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Exponential Decay for Kirchhoff Wave Equation with Nonlocal Condition in a Noncylindrical Domain

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Abstract—In this paper, we prove the exponential decay as time goes to infinity of regular solutions of the problem for the Kirchhoff wave equation with nonlocal condition and weak damping

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + \int_0^t g(t-s)\Delta u(\cdot, s)\, ds + \alpha u_t = 0, \qquad \text{in } \hat{Q},$$

where $\hat{Q}$ is a noncylindrical domain of $\mathbb{R}^{n+1}$ $(n \geq 1)$ with the lateral boundary $\hat{\Sigma}$ and α is a positive constant.

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1. INTRODUCTION

Let Ω be an open bounded domain of $\mathbb{R}^n$ containing the origin and having C^2 boundary. Let $\gamma : [0,\infty[\to \mathbb{R}$ be a continuously differentiable function. Consider the family of subdomains $\{\Omega_t\}_{0\le t<\infty}$ of $\mathbb{R}^n$ given by

$$\Omega_t = T(\Omega), \qquad T : y \in \Omega \mapsto x = \gamma(t)y,$$

whose boundaries are denoted by Γ_t, and let $\hat{Q}$ be the noncylindrical domain of $\mathbb{R}^{n+1}$ given by

$$\hat{Q} = \bigcup_{0\le t<\infty} \Omega_t \times \{t\}$$

with lateral boundary

$$\hat{\sum} = \bigcup_{0\le t<\infty} \Gamma_t \times \{t\}.$$

In this work, we study the existence of strong solutions as well as the exponential decay of the energy to the Kirchhoff wave equation with a nonlocal condition given by

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + \int_0^t g(t-s)\Delta u(\cdot,s)\,ds + \alpha u_t = 0, \qquad \text{in } \hat{Q}, \tag{1.1}$$

$$u = 0, \qquad \text{on } \hat{\sum}, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad \text{in } \Omega_0, \tag{1.3}$$

where u is the transverse displacement and α is a positive constant. We show the existence and uniqueness of strong solutions to the initial boundary value problem (1.1)–(1.3). The method we use to prove the result of existence and uniqueness is based on transforming our problem into another initial boundary value problem defined over a cylindrical domain whose sections are not time-dependent. This is done using a suitable change of variable. Then, we show the existence and uniqueness for this new problem. Our existence result on noncylindrical domain will follow by using the inverse transformation, that is, by using the diffeomorphism $\tau : \hat{Q} \to Q$ defined by

$$\tau : \hat{Q} \to Q, \qquad (x,t) \in \Omega_t \mapsto (y,t) = \left(\frac{x}{\gamma(t)}, t\right) \tag{1.4}$$

and $\tau^{-1} : Q \to \hat{Q}$ defined by

$$\tau^{-1}(y,t) = (x,t) = (\gamma(t)y, t). \tag{1.5}$$

Denoting by v the function

$$v(y,t) = u \circ \tau^{-1}(y,t) = u(\gamma(t)y, t), \tag{1.6}$$

the initial boundary value problem (1.1)–(1.3) becomes

$$\begin{gathered} v_{tt} - \gamma^{-2} M\left(\gamma^{n-2}\|\nabla v\|^2_{L^2(\Omega)}\right)\Delta v + \int_0^t g(t-s)\gamma^{-2}(s)\Delta v(\cdot,s)\,ds \\ +\alpha v_t - A(t)v + a_1\cdot\nabla\partial_t v + a_2\cdot\nabla v = 0, \qquad \text{in } Q, \end{gathered} \tag{1.7}$$

$$v|_\Gamma = 0, \tag{1.8}$$

$$v|_{t=0} = v_0, \quad v_t|_{t=0} = v_1, \qquad \text{in } \Omega, \tag{1.9}$$

where

$$A(t)v = \sum_{i,j=1}^{n} \partial_{y_i}\left(a_{ij}\partial_{y_j} v\right)$$

and

$$\begin{aligned} a_{ij}(y,t) &= -\left(\gamma'\gamma^{-1}\right)^2 y_i y_j, \qquad i,j=1,\dots,n, \\ a_1(y,t) &= -2\gamma'\gamma^{-1}y, \\ a_2(y,t) &= -\gamma^{-2}y\left(\gamma''\gamma+\gamma'\left(\alpha\gamma+(n-1)\gamma'\right)\right). \end{aligned} \tag{1.10}$$

