$H_q(4)$ Symmetry: The Linear q-Harmonic Oscillator Based on Generalized Irreps of the q-Deformed Heisenberg Algebra

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Received 19 June, 1998

An extended treatment of the one-dimensional q-harmonic oscillator, based on two examples of inequivalent representations of the Heisenberg quantum algebra which appeared recently in literature is presented. The dependence of several oscillator properties such as the energy spectrum, uncertainty relations and selection rules on the new parameters characterizing those generalized representations is also discussed.

I. Introduction

Quantum groups are among the recent mathematical developments in closest contact with physics, particularly interest for those showing a wider variety of applications. They have been applied to solvable statistical mechanics models, inverse scattering theory, molecular and nuclear physics, as well as particle and quantum field physics [1]. The structure of a quantum group was developed by mathematicians in order to define non-commutative geometry. By definition, a quantum group is characterized by a Hopf algebra (Drinfeld [2]). For a physicist, there are two essential new concepts related to a quantum group: that of co-multiplication (one of the Hopf operations) and that of the deformed quantum algebra. The deformation is obtained through a parameter q, which is introduced in the commutation relations defining the algebra. In the limit $q \to 1$, the original, undeformed Lie algebra, is reproduced.

In this work we will be mainly concerned with that algebraic part of the Heisenberg quantum group ($H_q(4)$), namely the Heisenberg q-algebra (also known as the Heisenberg-Weyl q-algebra $h_q(4)$ or q-oscillator algebra [1]). It is interesting to mention that there are several versions of the q-oscillator algebra. Oh and Singh [3] showed that they are not equivalent. As a consequence, the Hopf structure found for one of those versions does not hold for the others, just because of this inequivalence.

Our aim here is to perform a generalized treatment of the one-dimensional deformed harmonic oscillator. This will be done by making use of two different examples of generalized, inequivalent representations of the $h_q(4)$ algebra which appeared recently in literature. The first one, due to G. Rideau [4], is characterized by a real parameter v_0. In the limit $v_0 \to 0$ it goes to the ordinary q-deformed representation of the Heisenberg q-algebra.

It is in this sense that we will be dealing with a generalized representation. We note that Rideau's representation may be constructed from the ordinary representation of the Heisenberg q-algebra by means of a *dilatation,* as indicated in Ref. [1].

Our second example is that of Z. Chang and Song [5, 6] and is characterized by an additive constant $c \equiv b\alpha/2$, with $\alpha = ln\ q$. We note that in the two cases above we are dealing with inequivalent representations since they correspond to different values of the

Casimir operator of the Heisenberg q-algebra. On the other hand, the linear harmonic oscillator is a fundamental system in physics. The associated q-deformed linear harmonic oscillator to the algebra of $h_q(4)$, in terms of bosonic creation and annihilation operators, was introduced by MacFarlane [7] and Biedenharn [8], who calculated the corresponding spectrum, assuming the existence of a ground state. We believe that generalized inequivalent q-formulations of it may have phenomenological interest in possible applications. In this work we study the fundamental properties of such generalized q-oscillators.

This work is divided as follows. Section II is concentrated on the deformed linear harmonic oscillator constructed on the basis of the ordinary q-deformed Heisenberg algebra. Results for several q-oscillator properties such as the energy spectrum, uncertainty relations and selection rules are recollected. In the same direction, a realization of the "quantum group" $SU_q(2)$ based on a Jordan-Schwinger construction is carried out [1]. In section Ill, the results of the previous section are generalized using the two inequivalent representations referred to above. Section IV contains our final discussion and remarks. Finally, at the end of this work we have an appendix, where we present some useful q-number identities.

II. The Deformed Harmonic Oscillator

Construction and energy levels

In this section we briefly review the construction of the one-dimensional q-deformed harmonic oscillator. After it, in the following section, we will present the results for the inequivalent representations in a comparative way.

