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Monotonicity of the zeros of orthogonal polynomials through related measures ☆

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Abstract

Relation between two sequences of orthogonal polynomials, where the associated measures are related to each other by a first degree polynomial multiplication (or division), is well known. We use this relation to study the monotonicity properties of the zeros of generalized orthogonal polynomials. As examples, the Jacobi, Laguerre and Charlier polynomials are considered.

Keywords: Orthogonal polynomials; Related measures; Three term recurrence relations

1. Introduction

Let $d\phi$ be a (positive) measure defined over the real line. Let $P_n^{(\phi)}$ denote the monic orthogonal polynomial of degree n with respect to $d\phi$. Then it is known that (see [4,10])

$$P_{n+1}^{(\phi)}(z) = \big(z - \beta_{n+1}^{(\phi)}\big) P_n^{(\phi)}(z) - \alpha_{n+1}^{(\phi)} P_{n-1}^{(\phi)}(z), \quad n \geqslant 1,$$

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with $P_0^{(\phi)}(z) = 1$ and $P_1^{(\phi)}(z) = z - \beta_1^{(\phi)}$. For $n \geqslant 1$ the coefficients $\beta_n^{(\phi)}$ are real, $\alpha_{n+1}^{(\phi)}$ are all positive, and, furthermore, all the zeros of $\{P_n^{(\phi)}\}$ lie in the interior of $I^{(\phi)}$, the closed convex hull of the support of $d\phi$ (the smallest closed interval containing the support).

Let $d\phi_0$ and $d\phi_1$ be two measures related to each other by

$$(x-q)\,d\phi_1(x) = c\,d\phi_0(x). \tag{1.1}$$

Here if the support of $d\phi_0$ is $E \subset (-\infty, \infty)$ then q is such that $x - q$ does not change sign within E. Moreover, the support of $d\phi_1$ is either E or $E \cup \{q\}$, where in the latter case $d\phi_1$ has a jump at q. The non-zero constant c is arbitrary up to the point that $(x-q)/c$ is finite and non-negative on E.

Expressing $P_n^{(\phi_1)}$ as a linear combination of the polynomials of $\{P_n^{(\phi_0)}\}$, one obtains

$$P_n^{(\phi_1)}(z) = P_n^{(\phi_0)}(z) + b_{n-1} P_{n-1}^{(\phi_0)}(z), \quad n \geqslant 1, \tag{1.2}$$

where

$$b_{n-1} = \frac{\rho_n^{(\phi_1)}}{c\rho_{n-1}^{(\phi_0)}}, \quad n \geqslant 1, \tag{1.3}$$

with $\rho_0^{(\phi)} = \mu_0^{(\phi)}$ and $\rho_n^{(\phi)} = \int [P_n^{(\phi)}(x)]^2\,d\phi(x) = \alpha_{n+1}^{(\phi)}\alpha_n^{(\phi)}\ldots\alpha_2^{(\phi)}\mu_0^{(\phi)}$, $n \geqslant 1$. Here $\mu_r^{(\phi)} = \int x^r\,d\phi(x)$.

In this paper we use the results that follow from (1.1) and (1.2) to study the monotonicity behavior of the zeros of two classes of orthogonal polynomials.

2. Preliminary results

Let $X^{(\phi)} = (-\infty, \infty) \setminus I^{(\phi)}$ be the complement of the interval $I^{(\phi)}$ in $(-\infty, \infty)$. Let $\tilde{X}^{(\phi)}$ be the closure of $X^{(\phi)}$ within $(-\infty, \infty)$. Also we denote by $Z^{(\phi)}$ the complement $\overline{\mathbb{C}} \setminus I^{(\phi)}$ of $I^{(\phi)}$ in the extended complex plane.

