mathfrak{p}_n^{(b,\epsilon,s)}(q) - \overline{\mathfrak{q}}_n^{(b,\epsilon,s)}(q)\,\beta_n^{(b,\epsilon,s)}(q)}, \quad n \geq 1, \tag{2.30}$$

where $\mathfrak{q}_k^{(b,\epsilon,s)}(q)$ and $\mathfrak{q}_k^{(b,\epsilon,s)}(q)$ are given by (2.25)*. Also,*

$$\overline{\chi}_{n+1}^{(b,\epsilon)}(q)\,\beta_{n+1}^{(b,\epsilon,s)}(q) > 0, \quad n \geq 1.$$

Proof. By considering $\widetilde{\Phi}_{n+1}^{(b,\epsilon)}(q;\,.\,)$ as a linear combination of $\{\Psi_j^{(b,\epsilon,s)}(q;\,.\,)\}_{j=0}^{n+1}$ we find

$$\Psi_{n+1}^{(b,\epsilon,s)}(q;z) - \beta_{n+1}^{(b,\epsilon,s)}(q)\,\Psi_n^{(b,\epsilon,s)}(q;z) = \widetilde{\Phi}_{n+1}^{(b,\epsilon)}(q;z), \quad n \geq 0,$$

where

$$\overline{\beta}_{n+1}^{(b,\epsilon,s)}(q) = -\frac{\left\langle \widetilde{\Phi}_{n+1}^{(b,\epsilon)}(q;\,.\,), \Psi_n^{(b,\epsilon,s)}(q;\,.\,)\right\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}}}{\left\langle \Psi_n^{(b,\epsilon,s)}(q;\,.\,), \Psi_n^{(b,\epsilon,s)}(q;\,.\,)\right\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}}}, \quad n \geq 0. \tag{2.31}$$

Here, by $\overline{\beta}_{n+1}^{(b,\epsilon,s)}(q)$ we mean $\overline{\beta_{n+1}^{(b,\epsilon,s)}(q)}$. We now look at the numerators in the right hand side of (2.31). We have

$$\left\langle \widetilde{\Phi}_1^{(b,\epsilon)}(q;\,.\,), \Psi_0^{(b,\epsilon,s)}(q;\,.\,)\right\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}} = \left\langle \widetilde{\Phi}_1^{(b,\epsilon)}(q;\,.\,), 1\right\rangle_{\widetilde{\mu}_q^{(b,\epsilon)}} + s\left\langle 1, 0\right\rangle_{\mu_q^{(b+1)}} = 0.$$

Moreover, for $n \geq 1$,

$$\begin{aligned}&\left\langle \widetilde{\Phi}_{n+1}^{(b,\epsilon)}(q;\,.\,), \Psi_n^{(b,\epsilon,s)}(q;\,.\,)\right\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}}\\ &\quad= \left\langle \widetilde{\Phi}_{n+1}^{(b,\epsilon)}(q;\,.\,), \Psi_n^{(b,\epsilon,s)}(q;\,.\,)\right\rangle_{\widetilde{\mu}_q^{(b,\epsilon)}} + s\left\langle D_q[\widetilde{\Phi}_{n+1}^{(b,\epsilon)}(q;\,.\,)], D_q[\Psi_n^{(b,\epsilon,s)}(q;\,.\,)]\right\rangle_{\mu_q^{(b+1)}}\\ &\quad= s\left\langle D_q[\widetilde{\Phi}_{n+1}^{(b,\epsilon)}(q;\,.\,)], D_q[\Psi_n^{(b,\epsilon,s)}(q;\,.\,)]\right\rangle_{\mu_q^{(b+1)}}.\end{aligned}$$

Hence, from (2.23)

$$\begin{aligned}&\left\langle \widetilde{\Phi}_{n+1}^{(b,\epsilon)}(q;\,.\,), \Psi_n^{(b,\epsilon,s)}(q;\,.\,)\right\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}}\\ &\quad= s\left\langle \{n+1\}_q[\Phi_n^{(b+1)}(q;\,.\,) - \chi_{n+1}^{(b)}(q)\,\Phi_{n-1}^{(b+1)}(q;,.\,)], D_q[\Psi_n^{(b,\epsilon,s)}(q;\,.\,)]\right\rangle_{\mu_q^{(b+1)}},\end{aligned}$$

from which

$$\left\langle \widetilde{\Phi}_{n+1}^{(b,\epsilon)}(q;\,.\,), \Psi_n^{(b,\epsilon,s)}(q;\,.\,)\right\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}} = -s\,\{n\}_q\{n+1\}_q\,\overline{\chi}_{n+1}^{(b,\epsilon)}(q)\,A_{n-1}^{(b+1)}(q) = -\overline{\mathfrak{q}}_{n+1}^{(b,\epsilon,s)}(q), \quad n \geq 1.$$

Therefore, from (2.31) we find that $\beta_1^{(b,\epsilon,s)}(q) = 0$ and also that $\overline{\chi}_{n+1}^{(b,\epsilon)}(q)\beta_{n+1}^{(b,\epsilon,s)}(q) > 0$ for every $n \geq 1$.

Now we analyze the denominators in the right hand side of (2.31). We have

$$\left\langle \Psi_0^{(b,\epsilon,s)}(q;\,.\,), \Psi_0^{(b,\epsilon,s)}(q;\,.\,)\right\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}} = \left\langle \widetilde{\Phi}_0^{(b,\epsilon,s)}(q;\,.\,), \widetilde{\Phi}_0^{(b,\epsilon,s)}(q;\,.\,)\right\rangle_{\widetilde{\mu}_q^{(b,\epsilon)}} = 1$$

and, for $n \geq 1$,

$$\begin{aligned}\left\langle \Psi_n^{(b,\epsilon,s)}(q;\,.\,), \Psi_n^{(b,\epsilon,s)}(q;\,.\,)\right\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}} &= \left\langle \widetilde{\Phi}_n^{(b,\epsilon)}(q;\,.\,), \Psi_n^{(b,\epsilon,s)}(q;\,.\,)\right\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}}\\ &= \widetilde{A}_n^{(b,\epsilon)}(q) + s\left\langle D_q[\widetilde{\Phi}_n^{(b,\epsilon)}(q;\,.\,)], D_q[\Psi_n^{(b,\epsilon,s)}(q;\,.\,)]\right\rangle_{\mu_q^{(b+1)}}.\end{aligned}$$

Hence,

$$\left\langle \Psi_1^{(b,\epsilon,s)}(q;\,.\,), \Psi_1^{(b,\epsilon,s)}(q;\,.\,)\right\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}} = \widetilde{A}_1^{(b,\epsilon)}(q) + s\left\langle 1, 1\right\rangle_{\mu_q^{(b+1)}} = \widetilde{A}_1^{(b,\epsilon)}(q) + s.$$

For $n \geq 2$, from

$$\begin{aligned}D_q[\Psi_n^{(b,\epsilon,s)}(q;\,.\,)] &= D_q[\widetilde{\Phi}_n^{(b,\epsilon)}(q;\,.\,)] + \beta_n^{(b,\epsilon,s)}(q)\,D_q[\Psi_{n-1}^{(b,\epsilon,s)}(q;\,.\,)]\\ &= \{n\}_q\Big[\Phi_{n-1}^{(b+1)}(q;\,.\,) - \chi_n^{(b,\epsilon)}(q)\,\Phi_{n-2}^{(b+1)}(q;\,.\,)\Big] + \beta_n^{(b,\epsilon,s)}(q)\,D_q[\Psi_{n-1}^{(b,\epsilon,s)}(q;\,.\,)]\end{aligned}$$

and from (2.23), the above formula can be written as

$$\begin{aligned}\Big\langle \Psi_n^{(b,\epsilon,s)}(q;\,.\,), \Psi_n^{(b,\epsilon,s)}(q;\,.\,)\Big\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}} &\\ = \widetilde{A}_n^{(b,\epsilon)}(q) + s\,\{n\}_q^2\Big[A_{n-1}^{(b+1)}(q) + |\chi_n^{(b,\epsilon)}(q)|^2\,A_{n-2}^{(b+1)}(q)\Big] &\\ -s\,\{n\}_q\{n-1\}_q\,A_{n-2}^{(b+1)}(q)\,\overline{\chi}_n^{(b,\epsilon)}(q)\,\beta_n^{(b,\epsilon,s)}(q) &\\ = \mathfrak{p}_n^{(b,\epsilon,s)}(q) - \overline{\mathfrak{q}}_n^{(b,\epsilon,s)}(q)\,\beta_n^{(b,\epsilon,s)}(q). &\end{aligned} \tag{2.32}$$

Thus, the recurrence formula for $\beta_n^{(b,\epsilon,s)}(q)$ follows. ∎

For $n \geq 1$, using Theorem 2.6 we can also write

$$\Psi_n^{(b,\epsilon,s)}(q;z) - \beta_n^{(b,\epsilon,s)}(q)\Psi_{n-1}^{(b,\epsilon;s)}(q;z) = \frac{(q^{\lambda+1}\cos\eta_q;q)_n}{(q^{b+1};q)_n}\Big[R_n^{(b)}(q;z) - 2\Big(1-m_n^{(b,\epsilon)}(q)\Big)R_{n-1}^{(b)}(q;z)\Big].$$

2.5 Some Properties of the Sobolev-Type OPUC and Connection Coefficients

In this section we present some propierties of Sobolev-type orthogonal polynomials $\Psi_n^{(b,\epsilon,s)}(q;z)$ and of connection coefficients $\beta_n^{(b,\epsilon,s)}(q)$ given by (2.30).

The next theorem gives an upper bound and an asymptotic result for $\overline{\chi}_n^{(b,\epsilon)}(q)\beta_n^{(b,\epsilon,s)}(q)$, where $\chi_n^{(b,\epsilon)}(q)$ are given in (2.23).

Theorem 2.13. *The connection coefficients $\beta_n^{(b,\epsilon,s)}(q)$ in Theorem 2.12 satisfy*

(a) $$0 < \overline{\chi}_n^{(b,\epsilon)}(q)\beta_n^{(b,\epsilon,s)}(q) < \frac{\{n\}_q}{\{n-1\}_q}|\chi_n^{(b,\epsilon)}(q)|^2, \quad n \geq 2.$$

(b) $$\lim_{n\to\infty}\overline{\chi}_n^{(b,\epsilon)}(q)\beta_n^{(b,\epsilon,s)}(q) = q^{1/2}.$$

Here $\chi_n^{(b,\epsilon)}(q)$ are the connection coefficients in (2.23).

Proof. From

$$\begin{aligned}&\Big\langle \Psi_n^{(b,\epsilon,s)}(q;\,.\,), \Psi_n^{(b,\epsilon,s)}(q;\,.\,)\Big\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}}\\ &= \Big\langle \Psi_n^{(b,\epsilon,s)}(q;\,.\,), \Psi_n^{(b,\epsilon,s)}(q;\,.\,)\Big\rangle_{\widetilde{\mu}_q^{(b,\epsilon)}} + s\,\Big\langle D_q[\Psi_n^{(b,\epsilon,s)}(q;\,.\,)], D_q[\Psi_n^{(b,\epsilon,s)}(q;\,.\,)]\Big\rangle_{\mu_q^{(b+1)}},\end{aligned}$$

for $n \geq 2$ and from minimal norm characterization of monic orthogonal polynomials with respect to a probability measure supported on the unit circle, we get

$$\begin{aligned}&\Big\langle \Psi_n^{(b,\epsilon,s)}(q;\,.\,), \Psi_n^{(b,\epsilon,s)}(q;\,.\,)\Big\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}}\\ &> \Big\langle \widetilde{\Phi}_n^{(b,\epsilon,s)}(q;\,.\,), \widetilde{\Phi}_n^{(b,\epsilon,s)}(q;\,.\,)\Big\rangle_{\widetilde{\mu}_q^{(b,\epsilon)}} + s\{n\}_q^2\,\Big\langle \Phi_{n-1}^{(b+1)}(q;\,.\,), \Phi_{n-1}^{(b+1)}(q;\,.\,)\Big\rangle_{\mu_q^{(b+1)}}, \quad n \geq 2.\end{aligned}$$

Thus,

$$\Big\langle \Psi_n^{(b,\epsilon,s)}(q;\,.\,), \Psi_n^{(b,\epsilon,s)}(q;\,.\,)\Big\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}} > \widetilde{A}_n^{(b,\epsilon)}(q) + s\{n\}_q^2 A_{n-1}^{(b+1)}(q), \quad n \geq 2.$$

Hence, from (2.32) we have

$$s\{n\}_q\{n-1\}_q\, A_{n-2}^{(b+1)}(q)\overline{\chi}_n^{(b,\epsilon)}(q)\beta_n^{(b,\epsilon,s)}(q) < s\{n\}_q^2|\chi_n^{(b,\epsilon)}(q)|^2\, A_{n-2}^{(b+1)}(q), \quad n \geq 2.$$

This proves item (a) of the theorem.

Now to show item (b), we first observe from Theorem 2.12 that one can write $\mathfrak{d}_n^{(b,\epsilon,s)}(q) = [1-\mathfrak{m}_{n-1}^{(b,\epsilon,s)}(q)]\mathfrak{m}_n^{(b,\epsilon,s)}(q)$, where $\mathfrak{d}_n^{(b,\epsilon,s)}(q)$ are as in (2.28) and

$$\begin{aligned} \mathfrak{m}_n^{(b,\epsilon,s)}(q) &= \frac{\overline{\mathfrak{q}}_{n+1}^{(b,\epsilon,s)}(q)\beta_{n+1}^{(b,\epsilon,s)}(q)}{\mathfrak{p}_{n+1}^{(b,\epsilon,s)}(q)} \\ &= \frac{s\,\{n\}_q\{n+1\}_q A_{n-1}^{(b+1)}(q)\,\overline{\chi}_{n+1}^{(b,\epsilon)}(q)\,\beta_{n+1}^{(b,\epsilon,s)}(q)}{\widetilde{A}_{n+1}^{(b,\epsilon)}(q) + s\,\{n+1\}_q^2\Big[A_n^{(b+1)}(q) + A_{n-1}^{(b+1)}(q)\,|\chi_{n+1}^{(b,\epsilon)}(q)|^2\Big]}, \end{aligned} \tag{2.33}$$

for $n \geq 1$ with $\mathfrak{m}_0^{(b,\epsilon,s)}(q) = 0$. Since we can observe that $\mathfrak{m}_n^{(b,\epsilon,s)}(q) > 0$ and $\mathfrak{d}_n^{(b,\epsilon,s)}(q) > 0$ for $n \geq 1$, we also have $1-\mathfrak{m}_n^{(b,\epsilon,s)}(q) > 0$ for $n \geq 1$. Hence, we can state that $\{\mathfrak{d}_n^{(b,\epsilon,s)}(q)\}_{n\geq 1}$ is a positive chain sequence and $\{\mathfrak{m}_n^{(b,\epsilon,s)}(q)\}_{n\geq 0}$ is its minimal parameter sequence.

From (2.28) and the asymptotic behavior of $\widetilde{A}_n^{(b,\epsilon)}(q)$, $A_n^{(b+1)}(q)$ and $\chi_n^{(b,\epsilon)}(q)$ we now easily find

$$\lim_{n\to\infty} \mathfrak{d}_n^{(b,\epsilon,s)}(q) = \frac{q}{(1+q)^2}.$$

Since $0 < q < 1$, by Theorem 1.7 one finds

$$\lim_{n\to\infty} \mathfrak{m}_n^{(b,\epsilon)}(q) = \frac{1}{2}\left[1 - \sqrt{1 - 4\frac{q}{(1+q)^2}}\,\right] = \frac{q}{1+q}.$$

On the other hand, if $\lim_{n\to\infty} \overline{\chi}_n^{(b,\epsilon)}(q)\beta_n^{(b,\epsilon,s)}(q) = \ell_q$, then from (2.33),

$$\lim_{n\to\infty} \mathfrak{m}_n^{(b,\epsilon)}(q) = \frac{\ell_q\, q^{1/2}}{1+q}.$$

This immediately gives the asymptotic result for $\overline{\chi}_n^{(b,\epsilon)}(q)\beta_n^{(b,\epsilon,s)}(q)$ stated in the theorem.

■

From Theorems 2.9 and 2.13, we also have

$$\lim_{n\to\infty} \beta_n^{(b,\epsilon,s)}(q) = 1. \tag{2.34}$$

The coefficients $\beta_n^{(b,\epsilon,s)}(q)$ in (2.30) also turn out to be interesting from the point of view of orthogonal polynomials on the real line. We show that these coefficients can be expressed as rational functions involving a sequence of polynomials that satisfy a standard three term recurrence relation.

Definition 2.14. For given b and ϵ, let $\{S_n^{(b,\epsilon)}(q;y)\}_{n\geq 0}$ be such that

$$S_0^{(b,\epsilon)}(q;y) = 1 \quad \text{and} \quad \beta_{n+1}^{(b,\epsilon,\kappa_q^{(b,\epsilon)}/y)}(q) = \frac{S_{n-1}^{(b,\epsilon)}(q;y)}{S_n^{(b,\epsilon)}(q;y)}, \quad n \geq 1, \tag{2.35}$$

where

$$\kappa_q^{(b,\epsilon)} = \frac{2\tilde{m}_1^{(b,\epsilon)}(q)(1-q^{\lambda+1}\cos\eta_q)(1-q^{2\lambda+2})}{|1-q^{b+1}|^2}.$$

Then the following can be stated.

Theorem 2.15. *For $n \geq 0$, $S_n^{(b,\epsilon)}(y)$ in (2.35) is a polynomial in y of exact degree n. More precisely, if $\widehat{S}_n^{(b,\epsilon)}(q;y) = \mathfrak{t}_n^{(b,\epsilon)}(q)\, S_n^{(b,\epsilon)}(q;y)$, $n \geq 0$, with $\mathfrak{t}_0^{(b,\epsilon)}(q) = 1$ and*

$$\mathfrak{t}_n^{(b,\epsilon)}(q) = \frac{(1-\tilde{m}_{n+1}^{(b,\epsilon)}(q))}{(1-\tilde{m}_1^{(b,\epsilon)}(q))}\frac{(1-q^{\lambda+n+1}\cos\eta_q)}{(1-q^{\lambda+1}\cos\eta_q)}\frac{(q^{2\lambda+2};q)_n}{(q^{b+2};q)_n}\prod_{k=1}^{n}\{k\}_q^2, \quad n \geq 1,$$

then $\widehat{S}_n^{(b,\epsilon)}(q;y)$ are monic polynomials such that

$$\begin{aligned}
\widehat{S}_1^{(b,\epsilon)}(q;y) &= \Big[y \;+\; \mathfrak{a}_0^{(b,\epsilon)}(q)\Big]\widehat{S}_0^{(b,\epsilon)}(q;y),\\
\widehat{S}_n^{(b,\epsilon)}(q;y) &= \Big[y \;+\; \mathfrak{a}_{n-1}^{(b,\epsilon)}(q) + \mathfrak{c}_{n-1}^{(b,\epsilon)}(q)\Big]\widehat{S}_{n-1}^{(b,\epsilon)}(q;y) \;-\; \mathfrak{a}_{n-2}^{(b,\epsilon)}(q)\,\mathfrak{c}_{n-1}^{(b,\epsilon)}(q)\,\widehat{S}_{n-2}^{(b,\epsilon)}(q;y),
\end{aligned} \tag{2.36}$$

for $n \geq 2$. Here,

$$\begin{aligned}
\mathfrak{a}_{n-1}^{(b,\epsilon)}(q) &= \frac{2\{n\}_q^2\,(1-q^{\lambda+n+1}\cos\eta_q)\,\tilde{m}_{n+1}^{(b,\epsilon)}(q)}{(1-q^n)},\\
\mathfrak{c}_n^{(b,\epsilon)}(q) &= \frac{2\{n\}_q^2(1-q^{\lambda+n+1}\cos\eta_q)\,(1-\tilde{m}_{n+1}^{(b,\epsilon)}(q))}{(1-q^n)},
\end{aligned} \qquad n \geq 1.$$

Proof. Let us assume $\beta_{n+1}^{(b,\epsilon,\,\kappa_q^{(b,\epsilon)}/y)} = S_{n-1}^{(b,\epsilon)}(y)/S_n^{(b,\epsilon)}(q;y)$, with $S_0^{(b,\epsilon)}(q;y) = 1$. Hence, from (2.30), one finds $\mathfrak{q}_2^{(b,\epsilon,\,\kappa_q^{(b,\epsilon)}/y)}(q)S_1^{(b,\epsilon)}(q;y) = \mathfrak{p}_1^{(b,\epsilon,\,\kappa_q^{(b,\epsilon)}/y)}(q)S_0^{(b,\epsilon)}(q;y)$ and

$$\mathfrak{q}_{n+1}^{(b,\epsilon,\,\kappa_q^{(b,\epsilon)}/y)}(q)S_n^{(b,\epsilon)}(q;y) = \mathfrak{p}_n^{(b,\epsilon,\,\kappa_q^{(b,\epsilon)}/y)}(q)S_{n-1}^{(b,\epsilon)}(q;y) - \bar{\mathfrak{q}}_n^{(b,\epsilon,\,\kappa_q^{(b,\epsilon)}/y)}(q)S_{n-2}^{(b,\epsilon)}(q;y), \quad n \geq 2.$$

With the use of (2.25) this leads to

$$\begin{aligned}
\frac{y\mathfrak{q}_{n+1}^{(b,\epsilon,\,\kappa_q^{(b,\epsilon)}/y)}(q)}{\widetilde{A}_n^{(b,\epsilon)}(q)}S_n^{(b,\epsilon)}(q;y) = \Big[y + \frac{\kappa_q^{(b,\epsilon)}\{n\}_q^2}{\widetilde{A}_n^{(b,\epsilon)}(q)}\Big[A_{n-1}^{(b+1)}(q) + A_{n-2}^{(b+1)}(q)\,|\chi_n^{(b,\epsilon)}(q)|^2\Big]\Big]S_{n-1}^{(b,\epsilon)}(q;y)&\\
- \frac{y\bar{\mathfrak{q}}_n^{(b,\epsilon,\,\kappa_q^{(b,\epsilon)}/y)}(q)}{\widetilde{A}_n^{(b,\epsilon)}(q)}S_{n-2}^{(b,\epsilon)}(q;y),&
\end{aligned}$$

for $n \geq 2$ and $\dfrac{y\mathfrak{q}_2^{(b,\epsilon,\,\kappa_q^{(b,\epsilon)}/y)}(q)}{\widetilde{A}_1^{(b,\epsilon)}(q)}S_1^{(b,\epsilon)}(q;y) = \Big[y + \dfrac{\kappa_q^{(b,\epsilon)}}{\widetilde{A}_1^{(b,\epsilon)}(q)}A_0^{(b+1)}(q)\Big]S_0^{(b,\epsilon)}(q;y).$

Clearly, $y\mathfrak{q}_n^{(b,\epsilon,\,\kappa_q^{(b,\epsilon)}/y)}(q)$ are independent of y for all $n \geq 2$. Thus, if

$$\widehat{S}_0^{(b,\epsilon)}(q;y) = S_0^{(b,\epsilon)}(q;y) \quad \text{and} \quad \widehat{S}_n^{(b,\epsilon)}(q;y) = S_n^{(b,\epsilon)}(q;y)\prod_{k=1}^{n}\frac{y\mathfrak{q}_{k+1}^{(b,\epsilon,\,\kappa_q^{(b,\epsilon)}/y)}(q)}{\widetilde{A}_k^{(b,\epsilon)}(q)}, \quad n \geq 1,$$

then

$$\begin{aligned}
\widehat{S}_n^{(b,\epsilon)}(q;y) = \Big[y + \frac{\kappa_q^{(b,\epsilon)}\{n\}_q^2}{\widetilde{A}_n^{(b,\epsilon)}(q)}\Big[A_{n-1}^{(b+1)}(q) + A_{n-2}^{(b+1)}(q)\,|\chi_n^{(b,\epsilon)}(q)|^2\Big]\Big]\widehat{S}_{n-1}^{(b,\epsilon)}(q;y)&\\
- \frac{y\bar{\mathfrak{q}}_n^{(b,\epsilon,\,\kappa_q^{(b,\epsilon)}/y)}(q)}{\widetilde{A}_n^{(b,\epsilon)}(q)}\frac{y\mathfrak{q}_n^{(b,\epsilon,\,\kappa_q^{(b,\epsilon)}/y)}(q)}{\widetilde{A}_{n-1}^{(b,\epsilon)}(q)}\widehat{S}_{n-2}^{(b,\epsilon)}(q;y),&
\end{aligned}$$

for $n \geq 2$ and $\widehat{S}_1^{(b,\epsilon)}(y) = \Big[y + \dfrac{\kappa_q^{(b,\epsilon)}}{\widetilde{A}_1^{(b,\epsilon)}(q)} A_0^{(b+1)}(q)\Big]\widehat{S}_0^{(b,\epsilon)}(y)$.

First, we observe from (2.23) and (2.26) that

$$\frac{\kappa_q^{(b,\epsilon)}\{n\}_q^2 A_{n-1}^{(b+1)}(q)}{\widetilde{A}_n^{(b,\epsilon)}(q)} = \frac{\kappa_q^{(b,\epsilon)}|(1-q^{(b+1)}|^2}{2\tilde{m}_1^{(b,\epsilon)}(q)(1-q^{\lambda+1}\cos\eta_q)(1-q^{2\lambda+2})} \frac{2\{n\}_q^2\tilde{m}_{n+1}^{(b,\epsilon)}(q)(1-q^{\lambda+n+1}\cos\eta_q)}{(1-q^n)}$$

for $n \geq 1$. Thus, with the value of $\kappa_q^{(b,\epsilon)}$ in the theorem, we then find

$$\begin{aligned}
\mathfrak{a}_{n-1}^{(\lambda,\epsilon)}(q) = \frac{\kappa_q^{(b,\epsilon)}\{n\}_q^2 A_{n-1}^{(b+1)}(q)}{\widetilde{A}_n^{(b,\epsilon)}(q)} &= \frac{2\{n\}_q^2\,\tilde{m}_{n+1}^{(b,\epsilon)}(q)\,(1-q^{\lambda+n+1}\cos\eta_q)}{(1-q^n)} \\
&= \frac{\{n\}_q^2\,(1-q^{2\lambda+n+1})}{2(1-q^{\lambda+n}\cos\eta_q)(1-\tilde{m}_n^{(b,\epsilon)}(q))}, \quad n \geq 1.
\end{aligned}$$

For the latter formula for $\mathfrak{a}_{n-1}^{(\lambda,\epsilon)}(q)$ we use $\tilde{m}_{n+1}^{(b,\epsilon)}(q) = \tilde{d}_{n+1}^{(b,\epsilon)}(q)/(1-\tilde{m}_n^{(b,\epsilon)}(q))$. With the same choice for $\kappa_q^{(b,\epsilon)}$, we also find

$$\begin{aligned}
\mathfrak{c}_n^{(\lambda,\epsilon)}(q) &= \frac{\kappa_q^{(b,\epsilon)}\{n+1\}_q^2 A_{n-1}^{(b+1)}(q)|\chi_{n+1}^{(b,\epsilon)}(q)|^2}{\widetilde{A}_{n+1}^{(b,\epsilon)}(q)} \\
&= \frac{2\{n\}_q^2(1-q^{\lambda+n+1}\cos\eta_q)(1-\tilde{m}_{n+1}^{(b,\epsilon)}(q))}{1-q^n}, \quad n \geq 1.
\end{aligned}$$

Moreover, we can verify that

$$\frac{y\mathfrak{q}_{n+1}^{(b,\epsilon,\kappa_q^{(b,\epsilon)}/y)}(q)}{\widetilde{A}_n^{(b,\epsilon)}} = \{n\}_q^2\frac{(1-\tilde{m}_{n+1}^{(b,\epsilon)}(q))}{(1-\tilde{m}_n^{(b,\epsilon)}(q))}\frac{(1-q^{\lambda+n+1}\cos\eta_q)}{(1-q^{\lambda+n}\cos\eta_q)}\frac{(1-q^{2\lambda+n+1})}{(1-q^{b+n+1})}$$

and

$$\frac{y\overline{\mathfrak{q}}_{n+1}^{(b,\epsilon,\kappa_q^{(b,\epsilon)}/y)}(q)}{\widetilde{A}_{n+1}^{(b,\epsilon)}} = \{n\}_q^2\frac{(1-q^{b+n+1})}{(1-q^n)},$$

for $n \geq 1$. Thus, we also find

$$\frac{y\overline{\mathfrak{q}}_n^{(b,\epsilon,\kappa_q^{(b,\epsilon)}/y)}(q)}{\widetilde{A}_n^{(b,\epsilon)}}\frac{y\mathfrak{q}_n^{(b,\epsilon,\kappa_q^{(b,\epsilon)}/y)}(q)}{\widetilde{A}_{n-1}^{(b,\epsilon)}} = \mathfrak{a}_{n-2}^{(b,\epsilon)}(q)\,\mathfrak{c}_{n-1}^{(b,\epsilon)}(q), \quad n \geq 2,$$

and

$$\mathfrak{t}_n^{(b,\epsilon)}(q) = \prod_{k=1}^{n}\frac{y\mathfrak{q}_{k+1}^{(b,\epsilon,\,\kappa_q^{(b,\epsilon)}/y)}(q)}{\widetilde{A}_k^{(b,\epsilon)}(q)} = \frac{(1-\tilde{m}_{n+1}^{(b,\epsilon)}(q))}{(1-\tilde{m}_1^{(b,\epsilon)}(q))}\frac{(1-q^{\lambda+n+1}\cos\eta_q)}{(1-q^{\lambda+1}\cos\eta_q)}\frac{(q^{2\lambda+2};q)_n}{(q^{b+2};q)_n}\prod_{k=1}^{n}\{k\}_q^2,$$

for $n \geq 1$. Thus, we have shown the results of the theorem. ∎

From Theorem 2.15 and from (2.35), we can also write that

$$\begin{aligned}\beta_{n+1}^{(b,\epsilon,\kappa_q^{(b,\epsilon)}/y)}(q) &= \frac{\mathfrak{t}_n^{(b,\epsilon)}(q)\,\widehat{S}_{n-1}^{(b,\epsilon)}(q;y)}{\mathfrak{t}_{n-1}^{(b,\epsilon)}(q)\,\widehat{S}_n^{(b,\epsilon)}(q;y)} \\ &= \{n\}_q^2 \frac{(1-\tilde{m}_{n+1}^{(b,\epsilon)}(q))}{(1-\tilde{m}_n^{(b,\epsilon)}(q))} \frac{(1-q^{\lambda+n+1}\cos\eta_q)}{(1-q^{\lambda+n}\cos\eta_q)} \frac{(1-q^{2\lambda+n+1})}{(1-q^{b+n+1})} \frac{\widehat{S}_{n-1}^{(b,\epsilon)}(q;y)}{\widehat{S}_n^{(b,\epsilon)}(q;y)},\end{aligned} \tag{2.37}$$

for $n \geq 1$.