The hamiltonian of the one-dimensional q-harmonic oscillator reads

$$H_q = \frac{1}{2}\hbar\omega(a_q^+ a_q + a_q a_q^+) \tag{2.1}$$

and the relations among the operators a_q^+, a_q and N_q define the q-deformed Heisenberg algebra

$$\begin{aligned} [N_q, a_q^+] &= a_q^+ \\ [N_q, a_q] &= -a_q \\ a_q a_q^+ - q^{1/2} a_q^+ a_q &= q^{-N_q/2}. \end{aligned} \tag{2.2}$$

It is the action of these operators on the Fock-space states which defines the representation. One possible realization is given by

$$\begin{aligned} a_q|n> &= [n]_q^{1/2}|n-1> \\ a_q^+|n> &= [n+1]_q^{1/2}|n+1> \\ N_q|n> &= n|n>. \end{aligned} \tag{2.3}$$

Through eqs.(2.3) a_q, a_q^+ and N_q assume the roles of annihilation, creation and number operators, respectively.

We note the above relations were written in accordance with the definition of a q-number which employs $q^{1/2}$ factors

$$[x]_q = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}}. \tag{2.4}$$

The algebraic structure given by eqs.(2.2) and (2.3) is completed with one additional relation which defines the number operator in function of a_q and a_q^+. In the construction of MacFarlane [7] and Biedenharn [8] the q-deformed version of the number operator is defined by

$$[N_q] = a_q^+ a_q. \tag{2.5}$$

Then, acting on the ket eigenstates one obtains

$$\begin{aligned} a_q^+ a_q|n> &= [N_q]|n> = [n]|n>, \\ a_q a_q^+|n> &= [N_q+1]|n> = [n+1]|n>. \end{aligned} \tag{2.6}$$

Thus, the energy states associated to the hamiltonian (2.1) will be given by

$$E(n) = \frac{1}{2}\hbar\omega([n] + [n+1]) = \frac{1}{2}\hbar\omega\frac{[n+\frac{1}{2}]}{[1/2]}, \tag{2.7}$$

where to prove the last step we have made use of a pair of interesting q-number identities. We made the option to show them in an appendix at the end of this work.

From eq.(2.7) one sees that the energy of the ground-state is $\frac{1}{2}\hbar\omega$, as for the non-deformed oscillator. In the generalized cases of deformation of non-equivalent representations we show later, this situation changes.

From eq.(2.7) the separation between energy levels is

$$\Delta(n) = E(n+1) - E(n) = \frac{1}{2}\hbar\omega([n+2]-[n]) =$$
$$= \frac{1}{2}\hbar\omega\frac{1}{[1/2]}\left(\left[n+\frac{3}{2}\right] - \left[n+\frac{1}{2}\right]\right) = \hbar\omega\, cosh\left(\frac{n+1}{2}\alpha\right), \tag{2.8}$$

where $\alpha = ln\ q$. For $q \to 1$ one has equally-spaced levels: $\Delta(n) \to \hbar\omega$, as in the non-deformed oscillator.

Uncertainty relations and selection rules

Defining the q-position and the q-momentum operators by

$$Q_q = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a_q^+ + a_q) \tag{2.9}$$

$$P_q = i\left(\frac{m\omega\hbar}{2}\right)^{1/2}(a_q^+ - a_q) \tag{2.10}$$

the uncertainty principle assumes an analogous form of that of the non-deformed case

$$i[P_q, Q_q] = \hbar[a_q, a_q^+] = \hbar(a_q a_q^+ - a_q^+ a_q). \tag{2.11}$$

Under the action on ket states eq. (2.10) gives

$$i[P_q, Q_q] = \hbar([n+1]-[n]) = \hbar\frac{[\frac{1}{2}][2n+1]}{[n+\frac{1}{2}]}\ . \tag{2.12}$$

Transitions are ruled by

$$< n'|Q_q|n >= \left(\frac{\hbar}{2m\omega}\right)^{1/2}\left\{\begin{array}{ll} [n+1]^{1/2}, & \text{for:} n' = n+1, \\ [n]^{1/2} & \text{for:} n' = n-1; \end{array}\right. \tag{2.13}$$

$$< n'|P_q|n >= i\left(\frac{m\omega\hbar}{2}\right)^{1/2}\left\{\begin{array}{ll} [n+1]^{1/2}, & \text{for:} n' = n+1, \\ -[n]^{1/2}, & \text{for:} n' = n-1. \end{array}\right. \tag{2.14}$$

Thus we get as selection rules $n \to n \pm 1$, that is only transitions to neighbours levels are allowed.