Let $\{O_n^{(\phi)}\}$ be the monic associated polynomials given by

$$\mu_0^{(\phi)} O_n^{(\phi)}(z) = \int \frac{P_n^{(\phi)}(z) - P_n^{(\phi)}(x)}{z - x}\,d\phi(x), \quad n \geqslant 0.$$

The following results [4,10] are well known:

$$O_{n+1}^{(\phi)}(z) = \bigl(z - \beta_{n+1}^{(\phi)}\bigr) O_n^{(\phi)}(z) - \alpha_{n+1}^{(\phi)} O_{n-1}^{(\phi)}(z), \quad n \geqslant 1,$$

with $O_0^{(\phi)}(z) = 0$ and $O_1^{(\phi)}(z) = 1$, and

$$\frac{O_n^{(\phi)}(z)}{P_n^{(\phi)}(z)} = \frac{1}{z - \beta_1^{(\phi)}} - \frac{\alpha_2^{(\phi)}}{z - \beta_2^{(\phi)}} - \frac{\alpha_3^{(\phi)}}{z - \beta_3^{(\phi)}} - \cdots - \frac{\alpha_n^{(\phi)}}{z - \beta_n^{(\phi)}}.$$

If we also assume $d\phi$ to be a determinate measure (i.e. where the solution of the associated moment problem is unique), then $\mu_0^{(\phi)} O_n^{(\phi)}(z)/P_n^{(\phi)}(z) \to \int (z-x)^{-1}\,d\phi(x)$ uniformly on every compact subsets of $Z^{(\phi)}$.

Now let $S_n^{(\phi)}(z) = (z - \beta_1^{(\phi)})O_n^{(\phi)}(z)/P_n^{(\phi)}(z)$, $n \geqslant 0$. Hence $S_0^{(\phi)}(z) = 0$, $S_1^{(\phi)}(z) = 1$ and one can write

$$S_n^{(\phi)}(z) = \frac{1}{1} - \frac{a_1^{(\phi)}(z)}{1} - \frac{a_2^{(\phi)}(z)}{1} - \cdots - \frac{a_{n-1}^{(\phi)}(z)}{1}, \tag{2.1}$$

for $n \geqslant 2$, where $a_n^{(\phi)}(z) = \alpha_{n+1}^{(\phi)}/[(z - \beta_n^{(\phi)})(z - \beta_{n+1}^{(\phi)})]$, $n \geqslant 1$. For $S_n^{(\phi)}(z)$ the following results hold:

$$\left.\begin{aligned} S_n^{(\phi)}(z) - S_{n-1}^{(\phi)}(z) &= \frac{\alpha_2^{(\phi)}\alpha_3^{(\phi)}\ldots\alpha_n^{(\phi)}(z - \beta_1^{(\phi)})}{P_{n-1}^{(\phi)}(z)P_n^{(\phi)}(z)}, \\ S_n^{(\phi)}(z) &= 1 + \sum_{j=2}^{n} \frac{\alpha_2^{(\phi)}\alpha_3^{(\phi)}\ldots\alpha_j^{(\phi)}(z - \beta_1^{(\phi)})}{P_{j-1}^{(\phi)}(z)P_j^{(\phi)}(z)}, \end{aligned}\right\} \quad n \geqslant 2. \tag{2.2}$$

This means that for any $z \in \tilde{X}^{(\phi)}$ we have $S_n^{(\phi)}(z) > 1$ for $n > 1$ and, moreover, $\{S_n^{(\phi)}(z)\}$ is an increasing sequence. Assuming $d\phi$ to be determinate, we define $S^{(\phi)}(z)$ on $Z^{(\phi)}$ by

$$S^{(\phi)}(z) = \lim_{n\to\infty} S_n^{(\phi)}(z) = \frac{z - \beta_1^{(\phi)}}{\mu_0^{(\phi)}} \int \frac{1}{z - x}\, d\phi(x). \tag{2.3}$$

Then we can write,

$$1 < S_2^{(\phi)}(z) < \cdots < S_n^{(\phi)}(z) < \cdots < S^{(\phi)}(z),$$

for any $z \in \tilde{X}^{(\phi)}$.