Remark 2.16. The three-term recurrence relation (2.36) shows that $\{\widehat{S}_n^{(b,\epsilon)}(q;y)\}_{n\geq 0}$ forms a sequence of MOP with respect to some positive measure on the real line. Since the Sobolev inner product given by (2.24) is positive definite for $0 < s < \infty$ and the rational functions $\widehat{S}_{n-1}^{(b,\epsilon)}(q;y)/\widehat{S}_n^{(b,\epsilon)}(q;y)$ are well defined for $0 < y < \infty$, the support of this measure is, as expected, the negative half of the real axis. The identification of such a sequence of monic orthogonal polynomials is an open problem.

The following theorem gives bounds for $\beta_n^{(b,\epsilon,s)}(q)$.

Theorem 2.17. *The constants* $\beta_n^{(b,\epsilon,s)}(q)$, $n \geq 1$, *given by* (2.30) *for* $0 < s < \infty$, $0 \leq \epsilon < 1$ *and* $\mathrm{Re}(b) = \lambda > -1$, *are such that* $\beta_1^{(b,\epsilon,s)}(q) = 0$ *and*

$$\left|\beta_{n+1}^{(b,\epsilon,s)}(q)\right| < 2[1-\tilde{m}_{n+1}^{(b,\epsilon)}(q)]\,\frac{1-q^{\lambda+n+1}\cos\eta_q}{|1-q^{b+n+1}|}\,\frac{\mathfrak{a}_{n-1}^{(b,\epsilon)}(q)+\mathfrak{c}_{n-1}^{(b,\epsilon)}(q)}{\mathfrak{a}_{n-1}^{(b,\epsilon)}(q)+\mathfrak{c}_{n-1}^{(b,\epsilon)}(q)+\kappa_q^{(b,\epsilon)}/s}, \quad n \geq 1,$$

where $\kappa_q^{(b,\epsilon)}$, $\{\mathfrak{a}_{n-1}^{(b,\epsilon)}(q)\}_{n\geq 1}$ *and* $\{\mathfrak{c}_n^{(b,\epsilon)}(q)\}_{n\geq 1}$ *are as in* (2.35) *and Theorem 2.15. We also set* $\mathfrak{c}_0^{(b,\epsilon)}(q) = 0$.

Proof. From the TTRR (2.36) one can consider the sequences $\{\hat{d}_{n+1}^{(b,\epsilon)}(q;y)\}_{n\geq 1}$ and $\{\hat{m}_{n+1}^{(b,\epsilon)}(q;y)\}_{n\geq 0}$ such that

$$\left[1-\hat{m}_n^{(b,\epsilon)}(q;y)\right]\hat{m}_{n+1}^{(b,\epsilon)}(q;y) = \hat{d}_{n+1}^{(b,\epsilon)}(q;y) = \frac{\mathfrak{a}_{n-1}^{(b,\epsilon)}(q)\,\mathfrak{c}_n^{(b,\epsilon)}(q)}{\left[y+\mathfrak{a}_{n-1}^{(b,\epsilon)}(q)+\mathfrak{c}_{n-1}^{(b,\epsilon)}(q)\right]\left[y+\mathfrak{a}_n^{(b,\epsilon)}(q)+\mathfrak{c}_n^{(b,\epsilon)}(q)\right]}$$

and

$$1-\hat{m}_n^{(b,\epsilon)}(q;y) = \frac{\widehat{S}_n^{(b,\epsilon)}(q;y)}{\left[y+\mathfrak{a}_{n-1}^{(b,\epsilon)}(q)+\mathfrak{c}_{n-1}^{(b,\epsilon)}(q)\right]\widehat{S}_{n-1}^{(b,\epsilon)}(q;y)}, \tag{2.38}$$

for $n \geq 1$. For any $y \geq 0$ the sequence $\{\hat{d}_{n+1}^{(b,\epsilon)}(q;y)\}_{n\geq 1}$ is a positive chain sequence and $\{\hat{m}_{n+1}^{(b,\epsilon)}(q;y)\}_{n\geq 0}$ is its minimal parameter sequence. One way to verify this assertion is by observing the following:

By using Theorem 1.27, the sequence $\{\hat{d}_{n+1}^{(b,\epsilon)}(q;0)\}_{n\geq 1}$ is clearly a positive chain sequence and its minimal parameter sequence $\{\hat{m}_{n+1}^{(b,\epsilon)}(q;0)\}_{n\geq 0}$ is given by

$$\hat{m}_n^{(b,\epsilon)}(q;0) = \frac{\mathfrak{c}_{n-1}^{(b,\epsilon)}(q)}{\mathfrak{a}_{n-1}^{(b,\epsilon)}(q)+\mathfrak{c}_{n-1}^{(b,\epsilon)}(q)}, \quad n \geq 1.$$

Moreover,

$$\hat{d}_{n+1}^{(b,\epsilon)}(q;y) < \hat{d}_{n+1}^{(b,\epsilon)}(q;0), \quad n \geq 1,$$

for $y > 0$. Thus, our assertion follows from Theorem 1.6.

Furthermore, by Theorem 1.5, we also obtain that $\hat{m}_{n+1}^{(b,\epsilon)}(q;y) < \hat{m}_{n+1}^{(b,\epsilon)}(q;0)$ for $n \geq 1$ and $y > 0$. This means, from (2.38),

$$\frac{\widehat{S}_n^{(b,\epsilon)}(q;y)}{\Big[y + \mathfrak{a}_{n-1}^{(b,\epsilon)}(q) + \mathfrak{c}_{n-1}^{(b,\epsilon)}(q)\Big]\widehat{S}_{n-1}^{(b,\epsilon)}(q;y)} > \frac{\widehat{S}_n^{(b,\epsilon)}(q;0)}{\Big[0 + \mathfrak{a}_{n-1}^{(b,\epsilon)}(q) + \mathfrak{c}_{n-1}^{(b,\epsilon)}(q)\Big]\widehat{S}_{n-1}^{(b,\epsilon)}(q;0)} = \frac{\mathfrak{a}_{n-1}^{(b,\epsilon)}(q)}{\mathfrak{a}_{n-1}^{(b,\epsilon)}(q) + \mathfrak{c}_{n-1}^{(b,\epsilon)}(q)},$$

for $n \geq 2$. Thus, for $y \geq 0$,

$$\frac{\widehat{S}_{n-1}^{(b,\epsilon)}(q;y)}{\widehat{S}_n^{(b,\epsilon)}(q;y)} \leq \frac{\mathfrak{a}_{n-1}^{(b,\epsilon)}(q) + \mathfrak{c}_{n-1}^{(b,\epsilon)}(q)}{\mathfrak{a}_{n-1}^{(b,\epsilon)}(q)\,[y + \mathfrak{a}_{n-1}^{(b,\epsilon)}(q) + \mathfrak{c}_{n-1}^{(b,\epsilon)}(q)]}, \quad n > 1. \tag{2.39}$$

The equality holds when $y = 0$.

Substitution of this in (2.37) gives the required result of the theorem. ∎

The following theorem gives information about the convergence and some monotonicity properties associated with $\beta_n^{(b,\epsilon,s)}$.

Theorem 2.18. *For any given $n \geq 2$ the following results hold:*

$$\lim_{s\to\infty} \beta_n^{(b,\epsilon,s)}(q) = 2[1 - \tilde{m}_n^{(b,\epsilon)}(q)]\,\frac{1 - q^{\lambda+n}\cos\eta_q}{1 - q^{b+n}}\ .$$

Moreover, for b and ϵ fixed real numbers $|\beta_n^{(b,\epsilon,s)}(q)|$ is a strictly increasing function of s for $s \in [0,\infty)$.

Proof. We have from (2.35) and Theorem 2.15

$$\frac{\partial}{\partial y}|\beta_{n+1}^{(b,\epsilon,\,\kappa_q^{(b,\epsilon)}/y)}(q)| = -\frac{|\mathfrak{t}_n^{(b,\epsilon)}(q)|}{|\mathfrak{t}_{n-1}^{(b,\epsilon)}(q)|}\,\frac{\widehat{S}_n^{(b,\epsilon)\prime}(q;y)\widehat{S}_{n-1}^{(b,\epsilon)}(q;y) - \widehat{S}_n^{(b,\epsilon)}(q;y)\widehat{S}_{n-1}^{(b,\epsilon)\prime}(q;y)}{\Big(\widehat{S}_n^{(b,\epsilon)}(q;y)\Big)^2},$$

for $n \geq 1$. Observe that $\widehat{S}_n^{(b,\epsilon)\prime}(q;y)\widehat{S}_{n-1}^{(b,\epsilon)}(q;y) - \widehat{S}_n^{(b,\epsilon)}(q;y)\widehat{S}_{n-1}^{(b,\epsilon)\prime}(q;y)$ is positive for every real number y. This can be verified from the Theorem 1.22 which is the Christoffel-Darboux identity that follows from (2.36). Thus, $|\beta_{n+1}^{(b,\epsilon,\,\kappa_q^{(b,\epsilon)}/y)}(q)|$ is a strictly decreasing function of y and, consequently, $|\beta_{n+1}^{(b,\epsilon,s)}(q)|$ is a strictly increasing function of s.

Moreover, $\lim_{s\to\infty}\beta_{n+1}^{(b,\epsilon,s)}(q) = \lim_{y\to 0}\beta_{n+1}^{(b,\epsilon,\,\kappa_q^{(b,\epsilon)}/y)}(q)$ and the limit value in the theorem follows from (2.37) and (2.39). ∎

With the results in Theorem 2.18 we can now address the asymptotic behavior of $\Psi_n^{(b,\epsilon,s)}(q;z)$ with respect to the parameter s.

Theorem 2.19. *For any $n \geq 1$,*

$$\lim_{s\to\infty} \Psi_n^{(b,\epsilon,s)}(q;z) = \widehat{R}_n^{(b)}(q;z) - 2^n\frac{(q^{\lambda+1}\cos\eta_q;q)_n}{(q^{b+1};q)_n}\prod_{j=1}^{n}\Big(1 - \tilde{m}_j^{(b,\epsilon)}(q)\Big).$$

Proof. By considering the orthogonal polynomials that follow from the limit inner product $\lim_{s\to\infty} s^{-1}\langle f,g\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}} = \langle D_q[f], D_q[g]\rangle_{\mu_q^{(b+1)}}$, we first obtain from (2.15)

$$\lim_{s\to\infty}\Psi_n^{(b,\epsilon,s)}(z) = \Psi_n^{(b,\epsilon,\infty)}(z) = \widehat{R}_n^{(b)}(q;z) + \rho_n^{(b,\epsilon)}(q), \quad n \geq 1.$$

The constants $\rho_n^{(b,\epsilon)}(q)$ are still to be determined.

On the other hand, from (2.29)

$$\Psi_n^{(b,\epsilon,\infty)}(q;z) - \beta_n^{(b,\epsilon,\infty)}(q)\Psi_{n-1}^{(b,\epsilon,\infty)}(q;z) = \widetilde{\Phi}_n^{(b,\epsilon)}(q;z), \quad n \geq 1,$$

where $\beta_1^{(b,\epsilon,\infty)}(q) = 0$ and the values of $\beta_n^{(b,\epsilon,\infty)} = \lim_{s\to\infty} \beta_n^{(b,\epsilon,s)}$, $n \geq 2$, are found in Theorem 2.18.

Combining the above two formulas we get $\widehat{R}_1^{(b)}(q;z) + \rho_1^{(b,\epsilon)}(q) = \widetilde{\Phi}_1^{(b,\epsilon)}(q;z)$ and

$$\left[\widehat{R}_n^{(b)}(q;z) - \beta_n^{(b,\epsilon,\infty)}\widehat{R}_{n-1}^{(b)}(q;z)\right] + \left[\rho_n^{(b,\epsilon)}(q) - \beta_n^{(b,\epsilon,\infty)}\rho_{n-1}^{(b,\epsilon)}(q)\right] = \widetilde{\Phi}_n^{(b,\epsilon)}(z),$$

for $n \geq 2$. Thus, comparing the above with (2.19) we immediately find

$$\rho_1^{(b,\epsilon)}(q) = -2[1 - \tilde{m}_1^{(b,\epsilon)}(q)]\,\frac{1 - q^{\lambda+1}\cos\eta_q}{1 - q^{b+1}} \quad \text{and} \quad \rho_n^{(b,\epsilon)}(q) = \beta_n^{(b,\epsilon,\infty)}\rho_{n-1}^{(b,\epsilon)}(q), \quad n \geq 2.$$

The required result of the theorem follows. ∎

Finally, we give an outer relative asymptotic result.

Theorem 2.20.

$$\lim_{n\to\infty} \frac{\Psi_n^{(b,\epsilon,s)}(q;z)}{\widetilde{\Phi}_n^{(b,\epsilon)}(q;z)} = \frac{z}{z-1},$$

in every compact subset of $\mathbb{C} \setminus \overline{\mathbb{D}}$.

Proof. From (2.29) we obtain

$$\frac{\Psi_n^{(b,\epsilon,s)}(q;z)}{\widetilde{\Phi}_n^{(b,\epsilon)}(q;z)} - \beta_n^{(b,\epsilon,s)}(q)\frac{\widetilde{\Phi}_{n-1}^{(b,\epsilon)}(q;z)}{\widetilde{\Phi}_n^{(b,\epsilon)}(q;z)}\frac{\Psi_{n-1}^{(b,\epsilon,s)}(q;z)}{\widetilde{\Phi}_{n-1}^{(b,\epsilon)}(q;z)} = 1, \quad n \geq 1, \tag{2.40}$$

On the other hand, by using (2.20) $\lim_{n\to\infty} \widetilde{\Phi}_{n-1}^{(b,\epsilon)}(q;z)/\widetilde{\Phi}_n^{(b,\epsilon)}(q;z) = 1/z$. Thus, from (2.34), we obtain (see the proof of Theorem 5.1 in [49])

$$\lim_{n\to\infty} \frac{\Psi_n^{(b,\epsilon,s)}(q;z)}{\widetilde{\Phi}_n^{(b,\epsilon)}(q;z)} = U(z),$$

in every compact subset of $\mathbb{C} \setminus \overline{\mathbb{D}}$. Taking the limits in (2.40)

$$U(z) - \frac{1}{z}U(z) = 1$$

from which $U(z) = z/(z-1)$. This completes the proof of the theorem. ∎

Remark 2.21. According to the Hurwitz's theorem, for n large enough, the zeros of the polynomials $\Psi_n^{(b,\epsilon,s)}(q;z)$ are located in the interior of the unit disk.

3 Coherent Pair of Moment Functionals of Second Kind on the Real Line

The aim in this chapter is to consider a characterization of pairs of positive measures $\{\nu_0, \nu_1\}$ on the real line for which $\{\mathcal{P}_n(\nu_0;\cdot)\}_{n\geq 0}$ and $\{\mathcal{P}_n(\nu_1;\cdot)\}_{n\geq 0}$, respectively the corresponding sequences of MOP, satisfy

$$\mathcal{P}_n(\nu_1;x) - \tau_n \mathcal{P}_{n-1}(\nu_1;x) = \frac{1}{n+1}\mathcal{P}'_{n+1}(\nu_0;x), \quad n \geq 1, \tag{3.1}$$

where $\tau_n \neq 0$ for $n \geq 1$. We study this problem by dealing with a more general problem in the framework of quasi-definite moment functionals. We also present a matrix characterization.

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3.1 Coherent Pairs of the Second Kind

In order to arrive at information about pairs of positive measures $\{\nu_0, \nu_1\}$ for which there hold (3.1), in this section we study this problem in the framework of quasi-definite moment functionals, where we introduce the concept of coherence of second kind on the real line for quasi-definite moment functionals as follows.

Definition 3.1. Let $\mathbf{v}_0$ and $\mathbf{v}_1$ be quasi-definite moment functionals and let $\{P_n^{(0)}\}_{n\geq 0}$ and $\{P_n^{(1)}\}_{n\geq 0}$ be, respectively, the corresponding sequences of MOP. Then $\{\mathbf{v}_0, \mathbf{v}_1\}$ is said to be a *coherent pair of moment functionals of the second kind* (CPMF2K for short) if there exist non-zero constants τ_n such that

$$\frac{1}{n+1}P_{n+1}^{(0)\prime}(x) = P_n^{(1)}(x) - \tau_n P_{n-1}^{(1)}(x), \quad n \geq 1. \tag{3.2}$$

Let $\{\beta_n^{(0)}, \alpha_{n+1}^{(0)}\}_{n\geq 1}$ and $\{\beta_n^{(1)}, \alpha_{n+1}^{(1)}\}_{n\geq 1}$ be the coefficients in the TTRR

$$P_{n+1}^{(i)}(x) = (x - \beta_{n+1}^{(i)})P_n^{(i)}(x) - \alpha_{n+1}^{(i)}P_{n-1}^{(i)}(x), \quad n \geq 1, \quad i = 0, 1,$$

with $P_0^{(0)}(x) = P_0^{(1)}(x) = 1$ and $P_n^{(i)}(x) = x - \beta_1^{(i)}$, $i = 0, 1$. Also let $\mathcal{h}_n^{(0)} = \left\langle \mathbf{v}_0, (P_n^{(0)})^2 \right\rangle$ and $\mathcal{h}_n^{(1)} = \left\langle \mathbf{v}_1, (P_n^{(1)})^2 \right\rangle$ for $n \geq 0$.

Since $\{P_n^{(0)}\}_{n\geq 0}$ and $\{P_n^{(1)}\}_{n\geq 0}$ are a basis for $\mathbb{P}$, their corresponding dual basis exist and we denote, respectively, by $\{\mathbf{v}_n^{(0)}\}_{n\geq 0}$ and $\{\mathbf{v}_n^{(1)}\}_{n\geq 0}$. These sequences satisfy (1.10).

In the sequel we will write

$$\tilde{P}_n^{(0)}(x) = \frac{P_{n+1}^{(0)\prime}(x)}{n+1}, \quad n \geq 0,$$

and assume $\{\tilde{\mathbf{v}}_n^{(0)}\}_{n\geq 0}$ to be the dual basis given by

$$\left\langle \tilde{\mathbf{v}}_n^{(0)}, \tilde{P}_m^{(0)} \right\rangle = \delta_{n,m}.$$

In the following result we state a relation between the elements of the dual bases of the sequences $\{P_n^{(0)}\}_{n\geq 0}$, $\{P_n^{(1)}\}_{n\geq 0}$ and $\{P_{n+1}^{(0)\prime}(x)/(n+1)\}_{n\geq 0}$, when $\{\mathbf{v}_0, \mathbf{v}_1\}$ is a CPMF2K.

Proposition 3.2. *Let $\mathbf{v}_0$ and $\mathbf{v}_1$ be quasi-definite moment functionals and let $\{P_n^{(0)}\}_{n\geq 0}$ and $\{P_n^{(1)}\}_{n\geq 0}$ be, respectively, the corresponding sequences of MOP. If $\{\mathbf{v}_0, \mathbf{v}_1\}$ is a CPMF2K, then*

(a) $\quad \mathbf{v}_n^{(1)} = \tilde{\mathbf{v}}_n^{(0)} - \tau_{n+1}\tilde{\mathbf{v}}_{n+1}^{(0)}, \quad n \geq 0.$

(b) $\quad \mathcal{D}\left[\dfrac{P_n^{(1)}(x)}{\mathcal{h}_n^{(1)}}\,\mathbf{v}_1\right] = \left[(n+2)\tau_{n+1}\dfrac{P_{n+2}^{(0)}(x)}{\mathcal{h}_{n+2}^{(0)}} - (n+1)\dfrac{P_{n+1}^{(0)}(x)}{\mathcal{h}_{n+1}^{(0)}}\right]\mathbf{v}_0, \quad n \geq 0.$

Here, τ_n is as in (3.2).

Proof. For $n \geq 0$ if one writes

$$\mathbf{v}_n^{(1)} = \sum_{j=0}^{\infty} \lambda_{n,j}\,\tilde{\mathbf{v}}_j^{(0)},$$

then from (3.2)

$$\lambda_{n,j} = \left\langle \mathbf{v}_n^{(1)}, \tilde{P}_j^{(0)} \right\rangle = \left\langle \mathbf{v}_n^{(1)}, P_j^{(1)} \right\rangle - \tau_j \left\langle \mathbf{v}_n^{(1)}, P_{j-1}^{(1)} \right\rangle = \begin{cases} 1, & \text{if } \quad j = n, \\ -\tau_{n+1}, & \text{if } \quad j = n+1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the first item of the theorem follows.

On the other hand, by Lemma 1.14, $\mathbf{v}_n^{(0)}$ and $\mathbf{v}_n^{(1)}$ satisfy

$$\mathbf{v}_n^{(0)} = \frac{P_n^{(0)}(x)}{\mathcal{h}_n^{(0)}}\,\mathbf{v}_0, \quad \mathbf{v}_n^{(1)} = \frac{P_n^{(1)}(x)}{\mathcal{h}_n^{(1)}}\,\mathbf{v}_1, \qquad n \geq 0, \tag{3.3}$$

and by Lemma 1.15, $\tilde{\mathbf{v}}_n^{(0)}$ satisfies

$$\mathcal{D}(\tilde{\mathbf{v}}_n^{(0)}) = -(n+1)\mathbf{v}_{n+1}^{(0)} = -(n+1)\frac{P_{n+1}^{(0)}(x)}{\mathcal{h}_{n+1}^{(0)}}\,\mathbf{v}_0, \quad n \geq 1.$$

Thus, taking derivatives in the first item of the proposition and using the above equality there follows

$$\mathcal{D}(\mathbf{v}_n^{(1)}) = -(n+1)\mathbf{v}_{n+1}^{(0)} + (n+2)\tau_{n+1}\mathbf{v}_{n+2}^{(0)}, \quad n \geq 0.$$

Hence, the second result follows from (3.3). ■

In the next result we state a first characterization for the pair of quasi-definite moment functionals $\{\mathbf{v}_0, \mathbf{v}_1\}$ which is a CPMF2Ks.

Proposition 3.3. *Let $\mathbf{v}_0$ and $\mathbf{v}_1$ be quasi-definite moment functionals. Then, $\{\mathbf{v}_0, \mathbf{v}_1\}$ is a CPMF2K if and only if there exists an admissible pair of polynomials (A_3, A_2) with $\deg(A_3) \le 3$ and $\deg(A_2) = 2$ such that*

$$\mathcal{D}\mathbf{v}_1 = A_2\mathbf{v}_0 \quad \textit{and} \quad \mathbf{v}_1 = A_3\mathbf{v}_0. \tag{3.4}$$

Proof. First we assume that the pair of quasi-definite moment functionals $\{\mathbf{v}_0, \mathbf{v}_1\}$ is such that there exist polynomials

$$\begin{aligned} A_2(x) &= \mathfrak{d}_2 P_2^{(0)}(x) + \mathfrak{d}_1 P_1^{(0)}(x) + \mathfrak{d}_0 P_0^{(0)}(x) \qquad \text{and} \\ A_3(x) &= \mathfrak{c}_3 P_3^{(0)}(x) + \mathfrak{c}_2 P_2^{(0)}(x) + \mathfrak{c}_1 P_1^{(0)}(x) + \mathfrak{c}_0 P_0^{(0)}(x), \end{aligned} \tag{3.5}$$

such that $\mathcal{D}\mathbf{v}_1 = A_2\mathbf{v}_0$ and $\mathbf{v}_1 = A_3\mathbf{v}_0$. Then from

$$\frac{P_{n+1}^{(0)\prime}(x)}{n+1} = P_n^{(1)}(x) + \sum_{j=0}^{n-1} \lambda_{n,j}\, P_j^{(1)}(x), \quad n \ge 1,$$

where $(n+1)\hbar_j^{(1)}\lambda_{n,j} = \left\langle \mathbf{v}_1, P_{n+1}^{(0)\prime}\, P_j^{(1)} \right\rangle = \left\langle \mathbf{v}_1, (P_j^{(1)} P_{n+1}^{(0)})' - P_j^{(1)\prime} P_{n+1}^{(0)} \right\rangle$, we get

$$\lambda_{n,j} = -\frac{1}{(n+1)\hbar_j^{(1)}} \left[\left\langle \mathcal{D}\mathbf{v}_1, P_j^{(1)} P_{n+1}^{(0)} \right\rangle + \left\langle \mathbf{v}_1, P_j^{(1)\prime} P_{n+1}^{(0)} \right\rangle \right],$$

for $0 \le j \le n-1$ and $n \ge 1$. Here, with the use of $\mathcal{D}\mathbf{v}_1 = A_2\mathbf{v}_0$ and $\mathbf{v}_1 = A_3\mathbf{v}_0$, one finds

$$\lambda_{n,j} = -\frac{1}{(n+1)\hbar_j^{(1)}} \left[\left\langle \mathbf{v}_0, A_2 P_j^{(1)} P_{n+1}^{(0)} \right\rangle + \left\langle \mathbf{v}_0, A_3 P_j^{(1)\prime} P_{n+1}^{(0)} \right\rangle \right],$$

for $0 \le j \le n-1$ and $n \ge 1$. Therefore, by orthogonality,

$$\lambda_{n,n-1} = -\frac{\hbar_{n+1}^{(0)}}{(n+1)\hbar_{n-1}^{(1)}} [\mathfrak{d}_2 + (n-1)\mathfrak{c}_3] \quad \text{and} \quad \lambda_{n,j} = 0, \quad 0 \le j \le n-2,$$

for $n \ge 2$. Moreover, $\lambda_{1,0} = -\hbar_2^{(0)}\mathfrak{d}_2/(2\hbar_0^{(1)})$. Hence, by setting $\lambda_{n,n-1} = -\tau_n$ we find $(n+1)^{-1} P_{n+1}^{(0)\prime}(x) = P_n^{(1)}(x) - \tau_n\, P_{n-1}^{(1)}(x)$, $n \ge 1$, where

$$\tau_n = \frac{\hbar_{n+1}^{(0)}}{(n+1)\hbar_{n-1}^{(1)}} [\mathfrak{d}_2 + (n-1)\mathfrak{c}_3], \quad n \ge 1. \tag{3.6}$$

In particular, the polynomials A_2 and A_3 are such that

$$\mathfrak{d}_2 = \frac{2\hbar_0^{(1)}}{\hbar_2^{(0)}}\tau_1, \quad \mathfrak{c}_3 = \frac{3\hbar_1^{(1)}}{\hbar_3^{(0)}}\tau_2 - \frac{2\hbar_0^{(1)}}{\hbar_2^{(0)}}\tau_1. \tag{3.7}$$

We now use Proposition 3.2, which was derived under the assumption that $\{\mathbf{v}_0, \mathbf{v}_1\}$ is a CPMF2K, to find the precise set of polynomials (A_3, A_2) for which (3.7) holds. Successively setting $n = 0$ and $n = 1$ in the second item of Proposition 3.2 we find that the polynomials

$$\begin{aligned} A_2(x) &= \frac{2\hbar_0^{(1)}\tau_1}{\hbar_2^{(0)}} P_2^{(0)}(x) - \frac{\hbar_0^{(1)}}{\hbar_1^{(0)}} P_1^{(0)}(x) \qquad \text{and} \\ A_3(x) &= \frac{3\hbar_1^{(1)}\tau_2}{\hbar_3^{(0)}} P_3^{(0)}(x) - \frac{2\hbar_1^{(1)}}{\hbar_2^{(0)}} P_2^{(0)}(x) - A_2(x) P_1^{(1)}(x) \end{aligned} \tag{3.8}$$

are such that $\mathcal{D}\mathbf{v}_1 = A_2\mathbf{v}_0$ and $\mathbf{v}_1 = A_3\mathbf{v}_0$. With the requirement $\tau_1 \neq 0$ we must have $deg(A_2) = 2$. However, it is possible that $deg(A_3) \leq 3$.

From (3.6) we now observe that $\tau_n \neq 0$ for $n \geq 1$ if and only if $\mathfrak{d}_2 + (n-1)\mathfrak{c}_3 \neq 0$ for $n \geq 1$. That is, $\tau_n \neq 0$ for $n \geq 1$ if and only if (A_3, A_2) in (3.8) is also an admissible pair of polynomials. This completes the proof of the proposition. ■

Now let us assume that the pair of quasi-definite moment functionals $\{\mathbf{v}_0, \mathbf{v}_1\}$ satisfies the conditions

$$\mathcal{D}[A_3\mathbf{v}_0] = A_2\mathbf{v}_0 \quad \text{and} \quad \mathbf{v}_1 = A_3\mathbf{v}_0, \tag{3.9}$$

where A_2 and A_3 are as in (3.5). Hence, from the equality between (3.9) and (3.4), we find that $\mathfrak{d}_2$ and $\mathfrak{c}_3$ satisfy (3.7).