To show the existence of strong solution, we will use the following hypotheses:

$$\gamma' \leq 0, \quad \text{if } n>2, \qquad \gamma' \geq 0, \quad \text{if } n \leq 2, \tag{1.11}$$

$$\gamma(\cdot) \in L^{\infty}(0,\infty), \qquad \inf_{0\leq t<\infty}\gamma(t) = \gamma_0 > 0, \tag{1.12}$$

$$\gamma' \in W^{2,\infty}(0,\infty) \cap W^{2,1}(0,\infty). \tag{1.13}$$

Note that assumption (1.11) means that $\hat{Q}$ is decreasing if $n > 2$ and increasing if $n \leq 2$ in the sense that when $t > t'$ and $n > 2$, then the projection of $\Omega_{t'}$ on the subspace $t = 0$ contains the projection of Ω_t on the same subspace and the contrary when $n \leq 2$. The above method was introduced by Dal Passo and Ughi [1] to study a certain class of parabolic equations in noncylindrical domains. Concerning the function $M \in C^1([0,\infty[)$, we assume that

$$M(\tau) \geq m_0, \quad M(\tau)\tau \geq \hat{M}(\tau), \qquad \forall\tau \geq 0, \tag{1.14}$$

where $\hat{M}(\tau) = \int_0^{\tau} M(s)\,ds$.

Unlike the existing papers on stability for hyperbolic equations in noncylindrical domain, we do not use the penalty method but work directly in our noncylindrical domain $\hat{Q}$. To see the dissipative properties of the system, we have to construct a suitable functional whose derivative is negative and is equivalent to the first-order energy. This functional is obtained using the multiplicative technique following Rivera's method in [2]. We only obtained the exponential decay of solution for our problem for the case $n > 2$. The main difficulty to obtain the decay for the case $n \leq 2$ is due to the influence of the geometry of the noncylindrical domain and it is an important open problem. From the physics point of view, problem (1.1)–(1.3) describes the transverse displacements of a stretched viscoelastic membrane fixed in a moving boundary device. The viscoelasticity property of the material is characterized by the memory term

$$\int_0^t g(t-s)\Delta u(\cdot,s)\,ds.$$

In a fixed domain, it is well known that if the relaxation function g decays to zero, then the energy of the system also decays to zero, see [3–8]. But in a moving boundary setting, the axial tension exerted by the horizontal movement of the boundary yields nonlinear terms involving derivatives in the space variable. To control these nonlinearities, we add in the system a frictional damping, characterized by u_t. This term will play an important role in the dissipative nature of the problem. A quite complete discussion in the modelling of transverse vibrations of purely elastic membranes can be found in [9]. It seems to us that there is no result concerning the asymptotic stability of solutions for system (1.1)–(1.3) in the literature. So to fill this gap we study here this topic. The notations we use in this paper are standard and can be found in Lion's book [10,11]. In the sequel by C (sometimes $C_1, C_2, \dots$), we denote various positive constants which do not depend on t or on the initial data. The organization of this paper is as follows. In Section 2, we prove the existence of regular solutions, and finally in Section 3, we give the proof of the uniform exponential decay.

2. EXISTENCE AND REGULARITY

In this section, we shall study the existence and regularity of solutions for system (1.1)–(1.3). For this, we assume that the kernel $g : \mathbb{R}_+ \to \mathbb{R}_+$ is in $W^{2,1}(0,\infty)$ and satisfies

$$g, -g' \geq 0, \qquad m_0\gamma_1^{-2} - \int_0^\infty g(s)\gamma^{-2}(s)\,ds = \beta > 0, \tag{2.1}$$

where

$$\gamma_1 = \sup_{0 \leq t < \infty} \gamma(t).$$

To simplify our analysis, we define the binary operator

$$g\square\frac{\nabla u(t)}{\gamma(t)} = \int_\Omega \int_0^t g(t-s)\gamma^{-2}(s)\,|\nabla u(t) - \nabla u(\cdot,s)|^2\,ds\,dx.$$

It is convenient to establish the following.

LEMMA 2.1. *For $v \in C^1(0,T : H^1(\Omega))$, we have*

$$\begin{aligned}\int_\Omega \int_0^t g(t-s)\gamma^{-2}(s)\nabla v \cdot \nabla v_t\,ds\,dx = -\frac{1}{2}\frac{g(t)}{\gamma^2(0)}\int_\Omega |\nabla v|^2\,dx + \frac{1}{2}g'\square\frac{\nabla v}{\gamma} \\ -\frac{1}{2}\frac{d}{dt}\left[g\square\frac{\nabla v}{\gamma} - \left(\int_0^t \frac{g(s)}{\gamma^2(s)}\,ds\right)\int_\Omega |\nabla v|^2\,dx\right].\end{aligned}$$

The proof of this lemma follows by differentiating the term $g\square(\nabla u(t)/\gamma(t))$. The well posedness of system (1.7)–(1.9) is given by the following theorem.

