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Patterns in parabolic problems with nonlinear boundary conditions

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Abstract

We obtain existence of asymptotically stable nonconstant equilibrium solutions for semilinear parabolic equations with nonlinear boundary conditions on small domains connected by thin channels. We prove the convergence of eigenvalues and eigenfunctions of the Laplace operator in such domains. This information is used to show that the asymptotic dynamics of the heat equation in this domain is equivalent to the asymptotic dynamics of a system of two ordinary differential equations diffusively (weakly) coupled. The main tools employed are the invariant manifold theory and a uniform trace theorem.

Keywords: Semilinear parabolic problems; Nonlinear boundary conditions; Dumbbell domains; Stable nonconstant equilibria; Invariant manifolds

1. Introduction

In this paper we study the existence of (asymptotically) stable nonconstant equilibrium solutions for semilinear parabolic problems of the form

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$$\begin{cases} u_t = \dfrac{\delta}{\epsilon^{N-1}} \Delta u + f(u) & \text{in } \Omega_\epsilon, \\ \dfrac{\delta}{\epsilon^{N-1}} \dfrac{\partial u}{\partial n} = g(u) & \text{in } \partial\Omega_\epsilon. \end{cases} \tag{1.1}$$

Here Ω_ϵ is a bounded smooth domain consisting of two fixed disconnected parts and a channel (ϵ-dependent) connecting them, $\frac{\partial}{\partial n}$ is the outer normal derivative on the boundary $\partial\Omega_\epsilon$ of Ω_ϵ, $\epsilon \in (0, 1]$ is a parameter, $\delta > 0$ is a constant and f, g are nonlinear functions satisfying certain growth and dissipativeness conditions that will be specified later.

It is well known that stable nonconstant equilibrium solutions do not exist for parabolic problems of the form

$$\begin{cases} u_t = \Delta u + f(u) & \text{in } D, \\ \dfrac{\partial u}{\partial n} = 0 & \text{in } \partial D, \end{cases} \tag{1.2}$$

when D is convex (see [10,21]). Therefore it is natural to seek for existence of stable nonconstant equilibrium solutions of (1.2) when D is not convex. Of course stable nonconstant equilibrium to (1.2) exists when D is disconnected and it becomes natural to investigate the existence of stable nonconstant equilibrium for (1.2) in domains consisting of two disconnected parts connected by a thin channel (dumbbell type domains). In [21], H. Matano shows an important result on existence of patterns for (1.1) and Ω_ϵ is a dumbbell type domain (as in Fig. 1). It also becomes important to detect which is the limiting problem when the channel shrinks to a one-dimensional domain and which dynamics of the limiting problem can be observed in the dumbbell type domains. In this direction are the works of Hale and Vegas [14], Vegas [24] and Jimbo [16–18].

In [11], N. Cònsul and J. Solà-Morales extended Matano's results proving the existence of stable nonconstant equilibria for diffusion equations with nonlinear boundary conditions

$$\begin{cases} u_t = \Delta u & \text{in } D, \\ \dfrac{\partial u}{\partial n} = kf(u) & \text{in } \partial D. \end{cases} \tag{1.3}$$

Later, in [12], they obtain an abstract result on the stability of local minima of semilinear problems of the form $u_t = Au + F(u)$ and apply this result to obtain stable nonconstant equilibrium to (1.1), (1.2) and to strongly damped semilinear wave equations with homogeneous Neumann boundary conditions.

Jimbo and Morita in [19] studied the eigenvalue problem for the Laplacian with Neumann boundary conditions in a domain $\Omega \subset \mathbb{R}^N$ that consists of k fixed (disconnected) domains and thin channels joining them. The volume of each of thin channels is controlled by a small parameter $\epsilon > 0$, and these channels shrink to a line segment as ϵ approaches zero (some of the channels

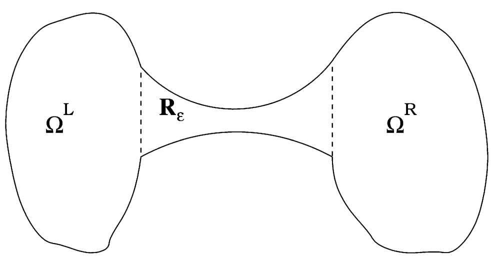

Fig. 1. Domain Ω_ϵ.

may be empty). They characterize the eigenvalues and eigenfunctions for the operator $-\Delta$ in this domain. Using the invariant manifold theory, they show in [23] (see also [22]) that the dynamics of (1.2) in such domains is equivalent to the dynamics of an explicitly given system of ordinary differential equations on an invariant manifold.

Also, there are several works where eigenvalue problems for elliptic operators on varying domains or with varying diffusivity are studied (see, for example, [5–8,14,18,24]).

Here, we use the invariant manifold theory and a uniform trace theorem to show that the dynamics of (1.1) is equivalent to the dynamics of a 2-dimensional ordinary differential equations on an invariant manifold.

Patterns obtained by nonlinear boundary conditions are specially interesting for, in a control problem involving reaction–diffusion models the boundary is, in general, the only accessible part of the domain.

The difference between our result and the results of Consul and Sóla-Morales [11,12], apart from the technique, is that the stable nonconstant equilibria obtained here are asymptotically stable while in [11,12] they are only stable.

Let us now introduce the assumptions on the nonlinearities f, g. Suppose that $f, g:\mathbb{R} \to \mathbb{R}$ are twice continuously differentiable functions satisfying

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} < 0 \quad \text{and} \quad \limsup_{|u|\to\infty} \frac{g(u)}{u} < 0.$$

In addition, assume some growth assumptions to ensure local well posedness of (1.1) (see [4]). Under these assumptions, problem (1.1) has a global attractor $\mathcal{A}_\epsilon$, $0 \leqslant \epsilon \leqslant \epsilon_0$, and $\sup_{0\leqslant\epsilon\leqslant\epsilon_0} \sup_{u\in\mathcal{A}_\epsilon} \|u\|_{L^\infty(\Omega_\epsilon)} < \infty$ (see [4]). This enables us to cut f and g in such a way that the attractors $\mathcal{A}_\epsilon$ remain the same and in such a way that f, g together with its derivatives up to second order are bounded. Hereafter we assume (without loss of generality) that f, g are bounded functions with bounded derivatives up to second order.

Next we specify the domain $\Omega_\epsilon \subset \mathbb{R}^N$ $(N \geqslant 2)$. It has a fixed part Ω and a parameter dependent part R_ϵ, that is, $\Omega_\epsilon = \Omega \cup R_\epsilon$, where $\Omega = \Omega^{\mathrm{L}} \cup \Omega^{\mathrm{R}}$. We assume that Ω^{L}, Ω^{R} and R_ϵ satisfy the following conditions:

(I) Ω^{L}, Ω^{R} are bounded smooth domains in $\mathbb{R}^N$ with disjoint closures.

(II) There is an orthogonal system of coordinates $x = (x_1, x_2, \ldots, x_N) = (x_1, x') \in \mathbb{R}^N$ such that the following conditions hold for some positive constant $\epsilon_0 > 0$:

$$\overline{\Omega}^{\mathrm{L}} \cap \big\{(x_1, x') \in \mathbb{R}^N\colon\ x_1 \geqslant 0,\ |x'| < \epsilon_0\big\} = \big\{(0, x') \in \mathbb{R}^N\colon\ |x'| < \epsilon_0\big\},$$

$$\overline{\Omega}^{\mathrm{R}} \cap \big\{(x_1, x') \in \mathbb{R}^N\colon\ x_1 \leqslant 1,\ |x'| < \epsilon_0\big\} = \big\{(1, x') \in \mathbb{R}^N\colon\ |x'| < \epsilon_0\big\}$$

and R_ϵ is expressed as follows

$$R_\epsilon = \big\{(x_1, x') \in \mathbb{R}^N\colon\ 0 < x_1 < 1,\ |x'| < \epsilon h(x_1)\big\},$$

where $h \in \mathcal{C}^0([0, 1]) \cap \mathcal{C}^1((0, 1))$ and $h(x_1) \geqslant 0$ for all $x_1 \in [0, 1]$.

Define the line segment

$$L = \bigcap_{\epsilon\in(0,1]} \overline{R}_\epsilon = \big\{(z, 0, \ldots, 0) \in \mathbb{R}^N\colon\ 0 \leqslant z \leqslant 1\big\} \cong \{z \in \mathbb{R}\colon\ 0 \leqslant z \leqslant 1\}$$

whose endpoints are $\mathbf{P}_0 = (0, 0, \ldots, 0)$ and $\mathbf{P}_1 = (1, 0, \ldots, 0)$.

Without loss of generality we assume that $|\Omega| = 1$ throughout the paper.

If we denote by $\{\lambda_n^\epsilon\}_{n=1}^\infty$ and $\{\varphi_n^\epsilon\}_{n=1}^\infty$ the set of eigenvalues and the orthonormalized eigenfunctions of the following eigenvalue problem

$$\begin{cases} -\Delta u = \lambda^\epsilon u & \text{in } \Omega_\epsilon = \Omega \cup R_\epsilon, \\ \dfrac{\partial u}{\partial n} = 0 & \text{in } \partial\Omega_\epsilon, \end{cases} \tag{1.4}$$

then we have the following result.

Theorem 1. *The following holds*:

$$\lambda_1^\epsilon = 0 \quad \textit{and} \quad \varphi_1^\epsilon = |\Omega_\epsilon|^{-1/2}, \quad \forall \epsilon > 0,$$

$$\lim_{\epsilon\to 0} \frac{\lambda_2^\epsilon}{\epsilon^{N-1}} = \mathrm{C_o}, \qquad \varphi_2^\epsilon \xrightarrow{\epsilon\to 0} \begin{cases} c_1^0 := -\sqrt{\dfrac{|\Omega^{\mathrm{R}}|}{|\Omega^{\mathrm{L}}|}}, & \textit{in } \Omega^{\mathrm{L}}, \\ c_2^0 := \sqrt{\dfrac{|\Omega^{\mathrm{L}}|}{|\Omega^{\mathrm{R}}|}}, & \textit{in } \Omega^{\mathrm{R}}, \end{cases} \quad \textit{in } L^2(\Omega) \textit{ and } L^2(\partial\Omega),$$

$$\sup_{\epsilon>0} \|\varphi_2^\epsilon\|_{L^\infty(\Omega_\epsilon)} < \infty,$$

$$\liminf_{\epsilon\to 0} \lambda_3^\epsilon > 0,$$

where

$$\mathrm{C_o} = \sigma_{N-1}\left(\frac{1}{|\Omega^{\mathrm{L}}|} + \frac{1}{|\Omega^{\mathrm{R}}|}\right)\theta, \quad \theta = \left\{\int_0^1 \frac{dx_1}{h^{N-1}(x_1)}\right\}^{-1}$$

and σ_{N-1} is the Lebesgue measure of the unit ball in $\mathbb{R}^{N-1}$.

A proof of this result (with stronger convergence properties for the eigenfunctions) is given in Appendix A.