From $\mathcal{D}[A_3\mathbf{v}_0] = A_2\mathbf{v}_0$ we have, for any polynomial p, that

$$-\langle \mathbf{v}_0, A_3p' \rangle = \langle \mathbf{v}_0, A_2p \rangle. \tag{3.10}$$

Here, the choice $p(x) = 1$ gives $\langle \mathbf{v}_0, A_2 \rangle = 0$ and hence the polynomial A_2 is a first order quasi-orthogonal polynomial of degree 2 with respect to the moment functional $\mathbf{v}_0$. Thus, A_2 takes the form

$$A_2(x) = \frac{2\hbar_0^{(1)}}{\hbar_2^{(0)}}\tau_1 P_2^{(0)}(x) + \mathfrak{d}_1 P_1^{(0)}(x).$$

Hence, from $\left\langle \mathbf{v}_0, A_2P_1^{(0)} \right\rangle = -\left\langle \mathbf{v}_0, A_3P_1^{(0)\prime} \right\rangle = -\langle \mathbf{v}_1, 1 \rangle$, which follows from $\mathbf{v}_1 = A_3\mathbf{v}_0$, we also obtain

$$\mathfrak{d}_1 = -\hbar_0^{(1)}/\hbar_1^{(0)}.$$

Thus, A_2 has to be the same polynomial given in (3.8).

Now, choosing $p(x) = P_1^{(1)}(x)$ and $p(x) = xP_1^{(1)}(x)$ in (3.10), we get respectively,

$$\left\langle \mathbf{v}_0, A_3 + A_2P_1^{(1)} \right\rangle = 0 \quad \text{and} \quad \left\langle \mathbf{v}_0, [A_3 + A_2P_1^{(1)}]x \right\rangle = 0.$$

To obtain the second equality above one also needs to use $\left\langle \mathbf{v}_0, A_3P_1^{(1)} \right\rangle = \left\langle \mathbf{v}_1, P_1^{(1)} \right\rangle = 0$, which again follows from $\mathbf{v}_1 = A_3\mathbf{v}_0$. Therefore, the polynomial $A_3 + A_2P_1^{(1)}$ is a first order quasi-orthogonal polynomial of degree 3 with respect to the moment functional $\mathbf{v}_0$. Thus, we can write

$$A_3(x) + A_2(x)P_1^{(1)}(x) = \frac{3\hbar_1^{(1)}\tau_2}{\hbar_3^{(0)}}P_3^{(0)}(x) - \mathfrak{a}_2P_2^{(0)}(x).$$

Hence, from

$$\begin{aligned}\left\langle \mathbf{v}_0, [A_3 + A_2P_1^{(1)}]P_2^{(0)} \right\rangle &= \left\langle \mathbf{v}_0, A_3P_2^{(0)} \right\rangle + \left\langle \mathcal{D}[A_3\mathbf{v}_0], P_1^{(1)}P_2^{(0)} \right\rangle \\ &= \left\langle \mathbf{v}_0, A_3P_2^{(0)} \right\rangle - \left\langle \mathbf{v}_0, A_3[P_1^{(1)}P_2^{(0)}]' \right\rangle \\ &= -\left\langle \mathbf{v}_0, A_3P_1^{(1)}P_2^{(0)\prime} \right\rangle = -2\hbar_1^{(1)},\end{aligned}$$

we find $\mathfrak{a}_2 = -2\hbar_1^{(1)}/\hbar_2^{(0)}$. Thus, A_3 is also the same polynomial given in (3.8).

Remark 3.4. Note that, a polynomial $R_n(x)$ is quasi-orthogonal polynomial of order r if it can be written as linear combination of orthogonal polynomials $\{P_k\}_{k=n-r}^{n}$ with respect to some quasi-definite moment functional $\mathbf{v}$.

We can now give the following more general statement regarding CPMF2K.

Theorem 3.5. *The pair of quasi-definite moment functionals* $\{\mathbf{v}_0, \mathbf{v}_1\}$ *is a CPMF2K if and only if there exists an admissible pair of polynomials* (A_3, A_2)*, with* $\deg(A_3) \le 3$ *and* $\deg(A_2) = 2$*, such that one of the following equivalent conditions holds:*

(a) $\mathcal{D}[A_3 \mathbf{v}_0] = A_2 \mathbf{v}_0$ *and* $\mathbf{v}_1 = A_3 \mathbf{v}_0$*;*

(b) $\mathcal{D}\mathbf{v}_1 = A_2 \mathbf{v}_0$ *and* $\mathcal{D}[A_3 \mathbf{v}_1] = (A_3' + A_2)\mathbf{v}_1$.

Moreover, the pair of polynomials (A_3, A_2) *must be as in* (3.8) *and that, with the additional admissibility assumption, both* $\mathbf{v}_0$ *and* $\mathbf{v}_1$ *are semiclassical moment functionals of class at most* $\mathsf{s} = 1$.

Proof. Item (a) has been verified above and also means that the pair (A_3, A_2) is as in (3.8).

Now to verify the equivalence of items (a) and (b), we first obtain for any polynomial $p \in \mathbb{P}$, that

$$\begin{aligned}\langle \mathcal{D}[A_3 \mathbf{v}_1], p \rangle = -\langle \mathbf{v}_1, A_3 p' \rangle &= -\langle \mathbf{v}_1, (A_3 p)' \rangle + \langle \mathbf{v}_1, A_3' p \rangle \\ &= \langle \mathcal{D}\mathbf{v}_1, A_3 p \rangle + \langle \mathbf{v}_1, A_3' p \rangle .\end{aligned}$$

Thus, if $\mathcal{D}\mathbf{v}_1 = A_2 \mathbf{v}_0$, then

$$\langle \mathcal{D}[A_3 \mathbf{v}_1], p \rangle = \langle A_2 \mathbf{v}_0, A_3 p \rangle + \langle \mathbf{v}_1, A_3' p \rangle = \langle A_3 \mathbf{v}_0, A_2 p \rangle + \langle \mathbf{v}_1, A_3' p \rangle .$$

Hence, $\mathcal{D}[A_3 \mathbf{v}_1] = (A_3' + A_2)\mathbf{v}_1$ if and only if $\mathbf{v}_1 = A_3 \mathbf{v}_0$. This gives the equivalence of items (a) and (b).

The admissibility of the pair (A_3, A_2) means $\mathfrak{d}_2 + (n-1)\mathfrak{c}_3 \ne 0$, $n \ge 1$, where $\mathfrak{d}_2$ and $\mathfrak{c}_3$ are the respective leading coefficients of A_2 and A_3. As we have observed from (3.6), this admissibility condition is necessary to guarantee $\tau_n \ne 0$, $n \ge 1$. Thus, if $\{\mathbf{v}_0, \mathbf{v}_1\}$ is a CPMF2K, then from item (a) we can also say that moment functional $\mathbf{v}_0$ is semiclassical of class at most $\mathsf{s} = 1$.

Observe also that the leading coefficient of $A_3' + A_2$ is $\tilde{\mathfrak{d}}_2 = 3\mathfrak{c}_3 + \mathfrak{d}_2$. Hence, $\tilde{\mathfrak{d}}_2 + (n-1)\mathfrak{c}_3 = \mathfrak{d}_2 + (n+2)\mathfrak{c}_3 \ne 0$, $n \ge 1$. Hence, the pair of polynomials $(A_3, A_3' + A_2)$ is also admissible and, as a consequence, from (b), the moment functional $\mathbf{v}_1$ is semiclassical of class at most $\mathsf{s} = 1$. This completes the proof of the theorem. ■

From item (b) of Theorem 3.5 we also find

$$\langle \mathbf{v}_1, A_3' + A_2 \rangle = \langle \mathcal{D}[A_3 \mathbf{v}_1], 1 \rangle = 0.$$

Thus, $A_3' + A_2$ is a first order quasi-orthogonal polynomial of degree 2 with respect to $\mathbf{v}_1$. By writing

$$A_3'(x) + A_2(x) = \mathfrak{b}_2 P_2^{(1)}(x) + \mathfrak{b}_1 P_1^{(1)}(x),$$

we also have

$$\mathfrak{b}_2 = \frac{9\hbar_1^{(1)}\tau_2}{\hbar_3^{(0)}} - \frac{4\hbar_0^{(1)}\tau_1}{\hbar_2^{(0)}} \quad \text{and} \quad \mathfrak{b}_1 = -\frac{1}{\hbar_1^{(1)}} \langle \mathbf{v}_1, A_3 \rangle = -\frac{1}{\hbar_1^{(1)}} \left\langle \mathbf{v}_0, A_3^2 \right\rangle .$$

Remark 3.6. We have observed that $A_3'+A_2$ is a first order quasi-orthogonal polynomial of degree 2 with respect to $\mathbf{v}_1$. Hence, if $\mathbf{v}_1$ is positive definite, then the zeros of $A_3'+A_2$ are real and at least one of these zeros must lie within the support of $\mathbf{v}_1$. Likewise, we have observed earlier that A_2 is a first order quasi-orthogonal polynomial of degree 2 with respect to $\mathbf{v}_0$. Hence, if $\mathbf{v}_0$ is positive definite then the zeros of A_2 are real and at least one of them lie within the support of $\mathbf{v}_0$.

From (3.8) and using (3.2), we can also write

$$A_3(x) = \mathfrak{c}_3 P_3^{(0)}(x) + \mathfrak{c}_2 P_2^{(0)}(x) + \mathfrak{c}_1 P_1^{(0)}(x) + \mathfrak{c}_0 P_0^{(0)}(x),$$

where

$$\mathfrak{c}_0 = \frac{\hbar_0^{(1)}}{\hbar_0^{(0)}} \neq 0, \qquad \mathfrak{c}_1 = \frac{\hbar_0^{(1)}}{2\hbar_1^{(0)}}[\beta_2^{(0)} - \beta_1^{(0)} - 2\tau_1] = \frac{\hbar_0^{(1)}}{\hbar_1^{(0)}}[\beta_1^{(1)} - \beta_1^{(0)}],$$

$$\mathfrak{c}_2 = \frac{\hbar_0^{(1)}}{\hbar_1^{(0)}} - \frac{2\hbar_1^{(1)}}{\hbar_2^{(0)}} + \frac{2\hbar_0^{(1)}\tau_1}{\hbar_2^{(0)}}[\beta_1^{(1)} - \beta_3^{(0)}], \qquad \mathfrak{c}_3 = \frac{3\hbar_1^{(1)}\tau_2}{\hbar_3^{(0)}} - \frac{2\hbar_0^{(1)}\tau_1}{\hbar_2^{(0)}}.$$

3.2 Some Special Cases

In this section we look at some examples of $\{\mathbf{v}_0, \mathbf{v}_1\}$ which are CPMF2Ks. These quasi-definite moment functionals can be given by an integral representation as

$$\int_E p(x)w(x)dx,$$

where $w(x)$ is weight function. In particular we look some examples of positive definite moment functionals.

We remind that, as shown in Theorem 3.5, in a pair $\{\mathbf{v}_0, \mathbf{v}_1\}$ of CPMF2K the associated pair (A_3, A_2) of admissible polynomials are such that $\deg(A_3) \leq 3$ and $\deg(A_2) = 2$. In what follows, results are presented in accordance with the degree of A_3 as in the next table.

$deg(A_3) = 3$	$deg(A_2) = 2$
$deg(A_3) = 2$	
$deg(A_3) = 1$	
$deg(A_3) = 0$	

A_3 is of degree 3

(i) $P_n^{(0)}$ are the Jacobi Polynomials:

We consider the CPMF2K $\{\mathbf{v}_0, \mathbf{v}_1\}$, in which $\mathbf{v}_0$ is the moment functional given by the Jacobi weight function. We look for information about the companion moment functional $\mathbf{v}_1$ by determining the associated polynomials A_2 and A_3.

Without any loss of generality we can assume $\mathbf{v}_0$ to be such that

$$\langle \mathbf{v}_0, p\rangle = \int_{-1}^{1} p(x)(1-x)^{\alpha}(1-x)^{\beta}dx = \int_{-1}^{1} p(x)d\nu^{(\alpha,\beta)}(x),$$

with $\alpha > -1$ and $\beta > -1$. Then the corresponding sequence of MOPs $\{P_n^{(0)}\}_{n\geq 0}$ is given by

$$P_n^{(0)}(x) = \mathcal{P}_n(\nu^{(\alpha,\beta)}; x) = \frac{2^n n!\,(\alpha+\beta+1)_n}{(\alpha+\beta+1)_{2n}}\, P_n^{(\alpha,\beta)}(x), \quad n \geq 0,$$

where

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\Big(-n, n+\alpha+\beta+1; \alpha+1; \frac{1}{2}(1-x)\Big), \quad n \geq 0,$$

are the Jacobi polynomials as usually defined. Also we remind that by $\mathcal{P}_n(\nu^{(\alpha,\beta)}; x)$ we mean the n-th degree monic orthogonal polynomial with respect to the Jacobi measure $\nu^{(\alpha,\beta)}$.

The coefficients in the three term recurrence of $\{P_n^{(0)}\}_{n\geq 0}$ and the values of $\hbar_n^{(0)} = \Big\langle \mathbf{v}_0, (P_n^{(0)})^2 \Big\rangle$ are known to be such that

$$\beta_n^{(0)} = c_n^{(\alpha,\beta)} \quad \text{and} \quad \alpha_{n+1}^{(0)} = \frac{\hbar_n^{(0)}}{\hbar_{n-1}^{(0)}} = d_{n+1}^{(\alpha,\beta)} \quad \text{for} \quad n \geq 1,$$

where

$$c_n^{(\alpha,\beta)} = \frac{\beta^2-\alpha^2}{(\alpha+\beta+2n-2)(\alpha+\beta+2n)},$$

$$d_{n+1}^{(\alpha,\beta)} = \frac{4n(\alpha+n)(\beta+n)(\alpha+\beta+n)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)}.$$

It is also well known that the moment functional $\mathbf{v}_0$ is classical and satisfies the Pearson's equation $\mathcal{D}[(x^2-1)\mathbf{v}_0] = [(\alpha+\beta+2)x + \alpha - \beta]\mathbf{v}_0$. However, with a $\mathfrak{q}$ that we choose here to be in $(-\infty,-1] \cup [1,\infty)$, we can also write the alternative Pearson's equation

$$\mathcal{D}[\mathcal{B}_3^{(\mathfrak{q})}\mathbf{v}_0] = \mathcal{C}_2^{(\alpha,\beta,\mathfrak{q})}\mathbf{v}_0, \tag{3.11}$$

where

$$\mathcal{B}_3^{(\mathfrak{q})}(x) = \operatorname{sgn}(\mathfrak{q})(x-\mathfrak{q})(x^2-1) \qquad \text{and}$$

$$\mathcal{C}_2^{(\alpha,\beta,\mathfrak{q})}(x) = \operatorname{sgn}(\mathfrak{q})\Big[(\alpha+\beta+3)x^2 + [\alpha-\beta-\mathfrak{q}(\alpha+\beta+2)]x - \mathfrak{q}(\alpha-\beta) - 1\Big].$$

This latter Pearson's equation can be easily obtained from

$$-\Big\langle \mathbf{v}_0, (x^2-1)p'(x) \Big\rangle = \langle \mathbf{v}_0, [(\alpha+\beta+2)x + \alpha - \beta]p(x) \rangle\,,$$

by substituting $p(x)$ by $(x-\mathfrak{q})p(x)$. Observe that, with our choice of $\mathfrak{q}$, the moment functional given by $\mathcal{B}_3^{(\mathfrak{q})}\mathbf{v}_0$ is also a positive definite moment functional.

The leading coefficient of $\mathcal{C}_2^{(\alpha,\beta,\mathfrak{q})}$ is $\mathfrak{d}_2 = sign(\mathfrak{q})(\alpha+\beta+3)$ and the leading coefficient of $\mathcal{B}_3^{(\mathfrak{q})}$ is $\mathfrak{c}_3 = \operatorname{sgn}(\mathfrak{q})$. Hence, $\mathfrak{d}_2 + (n-1)\mathfrak{c}_3 \neq 0$, $n \geq 1$, and $(\mathcal{B}_3^{(\mathfrak{q})}, \mathcal{C}_2^{(\alpha,\beta,\mathfrak{q})})$ is also an admissible pair of polynomials.

We can now use item (a) in Theorem 3.5, with $A_2(x) = \mathcal{C}_2^{(\alpha,\beta,\mathfrak{q})}(x)$ and $A_3(x) = \mathcal{B}_3^{(\mathfrak{q})}(x)$, to obtain information about the respective coherent pair $\{\mathbf{v}_0, \mathbf{v}_1\}$ of moment functionals of the second kind. We thus have

$$\begin{aligned}
\langle \mathbf{v}_1, p \rangle &= \int_{-1}^{1} p(x)\mathcal{B}_3^{(\mathfrak{q})}(x)(1-x)^{\alpha}(1+x)^{\beta}dx \\
&= \int_{-1}^{1} p(x)[-\operatorname{sgn}(\mathfrak{q})](x-\mathfrak{q})(1-x)^{\alpha+1}(1+x)^{\beta+1}dx \qquad (3.12)\\
&= \int_{-1}^{1} p(x)d\hat{\nu}^{(\alpha+1,\beta+1,\mathfrak{q})}(x)
\end{aligned}$$

and

$$\frac{1}{n+1}\mathcal{P}'_{n+1}(\nu^{(\alpha,\beta)};x) = P_n^{(1)}(x) - \tau_n P_{n-1}^{(1)}(x) \\ = \mathcal{P}_n(\hat{\nu}^{(\alpha+1,\beta+1,q)};x) - \tau_n\,\mathcal{P}_{n-1}(\hat{\nu}^{(\alpha+1,\beta+1,q)};x), \quad n \geq 1, \tag{3.13}$$

where from (3.6),

$$\mathrm{sgn}(q)\,\tau_n = \frac{\hbar_{n+1}^{(0)}}{(n+1)\hbar_{n-1}^{(1)}}(\alpha+\beta+n+2) \;>\; 0, \quad n \geq 1.$$

In particular, from

$$\mathcal{C}_2^{(\alpha,\beta,q)}(x) = A_2(x) = \frac{2\tau_1\hbar_0^{(1)}}{\hbar_2^{(0)}}\Big[P_2^{(0)}(x) - \frac{\hbar_2^{(0)}}{2\tau_1\hbar_1^{(0)}}P_1^{(0)}(x)\Big],$$

with the explicit expressions for the Jacobi polynomials $P_1^{(0)}$ and $P_2^{(0)}$, and that the value of $\hbar_2^{(0)}/\hbar_1^{(0)}$ is the same as $d_3^{(\alpha,\beta)}$, straightforward calculations show that the above equality holds if

$$\tau_1 = \frac{4(\alpha+2)(\beta+2)}{(\alpha+\beta+4)(\alpha+\beta+5)}\frac{1}{[q(\alpha+\beta+4)-(\beta-\alpha)]}.$$

Now observe that $\tilde{\mathfrak{d}}_2 + (n-1)\mathfrak{c}_3 = \mathfrak{d}_2 + (n+2)\mathfrak{c}_3 \neq 0$, $n \geq 1$, where $\tilde{\mathfrak{d}}_2$ is the leading coefficient of $\mathcal{B}_3^{(q)\prime} + \mathcal{C}_2^{(\alpha,\beta,q)}$. Hence, we obtain from item (b) of Theorem 3.5 that the moment functional $\mathbf{v}_1$ is semiclassical of class at most $\mathsf{s} = 1$.

We can also look at the connection coefficients τ_n in a alternative way. Since the monic Jacobi polynomials $P_n^{(0)}(x) = P_n(\nu^{(\alpha,\beta)};x)$ satisfy

$$P_{n+1}^{(0)\prime}(x) = (n+1)P_n(\nu^{(\alpha+1,\beta+1)};x)$$

we have from the coherence formula (3.13),

$$\mathcal{P}_n(\nu^{(\alpha+1,\beta+1)};x) = \mathcal{P}_n(\hat{\nu}^{(\alpha+1,\beta+1,q)};x) - \tau_n\,\mathcal{P}_{n-1}(\hat{\nu}^{(\alpha+1,\beta+1,q)};x), \quad n \geq 1.$$

This is the connection formula between the MOPs with respect to the Jacobi measure $\nu^{(\alpha+1,\beta+1)}$ and the MOPs with respect to the measure $\hat{\nu}^{(\alpha+1,\beta+1,q)}$. Thus, one can determine τ_n, $n \geq 1$, recursively using the following:

$$\frac{d_{n+2}^{(\alpha+1,\beta+1)}}{\tau_{n+1}} + c_{n+1}^{(\alpha+1,\beta+1)} + \tau_n = q, \quad n \geq 1, \tag{3.14}$$

with $\dfrac{d_2^{(\alpha+1,\beta+1)}}{\tau_1} + c_1^{(\alpha+1,\beta+1)} = q$. The identities in (3.14) were first observed in [54]. However, proofs of these can also be found in [7] and [45].

Remark 3.7. Note that in this example, if $q \neq -1$ or 1, then the polynomial A_3 has three simple zeros and further, the moment functional $\mathbf{v}_1$ is not classical. However, if q is either 1 or -1, then A_3 has a double zero and the moment functional $\mathbf{v}_1$ given by (3.12) is classical and is such that:

- $\langle \mathbf{v}_1, p\rangle = \int_{-1}^{1} p(x)(1-x)^{\alpha+2}(1-x)^{\beta+1}dx$, if $q = 1$.

- $\langle \mathbf{v}_1, p\rangle = \int_{-1}^{1} p(x)(1-x)^{\alpha+1}(1-x)^{\beta+2}dx$, if $q = -1$.

(ii) $P_n^{(0)}$ are the Jacobi Polynomials:

We now consider the CPMF2K $\{\mathbf{v}_0, \mathbf{v}_1\}$, where $\mathbf{v}_1$ is the moment functional given by the Jacobi weight function. That is, we set $\mathbf{v}_1$ to be such that

$$\langle \mathbf{v}_1, p\rangle = \int_{-1}^{1} p(x)(1-x)^{\alpha+1}(1-x)^{\beta+1}dx = \int_{-1}^{1} p(x)d\nu^{(\alpha+1,\beta+1)}(x),$$

with $\alpha > -1$ and $\beta > -1$ (in general) and

$$P_n^{(1)}(x) = \mathcal{P}_n(\nu^{(\alpha+1,\beta+1)}; x), \quad n \geq 0.$$

We have for the moment functional $\mathbf{v}_1$, which is classical,

$$\mathcal{D}[\mathcal{B}_3^{(q)}\mathbf{v}_1] = \mathcal{C}_2^{(\alpha+1,\beta+1,q)}\mathbf{v}_1,$$

where, $\mathcal{C}_2^{(\alpha,\beta,q)}$ and $\mathcal{B}_3^{(q)}$ are as in (3.11). Again, we assume $q \in (-\infty,-1] \cup [1,\infty)$.

We now use item (b) of Theorem 3.5 to obtain information about the companion moment functional $\mathbf{v}_0$ by letting $A_3(x) = \mathcal{B}_3^{(q)}(x)$ and $A_3'(x) + A_2(x) = \mathcal{C}_2^{(\alpha+1,\beta+1,q)}(x)$.

Observe that

$$\langle \mathcal{D}[A_3\mathbf{v}_1], p\rangle = -\langle \mathbf{v}_1, A_3p'\rangle = -\int_{-1}^{1} p'(x)\mathcal{B}_3^{(q)}(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}dx.$$

Since $d[\mathcal{B}_3^{(q)}(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}]/dx = \mathcal{C}_2^{(\alpha+1,\beta+1,q)}(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}$, using integration by parts we find

$$\langle \mathcal{D}[A_3\mathbf{v}_1], p\rangle = \int_{-1}^{1} p(x)\mathcal{C}_2^{(\alpha+1,\beta+1,q)}(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}dx.$$

On the other hand

$$\langle (A_3' + A_2)\mathbf{v}_1, p\rangle = \int_{-1}^{1} p(x)[\operatorname{sgn}(q)(3x^2 - 2qx - 1) + A_2(x)]\,(1-x)^{\alpha+1}(1+x)^{\beta+1}dx.$$

Hence, from the requirement $\mathcal{D}[A_3\mathbf{v}_1] = (A_3' + A_2)\mathbf{v}_1$ in item (b) of Theorem 3.5, we find

$$\int_{-1}^{1} p(x)A_2(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}dx$$
$$= \int_{-1}^{1} p(x)\widetilde{\mathcal{C}}_2^{(\alpha+1,\beta+1,q)}(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}dx,$$

where

$$\begin{aligned}\widetilde{\mathcal{C}}_2^{(\alpha+1,\beta+1,q)}(x) &= \mathcal{C}_2^{(\alpha+1,\beta+1,q)}(x) - \operatorname{sgn}(q)(3x^2 - 2qx - 1)\\ &= \operatorname{sgn}(q)(x-q)[(\alpha+\beta+2)x - (\beta-\alpha)].\end{aligned}$$

Hence, if we set $A_2(x) = \widetilde{\mathcal{C}}_2^{(\alpha+1,\beta+1,q)}(x)$, then from the other requirement $\mathcal{D}\mathbf{v}_1 = A_2\mathbf{v}_0$ in item (b) of Theorem 3.5,

$$-\int_{-1}^{1} p'(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}dx = \left\langle \mathbf{v}_0, \widetilde{\mathcal{C}}_2^{(\alpha+1,\beta+1,q)}p\right\rangle.$$

Using integration by parts we then have

$$-\int_{-1}^{1} p(x)[(\alpha+\beta+2)x - (\beta-\alpha)](1-x)^{\alpha}(1+x)^{\beta}dx = \left\langle \mathbf{v}_0, \widetilde{\mathcal{C}}_2^{(\alpha+1,\beta+1,q)}p\right\rangle. \quad (3.15)$$

Thus, if we choose $\mathbf{v}_0$ such that

$$\langle \mathbf{v}_0, p\rangle = \int_{-1}^{1} p(x)\frac{(1-x)^{\alpha}(1+x)^{\beta}}{\operatorname{sgn}(-q)(x-q)}dx + \epsilon\, p(q) = \int_{-1}^{1} p(x)d\tilde{\nu}^{(\alpha,\beta,q,\epsilon)}(x), \tag{3.16}$$

then it satisfies (3.15) and $\mathbf{v}_1 = A_3\mathbf{v}_0$. With $\epsilon > 0$ and with the range of values chosen for q, the moment functional $\mathbf{v}_0$ is also positive definite. However, it is important that if $q = 1$ then α must be such that $\alpha > 0$. Likewise, if $q = -1$ then β must be such that $\beta > 0$.

Thus, we conclude that

$$\begin{aligned}\frac{1}{n+1}\mathcal{P}'_{n+1}(\tilde{\nu}^{(\alpha,\beta,q,\epsilon)};x) &= \frac{1}{n+1}P^{(0)\prime}_{n+1}(x)\\ &= \mathcal{P}_n(\nu^{(\alpha+1,\beta+1)};x) - \tau_n\,\mathcal{P}_{n-1}(\nu^{(\alpha+1,\beta+1)};x), \quad n \geq 1,\end{aligned} \tag{3.17}$$

where from (3.6),

$$\operatorname{sgn}(q)\,\tau_n = \frac{\hbar^{(0)}_{n+1}}{(n+1)\hbar^{(1)}_{n-1}}[\alpha+\beta+n+1] > 0, \quad n \geq 1.$$

Since $\mathcal{P}'_{n+1}(\nu^{(\alpha,\beta)};x) = (n+1)\mathcal{P}_n(\nu^{(\alpha+1,\beta+1)};x)$, $n \geq 0$, observe also from the coherence formula (3.17),

$$\mathcal{P}_{n+1}(\tilde{\nu}^{(\alpha,\beta,q,\epsilon)};x) = \mathcal{P}_{n+1}(\nu^{(\alpha,\beta)};x) - \tilde{\tau}_n\mathcal{P}_n(\nu^{(\alpha,\beta)};x), \quad n \geq 1,$$

where $\tilde{\tau}_n = (n+1)\tau_n/n$, $n \geq 1$. This is the connection formula between the MOP with respect to the Jacobi measure $\nu^{(\alpha,\beta)}$ and the MOP with respect to the measure $\tilde{\nu}^{(\alpha,\beta,q,\epsilon)}$. Hence, the required values of $\tilde{\tau}_n$ can be generated from

$$q = \frac{d^{(\alpha,\beta)}_{n+1}}{\tilde{\tau}_{n-1}} + c^{(\alpha,\beta)}_{n+1} + \tilde{\tau}_n, \quad n \geq 1, \tag{3.18}$$

with $\tilde{\tau}_0 = -c^{(\alpha,\beta)}_1 + q + \langle \mathbf{v}_0, x-q\rangle / \hbar^{(0)}_0$. For the formulas in (3.18) we refer to [7], [45] and [54].

Remark 3.8. In this example, if $q \neq -1$ or 1, then A_3 has three simple zeros and $\mathbf{v}_0$ is semiclassical of class $\mathsf{s} = 1$. However, if q is either 1 or -1, this represents a situation in which the polynomial A_3 has a double zero and $\mathbf{v}_0$ given by (3.16) is classical if $\epsilon = 0$.

