ON THE HARDY SPACE $H^1$ ON PRODUCTS OF HALF-SPACES

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We show that the Hardy space $H^1_{anal}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ can be identified with the class of functions $f$ such that $f$ and all its double and partial Hilbert transforms $H_k f$ belong to $L^1(\mathbb{R}^2)$. A basic tool used in the proof is the bisubharmonicity of $|F|^q$, where $F$ is a vector field that satisfies a generalized conjugate system of Cauchy-Riemann type.

Introduction. The interest of a theory for the $H^p$ spaces on products of half-spaces was first raised by C. Fefferman and E. M. Stein in the now classic paper "$H^p$ spaces of several variables" [6]. Afterward several authors have contributed on this subject. It is worth mentioning the survey paper by C. Y. A. Chang and R. Fefferman [4], and the references quoted there. In particular, the $H^1$ spaces on products of half-spaces was studied by H. Sato [8] giving definitions via maximal functions and via the multiple Hilbert transform. On the other hand Merryfield [7] proves the equivalence of the definitions given via the area integrals and via the multiple Hilbert transforms. More recently S. Sato [9] proved the equivalence between the Lusin area integral and the nontangential maximal function.

The purpose of this paper is to derive directly the equivalence of the definitions of the $H^1$ space given via the multiple Hilbert transforms and via an $L^1$ condition on a biharmonic vector field $F = (u_1, u_2, u_3, u_4)$ which is a solution of a generalized Cauchy-Riemann system introduced by Bordin-Fernandez [3]. The main tool we shall use is the bi-subharmonicity of $|F|^q$, $0 < q < 1$. But the proof of this fact here is different from the classical one given by Stein-Weiss [10]. We rely on ideas of A. P. Calderón, R. Coiffman and G. Weiss (see [5]). We shall confine ourselves to the bidimensional case.

This paper is part of the author's doctoral thesis presented to UNICAMP in 1982, and the results are announced in [1] and [2].

NOTATION. We shall use the following notations throughout:

$$\square = \{k = (k_1, k_2), k_j = 0, 1, j = 1, 2\}$$
where $x_1 = x$, $x_2 = y$, $t_1 = s$, $t_2 = t$.

The generalized Cauchy-Riemann system was introduced by Bordin-Fernandez [3].

Let $P_r(x)$ and $Q_r(x)$ denote the Poisson and conjugate Poisson kernels in $\mathbb{R}^2_+$, i.e.,

$$P_r(x) = cr/(r^2 + x^2) \quad \text{and} \quad Q_r(x) = cx/(r^2 + x^2)$$

the vector field $(P_sP_t * f, Q_sP_t * f, P_sQ_t * f, Q_sQ_t * f)$, where $f \in L^p(\mathbb{R}^2)$, $1 \leq p < \infty$, is a generalized conjugate vector field.

1.2. DEFINITION. Let $F = (u_k; k \in \Box)$ be a conjugate vector field. We say that $F$ belongs to $H_{anal}^1(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ if

$$\|F\|_{H_{anal}^1} = \sup_{s,t>0} \int \int |F(x,s;y,t)| \, dx \, dy < \infty.$$ 

1.3. DEFINITION. The partial and double Hilbert transforms of $f \in L^1(\mathbb{R}^2)$ are the tempered distributions, $H_kf$, defined by

\begin{align*}
\mathcal{F}(H_{10}f)(x,y) &= i(\text{sign } x) \hat{f}(x,y), \\
\mathcal{F}(H_{01}f)(x,y) &= i(\text{sign } y) \hat{f}(x,y), \\
\mathcal{F}(H_{11}f)(x,y) &= i(\text{sign } x)i(\text{sign } y) \hat{f}(x,y),
\end{align*}

and

$$(H_{00}f)(x,y) = f(x,y)$$
or shortly by
(2) \( \mathcal{F}(H_k f)(x, y) \)
\[ = (i \text{sign } x)^{k_1} (i \text{sign } y)^{k_2} \hat{f}(x, y), \quad k = (k_1, k_2) \in \square, \]
where \( \mathcal{F} \) denotes the Fourier transformation and \( \hat{f} \) the Fourier transform of \( f \).

1.4. DEFINITION. By \( H_{1Hb}(\mathbb{R} \times \mathbb{R}) \) we mean all \( f \in L^1(\mathbb{R}^2) \) such that \( H_k f \in L^1(\mathbb{R}^2) \), for each \( k \in \square \). The norm of \( f \in H_{1Hb}^1 \) is defined by
\[ \| f \|_{Hb} = \sum_{k \in \square} \| H_k f \|_1. \]

2. The subharmonicity of \( |F|^q \). The basic fact which enables us to develop the theory of \( H^p \)-spaces on the product of half spaces is the existence of a positive \( q < 1 \) such that \( |F|^q \) is bisubharmonic. We shall show that every conjugate vector field has this property.