If u is a solution of problem (1.1), consider the following decomposition

$$u(t,x) = u_1(t)\varphi_1^\epsilon(x) + u_2(t)\varphi_2^\epsilon(x) + \omega(t,x),$$

where u_1, u_2 and w are given by

$$u_1 = \int_{\Omega_\epsilon} u\varphi_1^\epsilon, \qquad u_2 = \int_{\Omega_\epsilon} u\varphi_2^\epsilon, \qquad \omega = u - u_1\varphi_1^\epsilon - u_2\varphi_2^\epsilon.$$

This decomposition induces the following decomposition of (1.1)

$$\dot u_1 = \int_{\Omega_\epsilon} f(u)\varphi_1^\epsilon + \int_{\partial\Omega_\epsilon} \gamma\big(g(u)\big)\gamma\big(\varphi_1^\epsilon\big),$$

$$\dot u_2 = -\delta\frac{\lambda_2^\epsilon}{\epsilon^{N-1}}u_2 + \int_{\Omega_\epsilon} f(u)\varphi_2^\epsilon + \int_{\partial\Omega_\epsilon} \gamma\big(g(u)\big)\gamma\big(\varphi_2^\epsilon\big),$$

$$\omega_t = \frac{\delta}{\epsilon^{N-1}}\Delta\omega + f(u) - \left[\int_{\Omega_\epsilon} f(u)\varphi_1^\epsilon + \int_{\partial\Omega_\epsilon} \gamma\big(g(u)\big)\gamma\big(\varphi_1^\epsilon\big)\right]\varphi_1^\epsilon$$

$$- \Big[\int_{\Omega_\epsilon} f(u)\varphi_2^\epsilon + \int_{\partial\Omega_\epsilon} \gamma\big(g(u)\big)\gamma\big(\varphi_2^\epsilon\big) \Big] \varphi_2^\epsilon,$$

$$\frac{\delta}{\epsilon^{N-1}} \frac{\partial\omega}{\partial n} = g\big(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \omega\big), \tag{1.5}$$

where $\gamma(\varphi)$ denotes the trace of φ.

In what follows, we give a heuristic argument to find the limiting ode that should contain the asymptotic behavior of (1.5). To that end, we introduce a very simple lemma from semigroup theory. Its proof follows immediately from the variation of constants formula.

Lemma 2. *Let X be a Banach space and $A: D(A) \subset X \to X$ be the generator of a strongly continuous semigroup of bounded linear operators $\{e^{At}\colon\ t \geqslant 0\}$. If $f: X \to X$ is a locally Lipschitz continuous map which satisfies $\sup_{x\in X} M\|f(x)\|_X \leqslant \mathcal{N} < \infty$ and, for some $\nu > 0$, $\|e^{At}\|_{L(X)} \leqslant Me^{-\nu t}$ and $x(t, x_0)$ is the global solution to $\dot{x} = Ax + f(x)$, $x(0) = x_0$, then*

$$\big\|x(t, x_0)\big\|_X \leqslant Me^{-\nu t}\|x_0\|_X + \frac{\mathcal{N}}{\nu}.$$

After this lemma we see that, asymptotically, the norm of the solution is proportional to $\frac{1}{\nu}$ and, if ν is very large (compared to $\mathcal{N}$), all solutions are asymptotically small.

Since the third eigenvalue $\frac{\lambda_3^\epsilon}{\epsilon^{N-1}}$ goes to infinity when $\epsilon \to 0$, we guess that ω is not important in the asymptotic behavior (1.1) (for very small values of ϵ) and we have

$$\begin{cases} \dot{u}_1 \sim \displaystyle\int_{\Omega_\epsilon} f\big(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon\big)\varphi_1^\epsilon + \int_{\partial\Omega_\epsilon} g\big(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon\big)\varphi_1^\epsilon, \\ \dot{u}_2 \sim -\delta \frac{\lambda_2^\epsilon}{\epsilon^{N-1}} u_2 + \displaystyle\int_{\Omega_\epsilon} f\big(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon\big)\varphi_2^\epsilon + \int_{\partial\Omega_\epsilon} g\big(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon\big)\varphi_2^\epsilon. \end{cases}$$

According to Theorem 1, we should have the limit system given by

$$\begin{cases} \dot{u}_1 = \displaystyle\int_{\Omega} f(u_1 + u_2\phi_2) + \int_{\partial\Omega} g(u_1 + u_2\phi_2), \\ \dot{u}_2 = -\delta\, \mathrm{C_o} u_2 + \displaystyle\int_{\Omega} f(u_1 + u_2\phi_2)\phi_2 + \int_{\partial\Omega} g(u_1 + u_2\phi_2)\phi_2, \end{cases} \tag{1.6}$$

where $\phi_2 = c_1^0\chi_{\Omega^{\mathrm{L}}} + c_2^0\chi_{\Omega^{\mathrm{R}}}$. Or, in terms of c_1^0, c_2^0 and $\mathrm{C_o}$ given in Theorem 1,

$$\begin{cases} \dot{u}_1 = \big|\Omega^{\mathrm{L}}\big| f\big(u_1 + c_1^0 u_2\big) + \big|\Omega^{\mathrm{R}}\big| f\big(u_1 + c_2^0 u_2\big) \\ \qquad + \big|\partial\Omega^{\mathrm{L}}\big| g\big(u_1 + c_1^0 u_2\big) + \big|\partial\Omega^{\mathrm{R}}\big| g\big(u_1 + c_2^0 u_2\big), \\ \dot{u}_2 = -\delta \mathrm{C_o} u_2 + \big|\Omega^{\mathrm{L}}\big| c_1^0 f\big(u_1 + c_1^0 u_2\big) + \big|\Omega^{\mathrm{R}}\big| c_2^0 f\big(u_1 + c_2^0 u_2\big) \\ \qquad + \big|\partial\Omega^{\mathrm{L}}\big| c_1^0 g\big(u_1 + c_1^0 u_2\big) + \big|\partial\Omega^{\mathrm{R}}\big| c_2^0 g\big(u_1 + c_2^0 u_2\big). \end{cases} \tag{1.7}$$

The variables u_1 and u_2 may not be the best choice of variables to study this problem. A better choice of variables is probably that which reflects the averages over Ω^{L} and Ω^{R}. To relate u_1 and u_2 with these averages we consider

$$v_1 = \left|\Omega^{\mathrm{L}}\right|^{-1} \int\limits_{\Omega^{\mathrm{L}}} u(x)\,dx \quad \text{and} \quad v_2 = \left|\Omega^{\mathrm{R}}\right|^{-1} \int\limits_{\Omega^{\mathrm{R}}} u(x)\,dx.$$

Thus

$$\begin{cases} u_1 = \left|\Omega^{\mathrm{L}}\right| v_1 + \left|\Omega^{\mathrm{R}}\right| v_2, \\ u_2 = -\left(\left|\Omega^{\mathrm{L}}\right|\left|\Omega^{\mathrm{R}}\right|\right)^{1/2}(v_1 - v_2) \end{cases}$$

and

$$\begin{cases} v_1 = u_1 + c_1^0 u_2, \\ v_2 = u_1 + c_2^0 u_2. \end{cases} \tag{1.8}$$

With the variables (1.8), system (1.7) becomes

$$\begin{cases} \dot{v}_1 = -\delta \mathrm{C_o}\left|\Omega^{\mathrm{R}}\right|(v_1 - v_2) + f(v_1) + \dfrac{|\partial\Omega^{\mathrm{L}}|}{|\Omega^{\mathrm{L}}|} g(v_1), \\ \dot{v}_2 = \delta \mathrm{C_o}\left|\Omega^{\mathrm{L}}\right|(v_1 - v_2) + f(v_2) + \dfrac{|\partial\Omega^{\mathrm{R}}|}{|\Omega^{\mathrm{R}}|} g(v_2). \end{cases} \tag{1.9}$$

Now our aim is to show that the dynamics of (1.1) can be described by the dynamics of (1.9). With this in hand, we will try to produce asymptotically stable equilibrium solutions (v_1, v_2) for (1.9) with $v_1 \neq v_2$ and they should correspond to asymptotically stable equilibrium solutions to (1.1) which are nonconstant for v_1 reflects the average in Ω^{L} and v_2 reflects the average in Ω^{R}.

This paper is organized as follows. In Section 2 we state the main results of the paper, in Section 3 we prove the main results and in Section 4 we give an example for which we can obtain stable nonconstant equilibria from the boundary nonlinearity. At the end of the paper we include three appendixes where we prove Theorem 1 (Appendix A), an invariant manifold theorem to take into account the dependence on ϵ (Appendix B) and a uniform (with respect to ϵ) trace theorem (Appendix C).

2. Main results

In this section we will state the main results of this paper. The proofs will be given in Section 3.

Before we proceed, we need to introduce some terminology. For $\epsilon > 0$, let $X_\epsilon = L^2(\Omega_\epsilon)$ and $L_\epsilon : D(L_\epsilon) \subset X_\epsilon \to X_\epsilon$ be the operators defined by

$$D(L_\epsilon) = \left\{ u \in H^2(\Omega_\epsilon):\ \frac{\partial u}{\partial n} = 0 \right\},$$
$$L_\epsilon u = \frac{\delta}{\epsilon^{N-1}} \Delta u.$$

It is well known that L_ϵ is an unbounded, self adjoint, nonpositive definite operator which has compact resolvent. It follows that $-L_\epsilon$ is a sectorial operator and, for $\zeta > 0$ fixed, we can define the fractional powers $(-L_\epsilon + \zeta I)^\alpha$ and corresponding fractional power spaces X_ϵ^α

($X_\epsilon^\alpha = D((-L_\epsilon + \zeta I)^\alpha)$ endowed with the graph norm), $\alpha > 0$. X_ϵ^α is a Hilbert space with the inner product

$$\langle \varphi, \psi \rangle_\alpha = \int_{\Omega_\epsilon} (-L_\epsilon + \zeta I)^\alpha \varphi (-L_\epsilon + \zeta I)^\alpha \psi.$$

Then, $X_\epsilon^{1/2} = H^1(\Omega_\epsilon)$ (with norm $\|\psi\|^2_{X_\epsilon^{1/2}} = \frac{\delta}{\epsilon^{N-1}} \|\nabla \psi\|^2_{L^2(\Omega_\epsilon)} + \zeta \|\psi\|^2_{L^2(\Omega_\epsilon)}$) and $X_\epsilon^1 = D(L_\epsilon)$ with the graph norm.

Also, let

$$V = \operatorname{span}\bigl[\varphi_1^\epsilon, \varphi_2^\epsilon\bigr], \qquad V^\perp = \left\{ \varphi \in L^2(\Omega_\epsilon)\colon \int_{\Omega_\epsilon} \varphi \psi \, dx = 0, \text{ for all } \psi \in V \right\}$$

and $V_\alpha^\perp = V^\perp \cap X_\epsilon^\alpha = \{\varphi \in X_\epsilon^\alpha\colon \langle \varphi, \psi \rangle_\alpha = 0, \text{ for all } \psi \in V\}$. Define $A_\epsilon : D(A_\epsilon) \subset V^\perp \to V^\perp$ by $D(A_\epsilon) = V_1^\perp$ and $A_\epsilon \omega = L_\epsilon \omega$ for all $\omega \in D(A_\epsilon)$; that is, the restriction of L_ϵ to $V^\perp$. We also denote by A_ϵ its realization in $X_\epsilon^{1/2}$. Also define

$$B_\epsilon = \begin{bmatrix} 0 & 0 \\ 0 & -\delta \dfrac{\lambda_2^\epsilon}{\epsilon^{N-1}} \end{bmatrix}.$$

If $\rho = \sup_{0<\epsilon\leqslant 1} \delta \frac{\lambda_2^\epsilon}{\epsilon^{N-1}}$, $\beta(\epsilon) = \frac{\lambda_3^\epsilon}{\epsilon^{N-1}}$ and $\alpha \geqslant 0$, the operators A_ϵ and B_ϵ satisfy

$$\begin{aligned} &\bigl\|e^{A_\epsilon t} w\bigr\|_{V_\alpha^\perp} \leqslant e^{-\beta(\epsilon)t} \|w\|_{V_\alpha^\perp}, \quad t \geqslant 0, \\ &\bigl\|e^{A_\epsilon t} w\bigr\|_{V_\alpha^\perp} \leqslant t^{-\alpha} e^{-\beta(\epsilon)t} \|w\|_{V^\perp}, \quad t > 0, \\ &\bigl\|e^{B_\epsilon t} z\bigr\|_{\mathbb{R}^2} \leqslant e^{-\rho t} \|z\|_{\mathbb{R}^2}, \quad t \leqslant 0. \end{aligned} \tag{2.1}$$