(iii) $P^{(0)}_n$ are the Bessel Polynomials:

For $\alpha \notin \{0,-1,-2,\ldots\}$, let the moment functional $\mathbf{v}_0$ be given by

$$\langle \mathbf{v}_0, p\rangle = \int_{\mathbb{T}} p(x)w^{(\alpha,\beta)}(x)dx,$$

where $\mathbb{T}$ represents integration along the unit circle and

$$w^{(\alpha,\beta)}(x) = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha)}{\Gamma(k+\alpha-1)}\Big(-\frac{\beta}{x}\Big)^{k}.$$

Following Krall and Frink [39] (see also [11], [25], [73], among others), the MOP with respect $\mathbf{v}_0$ are the generalized monic Bessel polynomials. That is,

$$P_n^{(0)}(x) = \hat{y}_n(x;\alpha,\beta),$$

where $\hat{y}_0(x;\alpha,\beta) = 1$ and

$$\hat{y}_n(x;\alpha,\beta) = \beta^n \frac{\Gamma(n+\alpha-1)}{\Gamma(2n+\alpha-1)}\, {}_2F_0\Big(-n, n+\alpha-1; —; -\frac{x}{\beta}\Big), \quad n \geq 1.$$

Moreover, for the values of $\hbar_n^{(0)} = \Big\langle \mathbf{v}_0, (P_n^{(0)})^2 \Big\rangle$, $n \geq 0$, there follows

$$\hbar_0^{(0)} = -\beta \quad \text{and} \quad \hbar_n^{(0)} = (-1)^{n+1} n! \beta^{2n+1} \frac{\Gamma(\alpha)\Gamma(n+\alpha-1)}{\Gamma(2n+\alpha-1)\Gamma(2n+\alpha)}, \quad n \geq 1.$$

For the corresponding moments $(\mathbf{v}_0)_n^{(\alpha,\beta)}$, from

$$(\mathbf{v}_0)_n^{(\alpha,\beta)} = \int_{\mathbb{T}} x^n w^{(\alpha,\beta)}(x) dx = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{(-\beta)^{n+1}\Gamma(\alpha)}{\Gamma(\alpha+n)} x^{-1} dx, \quad n \geq 0,$$

we easily find

$$(\mathbf{v}_0)_n^{(\alpha,\beta)} = \frac{(-\beta)^{n+1}\Gamma(\alpha)}{\Gamma(\alpha+n)} \quad \text{and} \quad (\mathbf{v}_0)_n^{(\alpha+N,\beta)} = \frac{(\alpha)_N}{(-\beta)^N} (\mathbf{v}_0)_{n+N}^{(\alpha,\beta)},$$

for $n \geq 0$ and $N = 0, 1, 2, \ldots$. Therefore, for any polynomial p and for any non-negative integer N we have

$$\frac{(-\beta)^N}{(\alpha)_N} \int_{\mathbb{T}} p(x) w^{(\alpha+N,\beta)}(x) dx = \int_{\mathbb{T}} p(x)\, x^N w^{(\alpha,\beta)}(x) dx. \tag{3.19}$$

It was also observed in [39] that the function $w^{(\alpha,\beta)}$ satisfies

$$\frac{d[x^2 w^{(\alpha,\beta)}(x)]}{dx} = (\alpha x + \beta) w^{(\alpha,\beta)}(x) - \frac{(\alpha-1)(\alpha-2)}{2\pi} x,$$

and hence,

$$-\Big\langle \mathbf{v}_0, x^2 p' \Big\rangle = \langle \mathbf{v}_0, (\alpha x + \beta) p \rangle .$$

That is, $\mathbf{v}_0$ satisfies the well-known Pearson's equations $\mathcal{D}[x^2 \mathbf{v}_0] = [\alpha x + \beta]\mathbf{v}_0$.

Notice that, we can also write the alternative Pearson's equation $\mathcal{D}[A_3 \mathbf{v}_0] = A_2 \mathbf{v}_0$ for $\mathbf{v}_0$, where

$$A_3(x) = x^3 \quad \text{and} \quad A_2(x) = (\alpha+1)x^2 + \beta x.$$

This is obtained by replacing $p(x)$ with $xp(x)$ in $-\langle \mathbf{v}_0, x^2 p' \rangle = \langle \mathbf{v}_0, (\alpha x + \beta)p \rangle$.

For the leading coefficients $\mathfrak{d}_3$ and $\mathfrak{c}_2$ of A_3 and A_2, respectively, we have $\mathfrak{c}_2 + (n-1)\mathfrak{d}_3 = \alpha + 1 + (n-1) \neq 0$, $n \geq 1$. Hence, (A_3, A_2) is an admissible pair of polynomials and we can use item (a) of Theorem 3.5 to get the CPMF2K $\{\mathbf{v}_0, \mathbf{v}_1\}$, where $\mathbf{v}_1 = x^3 \mathbf{v}_0$.

From (3.19) there follows

$$P_n^{(1)}(x) = \hat{y}_n(x;\alpha+3,\beta), \quad n \geq 0,$$

and

$$\hbar_n^{(1)} = \frac{(-\beta)^3}{(\alpha)_3}(-1)^{n+1}n!\beta^{2n+1}\frac{\Gamma(\alpha+3)\Gamma(n+\alpha+2)}{\Gamma(2n+\alpha+2)\Gamma(2n+\alpha+3)}$$
$$= (-1)^n n!\beta^{2n+4}\frac{\Gamma(\alpha)\Gamma(n+\alpha+2)}{\Gamma(2n+\alpha+2)\Gamma(2n+\alpha+3)},$$

for $n \geq 0$. Consequently, from (3.6)

$$\frac{1}{n+1}\hat{y}'_{n+1}(x;\alpha,\beta) = \hat{y}_n(x;\alpha+3,\beta) - \tau_n\,\hat{y}_{n-1}(x;\alpha+3,\beta), \quad n \geq 1,$$

where

$$\tau_n = \frac{n+\alpha}{n+1}\frac{\hbar_{n+1}^{(0)}}{\hbar_{n-1}^{(1)}} = -\beta\frac{n}{(2n+\alpha)(2n+\alpha+1)}, \quad n \geq 1.$$

Remark 3.9. The above results provide a nice example of a CPMF2K where both moment functionals are classical but are quasi-definite. Moreover, this represents a situation in which A_3 has a triple zero.

A_3 is of degree 2

(i) $P_n^{(0)}$ are the Laguerre Polynomials:

We consider the CPMF2K $\{\mathbf{v}_0, \mathbf{v}_1\}$, in which $\mathbf{v}_0$ is given by the generalized Laguerre weight function. Without any loss of generality we can set $\mathbf{v}_0$ to be such that

$$\langle \mathbf{v}_0, p\rangle = \int_0^\infty p(x)x^\alpha e^{-x}dx = \int_0^\infty p(x)d\nu^{(\alpha)}(x),$$

where $\alpha > -1$. Then for the corresponding sequence of MOPs $\{P_n^{(0)}\}_{n\geq 0}$ we have

$$P_n^{(0)}(x) = \mathcal{P}_n(\nu^{(\alpha)};x) = (-1)^n n!\,L_n^{(\alpha)}(x), \quad n \geq 0,$$

where $L_n^{(\alpha)}$ are the generalized Laguerre polynomials as traditionally defined. That is,

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1\Big(-n, \alpha+1; x\Big), \quad n \geq 0.$$

The coefficients in the three term recurrence of $\{P_n^{(0)}\}$ and $\hbar_n^{(0)} = \Big\langle \mathbf{v}_0, (P_n^{(0)})^2\Big\rangle$ are such that

$$\beta_n^{(0)} = c_n^{(\alpha)} \quad \text{and} \quad \alpha_{n+1}^{(0)} = \frac{\hbar_n^{(0)}}{\hbar_{n-1}^{(0)}} = d_{n+1}^{(\alpha)} \quad \text{for} \quad n \geq 1,$$

where $c_n^{(\alpha)} = \alpha + 2n - 1$ and $d_{n+1}^{(\alpha)} = n(\alpha+n)$.

It is known that the moment functional $\mathbf{v}_0$, which is classical, satisfies the Pearson's equation $\mathcal{D}[x\mathbf{v}_0] = (\alpha+1-x)\mathbf{v}_0$. However, with a q which we choose here to be in $(-\infty, 0]$, we also have the alternative Pearson's equation

$$\mathcal{D}[\mathcal{B}_2^{(q)}\mathbf{v}_0] = \mathcal{C}_2^{(\alpha,q)}\mathbf{v}_0, \tag{3.20}$$

where $\mathcal{C}_2^{(\alpha,q)}(x) = -x^2 + (q+\alpha+2)x - q(\alpha+1)$ and $\mathcal{B}_2^{(q)}(x) = x(x-q)$. Observe that, with our choice of q, the moment functional given by $\mathcal{B}_2^{(q)}\mathbf{v}_0$ is also a positive definite moment functional.

With $A_2(x) = \mathcal{C}_2^{(\alpha,q)}(x)$ and $A_3(x) = \mathcal{B}_2^{(q)}(x)$, we can now use item (a) in Theorem 3.5 to determine information about the respective coherent pair of measures of the second kind.

We then have $\{\mathbf{v}_0, \mathbf{v}_1\}$ is a coherent pair of the second kind if

$$\begin{aligned}\langle \mathbf{v}_1, p\rangle = \langle A_3\mathbf{v}_0, p\rangle &= \int_0^\infty p(x)(x-q)x^{\alpha+1}e^{-x}dx \\ &= \int_0^\infty p(x)d\hat{\nu}^{(\alpha+1,q)}(x)\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}\frac{1}{n+1}\mathcal{P}'_{n+1}(\nu^{(\alpha)};x) &= P_n^{(1)}(x) - \tau_n P_{n-1}^{(1)}(x) \\ &= \mathcal{P}_n(\hat{\nu}^{(\alpha+1,q)};x) - \tau_n\mathcal{P}_{n-1}(\hat{\nu}^{(\alpha+1,q)};x), \quad n \geq 1.\end{aligned} \tag{3.22}$$

From item (b) of Theorem 3.5, the moment functional $\mathbf{v}_1$ satisfies

$$\mathcal{D}[x(x-q)\mathbf{v}_1] = [-x^2 + (q+\alpha+4)x - q(\alpha+2)]\mathbf{v}_1.$$

Hence, $\mathbf{v}_1$ is semiclassical of class $\mathsf{s} = 1$ if $q \in (-\infty, 0)$.

From (3.8)

$$A_2(x) = \frac{2\tau_1 \hbar_0^{(1)}}{\hbar_2^{(0)}}\left[P_2^{(0)}(x) - \frac{\hbar_2^{(0)}}{2\tau_1\hbar_1^{(0)}}P_1^{(0)}(x)\right] = \mathcal{C}_2^{(\alpha,q)}(x).$$

Hence, we can also verify that

$$\tau_1 = -\frac{(\alpha+2)}{\alpha+2-q}.$$

From (3.6),

$$\tau_n = -\frac{\hbar_{n+1}^{(0)}}{(n+1)\hbar_{n-1}^{(1)}} < 0, \quad n \geq 1.$$

We can look at the connection coefficients τ_n in a alternative way. Since the monic Laguerre polynomials $P_n^{(0)}(x) = \mathcal{P}_n(\nu^{(\alpha)};x)$ satisfy

$$\mathcal{P}'_{n+1}(\nu^{(\alpha)};x) = (n+1)\mathcal{P}_n(\nu^{(\alpha+1)};x),$$

we have from the associated coherence formula (3.22),

$$P_n(\nu^{(\alpha+1)};x) = \mathcal{P}_n(\hat{\nu}^{(\alpha+1,q)};x) - \tau_n\mathcal{P}_{n-1}(\hat{\nu}^{(\alpha+1,q)};x), \quad n \geq 1.$$

This is the connection formula between the MOPs with respect to the measure $\nu^{(\alpha+1)}$ and the MOPs with respect to the measure $\hat{\nu}^{(\alpha+1,q)}$. Thus, one can determine τ_n, $n \geq 1$, recursively using the following:

$$\frac{d_{n+2}^{(\alpha+1)}}{\tau_{n+1}} + c_{n+1}^{(\alpha+1)} + \tau_n = q, \quad n \geq 1, \tag{3.23}$$

with $\dfrac{d_2^{(\alpha+1)}}{\tau_1} + c_1^{(\alpha+1)} = q$. Again, for the formulas in (3.23) we refer to [7], [45] and [54].

Remark 3.10. In this example, if $q \neq 0$, then the polynomial A_3, which is of degree 2, has two simple zeros and further, the moment functional $\mathbf{v}_1$ is not classical. However, if $q = 0$ then A_3 has a double zero and $\mathbf{v}_1$ given by (3.21) is classical and is such that:

$$\langle \mathbf{v}_1, p\rangle = \int_0^\infty p(x)x^{\alpha+2}e^{-x}dx.$$

(ii) $P_n^{(1)}$ are the Laguerre Polynomials:

We consider the CPMF2K $\{\mathbf{v}_0, \mathbf{v}_1\}$, where $\mathbf{v}_1$ is given by the generalized Laguerre weight function. We will assume $\mathbf{v}_1$ to be such that

$$\langle \mathbf{v}_1, p \rangle = \int_0^\infty p(x) x^{\alpha+1} e^{-x} dx = \int_0^\infty p(x) d\nu^{(\alpha+1)}(x),$$

with (in general) $\alpha > -1$.

Then for the corresponding sequence of MOPs $\{P_n^{(1)}\}_{n\geq 0}$ we have

$$P_n^{(1)}(x) = \mathcal{P}_n(\nu^{(\alpha+1)}; x) = (-1)^n n!\, L_n^{(\alpha+1)}(x), \quad n \geq 0.$$

For the moment functional $\mathbf{v}_1$ we have $\mathcal{D}[\mathcal{B}_2^{(q)} \mathbf{v}_1] = \mathcal{C}_2^{(\alpha+1,q)} \mathbf{v}_1$, where $\mathcal{C}_2^{(\alpha,q)}$ and $\mathcal{B}_2^{(q)}$ are as in (3.20). Again, we assume $q \in (-\infty, 0]$.

We can now use item (b) of Theorem 3.5, with

$$A_3(x) = \mathcal{B}_2^{(q)}(x) \quad \text{and} \quad A_3'(x) + A_2(x) = \mathcal{C}_2^{(\alpha+1,q)}(x),$$

to obtain information about the companion functional $\mathbf{v}_0$. Observe that

$$\langle \mathcal{D}[A_3 \mathbf{v}_1], p \rangle = -\langle \mathbf{v}_1, A_3 p' \rangle = -\int_0^\infty p'(x) \mathcal{B}_2^{(q)}(x) x^{\alpha+1} e^{-x} dx.$$

Since $d[\mathcal{B}_2^{(q)}(x) x^{\alpha+1} e^{-x}]/dx = \mathcal{C}_2^{(\alpha+1,q)}(x) x^{\alpha+1} e^{-x}$, using integration by parts we then find

$$\langle \mathcal{D}[A_3 \mathbf{v}_1], p \rangle = \int_0^\infty p(x) \mathcal{C}_2^{(\alpha+1,q)}(x) x^{\alpha+1} e^{-x} dx.$$

On the other hand

$$\langle (A_3' + A_2) \mathbf{v}_1, p \rangle = \int_0^\infty p(x) [(2x - q) + A_2(x)] x^{\alpha+1} e^{-x} dx.$$

From $\mathcal{D}[A_3 \mathbf{v}_1] = (A_3' + A_2) \mathbf{v}_1$ we then find

$$\int_0^\infty p(x) A_2(x) x^{\alpha+1} e^{-x} dx \; = \; \int_0^\infty p(x) \widetilde{\mathcal{C}}_2^{(\alpha+1,q)}(x) x^{\alpha+1} e^{-x} dx,$$

where $\widetilde{\mathcal{C}}_2^{(\alpha+1,q)}(x) = \mathcal{C}_2^{(\alpha+1,q)}(x) - (2x - q) = (x - q)(-x + \alpha + 1)$. Thus, if we set $A_2(x) = \widetilde{\mathcal{C}}_2^{(\alpha+1,q)}(x)$, then from $\mathcal{D}\mathbf{v}_1 = A_2 \mathbf{v}_0$,

$$-\int_0^\infty p'(x) x^{\alpha+1} e^{-x} dx = \langle \mathbf{v}_0, (x - q)(-x + \alpha + 1) p \rangle.$$

Hence, using integration by parts

$$\int_0^\infty p(x)(-x + \alpha + 1) x^\alpha e^{-x} dx = \langle \mathbf{v}_0, (x - q)(-x + \alpha + 1) p \rangle.$$

Hence, for $\mathbf{v}_0$ given by

$$\langle \mathbf{v}_0, p \rangle = \int_0^\infty p(x) d\tilde{\nu}^{(\alpha,q,\epsilon)}(x) = \int_0^\infty p(x) \frac{x^\alpha e^{-x}}{(x - q)} dx + \epsilon\, p(q), \tag{3.24}$$

there hold $\mathbf{v}_1 = A_3 \mathbf{v}_0$ and $\mathcal{D}[A_3 \mathbf{v}_0] = A_2 \mathbf{v}_0$. Here, it is important that if $q = 0$, then α must be such that $\alpha > 0$.

From (3.6),

$$\tau_n = -\frac{\hbar_{n+1}^{(0)}}{(n+1)\hbar_{n-1}^{(1)}} \;<\; 0, \quad n \geq 1.$$

Since $P'_{n+1}(\nu^{(\alpha)};x) = (n+1)P_n(\nu^{(\alpha+1)};x)$, $n \geq 0$, observe also that the respective coherence formula

$$\frac{1}{n+1}\mathcal{P}'_{n+1}(\tilde{\nu}^{(\alpha,q,\epsilon)};x) = \mathcal{P}_n(\nu^{(\alpha+1)};x) - \tau_n\,\mathcal{P}_{n-1}(\nu^{(\alpha+1)};x), \quad n \geq 1,$$

is equivalent to

$$P_{n+1}(\tilde{\nu}^{(\alpha,q,\epsilon)};x) = P_{n+1}(\nu^{(\alpha)};x) - \tilde{\tau}_n P_n(\nu^{(\alpha)};x), \quad n \geq 1,$$

where $\tilde{\tau}_n = (n+1)\tau_n/n$, $n \geq 1$. This is the connection formula between the MOP with respect to the Laguerre measure $\nu^{(\alpha)}$ and the MOP with respect to the measure $\tilde{\nu}^{(\alpha,q,\epsilon)}$. Hence, the required values of $\tilde{\tau}_n$ can be generated from

$$q = \frac{d_{n+1}^{(\alpha)}}{\tilde{\tau}_{n-1}} + c_{n+1}^{(\alpha)} + \tilde{\tau}_n, \quad n \geq 1, \tag{3.25}$$

with $\tilde{\tau}_0 = -c_1^{(\alpha)} + q + \langle \mathbf{v}_0, x - q\rangle / \hbar_0^{(0)}$. For (3.25) we refer to [7], [45] and [54].

Remark 3.11. In this example, if $q \neq 0$, then A_3, which is of degree 2, has two simple zeros and $\mathbf{v}_0$ is semiclassical of class $\mathsf{s} = 1$. However, if $q = 0$, this represents a situation in which the polynomial A_3 has a double zero and $\mathbf{v}_0$ given by (3.24) is classical if $\epsilon = 0$.

(iii) $P_n^{(0)}$ are the Truncated Laguerre Polynomials:

We consider the CPMF2K $\{\mathbf{v}_0, \mathbf{v}_1\}$, where $\mathbf{v}_0$ is the moment functional given by

$$\langle \mathbf{v}_0, p\rangle = \int_0^t p(x)x^\alpha e^{-x}dx,$$

with $\alpha > -1$ and $t > 0$. This moment functional is associated with the so called truncated generalized Hermite linear functionals by using the symmetrization problem (see [20]) and a particular case when the truncated normal probability measure appears has been studied in [21].

In order to get he companion moment functional $\mathbf{v}_1$ we will obtain the associated polynomials A_2 and A_3. Observe that

$$\langle \mathcal{D}\left[x(t-x)\,\mathbf{v}_0\right], p\rangle = -\langle \mathbf{v}_0\,,\, x(t-x)p'\rangle = -\int_0^t p'(x)\,x(t-x)x^\alpha e^{-x}\,dx.$$

Thus, from integration by parts

$$\langle \mathcal{D}\left[x(t-x)\,\mathbf{v}_0\right], p\rangle = \int_0^t p(x)[x^2 - (\alpha+t+2)x + t(\alpha+1)]\,x^\alpha e^{-x}\,dx.$$

Hence,

$$\mathcal{D}[A_3\,\mathbf{v}_0] = A_2\,\mathbf{v}_0,$$

where $A_3(x) = x(t-x)$ and $A_2(x) = x^2 - (\alpha+t+2)x + t(\alpha+1)$. Thus, $\mathbf{v}_0$ is semiclassical of class $\mathsf{s} = 1$.

From item (a) of Theorem 3.5 there follows

$$\langle \mathbf{v}_1,\, p\rangle = \int_0^t p(x)\,(t-x)x^{\alpha+1}e^{-x}dx$$

and $\{\mathbf{v}_0, \mathbf{v}_1\}$ is a coherent pair of positive definite moment functionals of the second kind.

From item (b) of Theorem 3.5, we also have

$$\mathcal{D}[x(t-x)\mathbf{v}_1] = [x^2 - (\alpha+t+4)x + t(\alpha+2)]\mathbf{v}_1$$

and hence, $\mathbf{v}_1$ is also semiclassical of class $\mathsf{s}=1$.

(iv) $P_n^{(0)}$ are the MOP with Respect to the Weight Function $x^{\alpha}e^{-(x+\kappa/x)}$:

Let the moment functional $\mathbf{v}_0$ be given by

$$\langle \mathbf{v}_0, p\rangle = \int_0^\infty p(x)x^{\alpha}e^{-(x+\kappa/x)}dx = \int_0^\infty p(x)d\nu^{(\alpha,\kappa)}(x), \quad \kappa \geq 0,$$

where $\alpha \in \mathbb{R}$. The corresponding orthogonal polynomials $P_n^{(0)}(x) = \mathcal{P}_n(\nu^{(\alpha,\kappa)};x)$ and related formulas have been the subject of study in [79] and [80] as well as in Section 4.4 of [75], where the connection with alternate discrete Painlevé II equations for the coefficients involved in the structure relation is given.

Using integration by parts one can easily show that

$$\begin{aligned}\langle \mathcal{D}[x^2\mathbf{v}_0];p\rangle &= -\int_0^\infty p'(x)x^{\alpha+2}e^{-(x+\kappa/x)}dx \\ &= \int_0^\infty p(x)[-x^2+(\alpha+2)x+\kappa]x^{\alpha}e^{-(x+\kappa/x)}dx.\end{aligned}$$

Therefore, $\mathbf{v}_0$ satisfies the Pearson's equation $\mathcal{D}[A_3\mathbf{v}_0] = A_2\mathbf{v}_0$, where

$$A_3(x) = x^2 \quad \text{and} \quad A_2(x) = -x^2 + (\alpha+2)x + \kappa.$$

Thus, by item (a) of Theorem 3.5 we have the CPMF2K $\{\mathbf{v}_0, \mathbf{v}_1\}$, where

$$\langle \mathbf{v}_1, p\rangle = \int_0^\infty p(x)d\nu^{(\alpha+2,\kappa)}(x) \quad \text{and} \quad P_n^{(1)}(x) = \mathcal{P}_n(\nu^{(\alpha+2,\kappa)};x), \quad n\geq 0.$$

Moreover, $\mathbf{v}_1$ is semiclassical of class at most $\mathsf{s}=1$.

A_3 is of degree 1

With $\alpha > -1$ and κ real, let the moment functional $\mathbf{v}_0$ be given by

$$\langle \mathbf{v}_0, p\rangle = \int_0^{+\infty} p(x)x^{\alpha}e^{-x^2+\kappa x}dx.$$

Clearly, $\mathbf{v}_0$ is a positive definite moment functional and it represents a semiclassical extension of the Laguerre weight. The coefficients of the three term relation of the corresponding sequences of orthogonal polynomials satisfy an asymmetric Painlevé IV equation d-PIV (see, Section 5.1 in [75]).

Observe that

$$\langle \mathcal{D}\left[x\mathbf{v}_0\right], p\rangle = -\langle \mathbf{v}_0, xp'\rangle = -\int_0^{+\infty} p'(x)x^{\alpha+1}e^{-x^2+\kappa x}\,dx.$$

Thus, integration by parts gives

$$\langle \mathcal{D}\left[x\mathbf{v}_0\right], p\rangle = \int_0^{+\infty} p(x)[-2x^2 + \kappa x + \alpha + 1]\, x^\alpha e^{-x^2+\kappa x}\, dx.$$

That is, $\mathcal{D}[A_3\mathbf{v}_0] = A_2\mathbf{v}_0$, where $A_3(x) = x$ and $A_2(x) = -2x^2 + \kappa x + \alpha + 1$. Clearly, $A_3\mathbf{v}_0$ is also a positive definite moment functional.

Thus, by item (a) of Theorem 3.5, the pair of moment functionals $\{\mathbf{v}_0, \mathbf{v}_1\}$ is a CPMF2K if $\mathbf{v}_1 = A_3\mathbf{v}_0$.

A_3 is a Constant

We have the case in which $\deg(A_3) = 0$ and $\deg(A_2) = 2$. Thus, we can take without any loss of generality $\mathbf{v}_1 = A_3\mathbf{v}_0 = \mathbf{v}_0$. We now have a self coherent moment functional of the second kind $\mathbf{v}_0$ such that

$$\frac{1}{n+1} P_{n+1}^{(0)\prime}(x) = P_n^{(0)}(x) - \tau_n P_{n-1}^{(0)}(x), \quad n \geq 1.$$

Then, (see Prop. 4.2 in [48]) there exists a set of complex numbers $\{\varrho_n\}_{n\geq 1}$ with $\varrho_1 = 0$, $\varrho_n \neq 0$ for $n \geq 2$, such that

$$\frac{P'_{n+1}(x)}{n+1} + \varrho_{n+1}\frac{P'_n(x)}{n} = \left(x - \tilde{\beta}_{n+1} + \varrho_{n+1} + \frac{\tilde{\gamma}_{n+2}}{\varrho_{n+2}}\right) P_n(x), \quad n \geq 0.$$

The only example of moment functional known in this case, which was first obtained in [57], can be derived in the following form. With a, b, c complex, let

$$\langle \mathcal{D}\mathbf{v}_0, p\rangle = -\langle \mathbf{v}_0, p'\rangle = -\int_{-\infty}^{\infty} p'(x) e^{ax^3+bx^2+cx} dx.$$

Then by integration by parts

$$\langle \mathcal{D}\mathbf{v}_0, p\rangle = -\left[p(x)e^{ax^3+bx^2+cx}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} p(x) A_2(x) e^{ax^3+bx^2+cx} dx,$$

where $A_2 = 3ax^2 + 2bx + c$. Hence, for the existence of the respective integrals and that for A_2 to be a polynomial of exact degree 2 we must have either

- a purely complex and $\mathrm{Re}(b) < 0$;
- a purely complex, $\mathrm{Re}(b) = 0$ and c purely complex.

Hence, for example, by setting $a = i$ and $b = -1$ we can state that $\{\mathbf{v}_0, \mathbf{v}_0\}$ is a CPMF2K if

$$\langle \mathbf{v}_0, p\rangle = \int_{-\infty}^{\infty} p(x) e^{ix^3-x^2+cx} dx.$$

Hence, if c is such that $\mathbf{v}_0$ is quasi-definite, then $\mathbf{v}_0$ is self coherent of the second kind.

3.3 Matrix Characterization for Coherent Pairs of the Second Kind

In this section we focus the attention on the matrices associated with sequences of MOP corresponding to moment functionals $\mathbf{v}_0$ and $\mathbf{v}_1$ which are coherent pairs of the second kind.

We start with some preliminary results. In a matrix form the three-term recurrence relation (1.12) reads

$$x\mathbf{p}(x) = \mathbf{J}_P\,\mathbf{p}(x),$$

where $\mathbf{p}(x) = [P_0(x), P_1(x), \ldots]^\top$ and $\mathbf{J}_P$ is the infinite tridiagonal matrix

$$\mathbf{J}_P = \begin{bmatrix} \beta_1 & 1 & 0 & 0 & \ldots \\ \alpha_2 & \beta_2 & 1 & 0 & \ldots \\ 0 & \alpha_3 & \beta_3 & 1 & \ldots \\ 0 & 0 & \alpha_4 & \beta_4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

called the monic Jacobi matrix associated with $\{P_n\}_{n\geq 0}$.

In [76] a matrix characterization for the orthogonality of a sequence of polynomials has been introduced. Following [76], we consider the sequence of monic polynomials $\{P_n\}_{n\geq 0}$ given by

$$P_n(x) = \sum_{j=0}^{n} \mathsf{c}_{n,j} x^j, \quad \mathsf{c}_{n,n} = 1, \quad n \geq 0,$$

and define the infinite matrix $\mathbf{C}$ with entries $\mathsf{c}_{n,j}$, with $0 \leq j \leq n$, $n \geq 0$, and zero otherwise, i.e.,

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots \\ \mathsf{c}_{1,0} & 1 & 0 & 0 & \ldots \\ \mathsf{c}_{2,0} & \mathsf{c}_{2,1} & 1 & 0 & \ldots \\ \mathsf{c}_{3,0} & \mathsf{c}_{3,1} & \mathsf{c}_{3,2} & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Thus, $\mathbf{C}$ is a nonsingular lower triangular matrix whose nth row contains the coefficients of P_n with respect to the canonical basis $\{x^n\}_{n\geq 0}$. $\mathbf{C}$ is said to be the matrix associated with the sequence $\{P_n\}_{n\geq 0}$. We say that the entry $\mathsf{c}_{i,j}$ is in the diagonal of index $m = i-j$. Also, a matrix $\mathbf{C}$ is said to be (n,m)-banded if there exists a pair of integers (n,m), with $n \leq m$, and all the nonzero entries of $\mathbf{C}$ lie between the diagonals of indices n and m.

Let us define the matrices

$$\mathbf{X} = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & \ldots \\ 0 & 2 & 0 & 0 & \ldots \\ 0 & 0 & 3 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad \hat{\mathbf{D}} = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1/2 & 0 & \ldots \\ 0 & 0 & 0 & 1/3 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \quad (3.26)$$

Observe that $\mathbf{X}^\top$ is the right inverse of $\mathbf{X}$, since $\mathbf{X}\mathbf{X}^\top = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix,

and

$$\mathbf{X}^\top\mathbf{X} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The following matrix characterization was given in [76].