It is also known (see [4]) that $\{a_n^{(\phi)}(z)\}$ is a chain sequence for any $z \in \tilde{X}^{(\phi)}$. Therefore, we can write for any $z \in \tilde{X}^{(\phi)}$ that

$$v_n^{(\phi)}(z) = \big(\beta_n^{(\phi)} - z\big)\big(\beta_{n+1}^{(\phi)} - z\big) - \alpha_{n+1}^{(\phi)} > 0, \quad n \geqslant 1.$$

Now returning to (1.1) one has the following results.

Lemma 1. *Let $d\phi_0$ and $d\phi_1$ be such that* (1.1) *holds. Then the connecting coefficients b_n in the relation* (1.2) *satisfy*:

(A) $b_n/b_{n-1} = \alpha_{n+2}^{(\phi_1)}/\alpha_{n+1}^{(\phi_0)}$,
(B) $b_n - b_{n-1} = \beta_{n+1}^{(\phi_0)} - \beta_{n+1}^{(\phi_1)}$, *and*
(C) $(\beta_{n+1}^{(\phi_1)} - \beta_n^{(\phi_0)})b_{n-1} = \alpha_{n+1}^{(\phi_0)} - \alpha_{n+1}^{(\phi_1)}$,

for $n \geqslant 1$, where $b_0 = \beta_1^{(\phi_0)} - \beta_1^{(\phi_1)} = \alpha_2^{(\phi_1)}/(\beta_1^{(\phi_1)} - q)$. Moreover,

$$\frac{\alpha_{n+2}^{(\phi_1)}}{b_n} - \beta_{n+1}^{(\phi_1)} + b_{n-1} = \frac{\alpha_{n+1}^{(\phi_0)}}{b_{n-1}} - \beta_{n+1}^{(\phi_0)} + b_n = -q.$$

Consequently, the coefficients b_n, $n \geqslant 1$, can be generated by

$$b_n = \frac{\alpha_{n+2}^{(\phi_1)}}{-q + \beta_{n+1}^{(\phi_1)} - b_{n-1}} \quad \text{or} \quad b_n = \big(\beta_{n+1}^{(\phi_0)} - q\big) - \frac{\alpha_{n+1}^{(\phi_0)}}{b_{n-1}}.$$

The results of the above lemma have been given in Maroni [9] and Belmehdi [1]. These results were also observed in Marcellán and Petronilho [8]. For a proof of this lemma see also Berti et al. [2].

For the purpose of our study we also require the two following lemmas.

Lemma 2. *Let $q_n(z) = (z - x_1)\dots(z - x_n)$ and $q_{n-1}(z) = (z - y_1)\dots(z - y_{n-1})$ be real monic polynomials with real interlacing zeros. That is,*

$$x_n < y_{n-1} < x_{n-1} < \cdots < y_1 < x_1.$$

Then for any real constant c, the polynomial

$$Q_n(z) = q_n(z) + cq_{n-1}(z)$$

has n real zeros $\xi_n < \xi_{n-1} < \cdots < \xi_2 < \xi_1$ which interlace with the zeros of q_n and q_{n-1}. More precisely,

(A) *if $c < 0$ then $x_1 < \xi_1$ and $x_r < \xi_r < y_{r-1}$ for $r = 2, \dots, n$,*
(B) *if $c > 0$ then $y_r < \xi_r < x_r$ for $r = 1, \dots, n-1$ and $\xi_n < x_n$.*
(C) *Moreover, each ξ_r is a decreasing function of c.*

A proof of this lemma can be found in [3].

Lemma 3. *Let $\int_a^\infty f(x, y)\, d\phi(x)$ be convergent when $y_1 \leqslant y \leqslant y_2$ and $\int_a^\infty \frac{\partial f(x,y)}{\partial y}\, d\phi(x)$ be uniformly convergent in $y_1 \leqslant y \leqslant y_2$. Then in $y_1 \leqslant y \leqslant y_2$,*

$$\frac{d}{dy} \int_a^\infty f(x, y)\, d\phi(x) = \int_a^\infty \frac{\partial f(x, y)}{\partial y}\, d\phi(x).$$

The proof of this lemma (for $d\phi(x) = dx$) can be found, for example, in Ferrar [5] and Widder [11].