Note that there is a constant c, independent of ϵ, such that

$$\|\psi\|_{H^{2\alpha}(\Omega_\epsilon)} \leqslant c \epsilon^{\alpha(N-1)} \|\psi\|_{V_\alpha^\perp}, \qquad \frac{1}{2} \geqslant \alpha \geqslant 0, \tag{2.2}$$

for all $\psi \in V_\alpha^\perp$. Also note that, according to (1.5), if $\psi \in V_{1/2}^\perp$, then

$$\langle w_t, \psi \rangle = -\frac{\delta}{\epsilon^{N-1}} \langle \nabla w, \nabla \psi \rangle + \int_{\Omega_\epsilon} f\bigl(u_1 \varphi_1^\epsilon + v_2 \varphi_2^\epsilon + w\bigr)\psi + \int_{\partial\Omega_\epsilon} g\bigl(u_1 \varphi_1^\epsilon + v_2 \varphi_2^\epsilon + w\bigr)\psi.$$

Proceeding as in [9] we can now rewrite system (1.5) as

$$\begin{cases} \dot\omega = A_\epsilon \omega + H_\epsilon^\Omega(y, \omega) + H_\epsilon^{\partial\Omega}(y, \omega), \\ \dot y = B_\epsilon y + G_\epsilon(y, \omega), \end{cases} \tag{2.3}$$

where

$$\begin{aligned} &y = (u_1, u_2)^\perp, \\ &u = u_1 \varphi_1^\epsilon + u_2 \varphi_2^\epsilon + \omega, \\ &G_\epsilon(y, \omega) = \bigl(G_0^\epsilon(y, \omega), G_1^\epsilon(y, \omega)\bigr)^\perp, \\ &G_0^\epsilon(y, \omega) = \int_{\Omega_\epsilon} f(u) \varphi_1^\epsilon + \int_{\partial\Omega_\epsilon} g(u) \varphi_1^\epsilon, \end{aligned}$$

$$G_1^\epsilon(y,\omega)=\int\limits_{\Omega_\epsilon} f(u)\varphi_2^\epsilon+\int\limits_{\partial\Omega_\epsilon} g(u)\varphi_2^\epsilon,$$

$$\bigl\langle H_\epsilon^{\Omega}(y,\omega),\psi\bigr\rangle_{V_{-\alpha}^\perp,V_\alpha^\perp}=\int\limits_{\Omega_\epsilon} f\bigl(u_1\varphi_1^\epsilon+u_2\varphi_2^\epsilon+w\bigr)\psi,$$

$$\bigl\langle H_\epsilon^{\partial\Omega}(y,\omega),\psi\bigr\rangle_{V_{-\alpha}^\perp,V_\alpha^\perp}=\int\limits_{\partial\Omega_\epsilon} g\bigl(u_1\varphi_1^\epsilon+u_2\varphi_2^\epsilon+w\bigr)\psi.$$

We identify $u=u_1\varphi_\epsilon^1+u_2\varphi_\epsilon^2+w\in X_\epsilon^{1/2}$ with $(y,w)\in\mathbb{R}^2\times V_{1/2}^\perp$. This identification is an isomorphism which is bounded with bounded inverse uniformly with respect to ϵ. If f and g are bounded with bounded derivatives up to second order, $(y_0,w_0)\in\mathbb{R}^2\times V_{1/2}^\perp$, then there is a unique solution $(y,w)(t,(y_0,w_0))$ of (2.3) defined in $[0,\infty)$.

We can now state the main results of this paper.

Theorem 3. *There is a continuously differentiable map $\sigma_\epsilon:\mathbb{R}^2\to V_{1/2}^\perp$ such that*

$$S_\epsilon=\bigl\{(y,\omega):\ \omega=\sigma_\epsilon(y),\ y\in\mathbb{R}^2\bigr\}$$

is an exponentially attracting invariant manifold for (1.5). *The flow on S_ϵ is given by $u(t,x)=(y(t),\sigma_\epsilon(y(t)))$, where y is the solution of*

$$\dot y=B_\epsilon y+G_\epsilon\bigl(y,\sigma_\epsilon(y)\bigr). \tag{2.4}$$

Furthermore, $\sigma_\epsilon\to 0$ in $\mathcal{C}^1(\mathbb{R}^2,V_{1/2}^\perp)$, as $\epsilon\to 0$.

The following theorem tells us that the asymptotic dynamics of system (1.9) is equivalent to that of system (2.4). In particular, stable equilibria of (1.9) with $v_1\neq v_2$ corresponds to stable nonconstant equilibria of (1.1).

Theorem 4. *Assume that system* (1.9) *is structurally stable. Then for small enough ϵ, the flow on the invariant manifold given by* (2.4) *is topologically equivalent to the flow* (1.9).

Remark 5. Making $|\Omega^{\mathrm{L}}|=|\Omega^{\mathrm{R}}|$, $|\partial\Omega^{\mathrm{L}}|=|\partial\Omega^{\mathrm{R}}|$, $r:=\delta\mathrm{C_o}|\Omega^{\mathrm{L}}|=\delta\mathrm{C_o}|\Omega^{\mathrm{R}}|$ and

$$f(s)+\frac{|\partial\Omega^{\mathrm{L}}|}{|\Omega^{\mathrm{L}}|}g(s)=f(s)+\frac{|\partial\Omega^{\mathrm{R}}|}{|\Omega^{\mathrm{R}}|}g(s):=s-s^3$$

we can verify the conditions of [1, Theorem V, p. 395] for $r\neq\frac12$ and $r\neq\frac13$ (see [6, p. 400]). Hence, in this case, (1.9) is structurally stable. For the general case, see [13] for conditions ensuring that (1.9) is structurally stable.

3. Proof of the main results

In the proof of Theorem 3, we use the following uniform trace theorem whose proof is given in Appendix C.

Theorem 6 *(Uniform Trace Theorem). Let $\gamma : V_\alpha^\perp \to L^2(\partial\Omega_\epsilon)$ be the bounded linear operator which associates to each $w \in V_\alpha^\perp$ its trace $\gamma(w)$, $\alpha > \frac{1}{4}$. Then*

$$\|\gamma\|_{L(V_\alpha^\perp, L^2(\partial\Omega_\epsilon))} \leqslant \mathfrak{G}\epsilon^{(N-2)/2} \leqslant \mathfrak{G} \tag{3.1}$$

for some $\mathfrak{G} > 0$ independent of ϵ.

Proof of Theorem 3. Note that $\varphi_1^\epsilon(x) = |\Omega_\epsilon|^{-1/2} \leqslant 1$, $|\partial\Omega_\epsilon| \leqslant |\partial\Omega| + |\partial R_1|$ and that $m_2 := \sup_{\epsilon>0}\sup_{x\in\Omega_\epsilon}|\varphi_2^\epsilon(x)| < \infty$ (the last estimate follows from [4, Lemma B.1] and from Theorem 1). From our assumptions on f, g we have that

$$M_1 := \sup_{x\in\mathbb{R}}\bigl|f(x)\bigr| < \infty, \qquad L_1 := \sup_{x\in\mathbb{R}}\bigl|f'(x)\bigr| < \infty,$$
$$M_2 := \sup_{x\in\mathbb{R}}\bigl|g(x)\bigr| < \infty, \qquad L_2 := \sup_{x\in\mathbb{R}}\bigl|g'(x)\bigr| < \infty.$$

With this we have

$$\begin{aligned}\bigl|G_0^\epsilon(y,\omega)\bigr| &\leqslant \int_{\Omega_\epsilon}\bigl|f(y,w)\bigr|\bigl|\varphi_1^\epsilon\bigr| + \int_{\partial\Omega_\epsilon}\bigl|g(y,w)\bigr|\bigl|\varphi_1^\epsilon\bigr| \leqslant M_1|\Omega_\epsilon|^{1/2} + M_2\frac{|\partial\Omega_\epsilon|}{|\Omega_\epsilon|^{1/2}}\\ &\leqslant M_1\bigl(1+|R_1|\bigr)^{1/2} + M_2\bigl(|\partial\Omega| + |\partial R_1|\bigr) = N_1,\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\bigl|G_1^\epsilon(y,\omega)\bigr| &\leqslant M_1\int_{\Omega_\epsilon}\bigl|\varphi_2^\epsilon\bigr| + M_2\int_{\partial\Omega_\epsilon}\bigl|\varphi_2^\epsilon\bigr| \leqslant M_1|\Omega_\epsilon|^{1/2} + M_2 m_2|\partial\Omega_\epsilon|\\ &\leqslant M_1\bigl(1+|R_1|\bigr)^{1/2} + M_2 m_2\bigl(|\partial\Omega| + |\partial R_1|\bigr) = N_2.\end{aligned} \tag{3.3}$$

Hence, if $N_G = \sqrt{N_1^2 + N_2^2}$,

$$\bigl\|G_\epsilon(y,\omega)\bigr\|_{\mathbb{R}^2} \leqslant N_G.$$

To obtain global Lipschitz continuity, let $y_1 = (u_1,u_2)$, $y_2 = (v_1,v_2) \in \mathbb{R}^2$, $\omega_1,\omega_2 \in V_{1/2}^\perp$. Then

$$\begin{aligned}&\bigl|G_0^\epsilon(y_1,\omega_1) - G_0^\epsilon(y_2,\omega_2)\bigr|\\ &\quad\leqslant L_1\biggl[|u_1-v_1| + |u_2-v_2|\int_{\Omega_\epsilon}\bigl|\varphi_1^\epsilon\bigr|\bigl|\varphi_2^\epsilon\bigr| + \int_{\Omega_\epsilon}|\omega_1-\omega_2|\bigl|\varphi_1^\epsilon\bigr|\biggr]\\ &\qquad + L_2\biggl[|u_1-v_1|\int_{\partial\Omega_\epsilon}\bigl|\varphi_1^\epsilon\bigr|^2 + |u_2-v_2|\int_{\partial\Omega_\epsilon}\bigl|\varphi_1^\epsilon\bigr|\bigl|\varphi_2^\epsilon\bigr| + \int_{\partial\Omega_\epsilon}|\omega_1-\omega_2|\biggr]\\ &\quad\leqslant L_1\bigl[\sqrt{2}\,\|y_1-y_2\|_{\mathbb{R}^2} + c\epsilon^{(N-1)/2}\|\omega_1-\omega_2\|_{X_\epsilon^{1/2}}\bigr]\\ &\qquad + L_2\bigl[C_1\|y_1-y_2\|_{\mathbb{R}^2} + C_2\epsilon^{(N-2)/2}\|\omega_1-\omega_2\|_{X_\epsilon^{1/2}}\bigr],\end{aligned}$$

where $C_1 := \sqrt{2}(|\partial\Omega| + |\partial R_1|)\max\{1, m_2\}$, $C_2 = c\,(|\partial\Omega| + |\partial R_1|)^{1/2}\mathfrak{G}$ and we used Theorem 6 and (2.2). Hence there is a constant $\mathfrak{L}_1$, independent of ϵ, such that

$$\bigl|G_0^\epsilon(y_1,\omega_1) - G_0^\epsilon(y_2,\omega_2)\bigr| \leqslant \mathfrak{L}_1\bigl[\|y_1-y_2\|_{\mathbb{R}^2} + \|\omega_1-\omega_2\|_{V_{1/2}^\perp}\bigr].$$

Similarly, there is a constant $\mathfrak{L}_2$, independent of ϵ, such that

$$\left|G_1^\epsilon(y_1,\omega_1)-G_1^\epsilon(y_2,\omega_2)\right| \leqslant \mathfrak{L}_2\left[\|y_1-y_2\|_{\mathbb{R}^2}+\|\omega_1-\omega_2\|_{V_{1/2}^\perp}\right].$$