Theorem 3.12. *Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials and let $\mathbf{C}$ be the matrix associated with it. Then, $\{P_n\}_{n\ \geq 0}$ is a sequence of MOPs if and only if $\mathbf{J} = \mathbf{C}\mathbf{X}\mathbf{C}^{-1}$ is a $(-1,1)$-banded matrix whose entries in the diagonals of indices 1 and -1 are all nonzero.*

For an updated survey on the role of infinite matrices in the theory of orthogonal polynomials, see [77].

There are several characterizations of semiclassical orthogonal polynomials in terms of the so-called structure relations. We stated in the following the characterization given in [52], [55].

Theorem 3.13 (Structure relation)**.** *Let $\mathbf{v}$ be a quasi-definite moment functional satisfying* (1.13) *and let $\{P_n\}_{n\geq 0}$ be its corresponding sequence of MOPs. Then, $\mathbf{v}$ is semiclassical of class s if and only if $\{P_n\}_{n\geq 0}$ satisfies the following structure relation*

$$\phi(x)P'_{n+1}(x) = \sum_{j=n-\mathsf{s}}^{n+deg(\phi)} \mathsf{r}_{n,j}P_j(x), \quad n \geq \mathsf{s}, \quad \mathsf{r}_{n,n-\mathsf{s}} \neq 0, \quad n \geq \mathsf{s}+1. \tag{3.27}$$

In a matrix form, the structure relation (3.27) can be expressed as

$$\phi(x)\mathbf{p}'(x) = \mathbf{X}^\top\mathbf{R}\,\mathbf{p}(x),$$

where $\mathbf{p}'(x) = [P_0'(x), P_1'(x), \dots]^\top$, $\mathbf{X}$ is given in (3.26) and $\mathbf{R}$ is a $(-deg(\phi), s)$-banded matrix whose elements, starting from the row s, are the coefficients appearing in (3.27).

In [27] the authors deal with the symmetric tridiagonal (Jacobi) matrix associated with a positive definite semiclassical moment functional that constitutes a counterpart of the so called Laguerre-Freud equations that the coefficients of the three term recurrence relation of semiclassical moment functionals satisfy (see [6]). In the classical case, a matrix approach was given in [76]. The following result establishes a relation between $\mathbf{R}$ and $\mathbf{J}_P$ in the framework of quasi-definite semiclassical moment functionals and its proof can be found, for example, in [27].

Theorem 3.14. *Let $\mathbf{v}$ be a semiclassical moment functional satisfying* (1.13)*, $\{P_n\}_{n\geq 0}$ be its corresponding sequence of MOPs and let $\mathbf{R}$ be the $(-deg(\phi), s)$-banded matrix appearing in the structure relation* (3.27)*. Then,*

(i) $\left[\mathbf{J}_P, (\mathbf{X}^\top\mathbf{R}\hbar)_a\hbar^{-1}\right] = \phi(\mathbf{J}_P)$,

(ii) $(\mathbf{X}^\top\mathbf{R}\hbar)_s\hbar^{-1} = -\frac{1}{2}\psi(\mathbf{J}_P)$,

where $\left[\mathbf{A},\mathbf{B}\right] = \mathbf{AB} - \mathbf{BA}$ denotes the commutator of the matrices $\mathbf{A},\mathbf{B}$ and $\hbar = diag(\hbar_0, \hbar_1, \dots)$. On the other hand, $\mathbf{M}_s$ and $\mathbf{M}_a$ denote, respectively, the symmetric and antisymmetric components of the matrix $\mathbf{M}$.

The result given below provides a characterization of a coherent pairs of measures of the second kind in terms of the associated matrices representing orthogonal polynomials in terms of the monomial basis.

Theorem 3.15. *Let* $\mathbf{v}_0$ *and* $\mathbf{v}_1$ *be quasi-definite moment functionals and let* $\{P_n^{(0)}\}_{n\geq 0}$ *and* $\{P_n^{(1)}\}_{n\geq 0}$ *be, respectively, the corresponding sequences of MOPs. Consider, respectively,* $\mathbf{C}^{(0)}$ *and* $\mathbf{C}^{(1)}$ *as the matrices associated with* $\{P_n^{(0)}\}_{n\geq 0}$ *and* $\{P_n^{(1)}\}_{n\geq 0}$. *Denote* $\breve{\mathbf{C}}^{(0)} = \hat{\mathbf{D}}\mathbf{C}^{(0)}\mathbf{D}$.

Then, $\{\mathbf{v}_0, \mathbf{v}_1\}$ *is a CPMF2K if and only if the matrix* $\breve{\mathbf{C}}^{(0)}(\mathbf{C}^{(1)})^{-1}$ *is lower bidiagonal with ones in the main diagonal and nonzero entries in the subdiagonal.*

Proof. Suppose $\{\mathbf{v}_0, \mathbf{v}_1\}$ is a CPMF2K. Thus,

$$\frac{P_{n+1}^{(0)\prime}(x)}{n+1} = P_n^{(1)}(x) - \tau_n P_{n-1}^{(1)}(x), \quad n \geq 1,$$

where by convention $P_{-1}^{(1)}(x) = 0$ and τ_0 is a free parameter.

Consider the infinite matrices $\mathbf{X}$, $\mathbf{D}$ and $\hat{\mathbf{D}}$ given in (3.26). One can show that $\breve{\mathbf{C}}^{(0)}$ and $\mathbf{X}^\top\mathbf{C}^{(1)}$ are, respectively, the matrices associated with the sequences $\{P_{n+1}^{(0)\prime}/(n+1)\}_{n\geq 0}$ and $\{P_{n-1}^{(1)}\}_{n\geq 0}$. Then, the previous relation can be written in matrix form as

$$\hat{\mathbf{D}}\mathbf{C}^{(0)}\mathbf{D} = \breve{\mathbf{C}}^{(0)} = \mathbf{C}^{(1)} - \Lambda\mathbf{X}^\top\mathbf{C}^{(1)} = (\mathbf{I} - \Lambda\mathbf{X}^\top)\mathbf{C}^{(1)},$$

where $\mathbf{I}$ is the identity matrix and $\Lambda = diag(\tau_0, \tau_1, \tau_2, \ldots)$.

Since $\mathbf{C}^{(1)}$ is nonsingular, we have

$$\breve{\mathbf{C}}^{(0)}(\mathbf{C}^{(1)})^{-1} = \mathbf{I} - \Lambda\mathbf{X}^\top = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots \\ -\tau_1 & 1 & 0 & 0 & \ldots \\ 0 & -\tau_2 & 1 & 0 & \ldots \\ 0 & 0 & -\tau_3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The proof of the converse statement is analogous. ∎

The next theorem states a simple algebraic relation between the corresponding monic Jacobi matrices. Consider

$$\mathbf{p}^{(i)}(x) = [P_0^{(i)}(x), P_1^{(i)}(x), \ldots]^\top, \quad i = 0, 1.$$

Then we have

$$x\mathbf{p}^{(i)}(x) = \mathbf{J}_P^{(i)}\mathbf{p}^{(i)}(x), \quad i = 0, 1, \tag{3.28}$$

where $\mathbf{J}_P^{(i)}$ is the Jacobi matrix

$$\mathbf{J}_P^{(i)} = \begin{bmatrix} \beta_1^{(i)} & 1 & 0 & 0 & \ldots \\ \alpha_2^{(i)} & \beta_2^{(i)} & 1 & 0 & \ldots \\ 0 & \alpha_3^{(i)} & \beta_3^{(i)} & 1 & \ldots \\ 0 & 0 & \alpha_4^{(i)} & \beta_4^{(i)} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad i = 0, 1.$$

From Theorem 3.13, $\{P_n^{(i)}\}_{n\geq 0}$ satisfies (3.27) which can be expressed in a matrix form

$$A_3(x)\mathbf{p}^{(i)\prime}(x) = \mathbf{X}^\top \mathbf{R}^{(i)}\, \mathbf{p}^{(i)}(x), \quad i = 0, 1, \tag{3.29}$$

where $\mathbf{R}^{(i)}$ is a $(-deg(A_3), 1)$-banded matrix.

Writing (3.2) in a matrix form we get

$$\mathbf{p}^{(0)\prime}(x) = \mathbf{X}^\top \mathbf{B}\mathbf{p}^{(1)}(x), \tag{3.30}$$

where

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ -2\tau_1 & 2 & 0 & 0 & \dots \\ 0 & -3\tau_2 & 3 & 0 & \dots \\ 0 & 0 & -4\tau_3 & 4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Next we deduce a simple algebraic relation between the Jacobi matrices associated with a pair of moment functionals of the second kind.

Theorem 3.16. *Let $\mathbf{v}_0$ and $\mathbf{v}_1$ be quasi-definite moment functionals and let $\{P_n^{(0)}\}_{n\geq 0}$ and $\{P_n^{(1)}\}_{n\geq 0}$ be, respectively, the corresponding sequences of MOPs. If $\{\mathbf{v}_0, \mathbf{v}_1\}$ is a CPMF2K, then the monic Jacobi matrices associated with $\{P_n^{(0)}\}_{n\geq 0}$ and $\{P_n^{(1)}\}_{n\geq 0}$ satisfy*

$$\mathbf{B}\Big[A_3(\mathbf{J}_P^{(1)})(\mathbf{R}^{(1)})^{-1} + \mathbf{J}_P^{(1)}\mathbf{X}^\top\Big] = \tilde{\mathbf{J}}_P^{(0)}\mathbf{X}^\top\tilde{\mathbf{B}},$$

where $\tilde{\mathbf{J}}_P^{(0)}$ and $\tilde{\mathbf{B}}$ are, respectively, the matrices obtained by eliminating both the first row and the first column of $\mathbf{J}_P^{(0)}$ and $\mathbf{B}$.

Proof. Differentiating both sides in (3.28) and using (3.30), we get

$$\mathbf{p}^{(0)}(x) = \Big[\mathbf{J}_P^{(0)}\mathbf{X}^\top\mathbf{B} - \mathbf{X}^\top\mathbf{B}\mathbf{J}_P^{(1)}\Big]\mathbf{p}^{(1)}(x).$$

Thus, differentiating both sides and using (3.30), we have

$$\mathbf{X}^\top\mathbf{B}\mathbf{p}^{(1)}(x) = \Big[\mathbf{J}_P^{(0)}\mathbf{X}^\top\mathbf{B} - \mathbf{X}^\top\mathbf{B}\mathbf{J}_P^{(1)}\Big]\mathbf{p}^{(1)\prime}(x).$$

Multiplying both sides by $A_3(x)$ and using (3.29), we obtain

$$\mathbf{X}^\top\mathbf{B}A_3(\mathbf{J}_P^{(1)})\mathbf{p}^{(1)}(x) = \Big[\mathbf{J}_P^{(0)}\mathbf{X}^\top\mathbf{B} - \mathbf{X}^\top\mathbf{B}\mathbf{J}_P^{(1)}\Big]\mathbf{X}^\top\mathbf{R}^{(1)}\,\mathbf{p}^{(1)}(x),$$

or, equivalently,

$$\mathbf{B}A_3(\mathbf{J}_P^{(1)}) + \mathbf{B}\mathbf{J}_P^{(1)}\mathbf{X}^\top\mathbf{R}^{(1)} = \mathbf{X}\mathbf{J}_P^{(0)}\mathbf{X}^\top\mathbf{B}\mathbf{X}^\top\mathbf{R}^{(1)}.$$

Since $\mathbf{X}\mathbf{J}_P^{(0)}\mathbf{X}^\top = \tilde{\mathbf{J}}_P^{(0)}$ and $\mathbf{B}\mathbf{X}^\top = \mathbf{X}^\top\tilde{\mathbf{B}}$, this is equivalent to the formula of the statement of the Theorem. ∎

Remark 3.17. Notice that $\mathbf{J}_P^{(1)}\mathbf{X}^\top$ and $\tilde{\mathbf{J}}_P^{(0)}\mathbf{X}^\top$ are, respectively, monic $(0, 2)$-banded matrices obtained by eliminating the first column of the matrices $\mathbf{J}_P^{(1)}$ and $\tilde{\mathbf{J}}_P^{(0)}$.

4 Symmetric Coherent Pair of Moment Functionals of the Second Kind on the Real Line

The principal aim here is to study an extension to the concept of coherent pairs of measures on the real line studied on Chapter 3, now under the assumption that the measures are symmetric. We will, in particular, establish a characterization for pairs of symmetric positive measures $\{\nu_0, \nu_1\}$ on the real line for which $\{\mathcal{P}_n(\nu_0; \cdot)\}_{n\geq 0}$ and $\{\mathcal{P}_n(\nu_1; \cdot)\}_{n\geq 0}$, respectively, the corresponding sequences of MOP, satisfy

$$\mathcal{P}_n(\nu_1; x) - \tau_{n-1}\mathcal{P}_{n-2}(\nu_1; x) = \frac{1}{n+1}\mathcal{P}'_{n+1}(\nu_0; x), \quad n \geq 2,$$

where $\tau_n \neq 0$ for $n \geq 1$. In order to do this, we study this problem by dealing with a more general problem in the framework of symmetric and quasi-definite moment functionals.

4.1 Symmetric Coherent Pairs of the Second Kind

Notice that, the sequences of MOP $\{P_n^{(0)}\}_{n\geq 0}$ and $\{P_n^{(1)}\}_{n\geq 0}$ with respect to symmetric and quasi-definite moment functionals $\mathbf{v}_0$ and $\mathbf{v}_1$, respectively, satisfy the TTRR

$$\begin{aligned} P_{n+1}^{(0)} &= xP_n^{(0)} - \alpha_{n+1}^{(0)}P_{n-1}^{(0)}, \quad \alpha_{n+1}^{(0)} \neq 0, \\ P_{n+1}^{(1)} &= xP_n^{(1)} - \alpha_{n+1}^{(1)}P_{n-1}^{(1)}, \quad \alpha_{n+1}^{(1)} \neq 0, \end{aligned} \qquad n \geq 1.$$

Thus, if n is even the polynomials $P_n^{(0)}$ and $P_n^{(1)}$ are even functions and if n is odd they are odd functions. In this situation the property (3.2) only can be satisfied with $\tau_n = 0$ for all $n \geq 1$. Therefore, in the next definition we introduce the concept of coherence of the second kind for symmetric quasi-definite moment functionals.

Definition 4.1. Let $\mathbf{v}_0$ and $\mathbf{v}_1$ be symmetric and quasi-definite moment functionals and let $\{P_n^{(0)}\}_{n\geq 0}$ and $\{P_n^{(1)}\}_{n\geq 0}$ be, respectively, the corresponding sequences of MOP. Then $\{\mathbf{v}_0, \mathbf{v}_1\}$ is called a *symmetric coherent pair of moment functionals of the second kind* (SCPMF2K for short) if there exist non-zero constants τ_n such that

$$\frac{1}{n+1}P_{n+1}^{(0)\prime}(x) = P_n^{(1)}(x) - \tau_{n-1}P_{n-2}^{(1)}(x), \quad n \geq 2. \tag{4.1}$$

Therefore our objective is to find information regarding pairs $\{\mathbf{v}_0, \mathbf{v}_1\}$ of symmetric quasi-definite moment functionals which are SCPMF2K.

In what follows we also denote

$$\hbar_n^{(0)} = \left\langle \mathbf{v}_0, (P_n^{(0)})^2 \right\rangle \quad \text{and} \quad \hbar_n^{(1)} = \left\langle \mathbf{v}_1, (P_n^{(1)})^2 \right\rangle, \qquad n \geq 0. \tag{4.2}$$

4.2 The Main Results

In this section we present the main results of this chapter in which we establish a characterization for the pair of symmetric and quasi-definite moment functionals $\{\mathbf{v}_0, \mathbf{v}_1\}$ that is a SCPMF2K.

Now we denote, respectively, by $\{\mathbf{v}_n^{(0)}\}_{n\geq 0}$, $\{\mathbf{v}_n^{(1)}\}_{n\geq 0}$ and $\{\tilde{\mathbf{v}}_n^{(0)}\}_{n\geq 0}$ the corresponding basis dual of $\{P_n^{(0)}\}_{n\geq 0}$, $\{P_n^{(1)}\}_{n\geq 0}$ and $\{\tilde{P}_n^{(0)}\}_{n\geq 0}$, where

$$\tilde{P}_n^{(0)}(x) = \frac{P_{n+1}^{(0)\prime}(x)}{n+1}.$$

In the following result we state a relation between the elements of the dual bases of the sequences $\{P_n^{(0)}\}_{n\geq 0}$, $\{P_n^{(1)}\}_{n\geq 0}$ and $\{P_{n+1}^{(0)\prime}(x)(n+1)\}_{n\geq 0}$ when $\{\mathbf{v}_0, \mathbf{v}_1\}$ is a SCPMF2K.

Proposition 4.2. *Let* $\mathbf{v}_0$ *and* $\mathbf{v}_1$ *be symmetric and quasi-definite moment functionals and let* $\{P_n^{(0)}\}_{n\geq 0}$ *and* $\{P_n^{(1)}\}_{n\geq 0}$ *be, respectively, the corresponding sequences of MOP. If* $\{\mathbf{v}_0, \mathbf{v}_1\}$ *is a SCPMF2K, then*

(a) $\mathbf{v}_n^{(1)} = \tilde{\mathbf{v}}_n^{(0)} - \tau_{n+1}\tilde{\mathbf{v}}_{n+2}^{(0)}, \quad n \geq 0.$

(b) $\mathcal{D}\left[\dfrac{P_n^{(1)}(x)}{\hbar_n^{(1)}}\mathbf{v}_1\right] = \left[(n+3)\tau_{n+1}\dfrac{P_{n+3}^{(0)}(x)}{\hbar_{n+3}^{(0)}} - (n+1)\dfrac{P_{n+1}^{(0)}(x)}{\hbar_{n+1}^{(0)}}\right]\mathbf{v}_0, \quad n \geq 0.$

Proof. Analogous from Proposition 3.2. ■

In the next result we give the first characterization for the pair of symmetric and quasi-definite moment functionals $\{\mathbf{v}_0, \mathbf{v}_1\}$ to be a SCPMF2K.

Proposition 4.3. *Let* $\mathbf{v}_0$ *and* $\mathbf{v}_1$ *be symmetric and quasi-definite moment functionals. Then,* $\{\mathbf{v}_0, \mathbf{v}_1\}$ *is a SCPMF2K if and only if there exists an admissible pair of polynomials* (A_4, A_3), *where* A_4 *is an even polynomial with* $\deg(A_4) \leq 4$ *and* A_3 *is an odd polynomial with* $\deg(A_3) = 3$ *such that*

$$\mathcal{D}\mathbf{v}_1 = A_3\mathbf{v}_0 \quad \textit{and} \quad \mathbf{v}_1 = A_4\mathbf{v}_0. \tag{4.3}$$

Proof. First we assume that the pair of symmetric and quasi-definite moment functionals $\{\mathbf{v}_0, \mathbf{v}_1\}$ is such that there exist polynomials

$$\begin{aligned} A_3(x) &= \mathfrak{d}_3 P_3^{(0)}(x) + \mathfrak{d}_1 P_1^{(0)}(x) \qquad \text{and} \\ A_4(x) &= \mathfrak{c}_4 P_4^{(0)}(x) + \mathfrak{c}_2 P_2^{(0)}(x) + \mathfrak{c}_0 P_0^{(0)}(x), \end{aligned} \tag{4.4}$$

such that $\mathcal{D}\mathbf{v}_1 = A_3\mathbf{v}_0$ and $\mathbf{v}_1 = A_4\mathbf{v}_0$. From

$$\frac{P_{n+1}^{(0)\prime}(x)}{n+1} = P_n^{(1)}(x) + \sum_{j=0}^{n-2} \lambda_{n,j}\, P_j^{(1)}(x), \quad n \geq 2,$$

where $(n+1)\hbar_j^{(1)}\lambda_{n,j} = \left\langle \mathbf{v}_1, P_{n+1}^{(0)\prime} P_j^{(1)} \right\rangle = \left\langle \mathbf{v}_1, (P_j^{(1)} P_{n+1}^{(0)})' - P_j^{(1)\prime} P_{n+1}^{(0)} \right\rangle$, we get

$$\lambda_{n,j} = -\frac{1}{(n+1)\hbar_j^{(1)}} \left[\left\langle \mathcal{D}\mathbf{v}_1, P_j^{(1)} P_{n+1}^{(0)} \right\rangle + \left\langle \mathbf{v}_1, P_j^{(1)\prime} P_{n+1}^{(0)} \right\rangle \right],$$

for $0 \le j \le n-2$ and $n \ge 2$. Here, with the use of $\mathcal{D}\mathbf{v}_1 = A_3\mathbf{v}_0$ and $\mathbf{v}_1 = A_4\mathbf{v}_0$, one finds

$$\lambda_{n,j} = -\frac{1}{(n+1)\hbar_j^{(1)}} \left[\left\langle \mathbf{v}_0, A_3 P_j^{(1)} P_{n+1}^{(0)} \right\rangle + \left\langle \mathbf{v}_0, A_4 P_j^{(1)\prime} P_{n+1}^{(0)} \right\rangle \right],$$

for $0 \le j \le n-2$ and $n \ge 2$. Therefore, by orthogonality,

$$\lambda_{n,n-2} = -\frac{\hbar_{n+1}^{(0)}}{(n+1)\hbar_{n-2}^{(1)}}[\mathfrak{d}_3 + (n-2)\mathfrak{c}_4] \quad \text{and} \quad \lambda_{n,j} = 0, \quad 0 \le j \le n-3,$$

for $n \ge 3$. Moreover, $\lambda_{2,0} = -\hbar_3^{(0)}\mathfrak{d}_3/(3\hbar_0^{(1)})$. Hence, by setting $\lambda_{n,n-2} = -\tau_{n-1}$ we find

$$\frac{P_{n+1}^{(0)\prime}(x)}{n+1} = P_n^{(1)}(x) - \tau_{n-1} P_{n-2}^{(1)}(x), \quad n \ge 2,$$

where

$$\tau_{n-1} = \frac{\hbar_{n+1}^{(0)}}{(n+1)\hbar_{n-2}^{(1)}}[\mathfrak{d}_3 + (n-2)\mathfrak{c}_4], \quad n \ge 2. \tag{4.5}$$

In particular, the polynomials A_3 and A_4 are such that

$$\mathfrak{d}_3 = \frac{3\hbar_0^{(1)}}{\hbar_3^{(0)}}\tau_1, \quad \mathfrak{c}_4 = \frac{4\hbar_1^{(1)}}{\hbar_4^{(0)}}\tau_2 - \frac{3\hbar_0^{(1)}}{\hbar_3^{(0)}}\tau_1. \tag{4.6}$$

Conversely, we now use Proposition 4.2, which was derived under the assumption that $\{\mathbf{v}_0, \mathbf{v}_1\}$ is a SCPMF2K, to find the precise set of polynomials (A_4, A_3) for which (4.6) holds. Successively setting $n = 0$ and $n = 1$ in the second item of Proposition 4.2 we find that the polynomials

$$\begin{aligned} A_3(x) &= \frac{3\hbar_0^{(1)}\tau_1}{\hbar_3^{(0)}} P_3^{(0)}(x) - \frac{\hbar_0^{(1)}}{\hbar_1^{(0)}} P_1^{(0)}(x) \qquad \text{and} \\ A_4(x) &= \frac{4\hbar_1^{(1)}\tau_2}{\hbar_4^{(0)}} P_4^{(0)}(x) - \frac{2\hbar_1^{(1)}}{\hbar_2^{(0)}} P_2^{(0)}(x) - A_3(x) P_1^{(1)}(x) \end{aligned} \tag{4.7}$$

are such that $\mathcal{D}\mathbf{v}_1 = A_3\mathbf{v}_0$ and $\mathbf{v}_1 = A_4\mathbf{v}_0$. With the requirement $\tau_1 \neq 0$ we must have $deg(A_3) = 3$. However, it is possible that $deg(A_4) \le 4$.

From (4.5) we now observe that $\tau_{n-1} \neq 0$ for $n \ge 2$ if and only if $\mathfrak{d}_3 + (n-2)\mathfrak{c}_4 \neq 0$ for $n \ge 2$. That is, $\tau_{n-1} \neq 0$ for $n \ge 2$ if and only if (A_4, A_3) in (4.7) is also an admissible pair of polynomials. This completes the proof of the proposition. ∎

We can now give the following more general statement regarding SCPMF2K.

Theorem 4.4. *The pair of symmetric and quasi-definite moment functionals* $\{\mathbf{v}_0, \mathbf{v}_1\}$ *is a SCPMF2K if and only if there exists an admissible pair of polynomials* (A_4, A_3)*, where* A_4 *is an even polynomial with* $\deg(A_4) \le 4$ *and* A_3 *is an odd polynomial with* $\deg(A_3) = 3$*, such that one of the following equivalent conditions holds:*

(a) $\mathcal{D}[A_4\mathbf{v}_0] = A_3\mathbf{v}_0$ *and* $\mathbf{v}_1 = A_4\mathbf{v}_0$.

(b) $\mathcal{D}\mathbf{v}_1 = A_3\mathbf{v}_0$ *and* $\mathcal{D}[A_4\mathbf{v}_1] = (A_4' + A_3)\mathbf{v}_1$.

Moreover, the pair of polynomials (A_4, A_3) *must be as in* (4.7) *and that, with the additional admissibility assumption, both* $\mathbf{v}_0$ *and* $\mathbf{v}_1$ *are semiclassical symmetric moment functionals of class at most* $\mathsf{s} = 2$.

Proof. Item (a) in the theorem follows from Proposition 4.3 in which (4.3) can be rewritten as

$$\mathcal{D}[A_4\mathbf{v}_0] = A_3\mathbf{v}_0 \quad \text{and} \quad \mathbf{v}_1 = A_4\mathbf{v}_0,$$

where A_3 and A_4 are as in (4.4). Moreover, the pair (A_4, A_3) is as in (4.7). Indeed, from $\mathcal{D}[A_4\mathbf{v}_0] = A_3\mathbf{v}_0$ we have for any polynomial p,

$$-\left\langle \mathbf{v}_0, A_4p' \right\rangle = \left\langle \mathbf{v}_0, A_3p \right\rangle. \tag{4.8}$$

The choice $p(x) = x$ gives $\left\langle \mathbf{v}_0, A_3x \right\rangle = -\left\langle \mathbf{v}_0, A_4 \right\rangle = -\left\langle \mathbf{v}_1, 1 \right\rangle$, which follows from $\mathbf{v}_1 = A_4\mathbf{v}_0$. Therefore we obtain

$$\mathfrak{d}_1 = -\hbar_0^{(1)}/\hbar_1^{(0)}$$

and $\mathfrak{d}_3$ that satisfy (4.6), is obtained directly from the equality between (a) and (4.3). Thus, A_3 is the same polynomial given in (4.7).

Again, from the equality between (a) and (4.3), $\mathfrak{c}_4$ also satisfies (4.6). Since the polynomials A_3 and A_4 are odd and even polynomials, respectively, we can write

$$A_4(x) + xA_3(x) = \frac{4\hbar_1^{(1)}\tau_2}{\hbar_4^{(0)}}P_4^{(0)}(x) + \mathfrak{c}_2P_2^{(0)}(x) + \mathfrak{c}_0P_0^{(0)}(x). \tag{4.9}$$

Now, choosing $p(x) = x$ in (4.8), we get

$$\left\langle \mathbf{v}_0, A_4 + xA_3 \right\rangle = 0,$$

thus $\mathfrak{c}_0 = 0$. Therefore, (4.9) can be write as

$$A_4(x) + xA_3(x) = \frac{4\hbar_1^{(1)}\tau_2}{\hbar_4^{(0)}}P_4^{(0)}(x) + \mathfrak{c}_2P_2^{(0)}(x).$$

Hence, from

$$\begin{aligned}\left\langle \mathbf{v}_0, [A_4 + xA_3]P_2^{(0)} \right\rangle &= \left\langle \mathbf{v}_0, A_4P_2^{(0)} \right\rangle - \left\langle \mathbf{v}_0, A_4[xP_2^{(0)}]' \right\rangle \\ &= -\left\langle A_4\mathbf{v}_0, xP_2^{(0)\prime} \right\rangle = -2\left\langle \mathbf{v}_1, x^2 \right\rangle = -2\hbar_1^{(1)},\end{aligned}$$

we find $\mathfrak{c}_2 = -2\hbar_1^{(1)}/\hbar_2^{(0)}$. Thus, A_4 is also the same polynomial given in (4.7).

We now prove the equivalence of (a) and (b), we first obtain for any polynomial p,

$$\begin{aligned}\left\langle \mathcal{D}[A_4\mathbf{v}_1], p \right\rangle = -\left\langle \mathbf{v}_1, A_4p' \right\rangle &= -\left\langle \mathbf{v}_1, (A_4p)' \right\rangle + \left\langle \mathbf{v}_1, A_4'p \right\rangle \\ &= \left\langle \mathcal{D}\mathbf{v}_1, A_4p \right\rangle + \left\langle \mathbf{v}_1, A_4'p \right\rangle.\end{aligned}$$

Thus, if $\mathcal{D}\mathbf{v}_1 = A_3\mathbf{v}_0$ then

$$\left\langle \mathcal{D}[A_4\mathbf{v}_1], p \right\rangle = \left\langle A_3\mathbf{v}_0, A_4p \right\rangle + \left\langle \mathbf{v}_1, A_4'p \right\rangle = \left\langle A_4\mathbf{v}_0, A_3p \right\rangle + \left\langle \mathbf{v}_1, A_4'p \right\rangle.$$

Hence, $\mathcal{D}[A_4\mathbf{v}_1] = (A_4' + A_3)\mathbf{v}_1$ if and only if $\mathbf{v}_1 = A_4\mathbf{v}_0$. This gives the equivalence of items (a) and (b).