2.1. DEFINITION. Let \( (A_j)_{j=1, \ldots, n} \) be a family of matrices \( d \times m \). We say that \( (A_j) \) is an elliptic family provided that for an \( m \)-dimensional vector \( v \) and an \( n \)-tuple \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) we have
\[ \sum_{j=1}^n \lambda_j A_j v = 0 \]
only if either \( v \) or \( \lambda \) is zero.

2.2. LEMMA (Calderón). Let \( (A_j)_{j=1, \ldots, n} \) be an elliptic family; \( v \) and \( u^1, \ldots, u^n \) vectors of \( \mathbb{R}^m \) and \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \). Suppose that
\[ \sum_{j=1}^n A_j u^j = 0 \quad \text{and} \quad \sum_{j=1}^n \lambda_j A_j v = 0. \]
Then, there exists a positive \( \alpha < 1 \), depending only on \( A_1, \ldots, A_n \), such that
\[ \max_{j=1}^n (u^j \cdot v)^2 \leq \alpha \sum_{j=1}^n |u^j|^2. \]

Proof. See [5].

2.3. PROPOSITION. The generalized Cauchy-Riemann system 1.1(1) can be put in the form
\[ A_1 \frac{\partial F}{\partial s} + A_2 \frac{\partial F}{\partial x} + A_3 \frac{\partial F}{\partial t} + A_4 \frac{\partial F}{\partial y} = 0, \]
where
\[
\frac{\partial F}{\partial z} = \left( \frac{\partial u_k}{\partial z}; k \in \Box \right), \quad z = s, x, t, y
\]
and \( A_j, 1 \leq j \leq 4, \) are the \( 8 \times 4 \) matrices given by
\[
A_1 = \begin{pmatrix} B_1 \\ N \end{pmatrix}, \quad A_2 = \begin{pmatrix} B_2 \\ N \end{pmatrix}, \quad A_3 = \begin{pmatrix} N \\ B_3 \end{pmatrix}, \quad A_4 = \begin{pmatrix} N \\ B_4 \end{pmatrix}
\]
where
\[
B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},
\]
\[
B_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix},
\]
and \( N \) is the \( 4 \times 4 \) null matrix. Moreover the families \((B_1, B_2)\) and \((B_3, B_4)\) are elliptic.

**Proof.** We will first show that \((B_1, B_2)\) is an elliptic family. The proof that \((B_1, B_2)\) is an elliptic family is exactly the same.

Let \( \lambda = (\lambda_3, \lambda_4), \) \( v = (v_1, v_2, v_3, v_4) \) denote elements of \( \mathbb{R}^2 \) and \( \mathbb{R}^4 \), respectively, such that
\[
\sum_{j=3}^{4} \lambda_j B_j v = 0;
\]
we will show that \( \lambda = 0 \) or \( v = 0 \). Suppose \( v \neq 0 \) with \( v_1 \neq 0 \), for example; then we will show that \( \lambda = 0 \). Indeed, from (2) we have
\[
\lambda_3 v_1 + \lambda_4 v_3 = 0 \quad \text{and} \quad -\lambda_3 v_3 + \lambda_4 v_1 = 0.
\]
Since \( v_1 \neq 0 \), then \( \lambda_3 = \lambda_4 = 0 \); therefore \( \lambda = 0 \). In the same way, if \( v_j \neq 0, j \neq 1 \), we have that \( \lambda = 0 \). This proves the proposition.

2.4. **Theorem.** Let \( F = (u_k, k \in \Box) \) be a generalized conjugate vector field. Then, there exists a positive \( q < 1 \) such that \( |F|^q \) is bisubharmonic.

**Proof.** We shall use the following notation:
\[
F \cdot G = \sum_{k \in \Box} u_k \cdot v_k, \quad \text{where} \quad F = (u_k; k \in \Box) \quad \text{and} \quad G = (v_k; k \in \Box).
\]
We shall prove that there exists $0 < q_1 < 1$ such that

$$\Delta_{01}|F|^{q_1} = \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial t^2} \geq 0. \tag{1}$$

Since $F = (u_k; k \in \square)$ is a conjugate vector field, then

$$B_3 \frac{\partial F}{\partial t} + B_4 \frac{\partial F}{\partial y} = 0, \tag{2}$$

where $B_2$ and $B_4$ are defined in Proposition 2.3.