Thus if $\mathfrak{L}_G=\sqrt{\mathfrak{L}_1^2+\mathfrak{L}_2^2}$,

$$\left\|G_\epsilon(y_1,\omega_1)-G_\epsilon(y_2,\omega_2)\right\|_{\mathbb{R}^2} \leqslant \mathfrak{L}_G\left[\|y_1-y_2\|_{\mathbb{R}^2}+\|\omega_1-\omega_2\|_{V_{1/2}^\perp}\right].$$

It is easy to see that

$$\frac{\partial}{\partial u_1}G_0^\epsilon(u_1,u_2,\omega)=\int\limits_{\Omega_\epsilon} f'\left(u_1\varphi_1^\epsilon+u_2\varphi_2^\epsilon+w\right)\left(\varphi_1^\epsilon\right)^2+\int\limits_{\partial\Omega_\epsilon} g'\left(u_1\varphi_1^\epsilon+u_2\varphi_2^\epsilon+w\right)\left(\varphi_1^\epsilon\right)^2,$$

$$\frac{\partial}{\partial u_2}G_0^\epsilon(u_1,u_2,\omega)=\int\limits_{\Omega_\epsilon} f'\left(u_1\varphi_1^\epsilon+u_2\varphi_2^\epsilon+w\right)\varphi_1^\epsilon\varphi_2^\epsilon+\int\limits_{\partial\Omega_\epsilon} g'\left(u_1\varphi_1^\epsilon+u_2\varphi_2^\epsilon+w\right)\varphi_1^\epsilon\varphi_2^\epsilon,$$

$$\left(\frac{\partial}{\partial \omega}G_0^\epsilon(u_1,u_2,\omega)\right)\psi=\int\limits_{\Omega_\epsilon} f'\left(u_1\varphi_1^\epsilon+u_2\varphi_2^\epsilon+w\right)\varphi_1^\epsilon\psi+\int\limits_{\partial\Omega_\epsilon} g'\left(u_1\varphi_1^\epsilon+u_2\varphi_2^\epsilon+w\right)\varphi_1^\epsilon\psi$$

(for all $\psi\in V_{1/2}^\perp$) and, proceeding just as before, that they are Lipschitz continuous. Hence $G_0^\epsilon:\mathbb{R}^2\times V_{1/2}^\perp\to\mathbb{R}$ is continuously differentiable. Similarly $G_1^\epsilon:\mathbb{R}^2\times V_{1/2}^\perp\to\mathbb{R}$ is continuously differentiable.

Now for $H_\epsilon^\Omega:\mathbb{R}^2\times V_{1/2}^\perp\to V_{-\alpha}^\perp$, using (2.2), it is easy to see that

$$\left\|H_\epsilon^\Omega(y,\omega)\right\|_{V_{-\alpha}^\perp}\leqslant N_F\epsilon^{\alpha(N-1)}$$

and that

$$\left\|H_\epsilon^\Omega(y_1,\omega_1)-H_\epsilon^\Omega(y_2,\omega_2)\right\|_{V_{-\alpha}^\perp}\leqslant \epsilon^{\alpha(N-1)}\mathfrak{L}^\Omega\left(\|y_1-y_2\|_{\mathbb{R}^2}+\|\omega_1-\omega_2\|_{V_{1/2}^\perp}\right).$$

The derivatives of H_ϵ^Ω are given by

$$\left\langle\frac{\partial}{\partial u_1}H_\epsilon^\Omega(u_1,u_2,\omega)h,\psi\right\rangle_{V_{-\alpha}^\perp,V_\alpha^\perp}=h\int\limits_{\Omega_\epsilon} f'\left(u_1\varphi_1^\epsilon+u_2\varphi_2^\epsilon+w\right)\varphi_1^\epsilon\psi,\quad h\in\mathbb{R},$$

$$\left\langle\frac{\partial}{\partial u_2}H_\epsilon^\Omega(u_1,u_2,\omega)k,\psi\right\rangle_{V_{-\alpha}^\perp,V_\alpha^\perp}=k\int\limits_{\Omega_\epsilon} f'\left(u_1\varphi_1^\epsilon+u_2\varphi_2^\epsilon+w\right)\varphi_2^\epsilon\psi,\quad k\in\mathbb{R},$$

$$\left\langle\frac{\partial}{\partial \omega}H_\epsilon^\Omega(u_1,u_2,\omega)\xi,\psi\right\rangle_{V_{-\alpha}^\perp,V_\alpha^\perp}=\int\limits_{\Omega_\epsilon} f'\left(u_1\varphi_1^\epsilon+u_2\varphi_2^\epsilon+w\right)\xi\psi,\quad \xi\in V_{1/2}^\perp.$$

It is not difficult to see that these derivatives are continuous. Just to give an idea of the techniques involved, we observe that the continuity derivative with respect to ω (after the use of Hölder's inequality) requires that

$$\left\|f'\left(u_1\varphi_1^\epsilon+u_2\varphi_2^\epsilon+w_1\right)-f'\left(v_1\varphi_1^\epsilon+v_2\varphi_2^\epsilon+w_2\right)\right\|_{L^p(\Omega_\epsilon)}\to 0$$

as $\max\{|u_1-v_1|,|u_2-v_2|,\|w_1-w_2\|_{X_\epsilon^{1/2}}\}\xrightarrow{\epsilon\to 0}0$, for large values of p. This follows from the fact that it goes to zero when $p=2$ and it is bounded when $p=\infty$. Finally, for $H_\epsilon^{\partial\Omega}:\mathbb{R}^2\times V_{1/2}^\perp\to V_{-\alpha}^\perp$ it is easy to see that

$$\left\|H_\epsilon^{\partial\Omega}(y,\omega)\right\|_{V_{-\alpha}^\perp}\leqslant N_F\epsilon^{(N-2)/2}$$

and that

$$\begin{aligned}&\big\| H_\epsilon^{\partial\Omega}(y_1,\omega_1) - H_\epsilon^{\partial\Omega}(y_2,\omega_2)\big\|_{V_{-\alpha}^\perp}\\&\quad\leqslant \mathfrak{L}^{\partial\Omega}\epsilon^{(N-2)/2}\big(\|y_1-y_2\|_{\mathbb{R}^2}+\epsilon^{(N-2)/2}\|\omega_1-\omega_2\|_{V_{1/2}^\perp}\big).\end{aligned}$$

The derivatives of $H_\epsilon^{\partial\Omega}$ are given by

$$\left\langle \frac{\partial}{\partial u_1} H_\epsilon^{\partial\Omega}(u_1,u_2,\omega)h,\psi\right\rangle_{V_{-\alpha}^\perp,V_\alpha^\perp} = h\int_{\Omega_\epsilon} g'\big(u_1\varphi_1^\epsilon+u_2\varphi_2^\epsilon+w\big)\varphi_1^\epsilon\psi,\quad h\in\mathbb{R},$$

$$\left\langle \frac{\partial}{\partial u_2} H_\epsilon^{\partial\Omega}(u_1,u_2,\omega)k,\psi\right\rangle_{V_{-\alpha}^\perp,V_\alpha^\perp} = k\int_{\Omega_\epsilon} g'\big(u_1\varphi_1^\epsilon+u_2\varphi_2^\epsilon+w\big)\varphi_2^\epsilon\psi,\quad k\in\mathbb{R},$$

$$\left\langle \frac{\partial}{\partial \omega} H_\epsilon^{\partial\Omega}(u_1,u_2,\omega)\xi,\psi\right\rangle_{V_{-\alpha}^\perp,V_\alpha^\perp} = \int_{\Omega_\epsilon} g'\big(u_1\varphi_1^\epsilon+u_2\varphi_2^\epsilon+w\big)\xi\psi,\quad \xi\in V_{1/2}^\perp.$$

It is easy to see that these derivatives are continuous.

With this we see that F_ϵ and G_ϵ satisfy the hypotheses of Theorem B.2. It follows from (2.1) that the conditions on A_ϵ and B_ϵ of Theorem B.2 are also satisfied. The proof of Theorem 3 now follows from Theorem B.2. □

Proof of Theorem 4. For suitably small ϵ, the flow in the invariant manifold is given by $v(t,x)=u_1(t)\varphi_1^0(x)+u_2(t)\varphi_2^0(x)+\sigma_\epsilon(u_1(t),u_2(t))(x)$, where $(u_1(t),u_2(t))$ is a solution of

$$\begin{cases}\dot u_1 = G_0^\epsilon\big(u_1,u_2,\sigma_\epsilon(u_1,u_2)\big),\\ \dot u_2 = -\delta u_2 + G_1^\epsilon\big(u_1,u_2,\sigma_\epsilon(u_1,u_2)\big).\end{cases}\tag{3.4}$$

To obtain that the flow on the attractor of (1.6) (or equivalently (1.9)) is topologically equivalent to the flow on the attractor of (3.4), we only need to prove that the vector fields

$$X_\epsilon(u_1,u_2)=\big(X_0(\epsilon),X_1(\epsilon)\big)\quad\text{and}\quad X_0(u_1,u_2)=\big(X_0(0),X_1(0)\big),$$

where

$$X_0(\epsilon)=\int_{\Omega_\epsilon} f\big(u_1\varphi_1^\epsilon+u_2\varphi_2^\epsilon+\sigma_\epsilon(u_1,u_2)\big)\varphi_1^\epsilon+\int_{\partial\Omega_\epsilon} g\big(u_1\varphi_1^\epsilon+u_2\varphi_2^\epsilon+\sigma_\epsilon(u_1,u_2)\big)\varphi_1^\epsilon,$$

$$\begin{aligned}X_1(\epsilon)={}&-\delta\frac{\lambda_2^\epsilon}{\epsilon^{N-1}}u_2+\int_{\Omega_\epsilon} f\big(u_1\varphi_1^\epsilon+u_2\varphi_2^\epsilon+\sigma_\epsilon(u_1,u_2)\big)\varphi_2^\epsilon\\&+\int_{\partial\Omega_\epsilon} g\big(u_1\varphi_1^\epsilon+u_2\varphi_2^\epsilon+\sigma_\epsilon(u_1,u_2)\big)\varphi_2^\epsilon,\end{aligned}$$

$$X_0(0)=\int_\Omega f(u_1+u_2\phi_2)+\int_{\partial\Omega} g(u_1+u_2\phi_2),$$

$$X_1(0)=-\delta\,\mathrm{C_o}u_2+\int_\Omega f(u_1+u_2\phi_2)\phi_2+\int_{\partial\Omega} g(u_1+u_2\phi_2)\phi_2$$

are $\mathcal{C}^1$ close. This follows easily from the fact that σ_ϵ approaches 0 in the $\mathcal{C}^1(\mathbb{R}^2, V_{1/2}^{\perp})$ topology and the asymptotic properties of the eigenvalues and eigenfunctions of $-\frac{\delta}{\epsilon^{N-1}}\Delta$ as $\epsilon \to 0$. Just to give an idea of the techniques involved, let us prove that $\partial_{u_1} X_1(\epsilon)$ converges to $\partial_{u_1} X_1(0)$ as $\epsilon \to 0$. Note that

$$\begin{aligned}
&\partial_{u_1} X_1(\epsilon)(u_1, u_2) - \partial_{u_1} X_1(0)(u_1, u_2) \\
&\quad = -\delta\left(\frac{\lambda_2^\epsilon}{\epsilon^{N-1}} - \mathrm{C_o}\right)u_2 \\
&\qquad + \int_{\Omega_\epsilon} f'\big(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \sigma_\epsilon(u_1, u_2)\big)\varphi_2^\epsilon\varphi_1^\epsilon - \int_{\Omega} f'(u_1 + u_2\phi_2)\phi_2 \\
&\qquad + \int_{\partial\Omega_\epsilon} g'\big(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \sigma_\epsilon(u_1, u_2)\big)\varphi_2^\epsilon\varphi_1^\epsilon - \int_{\partial\Omega} g'(u_1 + u_2\phi_2)\phi_2 \\
&\qquad + \int_{\Omega_\epsilon} f'\big(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \sigma_\epsilon(u_1, u_2)\big)\varphi_2^\epsilon\partial_{u_1}\sigma_\epsilon(u_1, u_2) \\
&\qquad + \int_{\partial\Omega_\epsilon} g'\big(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \sigma_\epsilon(u_1, u_2)\big)\varphi_2^\epsilon\partial_{u_1}\sigma_\epsilon(u_1, u_2).
\end{aligned}$$