The admissibility of the pair (A_4, A_3) means $\mathfrak{d}_3 + (n-2)\mathfrak{c}_4 \neq 0$, $n \geq 2$, where $\mathfrak{d}_3$ and $\mathfrak{c}_4$ are the respective leading coefficients of A_3 and A_4. As we have observed from (4.5), this admissibility condition is necessary to guarantee $\tau_{n-1} \neq 0$, $n \geq 2$. Thus, if $\{\mathbf{v}_0, \mathbf{v}_1\}$ is a SCPMF2K, then from item (a) we can also say that the symmetric moment functional $\mathbf{v}_0$ is semiclassical of class at most $\mathsf{s} = 2$.

Notice that the leading coefficient of $A_4' + A_3$ is $\tilde{\mathfrak{d}}_3 = 4\mathfrak{c}_4 + \mathfrak{d}_3$. Hence, $\tilde{\mathfrak{d}}_3 + (n-2)\mathfrak{c}_4 = \mathfrak{d}_3 + (n+2)\mathfrak{c}_4 \neq 0$, $n \geq 2$. Hence, the pair of polynomials $(A_4, A_4' + A_3)$ is also admissible and, as a consequence, from (b), the symmetric moment functional $\mathbf{v}_1$ is semiclassical of class at most $\mathsf{s} = 2$. This completes the proof of the theorem. ∎

Since $\mathbf{v}_0$ and $\mathbf{v}_1$ are symmetric and quasi-definite moment functionals, using (1.14) let us define $\mathbf{u}_0$ and $\mathbf{u}_1$ by

$$\begin{aligned} \langle \mathbf{u}_0, x^n \rangle &= \left\langle \mathbf{v}_0, x^{2n} \right\rangle, \\ \langle \mathbf{u}_1, x^n \rangle &= \left\langle \mathbf{v}_1, x^{2n} \right\rangle, \end{aligned} \quad n \geq 0. \tag{4.10}$$

Then the sequences of monic polynomials $\{\mathcal{Q}_n^{(0)}\}_{n\geq 0}$, $\{\tilde{\mathcal{Q}}_n^{(0)}\}_{n\geq 0}$, $\{\mathcal{Q}_n^{(1)}\}_{n\geq 0}$ and $\{\tilde{\mathcal{Q}}_n^{(1)}\}_{n\geq 0}$ defined by

$$\begin{aligned} P_{2n}^{(0)}(x) &= \mathcal{Q}_n^{(0)}(x^2), \quad P_{2n+1}^{(0)}(x) = x\tilde{\mathcal{Q}}_n^{(0)}(x^2) \\ P_{2n}^{(1)}(x) &= \mathcal{Q}_n^{(1)}(x^2), \quad P_{2n+1}^{(1)}(x) = x\tilde{\mathcal{Q}}_n^{(1)}(x^2), \end{aligned} \tag{4.11}$$

are sequences of MOP with respect to $\mathbf{u}_0$, $x\mathbf{u}_0$, $\mathbf{u}_1$ and $x\mathbf{u}_1$, respectively. Consequently, from the above relations we can deduce that

$$\begin{aligned} \hbar_{2n}^{(0)} &= \left\langle \mathbf{v}_0, (P_{2n}^{(0)})^2 \right\rangle = \left\langle \mathbf{u}_0, (\mathcal{Q}_n^{(0)})^2 \right\rangle, \quad \hbar_{2n+1}^{(0)} = \left\langle \mathbf{v}_0, (P_{2n+1}^{(0)})^2 \right\rangle = \left\langle x\mathbf{u}_0, (\tilde{\mathcal{Q}}_n^{(0)})^2 \right\rangle, \\ \hbar_{2n}^{(1)} &= \left\langle \mathbf{v}_1, (P_{2n}^{(1)})^2 \right\rangle = \left\langle \mathbf{u}_1, (\mathcal{Q}_n^{(1)})^2 \right\rangle, \quad \hbar_{2n+1}^{(1)} = \left\langle \mathbf{v}_1, (P_{2n+1}^{(1)})^2 \right\rangle = \left\langle x\mathbf{u}_1, (\tilde{\mathcal{Q}}_n^{(1)})^2 \right\rangle, \end{aligned} \quad n \geq 0.$$

In the next theorem we look a connection between a SCPMF2K and CPMF2K.

Theorem 4.5. *If $\{\mathbf{v}_0, \mathbf{v}_1\}$ is a SCPMF2K, then $\{\mathbf{u}_0, x\mathbf{u}_1\}$ is a CPMF2K. That is, the respectives associated MOP satisfy*

$$\frac{1}{n+1}\mathcal{Q}_{n+1}^{(0)\prime}(x) = \tilde{\mathcal{Q}}_n^{(1)}(x) - \tau_{2n}\tilde{\mathcal{Q}}_{n-1}^{(1)}(x), \quad n \geq 1.$$

Moreover, the sequences of MPO $\{\mathcal{Q}_n^{(1)}\}_{n\geq 0}$ and $\{\tilde{\mathcal{Q}}_n^{(0)}\}_{n\geq 0}$ satisfy

$$\mathcal{Q}_n^{(1)}(x) - \tau_{2n-1}\mathcal{Q}_{n-1}^{(1)}(x) = \frac{1}{2n+1}\tilde{\mathcal{Q}}_n^{(0)}(x) + \frac{2x}{2n+1}\tilde{\mathcal{Q}}_n^{(0)\prime}(x).$$

Proof. Using (4.1) we have

$$\frac{1}{2n+2}P_{2n+2}^{(0)\prime}(x) = P_{2n+1}^{(1)}(x) - \tau_{2n}P_{2n-1}^{(1)}(x),$$

and from (4.11) we get

$$\frac{2x}{2n+2}\mathcal{Q}_{n+1}^{(0)\prime}(x^2) = \frac{1}{2n+2}\left[\mathcal{Q}_{n+1}^{(0)}(x^2)\right]' = x\tilde{\mathcal{Q}}_n^{(1)}(x^2) - \tau_{2n}x\tilde{\mathcal{Q}}_{n-1}^{(1)}(x^2).$$

Hence, the first equation of the theorem follows.

Again from (4.1) we have

$$\frac{1}{2n+1}P_{2n+1}^{(0)\prime}(x) = P_{2n}^{(1)}(x) - \tau_{2n-1}P_{2n-2}^{(1)}(x),$$

and from (4.11) we get

$$\mathcal{Q}_n^{(1)}(x^2) - \tau_{2n-1}\mathcal{Q}_{n-1}^{(1)}(x^2) = \frac{1}{2n+1}\left[x\tilde{\mathcal{Q}}_n^{(0)}(x^2)\right]' = \frac{1}{2n+1}[\tilde{\mathcal{Q}}_n^{(0)}(x^2) + 2x^2\tilde{\mathcal{Q}}_n^{(0)\prime}(x^2)].$$

Thus the second equation of the statement of the theorem follows. ∎

We will give an alternative proof of the above result in terms of moment functionals. Indeed, using (4.10) and (4.3) we get

$$\begin{aligned}\langle \mathbf{u}_1, x^n\rangle &= \langle \mathbf{v}_1, x^{2n}\rangle = \langle A_4\mathbf{v}_0, x^{2n}\rangle \\ &= \langle \mathbf{v}_0, x^{2n}A_4\rangle = \left\langle \mathbf{u}_0, x^n\hat{A}_2\right\rangle = \left\langle \hat{A}_2\mathbf{u}_0, x^n\right\rangle,\end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $\hat{A}_2(x^2) = A_4(x)$, i.e., $\hat{A}_2$ is a polynomial of degree at most 2. Thus, $\mathbf{u}_1 = \hat{A}_2(x)\mathbf{u}_0$.

On the other hand, by applying $\mathcal{D}$ to moment functional $x\mathbf{u}_1$ and using (4.10) we get

$$\langle \mathcal{D}[x\mathbf{u}_1], x^n\rangle = -n\left\langle x\mathbf{u}_1, x^{n-1}\right\rangle = -n\langle \mathbf{u}_1, x^n\rangle = -n\left\langle x\mathbf{v}_1, x^{2n-1}\right\rangle = \frac{1}{2}\left\langle \mathcal{D}[x\mathbf{v}_1], x^{2n}\right\rangle.$$

From (4.3) and $\mathcal{D}[x\mathbf{v}_1] = x\mathbf{v}_1 + \mathcal{D}\mathbf{v}_1$ the above expression reads as

$$\langle \mathcal{D}[x\mathbf{u}_1], x^n\rangle = \frac{1}{2}\left(\left\langle xA_3\mathbf{v}_0, x^{2n}\right\rangle + \left\langle \mathbf{v}_1, x^{2n}\right\rangle\right) = \frac{1}{2}\left(\left\langle \mathbf{u}_0, \hat{A}_3x^n\right\rangle + \langle \mathbf{u}_1, x^n\rangle\right),$$

where $\hat{A}_3(x^2) = xA_3(x)$, i.e., $\hat{A}_3$ is a polynomial of degree 2.

As a consequence, $D(x\mathbf{u}_1) = \frac{1}{2}(\hat{A}_3+\hat{A}_2)\mathbf{u}_0$. Hence, we can state the following theorem.

Theorem 4.6. *If* $\{\mathbf{v}_0, \mathbf{v}_1\}$ *is a SCPMF2K, then* $\{\mathbf{u}_0, x\mathbf{u}_1\}$ *is a CPMF2K and* $\mathbf{u}_1$ *is a Christoffel transformation of* $\mathbf{u}_0$ *of order 2.*

4.3 Some Special Cases

In this section we look at some examples of $\{\mathbf{v}_0, \mathbf{v}_1\}$ which are SCPMF2K. We remind that, as shown in Theorem 4.4, in a pair $\{\mathbf{v}_0, \mathbf{v}_1\}$ of SCPMF2K the associated pair (A_4, A_3) of admissible polynomials are such that $\deg(A_4) \le 4$ and $\deg(A_3) = 3$. In what follows, results are presented in accordance with the degree of A_4 as in the next table.

$deg(A_4) = 4$	
$deg(A_4) = 2$	$deg(A_3) = 3$
$deg(A_4) = 0$	

A_4 is of degree 4

(i) $P_n^{(0)}$ are the Gegenbauer Polynomials:

Let the moment functional $\mathbf{v}_0$ be given by

$$\langle \mathbf{v}_0, p\rangle = \int_{-1}^{1} p(x)(1-x^2)^{\lambda-1/2}dx = \int_{-1}^{1} p(x)d\nu^{(\lambda)}(x),$$

with $\lambda > -\frac{1}{2}$. The corresponding monic orthogonal polynomials $P_n^{(0)}(x) = C_n^{(\lambda)}(x)$ are known as monic Gegenbauer polynomials. Their properties have been studied extensively in the literature (see, for example, [1], [20], [33] and [74]).

It is also well known that the symmetric moment functional $\mathbf{v}_0$ is classical and satisfies the Pearson equation $\mathcal{D}[(x^2-1)\mathbf{v}_0] = [(2\lambda+1)x]\mathbf{v}_0$. However, with $q \in (-\infty,-1]\cup[1,\infty)$, we can also write the alternative Pearson's equation

$$\mathcal{D}[\mathcal{B}_4^{(q)}\mathbf{v}_0] = \mathcal{C}_3^{(\lambda,q)}\mathbf{v}_0, \tag{4.12}$$

where

$$\mathcal{B}_4^{(q)}(x) = (x^2-q^2)(x^2-1) \quad \text{and} \quad \mathcal{C}_3^{(\lambda,q)}(x) = (2\lambda+3)x^3 - [q^2(2\lambda+1)+2]x.$$

This latter Pearson equation can be easily obtained from

$$-\left\langle \mathbf{v}_0, (x^2-1)p' \right\rangle = \left\langle \mathbf{v}_0, [(2\lambda+1)x]p \right\rangle,$$

by replacing $p(x)$ by $(x^2-q^2)p(x)$. Observe that, with our choice of q, the moment functional given by $\mathcal{B}_4^{(q)}\mathbf{v}_0$ is also a positive definite moment functional.

The leading coefficient of $\mathcal{C}_3^{(\lambda,q)}$ is $\mathfrak{d}_3 = 2\lambda+3$ and the leading coefficient of $\mathcal{B}_4^{(q)}$ is $\mathfrak{c}_4 = 1$. Hence, $\mathfrak{d}_3 + (n-2)\mathfrak{c}_4 \neq 0$, $n \geq 2$, and $(\mathcal{B}_4^{(q)}, \mathcal{C}_3^{(\lambda,q)})$ is also an admissible pair of polynomials. We can now use item (a) in Theorem 4.4, with $A_3(x) = \mathcal{C}_3^{(\lambda,q)}(x)$ and $A_4(x) = \mathcal{B}_4^{(q)}(x)$, to obtain information about the respective symmetric coherent pair of moment functionals of the second kind $\{\mathbf{v}_0, \mathbf{v}_1\}$. We thus have

$$\begin{aligned}\langle \mathbf{v}_1, p\rangle &= \int_{-1}^{1} p(x)\mathcal{B}_4^{(q)}(x)(1-x^2)^{\lambda-1/2}dx \\ &= \int_{-1}^{1} p(x)(q^2-x^2)(1-x^2)^{\lambda+1/2}\,dx = \int_{-1}^{1} p(x)d\hat{\nu}^{(\lambda+1,q)}(x).\end{aligned}$$

Now observe that $\tilde{\mathfrak{d}}_3 + (n-2)\mathfrak{c}_4 = \mathfrak{d}_3 + (n+2)\mathfrak{c}_4 \neq 0$, $n \geq 2$, where $\tilde{\mathfrak{d}}_3$ is the leading coefficient of $\mathcal{B}_4^{(q)\prime} + \mathcal{C}_3^{(\lambda,q)}$. Hence, we obtain from item (b) of Theorem 4.4 that the moment functional $\mathbf{v}_1$ is semiclassical of class at most $\mathsf{s} = 2$.

(ii) $P_n^{(1)}$ are the Gegenbauer Polynomials:

Let the moment functional $\mathbf{v}_1$ be given by

$$\langle \mathbf{v}_1, p\rangle = \int_{-1}^{1} p(x)(1-x^2)^{\lambda+1/2}dx = \int_{-1}^{1} p(x)d\nu^{(\lambda+1)}(x),$$

with $\lambda > -1/2$. The corresponding MOP $P_n^{(1)}(x) = C_n^{(\lambda+1)}(x)$ are the monic Gegenbauer orthogonal polynomials with respect to the measure $\nu^{(\lambda+1)}$.

The moment functional $\mathbf{v}_1$ is classical and satisfies the Pearson equation

$$\mathcal{D}[\mathcal{B}_4^{(q)}\mathbf{v}_1] = \mathcal{C}_3^{(\lambda+1,q)}\mathbf{v}_1,$$

where $\mathcal{C}_3^{(\lambda,q)}(x)$ and $\mathcal{B}_4^{(q)}(x)$ are as in (4.12) and $q \in (-\infty,-1] \cup [1,\infty)$.

We now use item (b) of Theorem 4.4 to obtain information about the companion moment functional $\mathbf{v}_0$ by letting $A_4(x) = \mathcal{B}_4^{(q)}(x)$ and $A_4'(x) + A_3(x) = \mathcal{C}_3^{(\lambda+1,q)}(x)$.

Observe that

$$\langle \mathcal{D}[A_4\mathbf{v}_1], p\rangle = -\langle \mathbf{v}_1, A_4 p'\rangle = -\int_{-1}^{1} p'(x)\mathcal{B}_4^{(q)}(x)(1-x^2)^{\lambda+1}dx.$$

Since $d[\mathcal{B}_4^{(q)}(x)(1-x^2)^{\lambda+1/2}]/dx = \mathcal{C}_3^{(\lambda+1,q)}(x)(1-x^2)^{\lambda+1/2}$, using integration by parts we find

$$\langle \mathcal{D}[A_4\mathbf{v}_1], p\rangle = \int_{-1}^{1} p(x)\mathcal{C}_3^{(\lambda+1,q)}(x)(1-x^2)^{\lambda+1/2}dx.$$

On the other hand

$$\langle (A_4' + A_3)\mathbf{v}_1, p\rangle = \int_{-1}^{1} p(x)[4x^3 - 2(q^2+1)x + A_3(x)]\,(1-x^2)^{\lambda+1/2}dx.$$

Hence, from the requirement $\mathcal{D}[A_4\mathbf{v}_1] = (A_4' + A_3)\mathbf{v}_1$ in item (b) of Theorem 4.4, we find

$$\int_{-1}^{1} p(x)A_3(x)(1-x^2)^{\lambda+1/2}dx = \int_{-1}^{1} p(x)\widetilde{\mathcal{C}}_3^{(\lambda+1,q)}(x)(1-x^2)^{\lambda+1/2}dx,$$

where

$$\begin{aligned}\widetilde{\mathcal{C}}_3^{(\lambda+1,q)}(x) &= \mathcal{C}_3^{(\lambda+1,q)}(x) - 4x^3 + 2(q^2+1)x \\ &= (x^2 - q^2)[(2\lambda+1)x].\end{aligned}$$

Hence, if we set $A_3(x) = \widetilde{\mathcal{C}}_3^{(\lambda+1,q)}(x)$, then from the other requirement $\mathcal{D}\mathbf{v}_1 = A_3\mathbf{v}_0$ in item (b) of Theorem 4.4,

$$-\int_{-1}^{1} p'(x)(1-x^2)^{\lambda+1/2}dx = \left\langle \mathbf{v}_0, \widetilde{\mathcal{C}}_3^{(\lambda+1,q)} p\right\rangle.$$

Using integration by parts we then have

$$-\int_{-1}^{1} p(x)[(2\lambda+1)x](1-x^2)^{\lambda-1/2}dx = \left\langle \mathbf{v}_0, \widetilde{\mathcal{C}}_3^{(\lambda+1,q)} p\right\rangle. \tag{4.13}$$

Thus, if we choose $\mathbf{v}_0$ such that

$$\langle \mathbf{v}_0, p\rangle = \int_{-1}^{1} p(x)\frac{(1-x^2)^{\lambda-1/2}}{q^2 - x^2}dx + \epsilon\, p(q) + \epsilon\, p(-q) = \int_{-1}^{1} p(x)d\tilde{\nu}^{(\lambda,q,\epsilon)}(x),$$

then it satisfies (4.13) and $\mathbf{v}_1 = A_4\mathbf{v}_0$. With $\epsilon > 0$ and the range of values chosen for q, the symmetric moment functional $\mathbf{v}_0$ is also positive definite. However, it is important that if $q = \pm 1$, then λ must be such that $\lambda > 1/2$.

Note that, the pair $(\mathcal{B}_4^{(q)}, \widetilde{\mathcal{C}}_3^{(\lambda,q)})$ is an admissible pair of polynomials, since $\hat{\mathfrak{d}}_3 + (n-2)\mathfrak{c}_4 \neq 0$, $n \geq 2$, where $\hat{\mathfrak{d}}_3 = 2\lambda + 1$ and $\mathfrak{c}_4 = 1$ are the leading coefficient of $\widetilde{\mathcal{C}}_3^{(\lambda+1,q)}$ and $\mathcal{B}_4^{(q)}$, respectively. Hence, we obtain from item (a) of Theorem 4.4 that the moment functional $\mathbf{v}_0$ is semiclassical of class at most $\mathfrak{s} = 2$.

(iii) $P_n^{(0)}$ from a Symmetric Jacobi-type Moment Functional:

With $\alpha, \beta > -1$ and $\beta \neq 0$, let the symmetric moment functional $\mathbf{v}_0$ be given by

$$\langle \mathbf{v}_0, p \rangle = \int_{-1}^{1} p(x)|x|^{2\beta+1}(1-x^2)^{\alpha}\, dx + Np(0).$$

Clearly, $\mathbf{v}_0$ is a positive definite moment functional and it represents a symmetrized Jacobi-type moment functional, that is a particular case of the so-called Koornwinder moment functional [38] (see, also [36]).

Observe that, for every $p \in \mathbb{P}$, we have

$$\left\langle \mathcal{D}\left[x^2(1-x^2)\mathbf{v}_0\right], p \right\rangle = -\left\langle \mathbf{v}_0, x^2(1-x^2)p' \right\rangle = -\int_{-1}^{1} p'(x)|x|^{2\beta+3}(1-x^2)^{\alpha+1}\, dx.$$

Thus, integration by parts gives

$$\begin{aligned} \langle \mathcal{D}\left[A_4\mathbf{v}_0\right], p \rangle &= \int_{-1}^{1} p(x)[-(2\beta+2\alpha+5)x^3 + (2\beta+3)x]\,|x|^{2\beta+1}(1-x^2)^{\alpha}\, dx \\ &= \langle A_3\mathbf{v}_0, p \rangle, \end{aligned}$$

where $A_4(x) = x^2(1-x^2)$ and $A_3(x) = -(2\beta+2\alpha+5)x^3 + (2\beta+3)x$. Clearly, $A_4\mathbf{v}_0$ is also a positive definite moment functional and the moment functional $\mathbf{v}_0$ is semiclassical of class $\mathsf{s} = 2$ (see, [22]).

Therefore, by item (a) of Theorem 4.4, the pair of moment functionals $\{\mathbf{v}_0, \mathbf{v}_1\}$ is a SCPMF2K if $\mathbf{v}_1 = A_4\mathbf{v}_0$. Also, the symmetric moment functional $\mathbf{v}_1$ is semiclassical of class at most $\mathsf{s} = 2$.

A_4 is of degree 2

(i) $P_n^{(0)}$ are the Hermite Polynomials:

Let the moment functional $\mathbf{v}_0$ be given by

$$\langle \mathbf{v}_0, p \rangle = \int_{-\infty}^{\infty} p(x)e^{-x^2} dx = \int_{-\infty}^{\infty} p(x)d\nu(x),$$

The corresponding monic orthogonal polynomials $P_n^{(0)}(x) = H_n(x)$ and their properties also have been studied extensive in the basic literature (see, for example, [1], [20], [33] and [74]). The polynomials $H_n(x)$ are known as the monic orthogonal polynomials of Hermite the n-th degree with respect to the measure $\nu(x)$.

It is known that the symmetric moment functional $\mathbf{v}_0$, which is classical, satisfies the Pearson equation $\mathcal{D}[\mathbf{v}_0] = (-2x)\mathbf{v}_0$. However, with a $q \neq 0$ that we choose here to be in $\mathbb{R}$, we also have the alternative Pearson equation

$$\mathcal{D}[\mathcal{B}_2^{(q)}\mathbf{v}_0] = \mathcal{C}_3^{(\alpha,q)}\mathbf{v}_0, \tag{4.14}$$

where $\mathcal{C}_3^{(q)}(x) = -2x^3 - 2(q^2-1)x$ and $\mathcal{B}_2^{(q)}(x) = x^2 + q^2$. Observe that, with our choice of q, the moment functional given by $\mathcal{B}_2^{(q)}\mathbf{v}_0$ is also a positive definite symmetric moment functional.

With $A_3(x) = \mathcal{C}_3^{(\alpha,q)}(x)$ and $A_4(x) = \mathcal{B}_2^{(q)}(x)$, we can now use item (a) in Theorem 4.4 to determine information about the respective symmetric coherent pair of measures of the second kind $\{\mathbf{v}_0, \mathbf{v}_1\}$. We thus have

$$\langle \mathbf{v}_1, p \rangle = \langle A_4\mathbf{v}_0, p \rangle = \int_{-\infty}^{\infty} p(x)(x^2+q^2)e^{-x^2}dx$$
$$= \int_{-\infty}^{\infty} p(x)d\hat{\nu}^{(q)}(x).$$

From item (b) of Theorem 4.4, the moment functional $\mathbf{v}_1$ satisfies

$$\mathcal{D}[(x^2+q^2)\mathbf{v}_1] = [-2x^3 - 2(q^2-2)x]\mathbf{v}_1.$$

Hence, $\mathbf{v}_1$ is semiclassical of class $\mathsf{s} = 2$ if $q \neq 0$ but it is also semiclassical of class at most $\mathsf{s} = 1$ if $q = 0$.

(ii) $P_n^{(1)}$ are the Hermite Polynomials:

We will assume $\mathbf{v}_1$ to be such that

$$\langle \mathbf{v}_1, p \rangle = \int_{-\infty}^{\infty} p(x)e^{-x^2}dx = \int_{-\infty}^{\infty} p(x)d\nu(x).$$

Then for the corresponding sequence of MOP $\{P_n^{(1)}\}_{n\geq 0}$ we have

$$P_n^{(1)}(x) = H_n(x), \quad n \geq 0.$$

For the moment functional $\mathbf{v}_1$ we have $\mathcal{D}[\mathcal{B}_2^{(q)}\mathbf{v}_1] = \mathcal{C}_3^{(q)}\mathbf{v}_1$, where, $\mathcal{C}_3^{(q)}(x)$ and $\mathcal{B}_2^{(q)}(x)$ are as in (4.14). Again, we assume $0 \neq q \in \mathbb{R}$.

We can now use item (b) of Theorem 4.4, with

$$A_4(x) = \mathcal{B}_2^{(q)}(x) \quad \text{and} \quad A_4'(x) + A_3(x) = \mathcal{C}_3^{(q)}(x),$$

to obtain information about the companion symmetric functional $\mathbf{v}_0$. Observe that

$$\langle \mathcal{D}[A_4\mathbf{v}_1], p \rangle = -\langle \mathbf{v}_1, A_4p' \rangle = -\int_{-\infty}^{\infty} p'(x)\mathcal{B}_2^{(q)}(x)e^{-x^2}dx.$$

Since $d[\mathcal{B}_2^{(q)}(x)e^{-x^2}]/dx = \mathcal{C}_3^{(q)}(x)e^{-x^2}$, using integration by parts we then find

$$\langle \mathcal{D}(A_3\mathbf{v}_1), p \rangle = \int_{-\infty}^{\infty} p(x)\mathcal{C}_3^{(q)}(x)e^{-x^2}dx.$$

On the other hand

$$\langle (A_4' + A_3)\mathbf{v}_1, p \rangle = \int_{-\infty}^{\infty} p(x)[2x + A_3(x)]e^{-x^2}dx.$$

From $\mathcal{D}[A_4\mathbf{v}_1] = (A_4' + A_3)\mathbf{v}_1$ we then find

$$\int_{-\infty}^{\infty} p(x)A_3(x)e^{-x^2}dx = \int_{-\infty}^{\infty} p(x)\widetilde{\mathcal{C}}_3^{(q)}(x)e^{-x^2}dx,$$

where $\widetilde{\mathcal{C}}_3^{(q)}(x) = \mathcal{C}_3^{(q)}(x) - 2x = -2x(x^2+q^2)$. Thus, if we set $A_3(x) = \widetilde{\mathcal{C}}_3^{(q)}(x)$, then from $\mathcal{D}\mathbf{v}_1 = A_3\mathbf{v}_0$,

$$-\int_{-\infty}^{\infty} p'(x)e^{-x^2}dx = \left\langle \mathbf{v}_0, -2x(x^2+q^2)p \right\rangle.$$

Hence, using integration by parts

$$\int_{-\infty}^{\infty} p(x)(-2x)e^{-x^2}dx = \left\langle \mathbf{v}_0, -2x(x^2+q^2)p \right\rangle.$$

Therefore, for $\mathbf{v}_0$ given by

$$\langle \mathbf{v}_0, p \rangle = \int_{-\infty}^{\infty} p(x)d\tilde{\nu}^{(q)}(x) = \int_{-\infty}^{\infty} p(x)\frac{e^{-x^2}}{x^2+q^2}dx,$$

there hold $\mathbf{v}_1 = A_4\mathbf{v}_0$ and $\mathcal{D}[A_4\mathbf{v}_0] = A_3\mathbf{v}_0$. Hence, we obtain from item (a) of Theorem 4.4 that the pair $\{\mathbf{v}_0, \mathbf{v}_1\}$ is a SCPMF2K and the moment functional $\mathbf{v}_0$ is semiclassical of class at most $\mathsf{s} = 2$.

(iii) $P_n^{(0)}$ from a Generalized Hermite-type Moment Functional:

Let the moment functional $\mathbf{v}_0$ be given by

$$\langle \mathbf{v}_0, p \rangle = \int_{-\infty}^{\infty} p(x)|x|^{2\alpha+1}e^{-x^2}dx + \epsilon p(0) = \int_{-\infty}^{\infty} p(x)d\nu^{(\alpha)}(x) + \epsilon p(0),$$

where $\alpha > -1$. This is a generalized Hermite-type moment functional (see [42]).

