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Loop scattering in two-dimensional QCD

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Using the integrability conditions that we recently obtained in two-dimensional QCD with massless fermions we arrive at a sufficient number of conservation laws to fix the scattering amplitudes involving a local version of the Wilson loop operator.

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Two-dimensional QCD (QCD_2) has been studied by several authors (see Ref. [1] for a review). The theory presents, in a clear way, several features expected to characterize strong interactions, but in addition to such desirable physical features, it is possible to understand several of its properties analytically. In fact, we claim that it is possible to obtain its full on-shell solution, as far as the scattering of closed loop (white) operators is concerned.

The problem of two-dimensional QCD with massless fermions has evaded solution for 20 years, since 't Hooft's [2] first attempt to solve the model for large number of colors. Because of confinement we may integrate over the quarks, leaving an effective interaction for the gluons. We thus represent the fermionic determinant in terms of a bosonic functional integral over the (exponential of i times the) Wess-Zumino-Witten (WZW) action [3,4] and use the Polyakov-Wiegmann identity [3] (see below) to split the action of the product of two fields in terms of the WZW action of each single field and a contact term; this shows that, at the Lagrangian level, the dynamics factorizes in terms of several conformally invariant theories (WZW and ghost actions), and a nonconformally invariant piece, given by a WZW action perturbed off criticality, which contains the (main) dynamical information. Finally, these fields, decoupled at the Lagrangian level, are coupled via Becchi-Rouet-Stora-Tyutin (BRST) constraints [5] obeyed by the theory, implying that only white objects appear in the physical subspace. The most important result implying the computability of the S matrix is the integrability of the aforementioned off-critically perturbed WZW theory [6].

Let us briefly summarize the known results. The two-dimensional QCD partition function with massless quarks is given by

$$\mathscr{Z}[i_\mu,\eta,\bar\eta]=\int \mathscr{D}\psi\mathscr{D}\bar\psi\mathscr{D}A_\mu\mathscr{D}E\,\mathscr{D}[\text{ghosts}]\times\exp\left(iS_{\text{ghosts}}+i\int d^2x\left(\bar\psi i\not{D}\psi-\tfrac{1}{2}\,\mathrm{tr}E^2+\tfrac{1}{2}\,\mathrm{tr}EF_{+-}+i^\mu A_\mu+\bar\eta\psi+\bar\psi\eta\right)\right). \tag{1}$$

The fermion may be integrated out in terms of the gauge potentials U and V, introduced as

$$A_+=\frac{i}{e}U^{-1}\partial_+U,\quad A_-=\frac{i}{e}V\partial_-V^{-1}, \tag{2}$$

these in turn, are implemented in the quantum case via inclusion of the corresponding Jacobian in (1). As a result one obtains the WZW action upon use of

$$\det i\not{D}=e^{i\Gamma[UV]} \tag{3}$$

and

$$\det\left(\frac{\partial A_+}{\partial U}\frac{\partial A_-}{\partial V}\right)=\det i\not{D}^{\text{adj}}=e^{ic_V\Gamma[UV]}, \tag{4}$$

where a gauge-invariant regularization prescription has been chosen to evaluate the Jacobian, fixing the contact term A_+A_-. Here c_V is the second Casimir, defined in $f^{abc}f^{dbc}=\frac{1}{2}c_V\delta^{ad}$, where f^{abc} are the structure constants of the color group. Therefore, the result is a function of the gauge-invariant product UV; the well-known WZW functional reads

$$\Gamma[U]=\frac{1}{8\pi}\int d^2x\,\partial^\mu U^{-1}\partial_\mu U+\frac{1}{4\pi}\int_0^1 dr\int d^2x\,\epsilon^{\mu\nu}\tilde U^{-1}\dot{\tilde U}\tilde U^{-1}\partial_\mu\tilde U\tilde U^{-1}\partial_\nu\tilde U, \tag{5}$$

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and obeys the Polyakov-Wiegmann identity

$$\Gamma[UV]=\Gamma[U]+\Gamma[V]+\frac{1}{4\pi}\,\mathrm{tr}\int d^2xU^{-1}\partial_+UV\partial_-V^{-1}. \tag{6}$$

Above, the tilde denotes a field $\tilde{U}(r,x)$, where $\tilde{U}(0,x)=1$, and $\tilde{U}(1,x)=U(x)$.