THEOREM 2.1. *Let $v_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $v_1 \in H_0^1(\Omega)$ be functions satisfying*

$$\|\nabla v_0\|^2_{L^2(\Omega)} + \|\nabla v_1\|^2_{L^2(\Omega)} \leq \delta,$$

where δ is a small positive constant, and suppose that assumptions (1.11)–(1.14) and (2.1) hold. Then there exists a unique solution v of the problem (1.7)–(1.9) satisfying

$$\begin{aligned}v &\in L^\infty\left(0,\infty : H_0^1(\Omega) \cap H^2(\Omega)\right),\\ v_t &\in L^\infty\left(0,\infty : H_0^1(\Omega)\right),\\ v_{tt} &\in L^\infty\left(0,\infty : L^2(\Omega)\right).\end{aligned}$$

PROOF. Let us denote by B the operator

$$Bw = -\Delta w, \qquad D(B) = H_0^1(\Omega) \cap H^2(\Omega).$$

It is well known that B is a positive self-adjoint operator in the Hilbert space $L^2(\Omega)$ for which there exist sequences $\{w_n\}_{n\in\mathbb{N}}$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ of eigenfunctions and eigenvalues of B such that the set of linear combinations of $\{w_n\}_{n\in\mathbb{N}}$ is dense in $D(B)$ and $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \to \infty$ as $n \to \infty$. Let us denote by

$$v_0^m = \sum_{j=1}^m (v_0, w_j)w_j, \qquad v_1^m = \sum_{j=1}^m (v_1, w_j)w_j.$$

Note that for any $(v_0, v_1) \in D(B) \times H_0^1(\Omega)$, we have $v_0^m \to v_0$ strong in $D(B)$ and $v_1^m \to v_1$ strong in $H_0^1(\Omega)$.

Let us denote by V_m the space generated by $w_1, \dots, w_m$. Standard results on ordinary differential equations imply the existence of a local solution v^m of the form

$$v^m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j$$

to the system

$$\begin{aligned} &\int_\Omega v_{tt}^m w_j \, dy + \alpha \int_\Omega v_t^m w_j \, dy - \gamma^{-2} M\left(\gamma^{n-2} \|\nabla v^m\|_{L^2(\Omega)}^2\right) \int_\Omega \Delta v^m w_j \, dy \\ &\quad + \int_\Omega \int_0^t g(t-s)\gamma^{-2}(s) \nabla v^m(\cdot, s)\, ds \cdot \nabla w_j \, dy + \int_\Omega A(t) v^m w_j \, dy \\ &\quad + \int_\Omega a_1 \cdot \nabla v_t^m w_j \, dy + \int_\Omega a_2 \cdot \nabla v^m w_j \, dy = 0, \qquad j = 1, \dots, m, \end{aligned} \tag{2.2}$$

$$v^m(x,0) = v_0^m, \qquad v_t^m(x,0) = v_1^m. \tag{2.3}$$

The extension of these solutions to the whose interval $[0, \infty[$ is a consequence of the first estimate which we are going to prove below.

A Priori ESTIMATE I. Multiplying equation (2.2) by $g'_{jm}(t)$ and summing up the product result, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \|v_t^m\|_2^2 + \alpha \|v_t^m\|_2^2 + \frac{\gamma^{-2}}{2} M\left(\gamma^{n-2} \|\nabla v^m\|_2^2\right) \frac{d}{dt} \|\nabla v^m\|_2^2 \\ &\quad - \int_\Omega \int_0^t g(t-s)\gamma^{-2}(s) \nabla v^m(\cdot, s) \cdot \nabla v^m \, ds\, dy + \int_\Omega A(t) v^m v_t^m \, dy \\ &\quad + \int_\Omega a_1 \cdot \nabla v_t^m v_t^m \, dy + \int_\Omega a_2 \cdot \nabla v^m v_t^m \, dy = 0. \end{aligned} \tag{2.4}$$