3. Application to generalized orthogonal polynomials

Let $d\phi$ be a determinate measure with $I^\phi = [a, b]$, where $-\infty < a < b \leqslant \infty$. It is known that if $b < \infty$ then the measure is determinate. Let the real numbers N and κ be such that $0 \leqslant N < \infty$ and $\kappa < a$ (or exceptionally $\kappa \leqslant a$). We consider the monic orthogonal polynomials $P_n^{(\phi,\kappa,N)}$ associated with the measure $d\phi^{(\kappa,N)}$ given by

$$\int p(x)\, d\phi^{(\kappa,N)}(x) = \frac{1}{N+1}\left\{Np(\kappa) + \frac{\mathcal{M}^{(\phi,\kappa)}[p]}{\mathcal{M}^{(\phi,\kappa)}[1]}\right\}, \tag{3.1}$$

where p is any polynomial and

$$\mathcal{M}^{(\phi,\kappa)}[p] = \int_a^b p(x)(x-\kappa)^{-1}\, d\phi(x).$$

Here we allow $\kappa = a$ only if $\mathcal{M}^{(\phi,a)}[1]$ is convergent.

When $N > 0$ the measure $d\phi^{(\kappa,N)}$ has a jump at the point κ. In (3.1) the measure is given in a way so that the value of the jump at κ is $N/(N+1)$ and its total mass is equal to 1.

Now if we take $d\phi_1 = d\phi^{(\kappa,N)}$ and apply (1.1) with $q = \kappa$ and, for example, $c = a - \kappa + 1$, we obtain for $d\phi_0$,

$$\int p(x)\,d\phi_0(x) = \frac{1}{(N+1)(a-\kappa+1)\mathcal{M}^{(\phi,\kappa)}[1]} \int_a^b p(x)\,d\phi(x).$$

Note that $d\phi_0$ is in effect just the measure $d\phi$, given here with the total mass

$$\mu_0^{(\phi_0)} = \frac{\mu_0^{(\phi)}}{(N+1)(a-\kappa+1)\mathcal{M}^{(\phi,\kappa)}[1]},$$

where $\mu_0^{(\phi)} = \int_a^b d\phi(x)$.

Let us write the recurrence relation for $\{P_n^{(\phi,\kappa,N)}\}$ as

$$P_{n+1}^{(\phi,\kappa,N)}(z) = \left(z - \beta_{n+1}^{(\phi,\kappa,N)}\right)P_n^{(\phi,\kappa,N)}(z) - \alpha_{n+1}^{(\phi,\kappa,N)}P_{n-1}^{(\phi,\kappa,N)}(z),$$

for $n \geqslant 1$, with $P_0^{(\phi,\kappa,N)} = 1$ and $P_1^{(\phi,\kappa,N)}(z) = z - \beta_1^{(\phi,\kappa,N)}$.

From (1.2)

$$P_n^{(\phi,\kappa,N)}(z) = P_n^{(\phi)}(z) + b_{n-1}^{(\phi,\kappa,N)}P_{n-1}^{(\phi)}(z), \quad n \geqslant 1, \tag{3.2}$$

where from (1.3) $\operatorname{sign}(b_{n-1}^{(\phi,\kappa,N)}) = \operatorname{sign}(a-\kappa+1) = 1$, $n \geqslant 1$. Moreover, from Lemma 1,

$$\alpha_{n+2}^{(\phi,\kappa,N)} = \alpha_{n+1}^{(\phi)}\frac{b_n^{(\phi,\kappa,N)}}{b_{n-1}^{(\phi,\kappa,N)}}, \qquad \beta_{n+1}^{(\phi,\kappa,N)} = \beta_{n+1}^{(\phi)} - b_n^{(\phi,\kappa,N)} + b_{n-1}^{(\phi,\kappa,N)},$$