In terms of the gauge-invariant combination $\Sigma=UV$, the auxiliary E-field interaction becomes, upon introduction of F_{+-} as a function of U and V,

$$\begin{aligned}\mathrm{tr}EF_{+-}&=\frac{i}{e}\,\mathrm{tr}UEU^{-1}\partial_+(\Sigma\partial_-\Sigma^{-1})\\&\equiv\frac{i}{e}\,\mathrm{tr}\hat{E}\partial_+(\Sigma\partial_-\Sigma^{-1}).\end{aligned} \tag{7}$$

We change the auxiliary variable $\hat{E}=UEU^{-1}$ as

$$\partial_+\hat{E}=\frac{ie(c_V+1)}{2\pi}\beta^{-1}\partial_+\beta, \tag{8}$$

and use again the Polyakov-Wiegmann identity. This procedure leads to a complete separation of variables at the Lagrangian level, as we see from

$$\begin{aligned}\mathscr{Z}[i_\mu,\eta,\bar\eta]=&\int\mathscr{D}\hat{g}e^{i\Gamma[\hat{g}]}\mathscr{D}U\mathscr{D}[\mathrm{ghosts}]e^{iS_{\mathrm{ghosts}}}\int\mathscr{D}\hat{\Sigma}e^{-i(c_V+1)\Gamma[\hat{\Sigma}]}\int\mathscr{D}\beta\,\exp\left(i\Gamma[\beta]+\frac{\mu^2 i}{2}\,\mathrm{tr}\int d^2z[\partial_+^{-1}(\beta^{-1}\partial_+\beta)]^2\right)\\&\times\exp\left(i\int d^2z i_\mu A_\mu-i\int d^2z\,d^2w\bar\eta(z)(i\not{D})^{-1}(z,w)\eta(w)\right),\end{aligned} \tag{9}$$

where $\hat{g}=UgV$, $\hat{\Sigma}=\beta\Sigma$, and $\mu=(c_V+1)e/2\pi$. The gauge volume decouples. The quadratic and nonlocal piece, which we symbolize by Δ, can be linearized and made local by the introduction of the identity

$$\begin{aligned}e^{\frac{i}{2}\mu^2\Delta}=&\int\mathscr{D}C_-\,\exp\left(i\int d^2x\tfrac{1}{2}\,\mathrm{tr}(\partial_+C_-)^2\right.\\&\left.-\mu\,\mathrm{tr}\int d^2xC_-(\beta^{-1}\partial_+\beta)\right),\end{aligned} \tag{10}$$

where C_- plays the role of an auxiliary field. The theory contains hidden constraints, as discussed elsewhere [5,6]. Such constraints are essential to build the asymptotic states of the theory, which cannot have color.

The nontrivial dynamical content of the theory is described by the β field, which is the only nonconformally invariant piece of the theory. It is useful to discuss the model in parallel to its dual, obtained by rewriting the field C_- appearing in Eq. (10) as $C_-=(i/4\pi\mu)W\partial_-W^{-1}$. After a simple change of variables and use of (6), one obtains for the β,C_- integration the partition function (below, we use $\hat{\beta}=\beta W$)

$$\begin{aligned}\mathscr{Z}=&\int\mathscr{D}\hat{\beta}e^{i\Gamma[\hat{\beta}]}\int\mathscr{D}W\,\exp\left(-i(c_V+1)\Gamma[W]\right.\\&\left.-\frac{i}{2(4\pi\mu)^2}\,\mathrm{tr}\int d^2z[\partial_+(W\partial_-W^{-1})]^2\right),\end{aligned} \tag{11}$$