Noting that

$$\begin{aligned} \frac{\gamma^{-2}}{2} M\left(\gamma^{n-2} \|\nabla v^m\|_2^2\right) \frac{d}{dt} \|\nabla v^m\|_2^2 &= n\gamma^{-n-1}\gamma' \hat{M}\left(\gamma^{n-2} \|\nabla v^m\|_2^2\right) \\ &\quad + \frac{d}{dt}\left(\gamma^{-n} \hat{M}\left(\gamma^{n-2} \|\nabla v^m\|_2^2\right)\right) - (n-2)\gamma'\gamma^{-3} \|\nabla v^m\|_2^2 M\left(\gamma^{n-2} \|\nabla v^m\|_2^2\right) \end{aligned}$$

and using Lemma 2.1, equation (2.4) can be written as

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \|v_t^m\|_2^2 + \alpha \|v_t^m\|_2^2 + \frac{1}{2}\frac{d}{dt}\left(\gamma^{-n} \hat{M}\left(\gamma^{n-2} \|\nabla v^m\|_2^2\right)\right) \\ &- \frac{(n-2)\gamma'}{2\gamma^{n+1}} \left[\gamma^{n-2} \|\nabla v^m\|_2^2 M\left(\gamma^{n-2} \|\nabla v^m\|_2^2\right) - \hat{M}\left(\gamma^{n-2} \|\nabla v^m\|_2^2\right)\right] \\ &+ \frac{\gamma'}{\gamma^{n-2}} \hat{M}\left(\gamma^{n-2} \|\nabla v^m\|_2^2\right) + \frac{1}{2}\frac{d}{dt}\left[g \square \frac{\nabla v^m}{\gamma} - \left(\int_0^t \frac{g(s)}{\gamma^2(s)}\, ds\right) \|\nabla v^m\|_2^2\right] \\ &\quad + \frac{1}{2}\frac{g(t)}{\gamma^2(0)} \|\nabla v^m\|_2^2 - \frac{1}{2} g' \square \frac{\nabla v^m}{\gamma} + \int_\Omega A(t) v^m v_t^m \, dy \\ &\quad + \int_\Omega a_1 \cdot \nabla v_t^m v_t^m \, dy + \int_\Omega a_2 \cdot \nabla v^m v_t^m \, dy = 0. \end{aligned} \tag{2.5}$$

Taking into account (1.11)–(1.14) and (2.1), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} E_1^m(t) + \alpha \|v_t^m\|_2^2 - \frac{(n-2)\gamma'}{2\gamma^{n+1}} \left[\gamma^{n-2} \|\nabla v^m\|_2^2 M\left(\gamma^{n-2} \|\nabla v^m\|_2^2\right)\right. \\ &\left. - \hat{M}\left(\gamma^{n-2} \|\nabla v^m\|_2^2\right)\right] \le C\left(|\gamma'| + |\gamma''|\right)\left(E_1^m(t) + \|\nabla v^m\|_2^2\right), \end{aligned} \tag{2.6}$$

where

$$E_1^m(t) = \|v_t^m\|_2^2 + \gamma^{-n}\hat{M}\left(\gamma^{n-2}\|\nabla v^m\|_2^2\right) + g\Box\frac{\nabla v^m}{\gamma} - \left(\int_0^t g(s)\gamma^{-2}(s)\,ds\right)\|\nabla v^m\|_2^2.$$

From (1.11) and (1.14), it follows that inequality (2.6) can be written as

$$E_1^m(t) + \alpha\int_0^t \|v_t^m(s)\|_2^2\,ds \le C\left(\|v_1^m\|_2^2 + \frac{1}{\gamma^n(0)}\hat{M}\left(\gamma^{n-2}(0)\|\nabla v_0^m\|_2^2\right)\right) + C\int_0^t \left(|\gamma'| + |\gamma''|\right)E_1^m(s)\,ds.$$

Using Gronwall's lemma and taking into account (1.13), we conclude that

$$\|v_t^m\|_2^2 + \|\nabla v^m\|_2^2 + \int_0^t \|v_s^m\|_2^2\,ds \le C, \qquad \forall t\in[0,T], \quad \forall m\in\mathbb{N}. \tag{2.7}$$

A Priori ESTIMATE II. Now, if we multiply equation (2.2) by $\sqrt{\lambda_j}g'_{jm}(t)$ and summing up in $j = 1,\dots,m$ we get after some calculations

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla v_t^m\|_2^2 + \alpha\|\nabla v_t^m\|_2^2 + \frac{\gamma^{-2}}{2}M\left(\gamma^{n-2}\|\nabla v^m\|_2^2\right)\frac{d}{dt}\|\Delta v^m\|_2^2\\ &- \int_\Omega\int_0^t g(t-s)\gamma^{-2}(s)\Delta v^m(\cdot,s)\Delta v_t^m\,ds\,dy + \int_\Omega A(t)v^m\Delta v_t^m\,dy\\ &+ \int_\Omega a_1\cdot\nabla v_t^m\Delta v_t^m\,dy + \int_\Omega a_2\cdot\nabla v^m\Delta v_t^m\,dy = 0.\end{aligned} \tag{2.8}$$