All but the last two lines in the above expression go to zero, uniformly in bounded subsets of $\mathbb{R}^2$, because of the convergence properties of $\lambda_1^\epsilon, \lambda_2^\epsilon$, φ_1^ϵ and φ_2^ϵ and because $\|\sigma_\epsilon(u_1, u_2)\|_{V_{1/2}^{\perp}} \to 0$. For the last two lines we only need to observe that $\|\partial_{u_1}\sigma_\epsilon(u_1, u_2)\|_{V_{1/2}^{\perp}} \to 0$ and use Uniform Trace Theorem (Theorem 6) to conclude that $\|\partial_{u_1}\sigma_\epsilon(u_1, u_2)\|_{L^2(\partial\Omega_\epsilon)} \xrightarrow{\epsilon\to 0} 0$. □

4. Patterns

In this section we return to system (1.9). Now we want to obtain stable nonconstant equilibrium solutions for (1.1). For this purpose we consider δ sufficiently small, $g(u) = u - u^3$ and $f(u) = 0$. Thus system (1.9) has nine equilibrium points (see Fig. 2) which (for $\delta \ll 1$) are approximately equal to

$$\begin{aligned}
&P_1 = (0, 0), && P_2 = (0, 1), && P_3 = (0, -1), \\
&P_4 = (1, 0), && P_5 = (1, 1), && P_6 = (1, -1), \\
&P_7 = (-1, 0), && P_8 = (-1, 1), && P_9 = (-1, -1).
\end{aligned}$$

All these equilibrium points are hyperbolic. P_5, P_6, P_8, P_9 are stable. We have seen that the dynamics of (1.9) is equivalent to the dynamics of (1.1). The stable equilibrium points of the form $P = (v_1, v_2)$ with $v_1 \neq v_2$ correspond to stable nonconstant equilibrium for (1.1). Thus, if ϵ is suitably small, system (1.1) has asymptotically stable nonconstant equilibria.

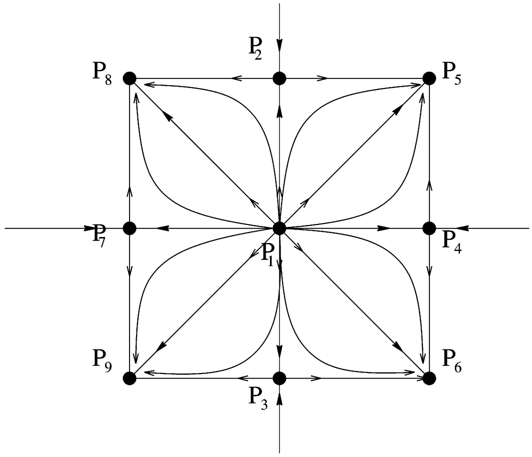

Fig. 2. Approximate phase portrait: $\delta \ll 1$, $g(u) = u - u^3$, $f \equiv 0$.

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Appendix A. Convergence of eigenvalues and eigenfunctions

In this appendix we show Theorem 1 concerning the eigenvalues λ_n^ϵ arranged in increasing order (counting multiplicity) and a complete system of orthonormalized eigenfunctions φ_n^ϵ associated to problem (1.4). Let $\phi_1 = |\Omega|^{-1/2} = 1$ and $\phi_2 = c_1^0 \chi_{\Omega^{\mathrm{L}}} + c_2^0 \chi_{\Omega^{\mathrm{R}}}$.

For the results in this section we follow the ideas in [2,3].

Lemma A.1.

$$\limsup_{\epsilon \to 0} \frac{\lambda_2^\epsilon}{\epsilon^{N-1}} \leqslant \sigma_{N-1} \int_0^1 h^{N-1}(x_1) \left|\xi'(x_1)\right|^2 dx_1 = \mathrm{C_o}, \tag{A.1}$$

where $\xi(x_1)$ is the solution of the boundary value problem

$$\begin{cases} (h^{N-1}\xi'(s))' = 0, & s \in (0,1), \\ \xi(0) = c_1^0, & \xi(1) = c_2^0, \end{cases} \tag{A.2}$$

where c_1^0 and c_2^0 are defined in Theorem 1.

Proof. From the variational characterization, we know that

$$\lambda_2^\epsilon = \inf\left\{ \frac{\int_{\Omega_\epsilon} |\nabla\varphi|^2 \, dx}{\int_{\Omega_\epsilon} |\varphi|^2 \, dx} \colon \varphi \in H^1(\Omega_\epsilon),\ \varphi \neq 0,\ \int_{\Omega_\epsilon} \varphi \, dx = 0 \right\}.$$

Defining $\tilde{\psi}(x_1, x')$ by

$$\tilde{\psi}(x_1, x') = \begin{cases} \xi(x_1) & \text{in } R_\epsilon, \\ c_1^0 & \text{in } \Omega^{\mathrm{L}}, \\ c_2^0 & \text{in } \Omega^{\mathrm{R}}, \end{cases}$$

and

$$\psi(x_1, x') = \tilde{\psi}(x_1, x') - \frac{1}{|\Omega_\epsilon|} \int\limits_{\Omega_\epsilon} \tilde{\psi}(x_1, x')\, dx_1\, dx',$$

since $\tilde{\psi} \in H^1(\Omega_\epsilon)$, then $\psi \in H^1(\Omega_\epsilon)$ and $\int_{\Omega_\epsilon} \psi\, dx_1\, dx' = 0$. So we have

$$\lambda_2^\epsilon \leqslant \frac{\int_{\Omega_\epsilon} |\nabla \psi|^2\, dx}{\int_{\Omega_\epsilon} |\psi|^2\, dx}.$$

From the definition of ψ, we have

$$\begin{aligned} \int\limits_{\Omega_\epsilon} |\nabla \psi|^2\, dx = \int\limits_{R_\epsilon} |\nabla \tilde{\psi}|^2\, dx &= \int\limits_{R_\epsilon} \bigl|\nabla \xi(x_1)\bigr|^2\, dx_1\, dx' \\ &= \sigma_{N-1}\, \epsilon^{N-1} \int\limits_0^1 h^{N-1}(x_1) \bigl|\xi'(x_1)\bigr|^2\, dx_1\, dy'. \end{aligned}$$

Also,

$$\int\limits_{\Omega_\epsilon} |\psi|^2\, dx = \int\limits_{\Omega_\epsilon} \bigl|\tilde{\psi}(x_1, x')\bigr|^2\, dx_1\, dx' - \frac{1}{|\Omega_\epsilon|} \biggl(\int\limits_{\Omega_\epsilon} \tilde{\psi}(x_1, x')\, dx_1\, dx' \biggr)^2$$

and since

$$\int\limits_{\Omega_\epsilon} \tilde{\psi}(x_1, x')\, dx_1\, dx' = \int\limits_{R_\epsilon} \tilde{\psi}(x_1, x')\, dx_1\, dx' = O\bigl(\epsilon^{N-1}\bigr),$$

we obtain

$$\begin{aligned} \int\limits_{\Omega_\epsilon} |\psi|^2\, dx &= \int\limits_{\Omega} \bigl|\tilde{\psi}(x_1, x')\bigr|^2\, dx_1\, dx' + \int\limits_{R_\epsilon} \bigl|\tilde{\psi}(x_1, x')\bigr|^2\, dx_1\, dx' + O\bigl(\epsilon^{2(N-1)}\bigr) \\ &= 1 + \sigma_{N-1} \epsilon^{N-1} \int\limits_0^1 h^{N-1}(x_1) \bigl|\xi(x_1)\bigr|^2\, dx_1 + O\bigl(\epsilon^{2(N-1)}\bigr). \end{aligned}$$

That gives us

$$\frac{\int_{\Omega_\epsilon} |\nabla \psi|^2\, dx}{\int_{\Omega_\epsilon} |\psi|^2\, dx} = \frac{\sigma_{N-1} \epsilon^{N-1} \int_0^1 h^{N-1}(x_1) |\xi'(x_1)|^2\, dx_1\, dy'}{1 + \sigma_{N-1} \epsilon^{N-1} \int_0^1 h^{N-1}(x_1) |\xi(x_1)|^2\, dx_1 + O(\epsilon^{2(N-1)})}$$

and consequently

$$\lambda_2^\epsilon \leqslant \sigma_{N-1} \epsilon^{N-1} \int\limits_0^1 h^{N-1}(x_1) \bigl|\xi'(x_1)\bigr|^2\, dx_1\, dy' + O\bigl(\epsilon^{2(N-1)}\bigr).$$

Hence,

$$\limsup_{\epsilon\to 0}\frac{\lambda_2^\epsilon}{\sigma_{N-1}\epsilon^{N-1}} \leqslant \int_0^1 h^{N-1}(x_1)\left|\xi'(x_1)\right|^2 dx_1. \tag{A.3}$$

Now note that the solution of (A.2) can be found explicitly by

$$\xi(x_1) = c_1^0 + \left(c_2^0 - c_1^0\right)\left\{\int_0^{x_1}\frac{dt}{h^{N-1}(t)}\right\}\left\{\int_0^1\frac{dx_1}{h^{N-1}(x_1)}\right\}^{-1} \tag{A.4}$$

and thus

$$\begin{aligned}\left|\xi'(x_1)\right|^2 &= \left(c_2^0 - c_1^0\right)^2\left\{\frac{1}{h^{N-1}(x_1)}\right\}^2\left\{\int_0^1\frac{dx_1}{h^{N-1}(x_1)}\right\}^{-2}\\ &= \left(\frac{1}{|\Omega^{\mathrm{L}}|}+\frac{1}{|\Omega^{\mathrm{R}}|}\right)\left\{\frac{1}{h^{N-1}(x_1)}\right\}^2\left\{\int_0^1\frac{dx_1}{h^{N-1}(x_1)}\right\}^{-2}.\end{aligned}$$

Replacing it in the right-hand side of Eq. (A.8), $\mathrm{C_o}$ appears. □

Theorem A.2 *(Convergence of eigenfunctions). Let $n \in \mathbb{N}$ and φ_n^ϵ be eigenfunctions for problem* (1.4)*, then*

(i) $\varphi_1^\epsilon \to \phi_1$ *in* $H^k(\Omega)$ *as* $\epsilon \to 0$, $k \geqslant 1$,
(ii) $\sup_\epsilon \|\varphi_2^\epsilon\|_{L^\infty(\Omega)} < \infty$ *and* $\varphi_2^\epsilon \to \phi_2$ *in* $H^1(\Omega)$ *as* $\epsilon \to 0$.

Proof. (i) Observe that $\varphi_1^\epsilon = |\Omega_\epsilon|^{-1/2}$ and $\phi_1 = |\Omega|^{-1/2} = 1$ are the corresponding eigenfunctions associated to $\lambda_1^\epsilon = \mu_1 = 0$. It is clear that $\varphi_1^\epsilon \to \phi_1$ in $H^k(\Omega)$ for all integer $k \geqslant 0$.

(ii) Let λ_2^ϵ be the second eigenvalue of (1.4) and φ_2^ϵ be a corresponding normalized eigenfunction. From Lemmas A.1 and [4, Lemma B.1], we have that $\varphi_2^\epsilon \in L^\infty(\Omega_\epsilon)$ and

$$\left\|\varphi_2^\epsilon\right\|_{L^\infty(\Omega_\epsilon)} \leqslant C\left\|\varphi_2^\epsilon\right\|_{L^2(\Omega_\epsilon)}$$

for some constant $C = C(|\Omega_\epsilon|, N)$.