The system (2) is elliptic, by Proposition 2.3. Therefore, by Lemma 2.2, there exists $0 < \alpha_1 < 1$ such that

$$\left(\frac{F}{|F|} \cdot \frac{\partial F}{\partial t}\right)^2 + \left(\frac{F}{|F|} \cdot \frac{\partial F}{\partial y}\right)^2 \leq \alpha_1 \left(\left|\frac{\partial F}{\partial t}\right|^2 + \left|\frac{\partial F}{\partial y}\right|^2\right). \tag{3}$$

Hence, since

$$\Delta_{01}|F|^{q_1} = q_1|F|^{q_1-2} \left\{\left|\frac{\partial F}{\partial t}\right|^2 + \left|\frac{\partial F}{\partial y}\right|^2 + (q_1-2) \left[\left(\frac{F}{|F|} \cdot \frac{\partial F}{\partial t}\right)^2 + \left(\frac{F}{|F|} \cdot \frac{\partial F}{\partial y}\right)^2\right]\right\}$$

we have, by (3),

$$\Delta_{01}|F|^{q_1} \geq 0 \tag{4}$$

with $q_1 \geq 2 - 1/\alpha_1$. In this way, for $q_2 \geq 2 - 1/\alpha_2$, with $\alpha_2$ given as in Lemma 2.2 we have

$$\Delta_{10}|F|^{q_2} \geq 0. \tag{5}$$

Hence, by (4) and (5) we have that there exists $0 < q < 1$ such that $|F|^q$ is bisubharmonic and therefore subharmonic.

3. The equivalence of $H_{Hb}^1(\mathbb{R} \times \mathbb{R})$ and $H_{\text{anal}}^1(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$.

3.1. **Theorem.** (i) If $F = (u_k; k \in \square)$ belongs to $H_{\text{anal}}^1(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$, there exists an $f \in L^1(\mathbb{R}^2)$ such that $H_k f \in L^1(\mathbb{R}^2)$ and $u_k = (P_s P_t)^* H_k f$ for each $k \in \square$. Moreover, there is a positive constant $C$, independent of $F$, such that

$$\sum_{k \in \square} \|H_k f\|_1 \leq C\|F\|_{H_{\text{anal}}^1}. \tag{1}$$
(ii) Let \( f \in L^1(\mathbb{R}^2) \). If \( H_k f \in L^1(\mathbb{R}^2) \), for each \( k \in \square \), then the conjugate vector field

\[
F = \left((P_sP_t) \ast H_k f; k \in \square\right)
\]

belongs to \( H^1_{\text{anal}}(\mathbb{R}^2_+ \times \mathbb{R}^2_+) \) and there exists a positive constant \( C \), independent of \( f \), such that

\[
\|F\|_{H^1_{\text{anal}}} \leq C \sum_{k \in \square} \|H_k f\|_1.
\]

Thus, \( H^1_{\text{anal}}(\mathbb{R}^2_+ \times \mathbb{R}^2_+) \) can be identified with \( H^1_{Hb}(\mathbb{R} \times \mathbb{R}) \) with equivalence of norms. In the proof of this theorem we will use the result stated in the next lemma.

3.2. Lemma. If \( F = (u_k; k \in \square) \) belongs to \( H^1_{\text{anal}}(\mathbb{R}^2_+ \times \mathbb{R}^2_+) \), there exists a positive constant \( C \), independent of \( F \), such that

\[
\int \int \sup_{s,t>0} |F(x,s;y,t)| \, dx \, dy \leq C \sup_{s,t>0} \int \int |F(x,s;y,t)| \, dx \, dy.
\]

Moreover,

\[
\lim_{s,t \to 0} F(x,s;y,t) = F(x,y)
\]

exists almost everywhere and in \( L^1(\mathbb{R}^2) \) norm.

Proof. Suppose that each \( u_k \) takes values in a fixed finite-dimensional Hilbert space, \( V_1 \). We take a conjugate vector field \( \varphi = (v_k; k \in \square) \), where each \( v_k \) takes its values in \( V_2 \) (\( V_2 \) is another finite-dimensional Hilbert space and we consider \( V = V_1 \oplus V_2 \)), satisfying:

\[
\begin{align*}
(2) & \quad |\varphi(x,s;y,t)|^2 = 2/\left[x^2 + (1 + s)^2\right]\left[y^2 + (1 + t)^2\right], \\
(3) & \quad \lim_{(x,s) \to \infty, (x,s) \in \mathbb{R}^2_+} |v_k(x,s;y,t)| = 0, \quad \text{for each pair } (y,t) \in \mathbb{R}_+^2,
\end{align*}
\]