Using Lemma 2.1, we obtain

$$\begin{aligned}&\int_\Omega\int_0^t g(t-s)\gamma^{-2}(s)\Delta v^m(\cdot,s)\Delta v_t^m\,dy = -\frac{1}{2}g(t)\gamma^{-2}(0)\|\Delta v^m\|_2^2\\ &+\frac{1}{2}g'\Box\frac{\Delta v^m}{\gamma} - \frac{1}{2}\frac{d}{dt}\left[g\Box\frac{\Delta v^m}{\gamma} - \left(\int_0^t g(s)\gamma^{-2}(s)\,ds\right)\|\Delta v^m\|_2^2\right].\end{aligned} \tag{2.9}$$

Substituting (2.9) into (2.8) and taking into account (1.14), we get

$$\frac{1}{2}\frac{d}{dt}E_2^m(t) + \frac{\alpha}{2}\|\nabla v_t^m\|_2^2 \le C\|\nabla v^m\|_2^2 + C\left(|\gamma'| + |\gamma''|\right)E_2^m(t), \tag{2.10}$$

where

$$E_2^m(t) = \|\nabla v_t^m\|_2^2 + \gamma^{-2}M\left(\gamma^{n-2}\|\nabla v^m\|_2^2\right)\|\Delta v^m\|_2^2 - \left(\int_0^t g(s)\gamma^{-2}(s)\,ds\right)\|\Delta v^m\|_2^2.$$

From (2.7), it follows that inequality (2.10) can be written as

$$E_2^m(t) + \alpha\int_0^t \|\nabla v_s^m\|_2^2\,ds \le C_1 + C_2\int_0^t \left(|\gamma'| + |\gamma''|\right)E_2^m(s)\,ds.$$

Using Gronwall's Lemma and taking into account (1.13), we get

$$E_2^m(t) + \alpha\int_0^t \|\nabla v_s^m\|_2^2\,ds \le C, \qquad \forall t\in[0,T], \quad \forall m\in\mathbb{N}. \tag{2.11}$$

A Priori ESTIMATE III. From equation (2.2), we get

$$\|v_{tt}^m(0)\|_2^2 \le C, \qquad \forall m\in\mathbb{N}. \tag{2.12}$$

Finally, we differentiate (2.2) with respect to t, multiply it by $g''_{jm}(t)$, and use similar arguments as (2.7) to obtain

$$E_3^m(t) + \int_0^t \|v_{ss}^m\|_2^2 \, ds \le C, \qquad \forall\, t \in [0,T], \quad \forall\, m \in \mathbb{N}, \tag{2.13}$$

where

$$E_3^m(t) = \|v_{tt}^m\|_2^2 + \gamma^{-2} M\left(\gamma^{n-2}\|\nabla v^m\|_2^2\right)\|\nabla v_t^m\|_2^2 + g\Box\frac{\nabla v_t^m}{\gamma} - \left(\int_0^t g(s)\gamma^{-2}(s)\, ds\right)\|\nabla v_t^m\|_2^2.$$

From estimates (2.7), (2.11), and (2.13), it follows that v^m converge strongly to $v \in L^2(0,\infty : H_0^1(\Omega))$. Moreover, since $M \in C^1(0,\infty)$ and ∇v^m is bounded in $L^\infty(0,\infty : L^2(\Omega)) \cap L^2(0,\infty : L^2(\Omega))$, we have

$$\int_0^t \left| M\left(\gamma^{n-2}\|\nabla v^m\|_2^2\right) - M\left(\gamma^{n-2}\|\nabla v\|_2^2\right)\right| ds \le C \int_0^t \|v^m - v\|_{H^1(\Omega)}^2 \, ds,$$

where C is a positive constant independent of m' at t, so that

$$M\left(\gamma^{n-2}\|\nabla v^m\|_2^2\right)(\Delta v^m, w_j) \to M\left(\gamma^{n-2}\|\nabla v\|_2^2\right)(\Delta v, w_j).$$

Therefore, v satisfies

$$\begin{aligned} v &\in L^\infty\left(0,\infty : H_0^1(\Omega)\right), \\ v_t &\in L^\infty\left(0,\infty : H_0^1(\Omega)\right), \\ v_{tt} &\in L^\infty\left(0,\infty : L^2(\Omega)\right). \end{aligned}$$