for $n \geqslant 1$, with $\alpha_2^{(\phi,\kappa,N)} = b_0^{(\phi,\kappa,N)}(\beta_1^{(\phi)} - b_0^{(\phi,\kappa,N)} - \kappa)$ and $\beta_1^{(\phi,\kappa,N)} = \beta_1^{(\phi)} - b_0^{(\phi,\kappa,N)}$. The $\{b_n^{(\phi,\kappa,N)}\}$ can be generated by

$$b_n^{(\phi,\kappa,N)} = \left(\beta_{n+1}^{(\phi)} - \kappa\right) - \frac{\alpha_{n+1}^{(\phi)}}{b_{n-1}^{(\phi,\kappa,N)}}, \quad n \geqslant 1, \tag{3.3}$$

with

$$b_0^{(\phi,\kappa,N)} = \beta_1^{(\phi)} - \beta_1^{(\phi,\kappa,N)} = \beta_1^{(\phi)} - \kappa - \frac{\mu_0^{(\phi)}}{(N+1)\mathcal{M}^{(\phi,\kappa)}[1]}. \tag{3.4}$$

If $\kappa < a$, then for $S_n^{(\phi)}(z)$ and $S^{(\phi)}(z)$ defined as in (2.2) and (2.3), we have

$$1 < S_2^{(\phi)}(\kappa) < \cdots < S_n^{(\phi)}(\kappa) < \cdots < S^{(\phi)}(\kappa). \tag{3.5}$$

This result also holds for $\kappa = a$ provided that $\mathcal{M}^{(\phi,a)}[1]$ is convergent.

Theorem 1. *Let $x_{n,r}^{(\phi,\kappa,N)}$ and $x_{n,r}^{(\phi)}$, $r = 1, 2, \ldots, n$, be the respective zeros of $P_n^{(\phi,\kappa,N)}$ and $P_n^{(\phi)}$ arranged in decreasing order. The following results hold.*

(1) $x_{n-1,r}^{(\phi)} < x_{n,r}^{(\phi,\kappa,N)} < x_{n,r}^{(\phi)}$, $r = 1, \dots, n-1$ *and* $\kappa < x_{n,n}^{(\phi,\kappa,N)} < x_{n,n}^{(\phi)}$.
(2) $x_{n,r}^{(\phi,\kappa,N)}$ *is a decreasing function of* N.
(3) *If* $N = 0$ *then* $x_{n,r}^{(\phi,\kappa,N)}$ *is a decreasing function of* κ.
(4) *If* $N > 0$ *then* $x_{n,r}^{(\phi,\kappa,N)}$ *is an increasing function of* κ *provided that* κ *varies in the range* $(-\infty, \check{\kappa}^{(\phi,N)}]$, *where*

$$\check{\kappa}^{(\phi,N)} = a - \frac{\sqrt{[N(\beta_2^{(\phi)}-a)-(\beta_1^{(\phi)}-a)]^2+4Nv_1^{(\phi)}(a)}-[N(\beta_2^{(\phi)}-a)-(\beta_1^{(\phi)}-a)]}{2N}.$$

Here, $v_1^{(\phi)}(a) = [(\beta_1^{(\phi)} - a)(\beta_2^{(\phi)} - a) - \alpha_2^{(\phi)}] > 0$.

Proof. Since $\operatorname{sign}(b_{n-1}^{(\phi,\kappa,N)}) = 1$, application of part (B) of Lemma 2 to the relation (3.2) leads to part (1) of this theorem.

Now since $\kappa < a$ (or $\kappa \leqslant a$) it follows that $\mathcal{M}^{(\phi,\kappa)}[1]$ is positive and hence from (3.4) $b_0^{(\phi,\kappa,N)}$ is an increasing function of N. Thus from (3.3) $b_n^{(\phi,\kappa,N)}$ is an increasing function of N for any $n \geqslant 0$. Application of the result in part (C) of Lemma 2 then leads to part (2) of the theorem.