Using integration by parts one can easily show that

$$\begin{aligned} \langle \mathcal{D}[x^2\mathbf{v}_0]; p \rangle = -\left\langle \mathbf{v}_0, x^2p' \right\rangle &= -\int_{-\infty}^{\infty} p'(x)|x|^{2\alpha+3}e^{-x^2}dx \\ &= \int_{-\infty}^{\infty} p(x)[2x^3 - (2\alpha+3)x]|x|^{2\alpha+1}e^{-x^2}dx. \end{aligned}$$

Thus, $\mathbf{v}_0$ is semiclassical of class $\mathsf{s} = 2$ (see [22]) which satisfies the Pearson equation $\mathcal{D}[A_4\mathbf{v}_0] = A_3\mathbf{v}_0$, where

$$A_4(x) = x^2 \quad \text{and} \quad A_3(x) = 2x^3 - (2\alpha+3)x.$$

Therefore, by item (a) of Theorem 4.4 we have the SCPMF2K $\{\mathbf{v}_0, \mathbf{v}_1\}$, where

$$\langle \mathbf{v}_1, p \rangle = \int_{\infty}^{\infty} p(x)|x|^{2\alpha+3}e^{-x^2}dx \quad \text{and} \quad P_n^{(1)}(x) = \mathcal{P}_n(\nu^{(\alpha+1)}; x), \quad n \geq 0.$$

A_4 is a Constant

Let the moment functional $\mathbf{v}_0$ be given by

$$\langle \mathbf{v}_0, p \rangle = \int_{-\infty}^{\infty} p(x)e^{-x^4}dx = \int_{-\infty}^{\infty} p(x)d\nu_0(x).$$

The corresponding orthogonal polynomials $P_n^{(0)}(x) = \mathcal{P}_n(\nu_0; x)$ satisfy a three-term recurrence relation (see, [64, 65])

$$x\mathcal{P}_n(\nu_0; x) = \mathcal{P}_{n+1}(\nu_0; x) + c_n\mathcal{P}_{n-1}(\nu_0; x), \quad n \geq 1, \tag{4.15}$$

with initial conditions $\mathcal{P}_0(\nu_0; x) = 1$ and $\mathcal{P}_1(\nu_0; x) = x$, where the parameters c_n satisfy a non-linear recurrence relation

$$n = 4c_n(c_{n+1} + c_n + c_{n-1}), \quad n \geq 1,$$

with initial conditions $c_0 = 0$, $c_1 = \Gamma(3/4)\Gamma(1/4)$, where $\Gamma(z)$ denotes the Euler's Gamma function. Furthermore, the sequence of MOP $\{\mathcal{P}_n(\nu_0;\cdot)\}_{n\geq 0}$ satisfies the structure relation

$$\mathcal{P}'_{n+1}(\nu_0;x) = (n+1)\mathcal{P}_n(\nu_0;x) + d_{n+1}\mathcal{P}_{n-2}(\nu_0;x), \quad n \geq 2, \tag{4.16}$$

where $d_{n+1} = 4c_{n+1}c_nc_{n-1}$, $n \geq 2$.

It is also well known that the moment functional $\mathbf{v}_0$ is a symmetric semiclassical moment functional of class $\mathsf{s} = 2$ (see [22]) and satisfies the Pearson equation

$$\mathcal{D}[A_4\mathbf{v}_0] = A_3\mathbf{v}_0,$$

where $A_4(x) = 1$ and $A_3(x) = -4x^3$. We have the case in which $\deg(A_4) = 0$ and $\deg(A_3) = 3$. Thus, we can take without any loss of generality $\mathbf{v}_1 = A_4\mathbf{v}_0 = \mathbf{v}_0$. and by (4.16), we have a self coherent symmetric moment functional of the second kind $\mathbf{v}_0$ such that

$$\frac{P^{(0)\prime}_{n+1}(x)}{n+1} = P^{(0)}_n(x) - \tau_{n-1}P^{(0)}_{n-2}(x), \quad n \geq 2,$$

where $\tau_{n-1} = -d_{n+1}/(n+1)$.

5 Sobolev Orthogonal Polynomials from SCPMF2K on the Real Line

The main objective in this chapter is to consider the sequence of orthogonal polynomials $\{\mathcal{S}_n\}_{n\geq 0}$ with respect to the positive definite Sobolev inner product

$$\langle f, g\rangle_{\mathfrak{S}} = \langle \mathbf{v}_0, fg\rangle + \tilde{\lambda}\,\langle \mathbf{v}_1, f'g'\rangle\,, \quad \tilde{\lambda} > 0, \tag{5.1}$$

where the pair of symmetric positive definite moment functionals $\{\mathbf{v}_0, \mathbf{v}_1\}$ is a SCPMF2K on the real line. The sequence $\{\mathcal{S}_n\}_{n\geq 0}$ is called the sequence of *monic Sobolev orthogonal polynomials* (MSOP for short). We study some properties of this sequence of polynomials and establish the connection formulas that these polynomials satisfy together with the sequence of MOP corresponding to moment functional $\mathbf{v}_0$.

5.1 Introduction

Let $\{\mathbf{v}_0, \mathbf{v}_1\}$ be a pair of symmetric and positive definite moment functional. If $\{\mathbf{v}_0, \mathbf{v}_1\}$ is a SCPMF2K from Proposition 4.3 there exists an admissible pair of polynomials (A_4, A_3) with $\deg(A_4) \leq 4$ and $\deg(A_3) = 3$ such that $\{\mathbf{v}_0, \mathbf{v}_1\}$ satisfy (4.3). Moreover, the constants τ_n in the coherence formula (4.1) satisfy

$$\tau_{n-1} = \frac{\mathscr{h}^{(0)}_{n+1}}{(n+1)\mathscr{h}^{(1)}_{n-2}}[\mathfrak{d}_3 + (n-2)\mathfrak{c}_4], \quad n \geq 2, \tag{5.2}$$

where $\mathfrak{d}_3$ is the leading coefficient of A_3 and $\mathfrak{c}_4$ is coefficient of degree 4 in A_4. Here, $\mathscr{h}^{(0)}_n$ and $\mathscr{h}^{(1)}_n$ are given as in (4.2).

We define the elements of sequences $\{\mathfrak{p}_n\}_{n\geq 1}$ and $\{\mathfrak{q}_n\}_{n\geq 1}$, which will play an important role in the sequel. Let

$$\left.\begin{aligned} \mathfrak{q}_{n+1} &= \tilde{\lambda}(n+1)(n-1)\tau_{n-1}\mathscr{h}^{(1)}_{n-2}, \\ \mathfrak{p}_{n+1} &= \mathscr{h}^{(0)}_{n+1} + \tilde{\lambda}(n+1)^2\Big[\mathscr{h}^{(1)}_{n} + \tau^2_{n-1}\mathscr{h}^{(1)}_{n-2}\Big], \end{aligned}\right\} \quad n \geq 2, \tag{5.3}$$

with $\mathfrak{q}_2 = 0$, $\mathfrak{p}_1 = \mathscr{h}^{(0)}_1 + \tilde{\lambda}\mathscr{h}^{(1)}_0$ and $\mathfrak{p}_2 = \mathscr{h}^{(0)}_2 + 4\tilde{\lambda}\mathscr{h}^{(1)}_1$.

Throughout in this section, we will assume that the moment functionals $\mathbf{v}_0$ and $\mathbf{v}_1$ are symmetric and positive definite. Thus, there exist symmetric positive Borel measures ν_0 and ν_1 supported on infinite subsets of the real line $E_0 \subseteq \mathbb{R}$ and $E_1 \subseteq \mathbb{R}$, respectively, such that

$$\langle \mathbf{v}_0, p\rangle = \int_{E_0} p(x)\, d\nu_0(x) \quad \text{and} \quad \langle \mathbf{v}_1, p\rangle = \int_{E_1} p(x)\, d\nu_1(x), \quad p \in \mathbb{P}.$$

In this case, the Sobolev inner product (5.1) can be rewritten as

$$\langle f, g\rangle_{\mathfrak{S}} = \langle f, g\rangle_{\nu_0} + \tilde{\lambda}\langle f', g'\rangle_{\nu_1}, \quad \tilde{\lambda} > 0, \tag{5.4}$$

where $\langle f, g\rangle_{\nu_0} = \int_{E_0} f(x)g(x)d\nu_0(x)$ and $\langle f, g\rangle_{\nu_1} = \int_{E_1} f(x)g(x)d\nu_1(x)$. Moreover, instead of saying $\{\mathbf{v}_0, \mathbf{v}_1\}$ is a SCPMF2K on the real line we will say $\{\nu_0, \nu_1\}$ is a symmetric coherent pair of positive measures of the second kind on the real line (as abbreviated, SCPPM2K on the real line).

Remark 5.1. The Sobolev inner product (5.1) when the pair of positive definite moment functionals $\{\mathbf{v}_0, \mathbf{v}_1\}$ is CPMF2K on the real line was consider in the recent papers [29] and [40].

The next result follows analogously from Theorem 2.10 which is a result on the unit circle. For the sake of completeness, we also provide its proof.

Theorem 5.2. *Let $\{\nu_0, \nu_1\}$ be a SCPPM2K on the real line. Then the sequence of MSOP $\{\mathcal{S}_n\}_{n\geq 0}$ with respect to the inner product* (5.4) *is such that*

$$\mathcal{S}_0(x) = 1, \quad \mathcal{S}_1(x) = P_1^{(0)}(x), \quad \mathcal{S}_2(x) = P_2^{(0)}(x) \quad \textit{and}$$

$$\left.\begin{aligned} \mathcal{S}_{2n+2}(x) &= P_{2n+2}^{(0)}(x) + \sum_{j=1}^{n} \mathfrak{a}_{2n+2,j} P_{2j}^{(0)}(x), \\ \mathcal{S}_{2n+1}(x) &= P_{2n+1}^{(0)}(x) + \sum_{j=1}^{n} \mathfrak{a}_{2n+1,j} P_{2j-1}^{(0)}(x), \end{aligned}\right\} \quad n \geq 1,$$

Here, $\mathbf{a}_n = [\mathfrak{a}_{2n+2,1}, \mathfrak{a}_{2n+2,2}, \ldots, \mathfrak{a}_{2n+2,n}]^\top$ is the solution of the system of linear equations

$$\mathbf{T}_{2n}\, \mathbf{a}_n = \mathfrak{q}_{2n+2}\, \mathbf{e}_n \quad \textit{and} \quad n \geq 1,$$

where $\mathbf{e}_n$ is the n-th column of the $n \times n$ identity matrix and $\mathbf{T}_{2n}$ is the $n \times n$ real symmetric tridiagonal matrix

$$\mathbf{T}_{2n} = \begin{bmatrix} \mathfrak{p}_2 & -\mathfrak{q}_4 & & & \\ -\mathfrak{q}_4 & \mathfrak{p}_4 & -\mathfrak{q}_6 & & \\ \ddots & \ddots & \ddots & & \\ & & -\mathfrak{q}_{2n-2} & \mathfrak{p}_{2n-2} & -\mathfrak{q}_{2n} \\ & & & -\mathfrak{q}_{2n} & \mathfrak{p}_{2n} \end{bmatrix},$$

$\mathbf{b}_n = [\mathfrak{a}_{2n+1,1}, \mathfrak{a}_{2n+1,2}, \ldots, \mathfrak{a}_{2n+1,n}]^\top$ is the solution of the system of linear equations

$$\mathbf{T}_{2n-1}\, \mathbf{b}_n = \mathfrak{q}_{2n+1}\, \mathbf{e}_n, \quad n \geq 1,$$

where $\mathbf{e}_n$ is the n-th column of the $n \times n$ identity matrix and $\mathbf{T}_{2n-1}$ is the $n \times n$ real symmetric tridiagonal matrix

$$\mathbf{T}_{2n-1} = \begin{bmatrix} \mathfrak{p}_1 & -\mathfrak{q}_3 & & & \\ -\mathfrak{q}_3 & \mathfrak{p}_3 & -\mathfrak{q}_5 & & \\ \ddots & \ddots & \ddots & & \\ & & -\mathfrak{q}_{2n-3} & \mathfrak{p}_{2n-3} & -\mathfrak{q}_{2n-1} \\ & & & -\mathfrak{q}_{2n-1} & \mathfrak{p}_{2n-1} \end{bmatrix}.$$

Proof. If we write

$$\mathcal{S}_{2n+2}(x) = \sum_{j=0}^{n+1} \mathfrak{a}_{2n+2,j} P_{2j}^{(0)}(x) \quad \text{with} \quad \mathfrak{a}_{2n+2,n+1} = 1,$$

then by considering the orthogonality of $\mathcal{S}_{2n+2}$ we can deduce

$$\begin{aligned} 0 &= \langle P_{2k}^{(0)}, \mathcal{S}_{2n+2} \rangle_{\mathfrak{S}} \\ &= \langle P_{2k}^{(0)}, \sum_{j=0}^{n+1} \mathfrak{a}_{2n+2,j} P_{2j}^{(0)} \rangle_{\nu_0} + \tilde{\lambda}\, \langle P_{2k}^{(0)\prime}, \sum_{j=0}^{n+1} \mathfrak{a}_{2n+2,j} P_{2j}^{(0)\prime} \rangle_{\nu_1} \\ &= \mathcal{h}_{2k}^{(0)} \mathfrak{a}_{2n+2,k} + \tilde{\lambda}\, \langle P_{2k}^{(0)\prime}, \sum_{j=0}^{n+1} \mathfrak{a}_{2n+2,j} P_{2j}^{(0)\prime} \rangle_{\nu_1} \end{aligned} \tag{5.5}$$

for $n \geq 0$ and $k = 0, 1, \ldots, n$. From this, with the observation $P_0^{(0)\prime}(x) = 0$ and $P_2^{(0)\prime}(x) = 2P_1^{(0)}(x)$ we find

$$\mathfrak{a}_{2n+2,0} = 0 \quad \text{and} \quad [\mathcal{h}_2^{(0)} + 4\tilde{\lambda}\, \mathcal{h}_1^{(1)}]\mathfrak{a}_{2n+2,1} - 8\tilde{\lambda} \mathcal{h}_1^{(1)} \tau_2 \mathfrak{a}_{2n+2,2} = 0 \quad \text{for} \quad n \geq 0.$$

Using (4.1) and substituting into (5.5) gives

$$\begin{aligned} 0 = \;& -4\tilde{\lambda}\, (k-1)k\tau_{2k-2} \mathcal{h}_{2k-3}^{(1)} \mathfrak{a}_{2n+2,k-1} \\ & + \Big[\mathcal{h}_{2k}^{(0)} + 4\tilde{\lambda} k^2 \left(\mathcal{h}_{2k-1}^{(1)} + \tau_{2k-2}^2\, \mathcal{h}_{2k-3}^{(1)}\right)\Big] \mathfrak{a}_{2n+2,k} - 4\tilde{\lambda}\, k(k+1)\tau_{2k} \mathcal{h}_{2k-1}^{(1)} \mathfrak{a}_{2n+2,k+1} \\ = \;& -\mathfrak{q}_{2k} \mathfrak{a}_{2n+2,k-1} + \mathfrak{p}_{2k} \mathfrak{a}_{2n+2,k} - \mathfrak{q}_{2k+2} \mathfrak{a}_{2n+2,k+1}, \end{aligned}$$

for $k = 2, 3, \ldots, n$. Hence, since $\mathfrak{a}_{2n+2,n+1} = 1$, we obtain the required first system of linear equations.

On the other hand, since $\mathcal{S}_{2n+1}(x)$ is an odd function we can write

$$\mathcal{S}_{2n+1}(x) = \sum_{j=1}^{n+1} \mathfrak{a}_{2n+1,j} P_{2j-1}^{(0)}(x) \quad \text{with} \quad \mathfrak{a}_{2n+1,n+1} = 1,$$

then by considering the orthogonality of $\mathcal{S}_{2n+1}(x)$ we can deduce

$$\begin{aligned} 0 &= \langle P_{2k-1}^{(0)}, \mathcal{S}_{2n+1} \rangle_{\mathfrak{S}} \\ &= \langle P_{2k-1}^{(0)}, \sum_{j=1}^{n+1} \mathfrak{a}_{2n+1,j} P_{2j-1}^{(0)} \rangle_{\nu_0} + \tilde{\lambda}\, \langle P_{2k-1}^{(0)\prime}, \sum_{j=1}^{n+1} \mathfrak{a}_{2n+1,j} P_{2j-1}^{(0)\prime} \rangle_{\nu_1} \\ &= \mathcal{h}_{2k-1}^{(0)} \mathfrak{a}_{2n+1,k} + \tilde{\lambda}\, \langle P_{2k-1}^{(0)\prime}, \sum_{j=1}^{n+1} \mathfrak{a}_{2n+1,j} P_{2j-1}^{(0)\prime} \rangle_{\nu_1} \end{aligned} \tag{5.6}$$

for $n \geq 1$ and $k = 1, 2, \ldots, n$. From this, with the observation $P_1^{(0)\prime}(x) = 1$ we find

$$[\mathcal{h}_1^{(0)} + \tilde{\lambda}\, \mathcal{h}_0^{(1)}]\mathfrak{a}_{2n+1,1} - 3\tilde{\lambda} \mathcal{h}_0^{(1)} \tau_1 \mathfrak{a}_{2n+1,2} = 0 \quad \text{for} \quad n \geq 1.$$

Using (4.1) and substituting into (5.6) gives

$$\begin{aligned} 0 = \;& -\tilde{\lambda}\, (2k-1)(2k-3)\tau_{2k-3} \mathcal{h}_{2k-4}^{(1)} \mathfrak{a}_{2n+1,k-1} + \Big[\mathcal{h}_{2k-1}^{(0)} + \tilde{\lambda}(2k-1)^2 \mathcal{h}_{2k-2}^{(1)} + \\ & \tilde{\lambda}(2k-1)^2 \tau_{2k-3}^2\, \mathcal{h}_{2k-4}^{(1)}\Big] \mathfrak{a}_{2n+1,k} - \tilde{\lambda}(2k-1)(2k+1)\tau_{2k-1} \mathcal{h}_{2k-2}^{(1)} \mathfrak{a}_{2n+1,k+1} \\ = \;& -\mathfrak{q}_{2k-1} \mathfrak{a}_{2n+1,k-1} + \mathfrak{p}_{2k-1} \mathfrak{a}_{2n+1,k} - \mathfrak{q}_{2k+1} \mathfrak{a}_{2n+1,k+1}, \end{aligned}$$

for $k = 2, 4, \ldots, n$. Hence, since $\mathfrak{a}_{2n+1,n+1} = 1$, we obtain the required second system of linear equations. This completes the proof of the theorem. ■

Since $\mathbf{T}_{2n}$ and $\mathbf{T}_{2n-1}$ are tridiagonal matrices, it is easily seen that

$$\begin{aligned} det(\mathbf{T}_{2n}) &= \mathfrak{p}_{2n} det(\mathbf{T}_{2n-2}) - \mathfrak{q}_{2n}^2 det(\mathbf{T}_{2n-4}), \\ det(\mathbf{T}_{2n-1}) &= \mathfrak{p}_{2n-1} det(\mathbf{T}_{2n-3}) - \mathfrak{q}_{2n-1}^2 det(\mathbf{T}_{2n-5}), \end{aligned} \quad n \geq 2,$$

with $det(\mathbf{T}_{-1}) = 1$, $det(\mathbf{T}_0) = 1$, $det(\mathbf{T}_1) = \mathfrak{p}_1$ and $det(\mathbf{T}_2) = \mathfrak{p}_2$. From this,

$$\frac{det(\mathbf{T}_{2n})}{\mathfrak{p}_{2n} det(\mathbf{T}_{2n-2})} \left[1 - \frac{det(\mathbf{T}_{2n+2})}{\mathfrak{p}_{2n+2} det(\mathbf{T}_{2n})}\right] = \frac{\mathfrak{q}_{2n+2}^2}{\mathfrak{p}_{2n}\mathfrak{p}_{2n+2}}, \quad n \geq 1,$$

and

$$\frac{det(\mathbf{T}_{2n-1})}{\mathfrak{p}_{2n-1} det(\mathbf{T}_{2n-3})} \left[1 - \frac{det(\mathbf{T}_{2n+1})}{\mathfrak{p}_{2n+1} det(\mathbf{T}_{2n-1})}\right] = \frac{\mathfrak{q}_{2n+1}^2}{\mathfrak{p}_{2n-1}\mathfrak{p}_{2n+1}}, \quad n \geq 1.$$

The sequences $\{\mathfrak{d}_n^{(2)}\}_{n\geq 1}$ and $\{\mathfrak{d}_n^{(1)}\}_{n\geq 1}$, where

$$\mathfrak{d}_n^{(2)} = \frac{\mathfrak{q}_{2n+2}^2}{\mathfrak{p}_{2n}\mathfrak{p}_{2n+2}} \quad \text{and} \quad \mathfrak{d}_n^{(1)} = \frac{\mathfrak{q}_{2n+1}^2}{\mathfrak{p}_{2n-1}\mathfrak{p}_{2n+1}}, \tag{5.7}$$

for $n \geq 1$, are positive chain sequences. This we confirm later in this chapter. This means the sequence $\{\mathfrak{m}_n^{(2)}\}_{n\geq 1}$, where

$$(1 - \mathfrak{m}_{n-1}^{(2)})\mathfrak{m}_n^{(2)} = \mathfrak{d}_n^{(2)} \quad \text{and} \quad \mathfrak{m}_{n-1}^{(2)} = 1 - \frac{det(\mathbf{T}_{2n})}{\mathfrak{p}_{2n} det(\mathbf{T}_{2n-2})} \quad n \geq 1, \tag{5.8}$$

is the minimal parameter of $\{\mathfrak{d}_n^{(2)}\}_{n\geq 1}$, and the sequence $\{\mathfrak{m}_n^{(1)}\}_{n\geq 1}$, where

$$(1 - \mathfrak{m}_{n-1}^{(1)})\mathfrak{m}_n^{(1)} = \mathfrak{d}_n^{(1)} \quad \text{and} \quad \mathfrak{m}_{n-1}^{(1)} = 1 - \frac{det(\mathbf{T}_{2n-1})}{\mathfrak{p}_{2n-1} det(\mathbf{T}_{2n-3})} \quad n \geq 1, \tag{5.9}$$

is the minimal parameter of $\{\mathfrak{d}_n^{(1)}\}_{n\geq 1}$. Therefore,

$$\begin{aligned} 0 &< \frac{det(\mathbf{T}_{2n})}{\mathfrak{p}_{2n} det(\mathbf{T}_{2n-2})} = 1 - \mathfrak{m}_{n-1}^{(2)} < 1, \\ 0 &< \frac{det(\mathbf{T}_{2n-1})}{\mathfrak{p}_{2n-1} det(\mathbf{T}_{2n-3})} = 1 - \mathfrak{m}_{n-1}^{(1)} < 1, \end{aligned} \quad n \geq 2.$$

Remark 5.3. Note that the matrices $\mathbf{T}_{2n}$ and $\mathbf{T}_{2n-1}$ are positive definite. This fact guarantees the existence of a unique solution for the systems $\mathbf{T}_{2n}\,\mathbf{a}_n = \mathfrak{q}_{2n+2}\,\mathbf{e}_n$ and $\mathbf{T}_{2n-1}\,\mathbf{b}_n = \mathfrak{q}_{2n+1}\,\mathbf{e}_n$. Consequently, one can use this to verify the existence of the sequence of MSOP $\{\mathcal{S}_n\}_{n\geq 0}$ with respect to the inner product (5.4).

5.2 The Simple Connection Formulas

The following result states that two consecutive Sobolev polynomials are connected to an orthogonal polynomial with respect to the measure ν_0.

Theorem 5.4. *Let $\{\nu_0, \nu_1\}$ be a SCPPM2K on the real line. Then the sequence of MSOP $\{\mathcal{S}_n\}_{n\geq 0}$ with respect to the inner product* (5.4) *satisfies the simple connection formulas*

$$\begin{aligned} \mathcal{S}_{2n}(x) - \gamma_{2n-2}\mathcal{S}_{2n-2}(x) &= P_{2n}^{(0)}(x), \\ \mathcal{S}_{2n-1}(x) - \gamma_{2n-3}\mathcal{S}_{2n-3}(x) &= P_{2n-1}^{(0)}(x), \end{aligned} \qquad n \geq 1, \tag{5.10}$$

where $\mathcal{S}_0(x) = 1$ and γ_n are such that

$$\gamma_{-1} = 0, \quad \gamma_0 = 0, \quad \gamma_{2n} = \frac{\mathfrak{q}_{2n+2}}{\mathfrak{p}_{2n} - \mathfrak{q}_{2n}\,\gamma_{2n-2}} \quad \text{and} \quad \gamma_{2n-1} = \frac{\mathfrak{q}_{2n+1}}{\mathfrak{p}_{2n-1} - \mathfrak{q}_{2n-1}\,\gamma_{2n-3}}, \tag{5.11}$$

for $n \geq 1$. Here, the elements $\mathfrak{p}_n$ and $\mathfrak{q}_n$ are as in (5.3).

Proof. Let us consider the expansion

$$P_n^{(0)}(x) = \mathcal{S}_n(x) + \sum_{j=0}^{n-2} a_{n,j}\mathcal{S}_j(x), \quad n \geq 3,$$

where $\mathcal{S}_0(x) = 1$, $\mathcal{S}_1(x) = x$, $\mathcal{S}_2(x) = P_2^{(0)}(x)$ and

$$a_{n,j} = \frac{\langle P_n^{(0)}, \mathcal{S}_j\rangle_{\mathfrak{S}}}{\langle \mathcal{S}_j, \mathcal{S}_j\rangle_{\mathfrak{S}}} = \frac{\langle P_n^{(0)}, \mathcal{S}_j\rangle_{\nu_0} + \tilde{\lambda}\langle P_n^{(0)\prime}, \mathcal{S}_j'\rangle_{\nu_1}}{\langle \mathcal{S}_j, \mathcal{S}_j\rangle_{\mathfrak{S}}} = \tilde{\lambda}\frac{\langle P_n^{(0)\prime}, \mathcal{S}_j'\rangle_{\nu_1}}{\langle \mathcal{S}_j, \mathcal{S}_j\rangle_{\mathfrak{S}}},$$

for $j = 0, 1, \ldots, n-2$ and $n \geq 3$. Hence, substituting $P_n^{(0)\prime}$ using (4.1) we get $a_{n,j} = 0$ for $n \geq 3$ and $j = 0, 1, \ldots, n-3$. Moreover,

$$-\gamma_{n-2} = a_{n,n-2} = \tilde{\lambda}\frac{\langle P_n^{(0)\prime}, \mathcal{S}_{n-2}'\rangle_{\nu_1}}{\langle \mathcal{S}_{n-2}, \mathcal{S}_{n-2}\rangle_{\mathfrak{S}}}, \quad n \geq 3.$$

Thus,

$$\gamma_{-1} = 0, \quad \gamma_0 = 0, \quad \gamma_{2n} = \frac{\mathfrak{q}_{2n+2}}{\langle \mathcal{S}_{2n}, \mathcal{S}_{2n}\rangle_{\mathfrak{S}}} \quad \text{and} \quad \gamma_{2n-1} = \frac{\mathfrak{q}_{2n+1}}{\langle \mathcal{S}_{2n-1}, \mathcal{S}_{2n-1}\rangle_{\mathfrak{S}}}, \quad n \geq 1. \tag{5.12}$$

With the observation that $\mathcal{S}_k$ is monic, we have

$$\begin{aligned} \langle \mathcal{S}_k, \mathcal{S}_k\rangle_{\mathfrak{S}} &= \langle \mathcal{S}_k, P_k^{(0)}\rangle_{\mathfrak{S}} \\ &= \mathfrak{h}_k^{(0)} + \tilde{\lambda}\,\langle \mathcal{S}_k', P_k^{(0)\prime}\rangle_{\nu_1}. \end{aligned}$$

Thus, from $\mathcal{S}_k'(x) = P_k^{(0)\prime}(x) + \gamma_{k-2}\,\mathcal{S}_{k-2}'(x)$ and (4.1),

$$\begin{aligned} \langle \mathcal{S}_k, \mathcal{S}_k\rangle_{\mathfrak{S}} &= \mathfrak{h}_k^{(0)} + \tilde{\lambda}\left[k^2\mathfrak{h}_{k-1}^{(1)} + k^2\tau_{k-2}^2\mathfrak{h}_{k-3}^{(1)}\right] - \tilde{\lambda}\,(k-2)k\tau_{k-2}\mathfrak{h}_{k-3}^{(1)}\gamma_{k-2} \\ &= \mathfrak{p}_k - \mathfrak{q}_k\,\gamma_{k-2}, \end{aligned} \tag{5.13}$$

which gives the statement of the theorem, when $k = 2n$ and $k = 2n-1$. ∎

The next theorem gives an upper bound for the ratio γ_n/τ_n.

Theorem 5.5. *The connection coefficients γ_n that appear in* (5.10) *and the coefficients τ_n that appear in* (5.2) *satisfy*

$$0 \; < \; \frac{\gamma_n}{\tau_n} \; < \; \frac{n+2}{n}, \quad n \geq 1.$$

Proof. Observe that (5.12) gives $\mathfrak{q}_{n+2}\gamma_n > 0$ for $n \geq 1$. Hence, from (5.3)

$$\tau_n \gamma_n > 0, \quad n \geq 1.$$

Now by combining (5.4) with the minimal norm properties of MOP, one finds

$$\langle \mathcal{S}_n, \mathcal{S}_n \rangle_{\mathfrak{S}} \; > \; \langle P_n^{(0)}, P_n^{(0)} \rangle_{\nu_0} + \tilde{\lambda} n^2 \, \langle P_{n-1}^{(1)}, P_{n-1}^{(1)} \rangle_{\nu_1}, \quad n \geq 1.$$

Then, from (5.13) we have

$$\tilde{\lambda}\, n^2 \tau_{n-2}^2 \mathcal{h}_{n-3}^{(1)} \; > \; \tilde{\lambda}\, (n-2) n \tau_{n-2} \mathcal{h}_{n-3}^{(1)} \gamma_{n-2}, \quad n \geq 3.$$

This gives the inequality result of the theorem. ∎

Now, by using (5.11) one can write

$$\begin{aligned} (1 - \mathfrak{g}_{n-1}^{(2)})\mathfrak{g}_n^{(2)} &= \mathfrak{d}_n^{(2)}, \\ (1 - \mathfrak{g}_{n-1}^{(1)})\mathfrak{g}_n^{(1)} &= \mathfrak{d}_n^{(1)}, \end{aligned} \qquad n \geq 1,$$

where

$$\mathfrak{g}_{n-1}^{(2)} = \frac{\mathfrak{q}_{2n}\gamma_{2n-2}}{\mathfrak{p}_{2n}}, \quad \mathfrak{g}_{n-1}^{(1)} = \frac{\mathfrak{q}_{2n-1}\gamma_{2n-3}}{\mathfrak{p}_{2n-1}}, \quad \mathfrak{d}_n^{(2)} = \frac{\mathfrak{q}_{2n+2}^2}{\mathfrak{p}_{2n+2}\mathfrak{p}_{2n}} \quad \text{and} \quad \mathfrak{d}_n^{(1)} = \frac{\mathfrak{q}_{2n+1}^2}{\mathfrak{p}_{2n+1}\mathfrak{p}_{2n-1}},$$

for $n \geq 1$. Note that $\{\mathfrak{d}_n^{(2)}\}_{n\geq 1}$ and $\{\mathfrak{d}_n^{(1)}\}_{n\geq 1}$ are the same sequences given by (5.7). Thus, from the knowledge that $\mathfrak{m}_0^{(2)} = \mathfrak{m}_0^{(1)} = 0$, $\mathfrak{m}_n^{(2)} > 0$, $\mathfrak{m}_n^{(1)} > 0$, $\mathfrak{d}_n^{(2)} > 0$ and $\mathfrak{d}_n^{(1)} > 0$ for $n \geq 1$, there also holds

$$0 < \mathfrak{m}_n^{(2)} < 1 \quad \text{and} \quad 0 < \mathfrak{m}_n^{(1)} < 1, \quad n \geq 1.$$

Hence, the sequences $\{\mathfrak{d}_n^{(2)}\}_{n\geq 1}$ and $\{\mathfrak{d}_n^{(1)}\}_{n\geq 1}$ are positive chain sequence and $\{\mathfrak{m}_n^{(2)}\}_{n\geq 0}$ and $\{\mathfrak{m}_n^{(1)}\}_{n\geq 0}$, respectively, are its minimal parameter sequences.