Letting $m \to \infty$ in equation (2.2), we conclude that v satisfies (1.7) in the sense of $L^\infty(0,\infty : L^2(\Omega))$. Therefore, we have

$$v \in L^\infty\left(0,\infty : H_0^1(\Omega) \cap H^2(\Omega)\right).$$

The uniqueness follows by using standard arguments. ∎

To show the existence in noncylindrical domain, we return to our original problem in the noncylindrical domain by using the change variable given in (1.4) by $(y,t) = \tau(x,t)$, $(x,t) \in \hat{Q}$. Let v be the solution obtained from Theorem 2.1 and u defined by (1.6), then u belongs to the class

$$u \in L^\infty\left(0,\infty : H_0^1(\Omega_t)\right), \tag{2.14}$$
$$u_t \in L^\infty\left(0,\infty : H_0^1(\Omega_t)\right), \tag{2.15}$$
$$u_{tt} \in L^\infty\left(0,\infty : L^2(\Omega_t)\right). \tag{2.16}$$

Denoting by

$$u(x,t) = v(y,t) = (v \circ \tau)(x,t),$$

then from (1.6) it is easy to see that u satisfies equation (1.1) in the sense of $L^\infty(0,\infty : L^2(\Omega_t))$. Let u_1, u_2 be two solutions of (1.1), and v_1, v_2 be the functions obtained through the diffeomorphism τ given by (1.4). Then v_1, v_2 are solutions of (1.7). By the uniqueness result Theorem 2.1, we have $v_1 = v_2$, so that $u_1 = u_2$. Therefore, we have the following result.

THEOREM 2.2. *Let us take $u_0 \in H_0^1(\Omega_0) \cap H^2(\Omega_0)$, $u_1 \in H_0^1(\Omega_0)$ and let us suppose that assumptions (1.11)–(1.13) and (2.1) hold. Then there exists a unique solution u of problem (1.1)–(1.3) satisfying (2.9)–(2.11) and equation (1.1) in the sense of $L^\infty(0,\infty : L^2(\Omega_t))$.* ∎

3. EXPONENTIAL DECAY

In this section, we show that the solution of system (1.1)–(1.3) decays exponentially. To this end, we will assume that the memory g satisfies

$$-c_1 g(t) \leq g'(t) \leq -c_2 g(t), \tag{3.1}$$

for all $t \geq 0$, with positive constants c_1, c_2. Additionally, we assume that the function $\gamma(\cdot)$ satisfies the conditions

$$\gamma' \leq 0, \qquad t \geq 0, \qquad n > 2, \tag{3.2}$$

$$0 < \max_{0 \leq t < \infty} |\gamma'(t)| \leq \frac{1}{d}, \tag{3.3}$$

where $d = \operatorname{diam}(\Omega)$. Condition (3.3) implies that our domain is "time like" in the sense that

$$|\underline{\nu}| < |\overline{\nu}|,$$

where $\underline{\nu}$ and $\overline{\nu}$ denote the t-component and x-component of the outer unit normal of $\hat{\sum}$. Let us suppose that

$$m_0 - \int_0^\infty g(s)\, ds = \beta_1 > 0. \tag{3.4}$$

To facilitate our calculations, we introduce the following notation:

$$(g \Box \nabla u)(t) = \int_{\Omega_t} \int_0^t g(t-s) |\nabla u(t) - \nabla u(\cdot, s)|^2 \, ds\, dx.$$

First of all, we will prove the following two lemmas that will be used in the sequel.

LEMMA 3.1. *Let $F(\cdot,\cdot)$ be the smooth function defined in $\Omega_t \times [0,\infty[$ $(t \in [0,\infty[)$. Then,*

$$\frac{d}{dt} \int_{\Omega_t} F(x,t)\, dx = \int_{\Omega_t} \frac{d}{dt} F(x,t)\, dx + \frac{\gamma'}{\gamma} \int_{\Gamma_t} F(x,t)(x \cdot \overline{\nu})\, d\Gamma_t, \tag{3.5}$$

where $\overline{\nu}$ is the x-component of the unit normal exterior ν.

PROOF. We have by a change variable $x = \gamma(t) y$, $y \in \Omega$,

$$\begin{aligned} \frac{d}{dt} \int_{\Omega_t} F(x,t)\, dx &= \frac{d}{dt} \int_\Omega F(\gamma(t) y, t) \gamma^n(t)\, dy \\ &= \int_\Omega \left(\frac{\partial F}{\partial t} \right) \gamma^n(t)\, dy + \sum_{i=1}^n \int_\Omega \frac{\gamma'}{\gamma} x_i \left(\frac{\partial F}{\partial t} \right) \gamma^n(t)\, dy \\ &\quad + n \int_\Omega \gamma'(t) \gamma^{n-1}(t) F(\gamma(t) y, t)\, dy. \end{aligned}$$

If we return to the variable x, we get

$$\frac{d}{dt} \int_{\Omega_t} F(x,t)\, dx = \int_{\Omega_t} \frac{\partial F}{\partial t}\, dx + \frac{\gamma'}{\gamma} \int_{\Omega_t} x \cdot \nabla F(x,t)\, dx + n \frac{\gamma'}{\gamma} \int_{\Omega_t} F(x,t)\, dx.$$