or, equivalently, the dual action

$$\begin{aligned}S=&-(c_V+1)\Gamma[W]\\&+\mathrm{tr}\,\frac{1}{2}\int d^2x\left(B^2+\frac{1}{2\pi\mu}\partial_+B\partial_-WW^{-1}\right),\end{aligned} \tag{12}$$

where the field B plays, in the dual case, the same role as C_- did before, namely, it is an auxiliary field. The theory (9) and its dual (11), (12) can be studied by simple methods. The β (W) equations of motion correspond to a conservation law

$$\partial_+(I_-^{\beta,W})=0, \tag{13}$$

where

$$\begin{aligned}I_-^\beta(x^-)=&4\pi\mu^2J_-^\beta(x^+,x^-)+\partial_+\partial_-J_-^\beta(x^+,x^-)\\&+[J_-^\beta(x^+,x^-),\partial_+J_-^\beta(x^+,x^-)],\end{aligned} \tag{14}$$

$$\begin{aligned}I_-^W(x^-)=&\frac{(c_V+1)}{4\pi}J_-^W+\frac{1}{(4\pi\mu)^2}\partial_+\partial_-J_-^W\\&+\frac{1}{(4\pi\mu)^2}[J_-^W,\partial J_-^W].\end{aligned} \tag{15}$$

Notice that for the current $I_-^W(x^-)$ we have only local expressions. Concerning the current I_-^β, the first-class constraints are given by

$$i\hat{g}\partial_-\hat{g}^{-1}-i(c_V+1)\Sigma\partial_-\Sigma^{-1}+J_-(\mathrm{ghosts})\sim0, \tag{16}$$

$$i\hat{g}^{-1}\partial_+\hat{g}-i(c_V+1)\Sigma^{-1}\partial_+\Sigma+J_+(\mathrm{ghosts})\sim0, \tag{17}$$

leading to two BRST nilpotent charges $Q^{(\pm)}$, as discussed in [5,6]. Therefore we find two first-class constraints. For the dual theory one finds a very similar structure (see [1,6]). Second-class constraints also show up, being given, respectively, by

$$\Omega_{ij}^\beta=(\beta\partial_-\beta^{-1})_{ij}+4i\pi\mu(\beta C_-\beta^{-1})_{ij}-(\hat{g}\partial_-\hat{g}^{-1})_{ij}, \tag{18}$$

$$\Omega_{ij}^{W}=(c_V+1)\Sigma^{-1}\partial_+\Sigma-(c_V+1)W^{-1}\partial_+W+\frac{1}{\mu}W^{-1}\partial_+BW$$
$$=0. \tag{19}$$

To obtain the consequences of such a huge set of conservation laws and constraints, we have to verify how the corresponding charges act on asymptotic states. This action can be unraveled once one knows the Lorentz transformation properties of the charges. A useful set of conserved charges is

$$Q^{(n)}=\int dx(t-x)^n I_-(x). \tag{20}$$

It is not difficult to obtain the commutation relation between the Lorentz generator and the charge

$$[T,Q^{(n)}]=nQ^{(n)}, \tag{21}$$

where the Lorentz generator T acts on an asymptotic one-particle state as a derivative with respect to the rapidity θ [i.e., $p_\mu=m(\cosh\theta, \sinh\theta)$]:

$$T=\frac{d}{d\theta}, \tag{22}$$

from which one obtains, for the asymptotic charge, the expression

$$Q^{(n)}\simeq(p_-)^n J, \tag{23}$$

where J is the generator of the right transformations for the β fields. This would mean that the first charge $Q^{(1)}$ is the generator of right-SU(N) transformations for the β fields. In the quantum theory there is no contribution from the short-distance expansion of J_- and $\partial_\mu J_-$, since divergences are too mild. The SU(N) transformation generators are simple to compute. In the β language one has, for left-SU(N) transformations,

$$J_+^{L\,\mathrm{SU}(N)}=0,$$
$$J_-^{L\,\mathrm{SU}(N)}=-\frac{1}{4\pi}\partial_-\beta\beta^{-1}+i\mu\beta C_-\beta^{-1}, \tag{24}$$

while for right-SU(N) transformations we obtain

$$J_-^{R\,\mathrm{SU}(N)}=i\mu C_-+[\partial_+C_-,C_-],$$
$$J_+^{R\,\mathrm{SU}(N)}=-\frac{1}{4\pi}\beta^{-1}\partial_+\beta. \tag{25}$$