To obtain part (3) of this theorem, note that $\{P_n^{(\phi,\kappa,0)}\}$ can be considered as the orthogonal polynomials associated with $(\kappa - x)^{-1} d\phi(x)$ defined on $[a, b]$. Hence the result follows from the application of an extension (Ismail [6]) of a theorem of A. Markoff [10, Theorem 6.12.1].

Now to obtain part (4) of this theorem, from (3.4) we have

$$\frac{\partial b_0^{(\phi,\kappa,N)}}{\partial \kappa} = -1 + \frac{\mu_0^{(\phi)}}{(N+1)(\mathcal{M}^{(\phi,\kappa)}[1])^2} \frac{\partial \mathcal{M}^{(\phi,\kappa)}[1]}{\partial \kappa}.$$

Hence for values of κ such that $a - \kappa \geqslant \epsilon > 0$, from Lemma 3,

$$\frac{\partial b_0^{(\phi,\kappa,N)}}{\partial \kappa} = -1 + \frac{\mu_0^{(\phi)}}{(N+1)(\mathcal{M}^{(\phi,\kappa)}[1])^2} \int_a^b \frac{1}{(x-\kappa)^2}\, d\phi(x).$$

From this,

$$\frac{\partial b_0^{(\phi,\kappa,N)}}{\partial \kappa} < -1 + \frac{\mu_0^{(\phi)}}{(N+1)(a-\kappa)\mathcal{M}^{(\phi,\kappa)}[1]}.$$

From (3.5), since $S_2^{(\phi)}(\kappa) < \frac{\beta_1^{(\phi)}-\kappa}{\mu_0^{(\phi)}} \mathcal{M}^{(\phi,\kappa)}[1]$, we then have

$$\frac{\partial b_0^{(\phi,\kappa,N)}}{\partial \kappa} < -1 + \frac{(\beta_1^{(\phi)} - \kappa)(\beta_2^{(\phi)} - \kappa) - \alpha_2^{(\phi)}}{(N+1)(a-\kappa)(\beta_2^{(\phi)} - \kappa)}.$$

We look for the values of κ for which the right-hand side of the above inequality is non-positive and we find the interval $(-\infty, \check{\kappa}^{(\phi,N)}]$ where this holds. Within this interval we then have $b_0^{(\phi,\kappa,N)}$ is a decreasing function of κ. Consequently, from (3.3), within this interval we also have $b_n^{(\phi,\kappa,N)}$ is a decreasing function of κ for any $n \geqslant 0$. Hence part (4) of the theorem follows from part (C) of Lemma 2. $\square$

4. Examples

In this section we apply the results of Theorem 1 with three selected examples. The examples are chosen so that we cover absolutely continuous measures on finite and semi-infinite intervals and also discrete measures.

1. Generalized Jacobi polynomials. Let the real numbers N and κ be such that $0 \leqslant N < \infty$ and $|\kappa| > 1$ (or exceptionally $|\kappa| \geqslant 1$). We consider the monic orthogonal polynomials $P_n^{(\alpha,\beta,\kappa,N)}$ associated with the measure $d\phi^{(\alpha,\beta,\kappa,N)}$ given by

$$\int p(x)\,d\phi^{(\alpha,\beta,\kappa,N)}(x) = \frac{1}{N+1}\left\{Np(\kappa) + \frac{\mathcal{M}_J^{(\alpha,\beta,\kappa)}[p]}{\mathcal{M}_J^{(\alpha,\beta,\kappa)}[1]}\right\}, \tag{4.1}$$

where p is any polynomial and

$$\mathcal{M}_J^{(\alpha,\beta,\kappa)}[p] = \int_{-1}^{1} p(x)\frac{(1-x)^\alpha(1+x)^\beta}{x-\kappa}\,dx.$$

Here, if $|\kappa| > 1$ then $\alpha > -1$, $\beta > -1$, if $\kappa = 1$ then $\alpha > 0$, $\beta > -1$ and if $\kappa = -1$ then $\alpha > -1$, $\beta > 0$.