and

\[
\begin{align*}
(4) & \quad \lim_{(y,t) \to \infty, (y,t) \in \mathbb{R}_+^2} |v_k(x,s;y,t)| = 0, \quad \text{for each pair } (x,s) \in \mathbb{R}_+^2.
\end{align*}
\]
We define
\[ v_{00} = \left( \frac{\partial^2 H}{\partial s^2}, \frac{\partial^2 H}{\partial t^2}, \frac{\partial^2 H}{\partial s \partial x}, \frac{\partial^2 H}{\partial t \partial y} \right), \]
\[ v_{10} = \left( \frac{\partial^2 H}{\partial s \partial x}, \frac{\partial^2 H}{\partial t^2}, \frac{\partial^2 H}{\partial x^2}, \frac{\partial^2 H}{\partial t \partial y} \right), \]
\[ v_{01} = \left( \frac{\partial^2 H}{\partial s^2}, \frac{\partial^2 H}{\partial t^2}, \frac{\partial^2 H}{\partial s \partial y}, \frac{\partial^2 H}{\partial x^2} \right), \]
\[ v_{11} = \left( \frac{\partial^2 H}{\partial s \partial x}, \frac{\partial^2 H}{\partial t \partial y}, \frac{\partial^2 H}{\partial x^2}, \frac{\partial^2 H}{\partial y^2} \right), \]
where \( H: \mathbb{R}_2^+ \times \mathbb{R}_+^2 \to \mathbb{R} \) for
\[ H(x, s; y, t) = \frac{1}{2} \log[(x^2 + (1 + s)^2)^{-1}(y^2 + (1 + t)^2)^{-1}]. \]
(2), (3) and (4) follow easily.

Now, we define for every \( \varepsilon > 0 \)
\[ F_\varepsilon(x, s; y, t) = F(x, s + \varepsilon; y, t + \varepsilon) + \varepsilon \varphi(x, s; y, t). \]
We can verify that \( F_\varepsilon \) is continuous in \( (x, s) \in \mathbb{R}_2^+ \), \((y, t) \in \mathbb{R}_+^2\) for each pair \((y, t) ((x, s)); F_\varepsilon \) tends to zero as \(|(x, s)| \) or \(|(y, t)| \) tends to \( \infty \), and \(|F_\varepsilon| > 0 \). Then by Theorem 2.4, there exists a \( q, 0 < q < 1 \), such that \(|F_\varepsilon|^q \) is bisubharmonic.

Next, we define \( g_\varepsilon(x, y) = |F_\varepsilon(x, 0; y, 0)|^q \). By (2) and from our assumptions on \( F \), for \( p = 1/q \), we have
\[ \|g_\varepsilon\|_p^p \leq \|F\|_{H_{ \text{anal}}^1} + \varepsilon \|\varphi\|_1. \]
Now let \( G_\varepsilon(x, s; y, t) \) be the iterated Poisson integral of \( g_\varepsilon \). By the properties of \( F_\varepsilon \), the properties of the iterated Poisson integral and the maximum principle, we get
\[ |F_\varepsilon(x, s; y, t)|^q \leq G_\varepsilon(x, s; y, t). \]
Hence, we can select a subsequence \( g_\varepsilon \) which converges weakly to a function \( g \in L^p(\mathbb{R}^2) \) and such that
\[ \|g\|_p^p \leq \|F\|_{H_{ \text{anal}}^1}. \]
Hence, this yields
\[ |F(x, s; y, t)|^q \leq G(x, s; y, t), \]
where \( G(x, s; y, t) \) is the iterated Poisson integral of \( g \). By the properties of the partial Hardy-Littlewood maximal functions \( M^{01} \) and \( M^{10} \).
[10], and of the maximal functions \( u^*(G)(x, y) = \sup_{s, t > 0} G(x, s; y, t) \), we have

\[
\int \int \sup_{s, t > 0} |F(x, s; y, t)| \, dx \, dy 
\leq C \int \int M^{01}(M^{10}(u^*(G)))(x, y) \, dx \, dy 
\leq C \|u^*(G)\|_P^p \leq C \|g\|_P^p \leq C \|F\|_{H_{\text{anal}}}^1.
\]

Hence, we have

\[
\int \int \sup_{s, t > 0} |F(x, s; y, t)| \, dx \, dy \leq C \|F\|_{H_{\text{anal}}}^1.
\]

This proves (1).