Comparing the above minimal parameter sequences with the minimal parameter sequences given by (5.8) and (5.9) we have

$$\frac{\mathfrak{q}_{2n}\gamma_{2n-2}}{\mathfrak{p}_{2n}} = 1 - \frac{det(\mathbf{T}_{2n})}{\mathfrak{p}_{2n} det(\mathbf{T}_{2n-2})} \quad \text{and} \quad \frac{\mathfrak{q}_{2n-1}\gamma_{2n-3}}{\mathfrak{p}_{2n-1}} = 1 - \frac{det(\mathbf{T}_{2n-1})}{\mathfrak{p}_{2n-1} det(\mathbf{T}_{2n-3})} \quad n \geq 1.$$

5.3 The Coefficients γ_n as Rational Functions

In this section, we will show that the connection coefficients γ_n that appear in (5.10), can be expressed as rational functions involving a sequence of polynomials that satisfy a simple TTRR. Let $\{\widetilde{\mathcal{R}}_n(t)\}_{n\geq 0}$ and $\{\widetilde{\mathcal{A}}_n(t)\}_{n\geq 0}$ such that

$$\gamma_{2n} = \frac{\widetilde{\mathcal{R}}_{n-1}(t)}{\widetilde{\mathcal{R}}_n(t)} \quad \text{and} \quad \gamma_{2n-1} = \frac{\widetilde{\mathcal{A}}_{n-1}(t)}{\widetilde{\mathcal{A}}_n(t)}, \quad n \geq 1, \tag{5.14}$$

where $\widetilde{\mathcal{R}}_0(t) = \widetilde{\mathcal{A}}_0(t) = 1$ and $t = \kappa/\tilde{\lambda}$. For the moment we assume $\kappa \neq 0$ to be arbitrary. Then from (5.11)

$$\begin{aligned} \mathfrak{q}_{2n+2}\widetilde{\mathcal{R}}_n(t) &= \mathfrak{p}_{2n}\widetilde{\mathcal{R}}_{n-1}(t) - \mathfrak{q}_{2n}\widetilde{\mathcal{R}}_{n-2}(t), \\ \mathfrak{q}_{2n+1}\widetilde{\mathcal{A}}_n(t) &= \mathfrak{p}_{2n-1}\widetilde{\mathcal{A}}_{n-1}(t) - \mathfrak{q}_{2n-1}\widetilde{\mathcal{A}}_{n-2}(t), \end{aligned} \quad n \geq 2. \tag{5.15}$$

with $\mathfrak{q}_4\widetilde{\mathcal{R}}_1(t) = \mathfrak{p}_2\widetilde{\mathcal{R}}_0(t)$ and $\mathfrak{q}_3\widetilde{\mathcal{A}}_1(t) = \mathfrak{p}_1\widetilde{\mathcal{A}}_0(t)$. Thus, from (5.3) we can state the following theorems.

Theorem 5.6. *For $n \geq 0$, $\widetilde{\mathcal{R}}_n(t)$ in (5.14) is a polynomial in t of exact degree n. Precisely, if $\mathcal{R}_n(t) = \kappa^n \sigma_n \widetilde{\mathcal{R}}_n(t)$, $n \geq 0$, with*

$$\sigma_0 = 1, \quad \sigma_n = 4n(n+1)\frac{\tau_{2n}\hbar^{(1)}_{2n-1}}{\hbar^{(0)}_{2n}}\sigma_{n-1}, \quad n \geq 1,$$

then $\mathcal{R}_n(t)$ are monic polynomials such that

$$\mathcal{R}_{n+1}(t) = [t + 4\kappa(\mathfrak{a}_{n+1} + \mathfrak{b}_{n+1})]\mathcal{R}_n(t) - 16\kappa^2\mathfrak{a}_n\mathfrak{b}_{n+1}\mathcal{R}_{n-1}(t), \quad n \geq 1, \tag{5.16}$$

with $\mathcal{R}_0(t) = 1$ and $\mathcal{R}_1(t) = t + 4\kappa\mathfrak{a}_1$, where

$$\mathfrak{a}_n = \frac{n^2\hbar^{(1)}_{2n-1}}{\hbar^{(0)}_{2n}}, \quad \mathfrak{b}_{n+1} = \frac{(n+1)^2\tau^2_{2n}\hbar^{(1)}_{2n-1}}{\hbar^{(0)}_{2n+2}}, \quad n \geq 1.$$

Proof. Manipulating the formulas in (5.3) and substituting in (5.15) gives

$$\frac{t\mathfrak{q}_4}{\hbar^{(0)}_2}\widetilde{\mathcal{R}}_1(t) = t + \frac{4\kappa\hbar^{(1)}_1}{\hbar^{(0)}_2}$$

$$\frac{t\mathfrak{q}_{2n+2}}{\hbar^{(0)}_{2n}}\widetilde{\mathcal{R}}_n(t) = \left[t + \frac{4n^2\kappa}{\hbar^{(0)}_{2n}}(\hbar^{(1)}_{2n-1} + \tau^2_{2n-2}\hbar^{(1)}_{2n-3})\right]\widetilde{\mathcal{R}}_{n-1}(t) - \frac{\hbar^{(0)}_{2n-2}}{\hbar^{(0)}_{2n}}\frac{t\mathfrak{q}_{2n}}{\hbar^{(0)}_{2n-2}}\widetilde{\mathcal{R}}_{n-2}(t), \quad n \geq 2.$$

Since $t\mathfrak{q}_{2n+2}$ for $n \geq 1$ are independent of t, with

$$\kappa^n\sigma_n = \prod_{k=1}^{n}\left[\frac{t\mathfrak{q}_{2k+2}}{\hbar^{(0)}_{2k}}\right], \quad n \geq 1,$$

we obtain the required recurrence relation for $\{\mathcal{R}_n\}_{n\geq 0}$. From this recurrence relation one can also observe that $\mathcal{R}_n$ are monic polynomials in t. ∎

Theorem 5.7. *For $n \geq 0$, $\widetilde{\mathcal{A}}_n(t)$ in (5.14) is a polynomial in t of exact degree n. Precisely, if $\mathcal{A}_n(t) = \kappa^n\tilde{\sigma}_n\widetilde{\mathcal{A}}_n(t)$, $n \geq 0$, with*

$$\tilde{\sigma}_0 = 1, \quad \tilde{\sigma}_n = (2n+1)(2n-1)\frac{\tau_{2n-1}\hbar^{(1)}_{2n-2}}{\hbar^{(0)}_{2n-1}}\tilde{\sigma}_{n-1}, \quad n \geq 1,$$

then $\mathcal{A}_n(t)$ *are monic polynomials such that*

$$\mathcal{A}_{n+1}(t) = [t + \kappa(e_{n+1} + f_{n+1})]\mathcal{A}_n(t) - \kappa^2 e_n f_{n+1}\mathcal{A}_{n-1}(t), \quad n \geq 1, \tag{5.17}$$

with $\mathcal{A}_0(t) = 1$ *and* $\mathcal{A}_1(t) = t + \kappa e_1$*, where*

$$e_n = \frac{(2n-1)^2 \mathfrak{h}^{(1)}_{2n-2}}{\mathfrak{h}^{(0)}_{2n-1}}, \quad f_{n+1} = \frac{(2n+1)^2 \tau^2_{2n-1}\mathfrak{h}^{(1)}_{2n-2}}{\mathfrak{h}^{(0)}_{2n+1}}, \quad n \geq 1.$$

Proof. Manipulating the formulas in (5.3) and substituting in (5.15) gives

$$\frac{t\mathfrak{q}_3}{\mathfrak{h}^{(0)}_1}\widetilde{\mathcal{A}}_1(t) = t + \frac{\kappa \mathfrak{h}^{(1)}_0}{\mathfrak{h}^{(0)}_1}$$

$$\frac{t\mathfrak{q}_{2n+1}}{\mathfrak{h}^{(0)}_{2n-1}}\widetilde{\mathcal{A}}_n(t) = \Big[t + \frac{\kappa(2n-1)^2}{\mathfrak{h}^{(0)}_{2n-1}}(\mathfrak{h}^{(1)}_{2n-2} + \tau^2_{2n-3}\mathfrak{h}^{(1)}_{2n-4})\Big]\widetilde{\mathcal{A}}_{n-1}(t) - \frac{\mathfrak{h}^{(0)}_{2n-3}}{\mathfrak{h}^{(0)}_{2n-1}}\frac{t\mathfrak{q}_{2n-1}}{\mathfrak{h}^{(0)}_{2n-3}}\widetilde{\mathcal{A}}_{n-2}(t), \quad n \geq 2.$$

Since $t\mathfrak{q}_{2n+1}$ for $n \geq 1$ are independent of t, with

$$\kappa^n \widetilde{\sigma}_n = \prod_{k=1}^{n}\left[\frac{t\mathfrak{q}_{2k+1}}{\mathfrak{h}^{(0)}_{2k-1}}\right], \quad n \geq 1,$$

we obtain the required recurrence relation for $\{\mathcal{A}_n\}_{n\geq 0}$. From this recurrence relation one can also observe that $\mathcal{A}_n$ are monic polynomials in t. ∎

Remark 5.8. Since the elements of TTRR (5.16) and (5.17) are such that $\mathfrak{a}_n\mathfrak{b}_{n+1} > 0$ and $e_n f_{n+1} > 0$ for $n \geq 1$, by using Favard's Theorem 1.19 we can conclude that $\{\mathcal{R}_n\}_{n\geq 0}$ and $\{\mathcal{A}_n\}_{n\geq 0}$ are sequences of MOP with respect to some positive measures supported on the real line.

5.4 An Example

Let $\{\mathcal{P}_n(\nu_0;\cdot)\}_{n\geq 0}$ the sequence of MOP with respect to the Freud weight function e^{-x^4}, that is

$$\langle p, q\rangle_{\nu_0} = \int_{-\infty}^{\infty} p(x)q(x)e^{-x^4}dx = \int_{-\infty}^{\infty} p(x)q(x)d\nu_0(x).$$

According to the Section 4.3 when A_4 is constant the pair $\{\nu_0, \nu_0\}$ is a SCPPM2K.

Now we consider the Sobolev inner product $\langle f, g\rangle_{\mathfrak{S}}$ given by

$$\langle f, g\rangle_{\mathfrak{S}} = \langle f, g\rangle_{\nu_0} + \tilde{\lambda}\langle f', g'\rangle_{\nu_0}, \quad \tilde{\lambda} > 0, \tag{5.18}$$

this inner product has been introduced in [12] and algebraic and asymptotic properties of $\{\mathcal{S}_n\}_{n\geq 0}$ sequence of MSOP with respect to (5.18) were obtained. Later on, in [63] was proved that the polynomials $\{\mathcal{S}_n\}_{n\geq 0}$ have all their zeros real and simple.

From Theorem 5.4 the sequence of MSOP $\{\mathcal{S}_n\}_{n\geq 0}$ satisfies the connection formula (see [12])

$$\mathcal{S}_n(x) - \gamma_{n-2}\mathcal{S}_{n-2}(x) = \mathcal{P}_n(\nu; x), \quad n \geq 3,$$

where

$$-\gamma_{n-2} = 4\tilde{\lambda}(n-2)\frac{\int_{-\infty}^{\infty}\mathcal{P}^2_n(\nu;x)e^{-x^4}dx}{\langle \mathcal{S}_{n-2}, \mathcal{S}_{n-2}\rangle_{\mathfrak{S}}} > 0,$$

and $\mathcal{S}_i(x) = \mathcal{P}_i(\nu; x)$, $i = 0, 1, 2$. Moreover, the connection coefficients γ_n satisfy a non-linear recurrence relation in terms of c_n given in (4.15), that is

$$-\gamma_n = \frac{4\tilde{\lambda} n c_{n+2} c_{n+1} c_n}{c_n + \tilde{\lambda}[n^2 + 16c_n^2 c_{n-1} c_{n-2}] + 4(n-2)\tilde{\lambda} c_n \gamma_{n-2}}, \quad n \geq 3,$$

with $\gamma_1 = -\dfrac{4\tilde{\lambda} c_3 c_2 c_1}{c_1 + \tilde{\lambda}}$ and $\gamma_2 = -\dfrac{8\tilde{\lambda} c_4 c_3 c_2}{c_2 + 4\tilde{\lambda}}$.

By using Theorems 5.6 and 5.7 we can write the connection coefficients γ_n as rational functions which are related to a sequence of MOP satisfying a simple TTRR (see [12]). Let $\kappa \neq 0$ be an arbitrary real number such that $t = \kappa/\tilde{\lambda}$ and write

$$\gamma_{2n} = \frac{\kappa \sigma_n}{\sigma_{n-1}} \frac{\mathcal{R}_{n-1}(t)}{\mathcal{R}_n(t)}, \quad \gamma_{2n-1} = \frac{\kappa \widetilde{\sigma}_n}{\widetilde{\sigma}_{n-1}} \frac{\mathcal{A}_{n-1}(t)}{\mathcal{A}_n(t)}, \quad n \geq 1,$$

where

$$\sigma_0 = \widetilde{\sigma}_0 = 1, \quad \sigma_n = -8n c_{2n+2} c_{2n+1} \sigma_{n-1} \quad \text{and} \quad \widetilde{\sigma}_n = -4(2n-1) c_{2n+1} c_{2n} \widetilde{\sigma}_{n-1}, \quad n \geq 1.$$

Then $\{\mathcal{R}_n\}_{n\geq 0}$ and $\{\mathcal{A}_n\}_{n\geq 0}$ are sequences of MOP satisfying the TTRR

$$\begin{aligned} \mathcal{R}_{n+1}(t) &= [t + 4\kappa(\mathfrak{a}_{n+1} + \mathfrak{b}_{n+1})]\mathcal{R}_n(t) - 16\kappa^2 \mathfrak{a}_n \mathfrak{b}_{n+1} \mathcal{R}_{n-1}(t), \\ \mathcal{A}_{n+1}(t) &= [t + \kappa(e_{n+1} + f_{n+1})]\mathcal{A}_n(t) - \kappa^2 e_n f_{n+1} \mathcal{A}_{n-1}(t), \end{aligned} \qquad n \geq 1.$$

with $\mathcal{R}_0(t) = \mathcal{A}_0(t) = 1$, $\mathcal{R}_1(t) = t + 4\kappa\mathfrak{a}_1$ and $\mathcal{A}_1(t) = t + \kappa e_1$, where

$$\begin{aligned} \mathfrak{a}_n &= \frac{n^2}{c_{2n}}, \quad \mathfrak{b}_{n+1} = 4c_{2n+2} c_{2n+1} c_{2n}, \\ e_n &= \frac{(2n-1)^2}{c_{2n-1}}, \quad f_{n+1} = 16 c_{2n+1} c_{2n} c_{2n-1}, \end{aligned} \qquad n \geq 1.$$

6 Concluding Remarks and Future Work

Conclusions

The topics studied in this work have been focussed to study a concept known as "coherent pairs of measures of the second kind" both on the unit circle and on the real line. The main results established can be summarized as follows:

• We have presented an extension to the study of the concept of coherence of the second kind on the unit circle, where in the formula that defines the concept of coherence we replace the derivative operator by a q-difference operator. Specifically, we have considered a special pair of measures on the unit circle $\{\widetilde{\mu}_q^{(b,\epsilon)}, \mu_q^{(b+1)}\}$ for which the corresponding sequences of orthogonal polynomials $\{\widetilde{\Phi}_n^{(b,\epsilon)}(q;\cdot)\}_{n\geq 0}$ and $\{\Phi_n^{(b+1)}(q;\cdot)\}_{n\geq 0}$ satisfy

$$D_q[\widetilde{\Phi}_{n+1}^{(b,\epsilon)}(q;z)] = \{n+1\}_q \left[\Phi_n^{(b+1)}(q;z) - \tau_n \Phi_{n-1}^{(b+1)}(q;z)\right], \quad \tau_n \neq 0, \quad n \geq 1,$$

where $D_q[F(z)] = \frac{F(q^{-1/2}z)-F(q^{1/2}z)}{q^{-1/2}z-q^{1/2}z}$ and $\{n\}_q$ is such that $D_q[z^n] = \{n\}_q\, z^{n-1}$. The probability measures $\mu_q^{(b+1)}$ and $\widetilde{\mu}_q^{(b,\epsilon)}$ where $b = \lambda + i\eta$ and $0 < q < 1$, are defined as follows:

(i) $\mu_q^{(b+1)}$ is the probability measure given by

$$d\mu_q^{(b+1)}(\zeta) = \frac{1}{i2\pi\zeta}\tau_q^{(b+1)} \frac{|(q^{1/2}\zeta;\, q)_\infty|^2}{|(q^{b+3/2}\zeta;\, q)_\infty|^2} d\zeta,$$

with

$$\tau_q^{(b+1)} = \frac{(q;\, q)_\infty (q^{b+\bar{b}+3};\, q)_\infty}{(q^{b+2};\, q)_\infty (q^{\bar{b}+2};\, q)_\infty} = \frac{|\Gamma_q(b+2)|^2}{\Gamma_q(b+\bar{b}+3)}.$$

(ii) $\widetilde{\mu}_q^{(b,\epsilon)}$ is such that

$$\left\langle f, g\right\rangle_{\widetilde{\mu}_q^{(b,\epsilon)}} = (1-\epsilon)\tilde{\tau}_q^{(b)} \int_{\mathbb{T}} f(\zeta)\overline{g(\zeta)}\, \frac{|(q\zeta;q)_\infty|^2}{|(q^{b+1}\zeta;q)_\infty|^2} \frac{1}{i2\pi\zeta}\, d\zeta \;+\; \epsilon\, f(1)\overline{g(1)},$$

where $0 \leq \epsilon < 1$ and

$$\tilde{\tau}_q^{(b)} = \frac{(1-q^{\bar{b}+1})(1-q^{\lambda+1}\cos\eta_q)(q;q)_\infty (q^{2\lambda+2};q)_\infty}{(1-q^{\lambda+1}\cos\eta_q)\, {}_2\phi_1(q,\, q^{-b};\, q^{\bar{b}+2};\, q,\, q^{b+1})} \frac{1}{|(q^{b+1};q)_\infty|^2}, \quad \eta_q = -\eta\ln(q).$$

• Motivated by this extension of the concept of coherence of the second kind on the unit circle, we have provided an extensive study of properties of Sobolev-type orthogonal polynomials $\Psi_n^{(b,\epsilon,s)}(q;z)$ with respect to Sobolev-type inner product

$$\left\langle f,g\right\rangle_{\mathfrak{S}_q^{(b,\epsilon,s)}} = \left\langle f,g\right\rangle_{\widetilde{\mu}_q^{(b,\epsilon)}} + s\left\langle D_q[f], D_q[g]\right\rangle_{\mu_q^{(b+1)}}, \quad s>0. \tag{6.1}$$

• By considering measures defined on the real line, we have established a characterization of pairs of positive measures that satisfy the concept of coherence of the second kind on the real line. We showed that a pair of positive measures $\{\nu_0,\nu_1\}$ supported on the real line is a *coherent pair of positive measures of the second kind* (CPPM2K) if and only if there exists an admissible pair of polynomials (A_3,A_2), with $\deg(A_3)\le 3$ and $\deg(A_2)=2$, such that

$$\begin{aligned} -\int p'(x)d\nu_1(x) &= \int p(x)A_2(x)d\nu_0(x) \quad \text{and} \\ \int p(x)d\nu_1(x) &= \int p(x)A_3(x)d\nu_0(x), \end{aligned} \tag{6.2}$$

for every polynomial p. Recall that the admissibility condition of (A_3,A_2) holds if $\mathfrak{d}_2+n\mathfrak{c}_3\neq 0$ for $n\ge 0$, where $\mathfrak{d}_2$ is the leading coefficient of A_2 and $\mathfrak{c}_3$ is coefficient of degree 3 in A_3. The characterization formula (6.2) shows that ν_0 and ν_1 are semiclassical positive measures of class at most $\mathsf{s}=1$. Many illustrative examples has been also given.

• An extension of the concept of coherent pairs of measures on the real line where the measures are assumed to be symmetric has also been treated. We showed that a pair of symmetric positive measures $\{\nu_0,\nu_1\}$ supported on the real line is a *symmetric coherent pair of positive measures of the second kind* (SCPPM2K) if and only if there exists an admissible pair of polynomials (A_4,A_3), with $\deg(A_4)\le 4$ and $\deg(A_3)=3$, such that

$$\begin{aligned} -\int p'(x)d\nu_1(x) &= \int p(x)A_3(x)d\nu_0(x) \quad \text{and} \\ \int p(x)d\nu_1(x) &= \int p(x)A_4(x)d\nu_0(x), \end{aligned} \tag{6.3}$$

for every polynomial p. Recall that the admissibility condition of (A_4,A_3) holds if $\mathfrak{d}_3+(n-1)\mathfrak{c}_4\neq 0$ for $n\ge 1$, where $\mathfrak{d}_3$ is the leading coefficient of A_3 and $\mathfrak{c}_4$ is coefficient of degree 4 in A_4. The characterization formula (6.3) shows that ν_0 and ν_1 are semiclassical positive measures of class at most $\mathsf{s}=2$. Many illustrative examples have also been given.

• The results obtained in this work also contribute to the study of the monic orthogonal polynomials $\mathcal{S}_n(\nu_0,\nu_1;x)$ with respect to Sobolev inner product

$$\langle f,g\rangle_{\mathfrak{S}} = \int f(x)g(x)d\nu_0(x) + \mathcal{s}\int f'(x)g'(x)d\nu_1(x), \quad \mathcal{s}>0,$$

where the pair of positive measures $\{\nu_0,\nu_1\}$ is a coherent pair or symmetric coherent pair of positive measures of second kind on the real line. Some properties of Sobolev orthogonal polynomials $\mathcal{S}_n(\nu_0,\nu_1;x)$ have been analyzed in the symmetric case.

As mentioned in the Introduction of this work, some of the main results contained in this thesis have also appeared in the following texts, which we have also listed within the bibliography at the end of this thesis:

[29] M. Hancco Suni, G. A. Marcato, F. Marcellán, A. Sri Ranga, Coherent pairs of moment functionals of the second kind and associated orthogonal polynomials and Sobolev orthogonal polynomials, *J. Math. Anal. Appl.*, **525** (2023), 127118. DOI: https://doi.org/10.1016/j.jmaa.2023.127118

[30] M. Hancco Suni, F. Marcellán, A. Sri Ranga, Pastro polynomials and Sobolev-type orthogonal polynomials on the unit circle based on a q-difference operator, *J. Differ. Equ. Appl.*, **29** (2023), 315–343. DOI: https://doi.org/10.1080/10236198.2023.2198041

Future Work

Here we discuss some future directions of research that we plan to undertake.

Zeros of Sobolev-Type OPUC Based on the D_q Operator

An interesting question for a future work is to analyze, for a fixed n the dynamical behavior of the zeros of the orthogonal polynomial $\Psi_n^{(b,\epsilon,s)}(q;z)$ with respect to Sobolev-type inner product (6.1) in terms of the parameters b, ϵ and s.

Associated Fourier Approximation

One of our object of studies for the near future is to carry out an analysis of the Fourier expansions of functions in terms of the sequences of polynomials orthogonal with respect to the Sobolev inner product $\langle \cdot\,, \cdot \rangle_{\mathfrak{S}}$ when $\{\nu_0, \nu_1\}$ is a CPPM2K on the real line.

To be more precise, assume f to be a function such that $\langle f, x^j \rangle_{\mathfrak{S}}$ exist for $0 \leq j \leq N$. Let $Q_N^{\mathfrak{S}}$ denote the best approximation of f by a polynomial of degree N with respect to the norm $\|f\|_{\mathfrak{S}} = (\langle f, f \rangle_{\mathfrak{S}})^{1/2}$. In other words,

$$\|f - Q_N^{\mathfrak{S}}\|_{\mathfrak{S}} = \min_{\pi \in \mathbb{P}_N} \|f - \pi\|_{\mathfrak{S}},$$

where $\mathbb{P}_N$ denotes the set of all polynomials of degree N with real coefficients.

Let $\{\hbar_n^{\mathfrak{S}}\}_{n \geq 0}$ be the positive constants such that

$$\langle \mathcal{S}_m, \mathcal{S}_n \rangle_{\mathfrak{S}} = \hbar_n^{\mathfrak{S}} \delta_{m,n}, \quad m, n = 0, 1, 2, \ldots.$$

By least squares method one finds

$$Q_N^{\mathfrak{S}}(x) = \sum_{j=0}^{N} b_j^{\mathfrak{S}} \mathcal{S}_j(x),$$

where

$$b_j^{\mathfrak{S}} = \frac{\langle f, \mathcal{S}_j \rangle_{\mathfrak{S}}}{\hbar_j^{\mathfrak{S}}}, \quad j = 0, 1, 2, \ldots, N.$$

We observe that $\hbar_j^{\mathfrak{S}}$ and $\langle f, \mathcal{S}_j \rangle_{\mathfrak{S}}$ can be recursively obtained using (see [29]):

$$\begin{aligned}
\hbar_1^{\mathfrak{S}} &= \hbar_1^{\nu_0} + s\hbar_0^{\nu_1}, \\
\hbar_j^{\mathfrak{S}} &= -\gamma_{j-1}^2 \hbar_{j-1}^{\mathfrak{S}} + \hbar_j^{\nu_0} + sj^2[\hbar_{j-1}^{\nu_1} + \tau_{j-1}^2 \hbar_{j-2}^{\nu_1}], \quad j \geq 2,
\end{aligned}$$

and

$$\langle f, \mathcal{S}_0\rangle_{\mathfrak{S}} = \langle f, 1\rangle_{\nu_0}, \quad \langle f, \mathcal{S}_1\rangle_{\mathfrak{S}} = \langle f, \mathcal{P}_1(\nu_0;\cdot)\rangle_{\nu_0} + s\langle f', 1\rangle_{\nu_1},$$

$$\begin{aligned}\langle f, \mathcal{S}_j\rangle_{\mathfrak{S}} = {} & \gamma_{j-1}\langle f, \mathcal{S}_{j-1}\rangle_{\mathfrak{S}} + \langle f, \mathcal{P}_j(\nu_0;\cdot)\rangle_{\nu_0} \\ & + sj\langle f', \mathcal{P}_{j-1}(\nu_1;\cdot)\rangle_{\nu_1} - sj\tau_{j-1}\langle f', \mathcal{P}_{j-2}(\nu_1;\cdot)\rangle_{\nu_1}, \quad j \geq 2.\end{aligned}$$

Thus, the easy determination of the Fourier coefficients $b_n^{\mathfrak{S}}$ makes this a convenient technique of approximation. This constitutes a nice future problem that should be further explored in which one can also consider the symmetric case, i.e., when $\{\nu_0, \nu_1\}$ is a SCPPM2K on the real line.

The SCPPM2K-H and Associated Hermite-Sobolev Orthogonal Polynomials

In Section 4.3 we have stated two special examples of a SCPPM2K on the real line $\{\nu_0, \nu_1\}$ in which one of the measures is given by the Hermite measure (see Table 1.1). Precisely, these examples are such that:

SCPPM2K-H1:

$$d\nu_0(x) = e^{-x^2}dx,$$

$$d\nu_1(x) = (x^2 + q^2)e^{-x^2}dx.$$

The real number q is such that $q \neq 0$. As shown, the pair SCPPM2K-H1 verifies (6.3) with

$$\begin{aligned} A_3(x) &= -2x^3 - 2(q^2 - 1)x \quad \text{and} \\ A_4(x) &= x^2 + q^2. \end{aligned}$$

SCPPM2K-H2: :

$$d\nu_0(x) = \frac{e^{-x^2}}{x^2 + q^2}dx,$$

$$d\nu_1(x) = e^{-x^2}dx.$$

As shown, the pair SCPPM2K-H2 verifies (6.3) with

$$\begin{aligned} A_3(x) &= -2x(x^2 + q^2) \quad \text{and} \\ A_4(x) &= x^2 + q^2. \end{aligned}$$

Another interesting and important future work will be to use the results contained in Chapter 5 of this thesis to study the Sobolev orthogonal polynomials and the associated connection coefficients that follow from the pairs SCPPM2K-H1 and SCPPM2K-H1.

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