When $|\kappa| = 1$ then the polynomials $P_n^{(\alpha,\beta,\kappa,N)}$ are particular cases of the Koornwinder [7] polynomials.

We can then study the zeros of $P_n^{(\alpha,\beta,\kappa,N)}$ using the Jacobi polynomials $P_n^{(\alpha,\beta)}$. We remind that for the monic Jacobi polynomials

$$P_{n+1}^{(\alpha,\beta)}(z) = \big(z - \beta_{n+1}^{(\alpha,\beta)}\big)P_n^{(\alpha,\beta)}(z) - \alpha_{n+1}^{(\alpha,\beta)}P_{n-1}^{(\alpha,\beta)}(z), \quad n \geqslant 1,$$

with $P_0^{(\alpha,\beta)} = 1$ and $P_1^{(\alpha,\beta)}(z) = z - \beta_1^{(\alpha,\beta)}$, where (see, for example, [4])

$$\beta_n^{(\alpha,\beta)} = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta-1)^2-1} \quad \text{and}$$
$$\alpha_{n+1}^{(\alpha,\beta)} = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2[(2n+\alpha+\beta)^2-1]}, \quad n \geqslant 1.$$

Application of Theorem 1 gives

Corollary 1.1. *Let $x_{n,r}^{(\alpha,\beta,\kappa,N)}$ and $x_{n,r}^{(\alpha,\beta)}$, $r = 1,2,\ldots,n$, be the respective zeros of $P_n^{(\alpha,\beta,\kappa,N)}$ and $P_n^{(\alpha,\beta)}$ arranged in decreasing order. The following results hold.*

(1) *If $\kappa > 1$ (or $\kappa \geqslant 1$) then*
$$x_{n,1}^{(\alpha,\beta)} < x_{n,1}^{(\alpha,\beta,\kappa,N)} < \kappa \quad \textit{and} \quad x_{n,r}^{(\alpha,\beta)} < x_{n,r}^{(\alpha,\beta,\kappa,N)} < x_{n-1,r-1}^{(\alpha,\beta)}, \quad r = 2,\ldots,n.$$

(2) *If $\kappa < -1$ (or $\kappa \leqslant -1$) then*
$$x_{n-1,r}^{(\alpha,\beta)} < x_{n,r}^{(\alpha,\beta,\kappa,N)} < x_{n,r}^{(\alpha,\beta)},\ r = 1,\ldots,n-1 \quad \textit{and} \quad \kappa < x_{n,n}^{(\alpha,\beta,\kappa,N)} < x_{n,n}^{(\alpha,\beta)}.$$

(3) *If $\kappa > 1$ (or $\kappa \geqslant 1$) then $x_{n,r}^{(\alpha,\beta,\kappa,N)}$ is an increasing function of N.*

(4) *If* $\kappa < -1$ (*or* $\kappa \leqslant -1$) *then* $x_{n,r}^{(\alpha,\beta,\kappa,N)}$ *is a decreasing function of* N.
(5) *If* $N = 0$ *then* $x_{n,r}^{(\alpha,\beta,\kappa,N)}$ *is a decreasing function of* κ.
(6) *If* $N > 0$ *then* $x_{n,r}^{(\alpha,\beta,\kappa,N)}$ *is an increasing function of* κ *provided that* κ *varies in the range* $(-\infty, \check{\kappa}^{(\alpha,\beta,N)}]$ *or in the range* $[\hat{\kappa}^{(\alpha,\beta,N)}, \infty)$, *where*

$$\hat{\kappa}^{(\alpha,\beta,N)} = 1 + \frac{\sqrt{[N(1-\beta_2^{(\alpha,\beta)})-(1-\beta_1^{(\alpha,\beta)})]^2+4Nv_1^{(\alpha,\beta)}(1)}\ -[N(1-\beta_2^{(\alpha,\beta)})-(1-\beta_1^{(\alpha,\beta)})]}{2N},$$