$SU_q(2)$ formulation

To close this section we would like to remind the realization of the "quantum group" $SU_q(2)$ from the one-dimensional q-oscillator. As described by Biedenharn and MacFarlane, an analogous q-deformed construction of that of Jordan-Schwinger can be defined to realize the algebra of the generators in the deformed case. Thus, the angular momentum operators of $SU_q(2)$ are defined in function of creation a_{iq}^+ and annihilation a_{iq} operators of two ($i = 1,\ 2$) linear, commuting, q-harmonic oscillators in the following way

$$J_+ = a_{1q}^+ a_{2q}, \qquad J_- = a_{2q}^+ a_{1q},$$
$$J_z = \frac{1}{2}(N_{1q} - N_{2q}). \tag{2.15}$$

Then, with $j = 1/2(n_1+n_2)$, $m = 1/2(n_1-n_2)$, the action of the above operators on states $|j,m>_q$ obey the relations

$$J_\pm|j,m>_q = ([j\mp m]_q[j\pm m+1]_q)^{1/2}|j,m\pm 1>_q$$
$$J_z|j,m>_q = m|j,m>_q\ . \tag{2.16}$$

These generators verify the commutation relations in the following way

$$[J_+, J_-]|j,m>_q = [2J_z]|j,m>_q$$
$$[J_z, J_\pm]|j,m>_q = \pm J_\pm j,m>_q \tag{2.17}$$

These relations are valid, however, only when acting on ket states. This situation is fundamentally different of that in the ordinary $su(2)$ algebra, where the complete realization of the algebra is not dependent on the way the vacuum is defined.

We recall that the Casimir operator for the oscillator in this case corresponds to the *null operator:*

$$C_{osc} = C(H_q(4)) \equiv 0, \tag{2.18}$$

what is garanteed by eq.(2.5). A distinct situation will occur in the cases we study ahead.

The Casimir operator of $SU_q(2)$ has the form

$$\begin{aligned} C(SU_q(2)) &= [J_z + 1/2]^2 + J_- J_+ = \\ &= [J_z][J_z + 1] + [1/2]^2 + J_- J_+, \end{aligned} \tag{2.19}$$

which has an eigenvalue

$$[j + 1/2]^2 = [j][j + 1] + [1/2]^2 \ . \tag{2.20}$$

In the limit $q \to 1$ this number becomes $(j + 1/2)^2$, recovering the usual result of the non-deformed case.

III. The inequivalent representations

Construction

In the previous section we saw the construction of the q-deformed analogue of the Heisenberg algebra. However, that realization of the q-algebra is not unique. As we have seen, the quantum (q-deformed) extensions of the algebraic relations depend on the way Fock-space states are defined, fixing the representation. Due to this fact, other generalized representations can be constructed in the form of *inequivalent representations.*

One well-known form of an inequivalent representation has been considered by Rideau [4] and by Biedenharn [1], who called it a "dilatation". This representation is defined by

$$\begin{aligned} a_q^+|n> &= q^{-\upsilon_0/4}[n+1]_q^{1/2}|n+1> \\ a_q|n> &= q^{-\upsilon_0/4}[n]_q^{1/2}|n-1> \end{aligned} \tag{3.1}$$

where υ_0, called the parameter of the inequivalent representation, expresses the possibility of a redefinition of the vacuum, conducting to a new realization of the $h_q(4)$ algebra.

In this case, the number operator is not equal to $a_q^+ a_q$, but is given by

$$[N_q] = a_q^+ a_q + q^{N_q/2} C_{osc}, \tag{3.2}$$

where the Casimir operator C_{osc} associated to the q-oscillator algebra, differently of what occurs in the usual, deformed, "equivalent" representations, *is not* a null operator anymore. Inverting eq.(3.2) one gets

$$C_{osc} = q^{-N_q/2}([N_q] - a_q^+ a_q), \tag{3.3}$$

and the Casimir eigenvalue is

$$C_{osc} = q^{-\upsilon_0/2}[\upsilon_0]_q. \tag{3.4}$$

Obviously, for $\upsilon_0 = 0$ one has $C_{osc} \equiv 0$, as in eq.(2.18), in the case of ordinary (deformed) representations. In the present case, the action of N_q under Fock-space states gives

$$N_q|n >= (\upsilon_0 + n)|n >, \tag{3.5}$$

closing this algebra. Commutation relations as defined by eqs.(2.2) are preserved by this redefinition of a_q^+, a_q and N_q operators.