For the first set, i.e., the left transformations, one finds that the currents are equivalent to analogous chiral currents corresponding to free fields. On the other hand, the right transformations lead to an infinite number of conservation laws due to the presence of the Lax pair, and reproduce Eq. (13); therefore we have confirmed the fact that (13) generates the right-SU(N) transformations. For the left transformations we obtain only left-moving currents, which explain constraints such as Eqs. (18) and (19), where they are expressed in terms of free WZW currents.

The structure is even clearer in the W language, where the right-SU(N) transformations are generated by the currents

$$J_-^{R\,\mathrm{SU}(N)}=0,$$
$$J_+^{R\,\mathrm{SU}(N)}=-\frac{c_V+1}{4\pi}W^{-1}\partial_+W+\frac{1}{4\pi\mu}W^{-1}\partial_+BW, \tag{26}$$

while the left-SU(N) transformations are derived from the currents

$$J_-^{L\,\mathrm{SU}(N)}=\frac{c_V+1}{4\pi}\partial_-WW^{-1}+\frac{1}{4\pi\mu}[B,\partial_-WW^{-1}],$$
$$J_+^{L\,\mathrm{SU}(N)}=-\frac{1}{4\pi\mu}\partial_+B. \tag{27}$$

In the β_{ai} language the left indices, such as a,b,c,d are thus free, described by a trivial S matrix, while the right indices, such as i,j,k,l are described by an SU(N) covariant integrable S matrix, which has a well-known classification [7]. Indeed, SU(N)-invariant S matrices of the integrable type can be classified within 5 very definite types [7]. The first one is trivial. The third one is O(N) invariant and does not concern us here. The fourth and fifth types have a rather strange form, but what decides for the S matrix of the second type is the fact in such a case the particle/antiparticle backward scattering vanishes: a characteristic of scatterings where antiparticles are bound states of particles, or a bound-state structure as in the Z_N model appears (see Chap. 8 of Ref. [8]). In the present case, for large N, the interpretation of β as a Wilson loop variable implies $\beta^2\sim\beta$. We thus write the (unique) ansatz for scattering of β particles obeying the above requirements:

$$\begin{aligned}\langle ai\theta_1,bj\theta_2|ck\theta_3 dl\theta_4\rangle &= \delta(\theta_1-\theta_3)\delta(\theta_2-\theta_4)\\ &\times\delta_{ac}\delta_{bd}(\sigma_1(\theta)\delta_{ik}\delta_{jl}+\sigma_2(\theta)\delta_{il}\delta_{jk})\\ &+\delta(\theta_1-\theta_4)\delta(\theta_2-\theta_3)\\ &\times\delta_{ad}\delta_{bc}(\sigma_1(\theta)\delta_{il}\delta_{jk}+\sigma_2(\theta)\delta_{ik}\delta_{jl}),\end{aligned} \tag{28}$$

where $\theta=\theta_1-\theta_2$ is the relative rapidity. A similar result for the dual theory exists, where the left and right definitions have to be interchanged; namely, the field is W_{ia} in the dual case. Notice that here we are not concerned with the other sector, describing the vacuum.