This implies $\int_{R_\epsilon}|\varphi_2^\epsilon|^2 \xrightarrow{\epsilon\to 0} 0$ and $\int_{R_\epsilon}\varphi_2^\epsilon \xrightarrow{\epsilon\to 0} 0$ and, consequently,

$$\int_\Omega\left|\varphi_2^\epsilon\right|^2 \xrightarrow{\epsilon\to 0} 1 \quad\text{and}\quad \int_\Omega \varphi_2^\epsilon \xrightarrow{\epsilon\to 0} 0.$$

Also, since

$$\int_\Omega\left|\nabla\varphi_2^\epsilon\right|^2 \leqslant \int_{\Omega_\epsilon}\left|\nabla\varphi_2^\epsilon\right|^2 = \lambda_2^\epsilon, \tag{A.5}$$

we have that

$$\int_\Omega\left|\nabla\varphi_2^\epsilon\right|^2 \xrightarrow{\epsilon\to 0} 0. \tag{A.6}$$

Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, φ_2^ϵ has a convergent subsequence in $L^2(\Omega)$, which we denote again by φ_2^ϵ. Let φ be its limit. Thus we have

$$\begin{cases} \varphi_2^\epsilon \xrightarrow{\epsilon\to 0} \varphi, & \text{strongly in } L^2(\Omega), \\ \nabla\varphi_2^\epsilon \xrightarrow{\epsilon\to 0} 0, & \text{strongly in } L^2(\Omega), \\ \displaystyle\int_\Omega \varphi = 0, & \displaystyle\int_\Omega \varphi^2 = 1. \end{cases}$$

Hence, $\nabla\varphi = 0$. From this we have that $\varphi = c_1\chi_{\Omega^{\mathrm{L}}} + c_2\chi_{\Omega^{\mathrm{R}}}$ where c_1 and c_2 must satisfy

$$\begin{cases} c_1|\Omega^{\mathrm{L}}| + c_2|\Omega^{\mathrm{R}}| = 0, \\ (c_1)^2|\Omega^{\mathrm{L}}| + (c_2)^2|\Omega^{\mathrm{R}}| = 1. \end{cases} \tag{A.7}$$

Thus,

$$c_1 = \pm\sqrt{\frac{|\Omega^{\mathrm{R}}|}{|\Omega^{\mathrm{L}}|}} \quad \text{and} \quad c_2 = \mp\sqrt{\frac{|\Omega^{\mathrm{L}}|}{|\Omega^{\mathrm{R}}|}}.$$

So,

$$\int_\Omega \varphi_2^\epsilon\phi_2 = \int_{\Omega^{\mathrm{L}}} \varphi_2^\epsilon\phi_2 + \int_{\Omega^{\mathrm{R}}} \varphi_2^\epsilon\phi_2 \xrightarrow{\epsilon\to 0} \int_{\Omega^{\mathrm{L}}} c_1c_1^0 + \int_{\Omega^{\mathrm{R}}} c_2c_2^0.$$

From the first equation of (A.7) we see that c_1 and c_2 have opposite signs. We consider c_1 with negative sign and c_2 with positive sign. In this case $c_1 = c_1^0$ and $c_2 = c_2^0$. Thus we have

$$\int_\Omega \varphi_2^\epsilon\phi_2 \xrightarrow{\epsilon\to 0} \left(c_1^0\right)^2|\Omega^{\mathrm{L}}| + \left(c_2^0\right)^2|\Omega^{\mathrm{R}}| = 1.$$

Now we show $\varphi_2^\epsilon \xrightarrow{\epsilon\to 0} \phi_2$. In fact,

$$\begin{aligned} \|\varphi_2^\epsilon - \phi_2\|_{L^2(\Omega)}^2 &= \|\varphi_2^\epsilon\|_{L^2(\Omega)}^2 + \|\phi_2\|_{L^2(\Omega)}^2 - 2\int_{\Omega_0} \varphi_2^\epsilon\phi_2 \\ &\leqslant \|\varphi_2^\epsilon\|_{L^2(\Omega_\epsilon)}^2 + 1 - 2\int_\Omega \varphi_2^\epsilon\phi_2 \leqslant 2 - 2\int_\Omega \varphi_2^\epsilon\phi_2 \xrightarrow{\epsilon\to 0} 0. \end{aligned}$$

From this and (A.6) we conclude that $\|\varphi_2^\epsilon - \phi_2\|_{H^1(\Omega)} \xrightarrow{\epsilon\to 0} 0$. □

The exact rate of convergence of λ_2^ϵ to 0 is given by the following proposition.

Proposition A.3.

$$\lim_{\epsilon\to 0} \frac{\lambda_2^\epsilon}{\epsilon^{N-1}} = \mathrm{C_o}. \tag{A.8}$$

Proof. From Lemma A.1 it remains to show

$$\liminf_{\epsilon\to 0} \frac{\lambda_2^\epsilon}{\sigma_{N-1}\epsilon^{N-1}} \geqslant \int_0^1 h^{N-1}(x_1)\left|\xi'(x_1)\right|^2 dx_1. \tag{A.9}$$

For this, we consider the following family of functions $\xi_\epsilon(x_1, y') = \varphi_2^\epsilon(x_1, \epsilon y')$, where $(x_1, y') \in R_1 = \{(x_1, y')\colon\ 0 \leqslant x_1 \leqslant 1,\ |y'| < h(x_1)\}$. If $x' = \epsilon y'$,

$$\epsilon^{N-1} \int\limits_{R_1} \left[\left(\frac{\partial \xi_\epsilon}{\partial x_1} \right)^2 + \frac{1}{\epsilon^2} |\nabla_{y'} \xi_\epsilon|^2 \right] dx_1\, dy' = \int\limits_{R_\epsilon} \left| \nabla \varphi_2^\epsilon \right|^2 dx_1\, dx' \leqslant \lambda_2^\epsilon. \tag{A.10}$$

Using (A.3) and (A.10), we have

$$\sup_{\epsilon > 0} \int\limits_{R_1} \left[\left(\frac{\partial \xi_\epsilon}{\partial x_1} \right)^2 + \frac{1}{\epsilon^2} |\nabla_{y'} \xi_\epsilon|^2 \right] dx_1\, dy' < \infty. \tag{A.11}$$

It follows that the family of functions $\{\xi_\epsilon\}_{\epsilon > 0}$ is uniformly bounded in $H^1(R_1)$. So we can find a subsequence $\{\epsilon_n\}$ with $\epsilon_n \xrightarrow{n \to \infty} 0$, and a function $\xi_0 \in H^1(R_1)$ such that $\xi_{\epsilon_n} \xrightarrow{n \to \infty} \xi_0$ weakly in $H^1(R_1)$ and strongly in $H^s(R_1)$ for all $s < 1$. Hence, from (A.11) we conclude that ξ_0 is independent of x'.

Let K be the convex and closed set of $H^1(0, 1)$ defined by

$$\mathrm{K} = \left\{ u \in H^1(0, 1)\colon\ u(0) = c_1^0, u(1) = c_2^0 \right\}.$$

Now we show that $\xi_0 \in \mathrm{K}$. In fact, since $\xi_0 \in H^1(R_1)$, we conclude immediately that $\xi_0 \in H^1(0, 1)$. To find the boundary value of ξ_0 we use the fact that $\varphi_2^\epsilon \xrightarrow{\epsilon \to 0} \phi_2$ strongly in $H^1(\Omega)$. Thus, from the continuity of the trace operator, it follows that

$$\xi_{\epsilon_n}|_{\partial R_1 \cap \Omega} \to \xi_0|_{\partial R_1 \cap \Omega} \quad \text{in } H^{1/2}(\partial R_1).$$

Thus $\xi_0(0) = c_1^0$ and $\xi_0(1) = c_2^0$, which proves that $\xi_0 \in \mathrm{K}$.

Note that, from (A.10),

$$\lambda_2^\epsilon \geqslant \epsilon^{N-1} \int\limits_{R_1} \left(\frac{\partial \xi_\epsilon}{\partial x_1} \right)^2 dx_1\, dy' \tag{A.12}$$

and that, if $J : K \to \mathbb{R}$ is given by

$$J(u) = \int\limits_0^1 h^{N-1}(x_1) u'(x_1)^2\, dx_1,$$

then $J(\xi) = \min_{u \in \mathrm{K}} J(u)$. Hence $J(\xi) \leqslant J(\xi_0)$. From (A.12), it follows that

$$\begin{aligned} \liminf_{n \to \infty} \frac{\lambda_2^{\epsilon_n}}{\epsilon^{N-1}} &\geqslant \liminf_{\epsilon_n \to 0} \int\limits_{R_1} \left(\frac{\partial \xi_{\epsilon_n}}{\partial x_1} \right)^2 dx_1\, dy' \geqslant \int\limits_{R_1} \left(\frac{\partial \xi_0}{\partial x_1} \right)^2 dx_1\, dy' \\ &\geqslant \sigma_{N-1} \int\limits_0^1 h^{N-1}(x_1) \left| \xi'(x_1) \right|^2 dx_1. \end{aligned}$$

This gives (A.9), which along with (A.3) implies (A.8). This concludes the proof. □

Finally, to conclude the proof of Theorem 1 we need to ensure that λ_3^ϵ is bounded away from zero.

Proposition A.4.

$$\liminf_{\epsilon\to 0}\lambda_3^\epsilon > 0.$$

Proof. Note that φ_3^ϵ satisfies

$$\int_{\Omega_\epsilon}\varphi_3^\epsilon = 0, \qquad \int_{\Omega_\epsilon}\varphi_3^\epsilon\varphi_2^\epsilon = 0, \qquad \|\varphi_3^\epsilon\|_{\mathrm{L}^2(\Omega_\epsilon)} = 1. \tag{A.13}$$

Suppose that there is a sequence $\{\epsilon_j\}_{j=1}^\infty$ with $\epsilon_j \xrightarrow{j\to\infty} 0$ such that $\lim_{j\to\infty}\lambda_3^{\epsilon_j} = 0$. Then

$$\int_{\Omega}|\nabla\varphi_3^{\epsilon_j}|^2 \leqslant \int_{\Omega_{\epsilon_j}}|\nabla\varphi_3^{\epsilon_j}|^2 = \lambda_3^{\epsilon_j}\|\varphi_3^{\epsilon_j}\|^2 = \lambda_3^{\epsilon_j} \xrightarrow{j\to\infty} 0.$$

Hence there are $\varphi\in H^1(\Omega)$ and a subsequence of $\{\varphi_3^{\epsilon_j}\}$, with $\varphi_3^{\epsilon_j} \xrightarrow{j\to\infty} \varphi$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Thus we have

$$\varphi_3^{\epsilon_j} \xrightarrow{s-L^2(\Omega)} \varphi, \qquad \nabla\varphi_3^{\epsilon_j} \xrightarrow{s-L^2(\Omega)} 0.$$

It follows that

$$\varphi = \begin{cases} c_1, & \text{in } \Omega^{\mathrm{L}},\\ c_2, & \text{in } \Omega^{\mathrm{R}}.\end{cases} \tag{A.14}$$

Now $\sup_{j\in\mathbb{N}}\lambda_3^{\epsilon_j} < \infty$ and [4, Lemma B.1] imply that $\sup_{j\in\mathbb{N}}\|\varphi_3^{\epsilon_j}\|_{L^\infty(\Omega_{\epsilon_j})} < \infty$. This together with (A.13) imply that c_1 and c_2 satisfy

$$\begin{cases} c_1|\Omega^{\mathrm{L}}| + c_2|\Omega^{\mathrm{R}}| = 0,\\ c_1c_1^0|\Omega^{\mathrm{L}}| + c_2c_2^0|\Omega^{\mathrm{R}}| = 0,\\ (c_1)^2|\Omega^{\mathrm{L}}| + (c_2)^2|\Omega^{\mathrm{R}}| = 1.\end{cases} \tag{A.15}$$

Since the above system does not have a solution, we have a contradiction and the result is proved. □

Appendix B. Invariant manifold theorem

In this section, we state and give hints of the proof for the invariant manifold theorem used to prove Theorem 3. The proof is adapted from the results in Henry [15, Chapter 6].