Integrating by part in the last equality, we obtain formula (3.5). ∎

LEMMA 3.2. *For any function $g \in C^1(\mathbb{R}_+)$ and $u \in C^1((0,T) : H^2(\Omega_t))$, we have*

$$\int_{\Omega_t}\int_0^t g(t-s)\nabla u(\cdot,s)\cdot\nabla u_t\,ds\,dx = -\frac{1}{2}g(t)\int_{\Omega_t}|\nabla u(t)|^2\,dx + \frac{1}{2}g'\Box\nabla u$$
$$-\frac{1}{2}\frac{d}{dt}\left[g\Box\nabla u - \left(\int_0^t g(s)\,ds\right)\int_{\Omega_t}|\nabla u|^2\right]$$
$$+\frac{\gamma'}{2\gamma}\int_{\Gamma_t}\int_0^t g(t-s)|\nabla u(t)-\nabla u(\cdot,s)|^2\,(\overline{\nu}\cdot x)\;d\Gamma_t.$$

The proof of this lemma follows by differentiating the term $g\Box\nabla u$ and applying Lemma 3.1.
Let us introduce the functional

$$E(t) = \|u_t\|^2_{L^2(\Omega_t)} + \hat{M}\left(\|\nabla u\|^2_{L^2(\Omega_t)}\right) - \left(\int_0^t g(s)\,ds\right)\|\nabla u\|^2_{L^2(\Omega_t)} + g\Box\nabla u.$$

LEMMA 3.3. *Let us take $u_0 \in H_0^1(\Omega_0)\cap H^2(\Omega_0)$, $u_1 \in H_0^1(\Omega_0)$ and let us suppose that assumptions (1.11)–(1.13) and (2.1) hold. Then any regular solution of system (1.1)–(1.3) satisfies*

$$\frac{d}{dt}E(t) + 2\alpha\|u_t\|^2_{L^2(\Omega_t)} - \int_{\Gamma_t}\frac{\gamma'}{\gamma}(\overline{\nu}\cdot x)\left(|u_t|^2 + M\left(\|\nabla u\|^2_{L^2(\Omega_t)}\right)|\nabla u|^2\right)\,d\Gamma_t$$
$$-\int_{\Gamma_t}\frac{\gamma'}{\gamma}(\overline{\nu}\cdot x)\int_0^t g(t-s)|\nabla u(t)-\nabla u(\cdot,s)|^2\,ds\,d\Gamma_t$$
$$= -\int_{\Omega_t} g(t)|\nabla u|^2\,dx + g'\Box\nabla u.$$

PROOF. Multiplying equation (1.1) by u_t, performing an integration by parts over Ω_t, and using Lemma 3.1, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_t\|^2_{L^2(\Omega_t)} + \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2_{L^2(\Omega_t)} + \alpha\|u_t\|^2_{L^2(\Omega_t)}\\ &\quad-\int_{\Omega_t}\int_0^t g(t-s)\nabla u(\cdot,s)\,ds\cdot\nabla u_t\,dx\\ &\quad-\int_{\Gamma_t}\frac{\gamma'}{2\gamma}(\overline{\nu}\cdot x)\left(|u_t|^2+|\nabla u|^2\right)\,d\Gamma_t = 0.\end{aligned}\tag{3.6}$$

Taking into account Lemma 3.1, we have

$$\begin{aligned}\frac{1}{2}M\left(\|\nabla u\|^2_{L^2(\Omega_t)}\right)\int_{\Omega_t}\frac{d}{dt}|\nabla u|^2\,dx &= \frac{1}{2}\frac{d}{dt}\hat{M}\left(\|\nabla u\|^2_{L^2(\Omega_t)}\right)\\ &\quad-\frac{\gamma'}{2\gamma}\int_{\Gamma_t}M\left(\|\nabla u\|^2_{L^2(\Omega_t)}\right)|\nabla u|^2\,(\overline{\nu}\cdot x)\;d\Gamma_t.\end{aligned}\tag{3.7}$$

Substituting equality (3.6) into (3.5) and taking into account Lemma 3.2, we conclude the proof. ∎