The S matrix of the SU(N) theory is such that $\sigma_2=(2\pi i/N\theta)\sigma_1$. A pole term describing the condition $\beta^2=\beta$, suitably interpreted as a bound-state structure is given by

$$B_{N=\infty}(\theta)=\frac{\sinh(\theta)+i\,\sin(\pi/3)}{\sinh(\theta)-i\,\sin(\pi/3)}. \tag{29}$$

Finally, the only asymptotic states compatible with gauge SU(N) color symmetry, are those confining such degrees of freedom; therefore we have the scattering elements [8]

$$\langle \theta_1 \theta_2 | \theta_3 \theta_4 \rangle = B_N(\theta)[\delta(\theta_1-\theta_3)\delta(\theta_2-\theta_4) + \delta(\theta_1-\theta_4)\delta(\theta_2-\theta_3)]. \quad (30)$$

For large N, the amplitude is given by (29), since in this limit $\sigma_1 = 1 + O(1/N)$, and $\sigma_2 = O(1/N)$, and we write

$$\langle \theta_1 \theta_2 | \theta_3 \theta_4 \rangle = \frac{\sinh(\theta) + i \sin(\pi/3)}{\sinh(\theta) - i \sin(\pi/3)} [\delta(\theta_1-\theta_3)\delta(\theta_2-\theta_4) + \delta(\theta_1-\theta_4)\delta(\theta_2-\theta_3)], \quad (31)$$

whereas for finite N the coefficient needs further information to be fixed. Expression (31) is a highly nontrivial result, representing our main achievement. However, it is clear that Eq. (30) is valid for arbitrary N. Although we cannot compute $B_N(\theta)$ for arbitrary N, the information contained there, is of importance, due to the possibility of writing any scattering process in two-dimensional QCD in terms of a single function.

Notice that in our formulation the Fermi fields disappeared completely from the partition function, as, e.g., in Eq. (12). This is due to the fact that our present results are entirely based on the bosonized form of the theory, where the fermions, or more exactly the bound states of fermions are traded for a set of bosonic fields $\hat{g}$ and β, where $\hat{g}$ (together with Σ and the ghosts) describes the vacuum sector (see Ref. [9]) while the β field describes the massive sector. Thus, the physical content of the theory is obtained from the β field. However, an explicit interpolating operator relating it to the quark-antiquark bound states cannot be obtained in a simple way. There is a simpler picture in the Abelian case, where β is the exponential of the field strength, and directly describes the meson. Along these lines, a more physical insight about the significance of the β field was later obtained in Ref. [9], where it was understood that after bosonizing the theory, a change of variables as discussed in the steps going from (5) to (12) was essential to separate the physical degrees of freedom. Thus, part of the quark-antiquark degrees of freedom percolates to the vacuum sector, conferring to it a nontrivial structure, as discussed in Ref. [9], while the massive part is described by the β field. However, due to the infinite gauge tails involved in the transformation, a direct interpretation in terms of the original fields cannot be given at present. In this direction we would have to rewrite the conservation laws in terms of quark bound states, and find their action on hadronic asymptotic states. We are presently involved in such a project, but a definite answer is still beyond reach, since we need a full comprehension of the problem of confinement, and of how to build the relevant hadronic bound states, respecting the confinement constraints.

Concerning the physical interpretation of the fields we have considered in this paper we have to elaborate on the above description. Indeed, β was introduced in Eq. (8), meaning that β is a loop-type variable, namely,

$$\beta \approx e^{i \frac{e}{2\pi}(c_V+1)\tilde{E}} + \text{non-Abelian corrections.}$$

Since the E equation of motion is

$$E \approx F_{+-},$$

the loop character of β is obtained, although we still have to take into account the dressing arising from the transformation

$$\tilde{E} = UEU^{-1},$$

implicit in the above. Such a loop is expected to be described as physical variable, connected, as discussed in [9] and above, to the physical (dressed) part of the quark-antiquark bound states.

Finally, concerning the large-N limit, we have for colorless states to take the trace of the β field, namely, $\mathrm{tr}\beta$. In the large-N limit the scattering for such objects is extremely simple, being described by the phase (29), consequence of the bound-state description of the constraint. This is also compatible with the fact that the large-N limit accommodates a string-type description, which for pure QCD_2 was discussed in a series of papers by Gross [10].

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