Let $\epsilon\in(0,1]$, X_ϵ and Y_ϵ be Banach spaces, $A_\epsilon : D(A_\epsilon)\subset X_\epsilon \to X_\epsilon$ be a sectorial operator and $B_\epsilon \in L(Y_\epsilon)$. Denote by X_ϵ^α the fractional power spaces associated to A_ϵ, $\alpha\in[0,1)$. Let $F_\epsilon : X_\epsilon^\alpha\times Y_\epsilon \to X_\epsilon$ and $G_\epsilon : X_\epsilon^\alpha\times Y_\epsilon\to Y_\epsilon$ be Lipschitz continuous functions and consider the following system of weakly coupled semilinear differential equations in $X_\epsilon^\alpha\times Y_\epsilon$

$$\begin{cases} \dot{x} = A_\epsilon x + F_\epsilon(x,y),\\ \dot{y} = B_\epsilon y + G_\epsilon(x,y).\end{cases} \tag{B.1}$$

Definition B.1. A set $S_\epsilon\subset X_\epsilon^\alpha\times Y_\epsilon$ is an invariant manifold for (B.1) if there exists $\sigma_\epsilon : Y_\epsilon \to X_\epsilon^\alpha$ such that $S_\epsilon = \{(x,y)\in X_\epsilon^\alpha\times Y_\epsilon:\ x = \sigma_\epsilon(y)\}$ and, for each $(x_0^\epsilon, y_0^\epsilon)\in S_\epsilon$ there exists a solution $(x^\epsilon(\cdot), y^\epsilon(\cdot))$ of $(B.1)$, $(x^\epsilon(0), y^\epsilon(0)) = (x_0^\epsilon, y_0^\epsilon)$, defined on $\mathbb{R}$ such that $(x^\epsilon(t), y^\epsilon(t))\in S_\epsilon$,

$\forall t \in \mathbb{R}$. An invariant manifold S_ϵ is *exponentially attracting* if there are positive constants ν and K such that

$$\big\|x^\epsilon(t) - \sigma_\epsilon\big(y^\epsilon(t)\big)\big\|_{X_\epsilon^\alpha} \leqslant K e^{-\nu t} \big\|x_0^\epsilon - \sigma_\epsilon\big(y_0^\epsilon\big)\big\|_{X_\epsilon^\alpha},$$

whenever $(x^\epsilon(t), y^\epsilon(t))$ is a solution to (B.1) with $(x^\epsilon(0), y^\epsilon(0)) = (x_0^\epsilon, y_0^\epsilon) \in X_\epsilon^\alpha \times Y_\epsilon$.

Theorem B.2. *Assume that*

$$\begin{aligned}
&\big\|F_\epsilon(x,y) - F_\epsilon(z,w)\big\|_{X_\epsilon} \leqslant L_F(\epsilon)\big(\|x-z\|_{X_\epsilon^\alpha} + \|y-w\|_{Y_\epsilon}\big),\\
&\big\|F_\epsilon(x,y)\big\|_{X_\epsilon} \leqslant N_F(\epsilon),\\
&\big\|G_\epsilon(x,y) - G_\epsilon(z,w)\big\|_{Y_\epsilon} \leqslant L_G(\epsilon)\big(\|x-z\|_{X_\epsilon^\alpha} + \|y-w\|_{Y_\epsilon}\big),\\
&\big\|G_\epsilon(x,y)\big\|_{Y_\epsilon} \leqslant N_G(\epsilon),
\end{aligned}$$

for each (x,y), (z,w) in $X_\epsilon^\alpha \times Y_\epsilon$ where $N_F(\epsilon) > 0$, $L_F(\epsilon) > 0$, $N_G(\epsilon) > 0$ and $L_G(\epsilon) > 0$. Also, assume that there are positive constants M_A, M_B and ρ, independent of ϵ and $\beta(\epsilon) > 0$ such that

$$\begin{aligned}
&\big\|e^{-A_\epsilon t} w\big\|_{X_\epsilon^\alpha} \leqslant M_A e^{-\beta(\epsilon)t} \|w\|_{X_\epsilon^\alpha}, \quad t \geqslant 0,\\
&\big\|e^{-A_\epsilon t} w\big\|_{X_\epsilon^\alpha} \leqslant M_A t^{-\alpha} e^{-\beta(\epsilon)t} \|w\|_{X_\epsilon}, \quad t > 0,\\
&\big\|e^{-B_\epsilon t} z\big\|_{Y_\epsilon} \leqslant M_B e^{-\rho t} \|z\|_{Y_\epsilon}, \quad t \leqslant 0,
\end{aligned}$$

for any $w \in X_\epsilon^\alpha$ and $z \in Y_\epsilon$. If

$$\frac{N_F(\epsilon)}{\beta(\epsilon)^{\alpha-1}} \xrightarrow{\epsilon\to 0} 0, \qquad \frac{L_F(\epsilon)}{\beta(\epsilon)^{\alpha-1}} \xrightarrow{\epsilon\to 0} 0 \quad \textit{and} \qquad \frac{L_F(\epsilon)L_G(\epsilon)}{\beta(\epsilon)^{\alpha-1}} \xrightarrow{\epsilon\to 0} 0,$$

then, for small enough ϵ, there is an exponentially attracting invariant manifold

$$S_\epsilon = \big\{(x,y)\colon\ x = \sigma_\epsilon(y),\ y \in Y_\epsilon\big\}$$

for (B.1)*, where $\sigma_\epsilon : Y_\epsilon \to X_\epsilon^\alpha$ satisfies*

$$\begin{aligned}
&s(\epsilon) = \sup_{y\in Y_\epsilon} \big\|\sigma_\epsilon(y)\big\|_{X_\epsilon^\alpha} \xrightarrow{\epsilon\to 0} 0,\\
&\sup\left\{\frac{\|\sigma_\epsilon(y) - \sigma_\epsilon(z)\|_{X_\epsilon^\alpha}}{\|y-z\|_{Y_\epsilon}}\colon\ y,z \in Y_\epsilon,\ y \neq z\right\} \xrightarrow{\epsilon\to 0} 0.
\end{aligned}$$

If F_ϵ, G_ϵ are smooth, then σ_ϵ is smooth and its derivative $D\sigma_\epsilon$ satisfies

$$\sup_{y\in Y_\epsilon} \big\|D\sigma_\epsilon(y)\big\|_{L(Y_\epsilon, X_\epsilon^\alpha)} \xrightarrow{\epsilon\to 0} 0.$$

Proof. The first and crucial step is to obtain the existence of the invariant manifold. Given $D > 0$ and $\Delta > 0$, let $\sigma_\epsilon : Y_\epsilon \to X_\epsilon^\alpha$ be a function satisfying

$$|||\sigma_\epsilon||| := \sup_{y\in Y_\epsilon} \big\|\sigma_\epsilon(y)\big\|_{X_\epsilon^\alpha} \leqslant D, \qquad \big\|\sigma_\epsilon(y) - \sigma_\epsilon(y')\big\|_{X_\epsilon^\alpha} \leqslant \Delta \|y - y'\|_{Y_\epsilon}. \tag{B.2}$$

Denote by $y_\epsilon(t) = \psi(t, \tau, \eta, \sigma_\epsilon)$ the solution of

$$\begin{aligned}
&\dot y_\epsilon = -B_\epsilon y_\epsilon + G_\epsilon\big(\sigma_\epsilon(y_\epsilon), y_\epsilon\big), \quad t < \tau,\\
&y_\epsilon(\tau) = \eta
\end{aligned} \tag{B.3}$$

and define

$$\Theta(\sigma_\epsilon)(\eta) = \int_{-\infty}^{\tau} e^{-A_\epsilon(\tau-s)} F_\epsilon\big(\sigma_\epsilon\big(y(s)\big), y(s)\big)\, ds. \tag{B.4}$$

Note that

$$\big\|\Theta(\sigma_\epsilon)(\eta)\big\|_{X_\epsilon^\alpha} \leqslant \int_{-\infty}^{\tau} N_F M_A (\tau-s)^{-\alpha} e^{-\beta(\epsilon)(\tau-s)}\, ds \leqslant \frac{N_F M_A \Gamma(1-\alpha)}{\beta(\epsilon)^{1-\alpha}}. \tag{B.5}$$

Let $\epsilon_0 > 0$ be such that, for $\epsilon \leqslant \epsilon_0$, $\sup_{\eta\in Y_\epsilon} \|\Theta(\sigma_\epsilon)(\eta)\|_{X_\epsilon^\alpha} \leqslant D$. Suppose that σ_ϵ and σ_ϵ' are functions satisfying (B.2), $\eta, \eta' \in Y_\epsilon$, $y_\epsilon(t) = \psi(t,\tau,\eta,\sigma_\epsilon)$ and $y_\epsilon'(t) = \psi(t,\tau,\eta',\sigma_\epsilon')$. Then

$$y_\epsilon(t) - y_\epsilon'(t) = e^{-B_\epsilon(t-\tau)}(\eta-\eta') + \int_{\tau}^{t} e^{-B_\epsilon(t-s)}\big(G_\epsilon\big(\sigma_\epsilon(y_\epsilon), y_\epsilon\big) - G_\epsilon\big(\sigma_\epsilon'\big(y_\epsilon'\big), y_\epsilon'\big)\big)\, ds.$$

Now,

$$\begin{aligned}
&\big\|y_\epsilon(t) - y_\epsilon'(t)\big\|_{Y_\epsilon} \\
&\quad\leqslant M_B e^{\rho(\tau-t)}\|\eta-\eta'\|_{Y_\epsilon} \\
&\qquad + M_B \int_{t}^{\tau} e^{\rho(s-t)}\big\|G_\epsilon\big(\sigma_\epsilon(y_\epsilon), y_\epsilon\big) - G_\epsilon\big(\sigma'\big(y_\epsilon'\big), y_\epsilon'\big)\big\|_{Y_\epsilon}\, ds \\
&\quad\leqslant M_B e^{\rho(\tau-t)}\|\eta-\eta'\|_{Y_\epsilon} \\
&\qquad + M_B L_g \int_{t}^{\tau} e^{\rho(s-t)}\big(\big\|\sigma_\epsilon(y_\epsilon) - \sigma'\big(y_\epsilon'\big)\big\|_{X_\epsilon^\alpha} + \big\|y_\epsilon - y_\epsilon'\big\|_{Y_\epsilon}\big)\, ds \\
&\quad\leqslant M_B e^{\rho(\tau-t)}\|\eta-\eta'\|_{Y_\epsilon} \\
&\qquad + M_B L_g \int_{t}^{\tau} e^{\rho(s-t)}\big(\big\|\sigma_\epsilon(y_\epsilon) - \sigma'(y_\epsilon)\big\|_{X_\epsilon^\alpha} + (1+\Delta)\big\|y_\epsilon - y_\epsilon'\big\|_{Y_\epsilon}\big)\, ds \\
&\quad\leqslant M_B e^{\rho(\tau-t)}\|\eta-\eta'\|_{Y_\epsilon} \\
&\qquad + M_B L_g \int_{t}^{\tau} e^{\rho(s-t)}\big(|\!|\!|\sigma_\epsilon - \sigma'|\!|\!| + (1+\Delta)\big\|y_\epsilon - y_\epsilon'\big\|_{Y_\epsilon}\big)\, ds \\
&\quad\leqslant M_B e^{\rho(\tau-t)}\|\eta-\eta'\|_{Y_\epsilon} \\
&\qquad + M_B L_g (1+\Delta) \int_{t}^{\tau} e^{\rho(s-t)}\big\|y_\epsilon - y_\epsilon'\big\|_{Y_\epsilon}\, ds + \frac{e^{\rho(\tau-t)} M_B L_g}{\rho} |\!|\!|\sigma_\epsilon - \sigma'|\!|\!|.
\end{aligned}$$

Let $\phi(t) = e^{\rho(t-\tau)}\|y_\epsilon(t) - y_\epsilon'(t)\|_{Y_\epsilon}$. Then

$$\phi(t) \leqslant M_B\bigg[\|\eta-\eta'\|_{Y_\epsilon} + \frac{L_g}{\rho}|\!|\!|\sigma_\epsilon - \sigma'|\!|\!|_{X_\epsilon^\alpha}\bigg] + M_B L_g (1+\Delta)\int_{t}^{\tau} \phi(s)\, ds.$$

By Gronwall's lemma we have

$$\left\| y_\epsilon(t) - y'_\epsilon(t) \right\|_{Y_\epsilon} \leqslant \left[M_B \|\eta - \eta'\|_{Y_\epsilon} + \frac{L_G}{\rho} |||\sigma_\epsilon - \sigma'|||_{X_\epsilon^\alpha} \right] e^{[\rho + c_\Gamma](\tau - t)},$$

where $c_\Gamma = M_B L_g(1+\Delta)$.