Another interesting example of a quantum-deformed realization of the Heisenberg algebra found in literature is that used by Chang [5] and Song [6]. At first sight their approach seems to be the same as the one of the previous section, with a single modification of a redefinition of the number operator N by $N' = N + c$, where c is a constant. However, in a deformed algebra this change provides not only a shift on the spectrum of Fock states labeled by the quantum number n, but also conducts to the construction of a new representation, non-equivalent to the other.

A fundamental difference lies on the definition of the number operator. For them, the q-deformed N must equate the non-deformed operator

$$N'_q = a_q^+ a_q = N = a^+ a. \tag{3.6}$$

Here and henceforth we shall be using a prime ' to differentiate between the operators defined in this case from the previous ones.

Following Chang and Song [5, 6], the representation is defined by

$$a'_q|n>=[n+c]_q^{1/2}|n-1>,$$
$$a'^{+}_q|n>=[n+1+c]_q^{1/2}|n+1>,$$
$$N'_q|n>=n|n>. \qquad (3.7)$$

Then, provided the relations between deformed and non-deformed operators are given by

$$a'_q=\sqrt{\frac{[N+1+c]}{N+1}}a,$$
$$a'^{+}_q=a^{+}\sqrt{\frac{[N+1+c]}{N+1}}, \qquad (3.8)$$

the q-deformed Heisenberg algebra, as defined by eqs.(2.2), is realized. Notice that now the quantum number n differs from the non-deformed one by a constant. This is a distinct situation of that of the Rideau's representation. Thus, we can distinguish between *two* basic types of non-equivalent representations. Those of the *dilatation (or dilation) type*, with a multiplying factor, as in Rideau's representation, and those of the *dislocation (or displacement) type,* like the one of Chang, with an additional constant term.

The Casimir operator for the case of generalized, inequivalent representations is given by eq.(3.3). In the case of Song's representation, it leads to the following operator (reminding that, in this case, $N'_q=N$)

$$C_{osc}=q^{-N/2}([N]-[N+c]). \qquad (3.9)$$

Acting on the states $|n>$ gives

$$C_{osc}|n'>=q^{-n/2}([n]-[n+c])|n>. \qquad (3.10)$$

Therefore, C_{osc} is again a non-null operator and we have two different, inequivalent representations. In the above expressions, when $c=0$, C_{osc} becomes the null operator, as in eq.(2.18) again.

Energy levels and uncertainty relations

Now, we wish to comment about the q-oscillator energy for non-equivalent representations. Starting from the oscillator hamiltonian, eq.(2.1), one gets new expressions.

For the dilatation case of Rideau's representation the oscillator hamiltonian has eigenenergies

$$E(n)=\frac{1}{2}\hbar\omega q^{-v_0/2}([n]+[n+1])=\frac{1}{2}\hbar\omega q^{-v_0/2}\frac{[n+1/2]}{[1/2]}, \qquad (3.11)$$

which is the same as in eq.(2.7), except for the factor $q^{-v_0/2}$.

One sees that this multiplying factor has its physical consequences. First, it shifts the zero-point energy. In this case

$$E_0=\frac{1}{2}\hbar\omega e^{-\alpha v_0/2}. \qquad (3.12)$$

And second, it modifies the level spacement. Instead of eq.(2.8), now

$$\Delta(n)=\frac{1}{2}\hbar\omega\frac{e^{-\alpha v_0/2}}{[1/2]}([n+3/2]-[n+1/2]). \qquad (3.13)$$

In this expression the exponential factor offers the possibility of "balancing" the influence of the q-factor contained in the terms inside the parenthesis, which didn't happen before.

Selection rules are the same as in eqs.(2.13) and (2.14), as the definition of P_q and Q_q are still the same, eqs.(2.9) and (2.10). But now, due to the factor $q^{-v_0/4}$ eqs.(3.1), one also has this multiplying factor in the right-hand side of the selection rules equations.

In the non-equivalent representation of the *dilatation type*, uncertainty relation gives

$$i[P_q,Q_q]|n>=\hbar q^{-v_0/2}([n+1]-[n])|n>, \qquad (3.14)$$

which can also be rewritten in the following interesting form

$$i[P_q,Q_q]=\hbar\frac{cosh(1/4(2n+1)\alpha)}{cosh(\alpha/4)}, \qquad (3.15)$$

with $\alpha=lnq$.