Next, we shall prove that \( \lim_{s, t \to 0} F(x, s; y, t) \) exists almost everywhere and in the \( L^1(\mathbb{R}^2) \) norm. We have that

\[
|u_k(x, s; y, t)| \leq G(x, s; y, t)^p, \quad k \in \Box.
\]

Since \( G \) is nontangentially bounded, each \( u_k \) is nontangentially bounded, \( \lim_{s, t \to 0} u_k(x, s; y, t) \) exists almost everywhere. On the other hand, the dominated convergence theorem implies the convergence in the \( L^1(\mathbb{R}^2) \) norm.

**Proof of Theorem 3.1.** Step 1. Let \( F = (u_k; k \in \Box) \) in \( H_{\text{anal}}^1 \). Then, there exists finite Borel measures \( \mu_k \) such that

\[
u_k(x, s; y, t) = (P_s P_t) * \mu_k(x, y).
\]

Now, by the Lemma 3.2, the limits

(1) \( \lim_{s, t \to 0} u_k(x, s; y, t) = f_k(x, y), \quad k \in \Box, \)

exists in the \( L^1(\mathbb{R}^2) \) norm, and by the Fourier transform we have

(2) \( u_k(x, s; y, t) = \int \int (P_s P_t) \hat{f}(x', y') e^{-2\pi i (xx' + yy')} \, dx' \, dy'. \)

Then, as \( F = (u_k; k \in \Box) \) is a conjugate vector field, we have from (1) and (2) that

(3) \( f_k(x, y) = (H_k f_00)(x, y), \quad k \in \Box. \)

Since \( f_k \in L^1(\mathbb{R}^2) \), from (3) we have \( f_00 \in H_{Hb}^1 \) and

\[
\|f_00\|_{H_{Hb}^1} = \sum_{k \in \Box} \|H_k f_00\|_1
\leq 4 \sup_{s, t > 0} \int \int |F(x, s; y, t)| \, dx \, dy.
\]
Therefore $f = f_0 \in H^1_{Hb}$ and $\|f\|_{H^1_{Hb}} \leq C\|F\|_{H^1_{\text{anal}}}$. This proves (i).

**Step 2.** Let $f$ be a function in $L^1(\mathbb{R}^2)$ such that $H_k f \in L^1(\mathbb{R}^2)$, $k \in \square$. We will show that the vector field given by

$$(4) \quad F = ((P_s P_t)_{H_k} f; k \in \square)$$

belongs to $H^1_{\text{anal}}$.

By Fourier transform we see that $F$ is a conjugate vector field. Indeed,

$$\hat{u}_k(x, s; y, t) = (P_s P_t)\hat{f}(x, y)(H_k f)^*(x, y).$$

Since $H_k f \in L^1$, we have $(H_k f)^* \in L^\infty$ and

$$\int \int |\hat{u}_k(x, s; y, t)| \, dx \, dy \leq \|(H_k f)^*\|_{L^\infty} \int \int (P_s P_t)\hat{f}(x, y) \, dx \, dy.$$ 

Then, for each $k \in \square$, we have,

$$u_k(x, s; y, t) = \int \int \hat{u}_k(x', s; y', t) e^{-2\pi i (x' s + y' t)} \, dx' \, dy'$$

and consequently

$$u_{00}(x, s; y, t) = \int \int e^{-2\pi \|x||s} e^{-2\pi \|y||t} \hat{f}(x', y') e^{-2\pi i (x' s + y' t)} \, dx' \, dy',$$

$$u_{10}(x, s; y, t) = \int \int e^{-2\pi \|x||s} e^{-2\pi \|y||t} (i \text{ sign } x') \hat{f}(x', y') e^{-2\pi i (x' s + y' t)} \, dx' \, dy',$$

$$u_{01}(x, s; y, t) = \int \int e^{-2\pi \|x||s} e^{-2\pi \|y||t} (i \text{ sign } y') \hat{f}(x', y') e^{-2\pi i (x' s + y' t)} \, dx' \, dy',$$

$$u_{11}(x, s; y, t) = \int \int e^{-2\pi \|x||s} e^{-2\pi \|y||t} (i \text{ sign } x') (i \text{ sign } y') \hat{f}(x', y') e^{-2\pi i (x' s + y' t)} \, dx' \, dy'.$$

Henceforth $(u_k; k \in \square)$ is a conjugate vector field. Moreover, from (4) and Young's inequality we get

$$\int \int |F(x, s; y, t)| \, dx \, dy \leq \sum_{k \in \square} \| (P_s P_t)_{H_k} f \|_1 \leq \sum_{k \in \square} \| H_k f \|_1 = \| f \|_{H^2_{Hb}}.$$ 

This proves that $F \in H^1_{\text{anal}}(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$ and (2).
Acknowledgment. I would like to thank Dicesar Lass Fernandez for his guidance, encouragement and patience.

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Received December 30, 1987.

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