Therefore

$$\begin{aligned}
&\left\| \Theta(\sigma_\epsilon)(\eta) - \Theta(\sigma'_\epsilon)(\eta') \right\|_{X_\epsilon^\alpha} \\
&\quad \leqslant M_A \int_{-\infty}^{\tau} (\tau - s)^{-\alpha} e^{-\beta(\epsilon)(\tau - s)} \left\| F_\epsilon(\sigma_\epsilon(y), y) - F_\epsilon(\sigma'(y'), y') \right\|_{X_\epsilon} ds \\
&\quad \leqslant M_A L_F \int_{-\infty}^{\tau} (\tau - s)^{-\alpha} e^{-\beta(\epsilon)(\tau - s)} \left(\left\| \sigma_\epsilon(y) - \sigma'(y') \right\|_{X_\epsilon^\alpha} + \|y - y'\|_{Y_\epsilon} \right) ds \\
&\quad \leqslant M_A L_F \int_{-\infty}^{\tau} (\tau - s)^{-\alpha} e^{-\beta(\epsilon)(\tau - s)} \left(|||\sigma_\epsilon - \sigma'||| + (1+\Delta) \|y - y'\|_{Y_\epsilon} \right) ds
\end{aligned}$$

and consequently

$$\begin{aligned}
&\left\| \Theta(\sigma_\epsilon)(\eta) - \Theta(\sigma'_\epsilon)(\eta') \right\|_{X_\epsilon^\alpha} \\
&\quad \leqslant M_A L_F \int_{-\infty}^{\tau} (\tau - s)^{-\alpha} e^{-\beta(\epsilon)(\tau - s)} \left(1 + \frac{L_G(1+\Delta)}{\rho} e^{[\rho + c_\Gamma](\tau - s)} \right) ds \, |||\sigma_\epsilon - \sigma'||| \\
&\qquad + M_A M_B L_F (1+\Delta) \int_{-\infty}^{\tau} (\tau - s)^{-\alpha} e^{-[\beta(\epsilon) - \rho - c_\Gamma](\tau - s)} ds \, \|\eta - \eta'\|_{Y_\epsilon} \\
&\quad = I_\eta(\epsilon) \|\eta - \eta'\|_{Y_\epsilon} + I_\sigma(\epsilon) |||\sigma_\epsilon - \sigma'|||,
\end{aligned}$$

where

$$\begin{aligned}
I_\sigma(\epsilon) &= \frac{M_A L_F \Gamma(1-\alpha)}{\beta(\epsilon)^{1-\alpha}} + \frac{M_A L_F L_G (1+\Delta) \Gamma(1-\alpha)}{\rho(\beta(\epsilon) - \rho - c_\Gamma)^{1-\alpha}} \\
I_\eta(\epsilon) &= \frac{M_A M_B L_F (1+\Delta) \Gamma(1-\alpha)}{(\beta(\epsilon) - \rho - c_\Gamma)^{1-\alpha}}.
\end{aligned}$$

It is easy to see that, given $\theta < 1$, there exists $\epsilon_0 > 0$ such that, for $\epsilon \leqslant \epsilon_0$, $I_\sigma(\epsilon) \leqslant \theta$, $I_\eta(\epsilon) \leqslant \Delta$ and

$$\left\| \Theta(\sigma_\epsilon)(\eta) - \Theta(\sigma')(\eta') \right\|_{X_\epsilon^\alpha} \leqslant I_\eta(\epsilon) \|\eta - \eta'\|_{Y_\epsilon} + I_\sigma(\epsilon) \left|\left|\left| \sigma_\epsilon - \sigma'_\epsilon \right|\right|\right|. \tag{B.6}$$

The inequalities (B.5) and (B.6) imply that Θ is a contraction from the class of functions that satisfy (B.2) into itself. Hence it has a unique fixed point $\sigma_\epsilon^* = \Theta(\sigma_\epsilon^*)$ in this class.

The rest of the proof follows in the same manner as in [15, Chapter 6]. □

Appendix C. Proof of Theorem 6

We remark that this result is similar in nature to those presented in [20, Section 4.2]. The main differences being that, in our case, the cylinder is not straight (each section has a different radius), we consider the trace of less regular functions and we embed the trace space into fractional power spaces.

For $\omega \in V_\alpha^\perp$, $\alpha > \frac{1}{4}$, let $\gamma(\omega)$ be its trace. Our aim is to show that there is a constant $\mathfrak{G}$, independent of ϵ, such that

$$\|\gamma(\omega)\|_{L^2(\partial\Omega_\epsilon)} \leqslant \mathfrak{G}\|\omega\|_{X_\epsilon^\alpha}.$$

For simplicity of notation we also denote by ω its trace $\gamma(\omega)$. Note that

$$\|\omega\|_{L^2(\partial\Omega_\epsilon)} \leqslant \left(\int_{\partial\Omega} |\omega|^2\,ds + \int_{\partial R_\epsilon\setminus\partial\Omega} |\omega|^2\,ds\right)^{1/2} \leqslant \|\omega\|_{L^2(\partial\Omega)} + \|\omega\|_{L^2(\partial R_\epsilon\setminus\partial\Omega)}.$$

Since Ω is a fixed domain, it is easy to estimate the norm $\|\omega\|_{L^2(\partial\Omega)}$ in terms of the norm $\|\omega\|_{X_\epsilon^{1/2}}$. Hence we only need to estimate $\|w\|_{L^2(\partial R_\epsilon\setminus\partial\Omega)}$.

Before we proceed, let us introduce some notation:

$$\begin{aligned}
&R_\epsilon = \{(x,y,z) \in \mathbb{R}\times\mathbb{R}^{N-2}\times\mathbb{R}\colon\ 0<x<1,\ \|(y,z)\|_{\mathbb{R}^{N-1}} < \epsilon h(x)\},\\
&D_\epsilon = \{(x,y) \in \mathbb{R}\times\mathbb{R}^{N-2}\colon\ 0<x<1,\ |y| < \epsilon h(x)\},\\
&\Theta_\epsilon : D_\epsilon \to \mathbb{R}^+, \qquad x^2+|y|^2+\Theta_\epsilon(x,y)^2 = \epsilon^2 h(x)^2, \qquad \Theta_\epsilon(x,y) = \epsilon\Theta_1\left(x,\frac{y}{\epsilon}\right),\\
&\Gamma_\epsilon^\pm := \{(x,y,\pm\Theta_\epsilon(x,y))\colon\ (x,y)\in D_\epsilon\}, \qquad \partial R_\epsilon\setminus\partial\Omega = \Gamma_\epsilon^+ \cup \Gamma_\epsilon^-.
\end{aligned}$$

If $w : R_\epsilon \to \mathbb{R}$ we denote by $\tilde{w} : R_1 \to \mathbb{R}$ the function defined by $\tilde{w}(x,y',z') = w(x,\epsilon y',\epsilon z')$. Now, for $0<\epsilon\leqslant 1$, we have that

$$\epsilon^{(N-1)/2}\|\tilde{w}\|_{L^2(R_1)} = \|w\|_{L^2(R_\epsilon)}, \tag{C.1}$$

$$\epsilon^{(N-1)/2}\|\tilde{w}\|_{H^1(R_1)} \leqslant \|w\|_{H^1(R_\epsilon)} \leqslant \epsilon^{(N-1)/2}\epsilon^{-1}\|\tilde{w}\|_{H^1(R_1)}, \tag{C.2}$$

$$\epsilon^{(N-1)/2}\|\tilde{w}\|_{H^s(R_1)} \leqslant \|w\|_{H^s(R_\epsilon)} \leqslant \epsilon^{(N-1)/2}\epsilon^{-s}\|\tilde{w}\|_{H^s(R_1)}, \quad 0<s<1, \tag{C.3}$$

where (C.3) is obtained from (C.1) and (C.2) by interpolation.

Let $I^\pm = \|w\|_{L^2(\Gamma_\epsilon^\pm)}$ and $I = I^+ + I^- = \|w\|_{L^2(\partial R_\epsilon\setminus\partial\Omega)}$. Hence, for $\frac{1}{2} < s \leqslant 1$,

$$\begin{aligned}
(I^\pm)^2 &= \int_{\Gamma_\epsilon^\pm} |\omega|^2\,d\Gamma_\epsilon\\
&= \int_{D_\epsilon} \left|\omega(x,y,\pm\Theta_\epsilon(x,y))\right|^2 \sqrt{1+\left(\partial_x\Theta_\epsilon(x,y)\right)^2+\left(\partial_y\Theta_\epsilon(x,y)\right)^2}\,dy\\
&= \int_{D_\epsilon} \left|\omega\left(x,y,\pm\epsilon\Theta_1\left(x,\frac{y}{\epsilon}\right)\right)\right|^2 \sqrt{1+\left(\epsilon\partial_x\Theta_1\left(x,\frac{y}{\epsilon}\right)\right)^2+\left(\epsilon\partial_y\Theta_1\left(x,\frac{y}{\epsilon}\right)\right)^2}\,dy\\
&= \epsilon^{N-2}\int_{D_1} \left|\omega(x,\epsilon y',\pm\epsilon\Theta_1(x,y'))\right|^2 \sqrt{1+\left(\epsilon\partial_x\Theta_1(x,y')\right)^2+\left(\partial_{y'}\Theta_1(x,y')\right)^2}\,dy'
\end{aligned}$$

$$\leqslant \epsilon^{N-2} \int_{D_1} \left|\omega\big(x, \epsilon y', \pm\epsilon\Theta_1(x, y')\big)\right|^2 \sqrt{1 + \big(\partial_x \Theta_1(x, y')\big)^2 + \big(\partial_{y'} \Theta_1(x, y')\big)^2}\, dy'$$

$$= \epsilon^{N-2} \int_{\Gamma_1^{\pm}} |\tilde{\omega}_\epsilon|^2 \, d\Gamma_1 \leqslant \epsilon^{N-2} c \|\tilde{w}_\epsilon\|^2_{H^s(R_1)} \leqslant \epsilon^{-1} c \|w_\epsilon\|^2_{H^s(R_\epsilon)}$$

$$\leqslant c\epsilon^{N-2} \|\omega\|^2_{X_\epsilon^{s/2}} \leqslant c\|\omega\|^2_{X_\epsilon^{s/2}},$$

where $y = \epsilon y'$, $z = \epsilon z'$ and we have used that $0 < \epsilon \leqslant 1$, $N \geqslant 2$, and the Trace Theorem for functions in $H^s(R_1)$, $\frac{1}{2} < s \leqslant 1$, which are orthogonal to a constant. The proof is complete. □

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