Let us consider the functional

$$\psi(t) = 2\int_{\Omega_t} u_t u\,dx + \alpha\|u_t\|^2_{L^2(\Omega_t)}.$$

LEMMA 3.4. *Let us take $u_0 \in H_0^1(\Omega_0) \cap H^2(\Omega_0)$, $u_1 \in H_0^1(\Omega_0)$ and let us suppose that assumptions (1.11)–(1.13) and (2.1) hold. Then any regular solution of system (1.1)–(1.3) satisfies*

$$\frac{1}{2}\frac{d}{dt}\psi(t) \leq \|u_t\|^2_{L^2(\Omega_t)} - \hat{M}\left(\|\nabla u\|^2_{L^2(\Omega_t)}\right) + \left(\int_0^t g(s)\,ds\right)\|\nabla u\|^2_{L^2(\Omega_t)} + \|\nabla u\|_{L^2(\Omega_t)}\left(\int_0^t g(s)\,ds\right)^{1/2}(g\Box\nabla u)^{1/2}.$$

PROOF. Multiplying (1.1) by u, integrating over Ω_t, and using (1.14), we obtain

$$\frac{1}{2}\frac{d}{dt}\psi(t) = \|u_t\|^2_{L^2(\Omega_t)} - \hat{M}\left(\|\nabla u\|^2_{L^2(\Omega_t)}\right) + \int_{\Omega_t}\int_0^t g(t-s)\nabla u(\cdot,s)\cdot\nabla u\,ds\,dx.$$

Noting that

$$\int_{\Omega_t}\int_0^t g(t-s)\nabla u(\cdot,s)\cdot\nabla u\,ds\,dx = \int_{\Omega_t}\int_0^t g(t-s)(\nabla u(\cdot,s)-\nabla u(t))\cdot\nabla u\,ds\,dx + \int_{\Omega_t}\left(\int_0^t g(s)\,ds\right)|\nabla u|^2\,dx,$$

and taking into account that

$$\left|\int_{\Omega_t}\int_0^t g(t-s)(\nabla u(\cdot,s)-\nabla u(t))\cdot\nabla u\,dx\right| \leq \|\nabla u\|_{L^2(\Omega_t)}\left(\int_0^t g(s)\,ds\right)^{1/2}(g\Box\nabla u)^{1/2},$$

we conclude the proof. ∎

Let us introduce the functional

$$\mathcal{L}(t) = NE(t) + \psi(t), \tag{3.8}$$

with $N > 0$. It is not difficult to see that $\mathcal{L}(t)$ verifies

$$k_0E(t) \leq \mathcal{L}(t) \leq k_1E(t), \tag{3.9}$$

for k_0 and k_1 positive constants. Now we are in a position to show the main result of this paper.

THEOREM 3.1. *Let us take $u_0 \in H_0^1(\Omega_0)\cap H^2(\Omega_0)$, $u_1 \in H_0^1(\Omega_0)$, and let us suppose that assumptions (1.12), (1.13), (2.1), (3.2), and (3.3) hold. Then any regular solution of system (1.1)–(1.3) satisfies*

$$E(t) \leq Ce^{-\xi t}E(0), \qquad \forall\, t \geq 0,$$

where C and ξ are positive constants.

PROOF. Using Lemmas 3.3 and 3.4, we get

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \leq{}& -2N\alpha\|u_t\|^2_{L^2(\Omega_t)} - C_1Ng\Box\nabla u + \|u_t\|^2_{L^2(\Omega_t)}\\ &-\hat{M}\left(\|\nabla u\|^2_{L^2(\Omega_t)}\right) + \left(\int_0^t g(s)\,ds\right)\|\nabla u\|^2_{L^2(\Omega_t)}\\ &+\|\nabla u\|^2_{L^2(\Omega_t)}\left(\int_0^t g(s)\,ds\right)^{1/2}(g\Box\nabla u)^{1/2}.\end{aligned}$$

Using Young inequality, we obtain for $\epsilon > 0$

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \leq & -2N\alpha\|u_t\|^2_{L^2(\Omega_t)} - C_1 N g\Box\nabla u + \|u_t\|^2_{L^2(\Omega_t)} \\ & -\hat{M}\left(\|\nabla u\|^2_{L^2(\Omega_t)}\right) + \left(\int_0^t g(s)\,ds\right)\|\nabla u\|^2_{L^2(\Omega_t)} \\ & +\frac{\epsilon}{2}\|\nabla u\|^2_{L^2(\Omega_t)} + \frac{\|g\|_{L^1(0,\infty)}}{2\epsilon} g\Box\nabla u.\end{aligned}$$

Choosing N large enough and ϵ small, we obtain

$$\frac{d}{dt}\mathcal{L}(t) \leq -\lambda_0 E(t), \tag{3.10}$$

where λ_0 is a positive constant independent of t. From (3.9) and (3.10) it follows that

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-(\lambda_0/k_1)t}, \qquad \forall\, t \geq 0.$$

From equivalence relation (3.9), our conclusion follows. ∎

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