In the case of non-equivalent representations of the *dislocation type,* rather different expressions will come out. The energy states obtained from the one-dimensional hamiltonian of the q-harmonic oscillator are now (in Chang's notation, with $c=b\gamma$ and $\gamma=\alpha/2$)

$$E(n)=\frac{1}{2}\hbar\omega([n+b\gamma]+[n+1+b\gamma])=$$
$$=\frac{1}{2}\hbar\omega\frac{sinh\left(\gamma\left(n+b\gamma+\frac{1}{2}\right)\right)}{sinh(\gamma/2)}. \qquad (3.16)$$

Eq.(3.16) can be rewritten as

$$E(n) = \frac{1}{2}\hbar\omega\frac{[n+\frac{1}{2}+c]}{[\frac{1}{2}]} =$$
$$= \frac{1}{2}\hbar\omega\frac{sinh\left(\frac{\alpha}{2}\left(n+b\frac{\alpha}{2}+\frac{1}{2}\right)\right)}{sinh(\alpha/4)}, \qquad (3.17)$$

with $c = b\alpha/2$.

And the separation of energy levels in this case is given by

$$\Delta(n) = \frac{1}{2}\hbar\omega\frac{1}{[1/2]}\left(\left[n+\frac{3}{2}+b\frac{\alpha}{2}\right] - \left[n+\frac{1}{2}+b\frac{\alpha}{2}\right]\right) =$$
$$= \frac{1}{2}\hbar\omega\left(\left[n+2+b\frac{\alpha}{2}\right] - \left[n+b\frac{\alpha}{2}\right]\right) =$$
$$= \hbar\omega\ cosh\left(\frac{\alpha}{2}\left(n+b\frac{\alpha}{2}+1\right)\right). \qquad (3.18)$$

Again, for $q \to 1$ one recovers $\Delta(n) = \hbar\omega$, as expected. The deformation in this case introduces, however, not only a non-equally spacing of the energy levels, but also shifts the zero-point energy

$$E_0 = \frac{1}{2}\hbar\omega\frac{[\frac{1}{2}+b\frac{\alpha}{2}]}{[\frac{1}{2}]}. \qquad (3.19)$$

The influence of the constant, additional term $b\frac{\alpha}{2}$ will also reflect in the uncertainty principle. Again, the expression is defined by eq.(2.11), but this time gives (reminding once more that $N_q' = N$)

$$i[P_q, Q_q] = \hbar\left(\left[N+b\frac{\alpha}{2}+1\right] - \left[N+b\frac{\alpha}{2}\right]\right) , \qquad (3.20)$$

which produces, under the action on states, the eigenvalue

$$i[P_q, Q_q] = \hbar\left(\left[n+b\frac{\alpha}{2}+1\right] - \left[n+b\frac{\alpha}{2}\right]\right) = \hbar\frac{[1/2][2n+b\alpha+1]}{[n+b\frac{\alpha}{2}+\frac{1}{2}]}. \qquad (3.21)$$

In this case, the selection rules are the same as in eqs.(2.13) and (2.14), recalling that n is replaced by $n + b\frac{\alpha}{2}$.

$SU_q(2)$ formulation and the Casimir operator

From the results we have shown, one notices a variety of possibilities, depending on the different type of inequivalent representation that is used. These fundamental differences, related to the inequivalence of representations, are also noticed in what concerns the Jordan-Schwinger realization of the $SU_q(2)$ group in both cases.

For inequivalent representations of the dislocation type one may use the same set of generators that we have reminded in the preceding section. If, as before, one defines J_+, J_- and J_z through eqs.(2.15), in this case, due to application of eqs.(3.7), one obtains

$$J_\pm|j,m>_q = [j \mp m + c]_q^{1/2}[j \pm m + 1 + c]_q^{1/2}|j, m \pm 1>_q,$$
$$J_z|j,m>_q = m|j,m>_q, \qquad (3.22)$$

where the first equations *depend* on c, in constrast with eq.(2.16).

Furthermore, the algebra of the $SU_q(2)$ generators, as given by eqs.(2.17), is sustained by this generalization. The Casimir operator for the $SU_q(2)$ is defined in the usual way, by eq.(2.19), but this time the action of J-operators on states $|j,m>$ will give

$$C|j,m> = ([j-m+c][j+m+1+c] + [m][m+1] + [1/2]^2)|j,m> . \qquad (3.23)$$

Using the q-number identity

$$[a+1/2]^2 = [a][a+1] + [1/2]^2 \qquad (3.24)$$

and applying it to

$$[j+m+1][j-m] = [j][j+1] - [m][m+1] =$$
$$= [j+1/2]^2 - [m+1/2]^2, \qquad (3.25)$$

one then writes

$$C|j,m> = ([j+c][j+c+1] + [1/2]^2)|j,m> . \quad (3.26)$$

In this expression we notice the dependence on c, which is equivalent to put $j+c=j'$. In this way, we reach to the result

$$C|j,m> = ([j'][j'+1]+[1/2]^2)|j,m> = [j'+1/2]^2|j,m> =$$
$$= [j+c+1/2]^2|j,m> \ . \qquad (3.27)$$

Then, it is clear that the present definition of N, done in this case by eq.(3.6), leads to the need of an associated redefinition of $\vec{J}$. We also call attention to the fact that the term $[1/2]^2$ on the above equations is sometimes dropped by some authors, depending on the application, as it is an additive constant. Here we keep it, as it is fundamental in the last step in eq.(3.27), and in the use of identity eq.(3.24), to obtain $[j'+1/2]^2$, with the correct limit $(j'+1/2)^2$ when $q \to 1$.

Other interesting consequences are found associated to the $su_q(2)$ algebra realization in the dilatation type of inequivalent representations. The Jordan-Schwinger construction with eqs.(2.15) would give an extra factor $q^{-v_0/2}$ in eqs.(2.16). Then, instead of eq.(2.15) one should use the prescription (Rideau [4])

$$J_+ = q^{v_0/2} a^+_{1q} a_{2q},$$
$$J_- = q^{v_0/2} a^+_{2q} a_{1q},$$
$$J_z = \frac{1}{2}(N_{1q} - N_{2q}), \qquad (3.28)$$

where the factors $q^{v_0/2}$ compensate for those in eqs.(3.1) and, consequently, relations (2.16) are reobtained.

Proceeding in this way, one gets the following Casimir operator for the $SU_q(2)$

$$C(SU_q(2)) = q^{-v_0}([J_z+1/2]^2 + J_- J_+). \qquad (3.29)$$

This operator is the same as that of eq.(2.19), with a factor q^{-v_0} in front. It now leads to a different eigenvalue, given by

$$C|j,m> = q^{-v_0}[j+1/2]^2|j,m> . \qquad (3.30)$$

Note that, for $v_0 = 0$ one obtains $[j+1/2]^2$, as from eq.(2.19). However, for $q \to 1$ one gets $(j+1/2)^2$-directly, as expected.

We would like to call attention to the fact that, if the redefinition of J_+ and J_- in eqs.(3.28) with the factor $q^{v_0/2}$ was not done, one would reach to the same Casimir operator, eq.(3.29), but with a different eigenvalue

$$C(SU_q(2)) = (1-e^{-\alpha v_0})[m+1/2]^2 + e^{-\alpha v_0}[j+1/2]^2. \qquad (3.31)$$

We want to note that the same limiting situations for $v_0 \to 0$ and $q \to 1$ are maintained correct by this expression, but this time $C(SU_q(2))$ of eq.(3.31) would keep an odd dependence in m.

In reference [10], we have treated triplet states for quarks and leptons, using the $su_q(2)$ as a spectrum generating algebra. We have applied eq.(3.31), more successfully than with eq.(3.30), to describe the masses of the fundamental fermions. This fact would be comprehensible, if the particle states we find in the Standard Model are defined according to a construction which follows equations (2.3) and (2.15), instead of equations (3.1) and (3.28). In other words, the particle states of the Standard Model are associated to a vacuum which, however, in this line of work, *depends* on the construction scheme adopted for a certain type of representation. This point may be relevant, if the physical states are connected somehow to an underlying algebraic structure of any form which leads to non-equivalent representations.

IV. Conclusions

It is well known that the Heisenberg algebra h(4) plays a fundamental role in the analysis of the linear harmonic oscillator. Recently, models based on the q-deformed $h(4)$ algebra ($h_q(4)$, of the group $H_q(4)$) have been studied to describe vibrational spectra of diatomic molecules [5,9]. We note that when the deformation parameter q is near to 1, deviations of the $H(4)$ symmetry will occur, which may contain important physics. In a similar way, models based on $SU_q(2)$ have been constructed to study the rotational spectra of diatomic molecules. It was also suggested that the coupling between vibrational and rotational motions in diatomic molecules is described by $H_q(4) \otimes SU_q(2)$, [5,9]. It is interesting to point out that similar situations occur in nuclear spectroscopy in connection with nuclear rotations and vibrations [11].

In the present work we performed an extended treatment of the deformed linear harmonic oscillator making use of two classes of inequivalent irreps of the $h_q(4)$ algebra, introduced independently by Rideau [4] and by Chang and Yan [9]. It is worthwhile to mention that Rideau's case found interesting applications in an algebraic treatment of q-deformed Bose gases [12]. Another application was done in a calculation of the fundamental fermion masses from deformed $SU_q(2)$ triplets [10]. Expressions of the $SU_q(2)$ Casimir operators involved in our analysis, for $v_0 \neq 0$, were discussed at the end of section III. In the context of $H_q(4)$ we note that the corresponding Hopf algebra has been explicitly constructed, including the quantum Yang-Baxter equation [5,9].

Last, but not least, we wish to point out two different directions for further work: (i) the q-analogue of the isotropic n-dimensional harmonic oscillator, as defined in Ref.[1] and (ii) the extension of the present analysis of the linear harmonic oscillator for values of q beyond q real. An attractive situation corresponds, for instance, to q being a *primitive root of unity*. Study of models which correspond to quantum mechanical analogues of q-oscillator problems [13] represent a very promising and interesting line of work, including the case of imaginary q. We also have in mind the analysis of statistical properties of systems of such q-deformed oscillators, like the study of Planck distributions for q-boson gases [14]. This work is in progress and shall appear in a forthcoming publication.

Appendix

In this appendix we present some q-number identities which showed to be very useful in rewriting our main results. Of course, from the ones we have here, others can be derived and readily applied.

In the last step of eq.(2.7) we have made use of the following q-number identity

$$[a][b] = \left[\frac{a+b}{2}\right]^2 - \left[\frac{a-b}{2}\right]^2 \qquad (A1)$$

written for $a = n$, $b = 1/2$ and $a = n+1$, $b = 1/2$, and then adding up the two relations which are obtained.

Next, we have used the q-number identity

$$[a]^2 - [b]^2 = [a+b][a-b], \qquad (A2)$$

with $a = 1/2(n+3/2)$ and $b = 1/2(n-1/2)$ to end up with

$$[n] + [n+1] = \frac{[n+1/2]}{[1/2]}. \qquad (A3)$$

To write the second equality in eq.(2. 12) we have used the relation

$$[a]^2 - [b]^2 = ([a]+[b])([a]-[b]), \qquad (A4)$$

with $a = n+1$, $b = n$, to obtain

$$[n+1]^2 - [n]^2 = ([n+1]+[n])([n+1]-[n]) = [2n+1]. \qquad (A5)$$

Then, combining eqs.(A3) and (A5) leads to eq.(2.12). Furthermore, from (A3) one finds an expression for the uncertainty of the n-th energy level state in terms of the energy

$$\frac{i}{\hbar}[P_q, Q_q] = \frac{1}{2}\hbar\omega\frac{[2n+1]}{E(n)} \qquad (A6)$$

Finally, we would like to note that in the same way identity (A3) can be used to rewrite the q-deformed energy expressions, the identities

$$\frac{1}{[1/2]}([n+3/2]-[n+1/2])=[n+2]-[n] \qquad (A7)$$

and

$$[n+2]-[n]=2\ \cosh\left(\frac{\alpha}{2}(n+1)\right) \qquad (A8)$$

can be very useful in rewriting the expressions of the separation of energy levels $\Delta(n)$, leading to equations like eqs.(2.8) and (3.18) of the text.

Acknowledgments

One of us (P.L.F.) thanks the financial support of the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil.

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