$$\check{\kappa}^{(\alpha,\beta,N)} = -1 - \frac{\sqrt{[N(1+\beta_2^{(\alpha,\beta)})-(1+\beta_1^{(\alpha,\beta)})]^2+4Nv_1^{(\alpha,\beta)}(-1)}-[N(1+\beta_2^{(\alpha,\beta)})-(1+\beta_1^{(\alpha,\beta)})]}{2N}.$$

Here, $v_1^{(\alpha,\beta)}(z) = [(\beta_1^{(\alpha,\beta)} - z)(\beta_2^{(\alpha,\beta)} - z) - \alpha_2^{(\alpha,\beta)}]$ *and therefore* $v_1^{(\alpha,\beta)}(\pm 1) > 0$.

Proof. The results of parts (2) and (4) follow from Theorem 1. Also the results of parts (5) and (6) that correspond to the possible negative values of κ follow from Theorem 1.

To obtain the remaining results, we note that

$$\int p(x)\, d\phi^{(\alpha,\beta,\kappa,N)}(x) = \frac{1}{N+1}\left\{N\tilde{p}(-\kappa) + \frac{\mathcal{M}_J^{(\beta,\alpha,-\kappa)}[\tilde{p}]}{\mathcal{M}_J^{(\beta,\alpha,-\kappa)}[1]}\right\}$$
$$= \int \tilde{p}(x)\, d\phi^{(\beta,\alpha,-\kappa,N)}(x),$$

where $\tilde{p}(-x) = p(x)$. Hence, $P_n^{(\alpha,\beta,\kappa,N)}(x) = P_n^{(\beta,\alpha,-\kappa,N)}(-x)$, $n \geqslant 1$. Moreover, it is known that $P_n^{(\alpha,\beta)}(x) = P_n^{(\beta,\alpha)}(-x)$ and $\beta_n^{(\alpha,\beta)} = -\beta_n^{(\beta,\alpha)}$ for $n \geqslant 1$.

Hence for $\kappa \geqslant 1$, from part (2) of this corollary

$$x_{n-1,r}^{(\beta,\alpha)} < x_{n,r}^{(\beta,\alpha,-\kappa,N)} < x_{n,r}^{(\beta,\alpha)}, \quad 1 \leqslant r < n, \quad \text{and}$$
$$-\kappa < x_{n,n}^{(\beta,\alpha,-\kappa,N)} < x_{n,n}^{(\alpha,\beta)}.$$

Since

$$x_{n,r}^{(\beta,\alpha,-\kappa,N)} = -x_{n,n-r+1}^{(\alpha,\beta,\kappa,N)} \quad \text{and} \quad x_{n,r}^{(\beta,\alpha)} = -x_{n,n-r+1}^{(\alpha,\beta)} \quad \text{for } 1 \leqslant r \leqslant n,$$

we then obtain part (1) of the corollary.

Now from part (4) of this corollary, if $\kappa \geqslant 1$ then $x_{n,r}^{(\beta,\alpha,-\kappa,N)}$ is a decreasing function of N. Therefore $x_{n,r}^{(\alpha,\beta,\kappa,N)}$ is an increasing function of N, proving part (3) of the corollary.

To prove part (5) of this corollary whenever $\kappa \geqslant 1$, we have from part (3) of Theorem 1 that $x_{n,r}^{(\beta,\alpha,\tilde{\kappa},0)}$ is a decreasing function of $\tilde{\kappa} = -\kappa$. Therefore $x_{n,r}^{(\alpha,\beta,\kappa,0)}$ is a decreasing function of κ.

Finally, using part (6) of this corollary for $\kappa < -1$ we have that $x_{n,r}^{(\beta,\alpha,\tilde{\kappa},N)}$ is an increasing function of $\tilde{\kappa} = -\kappa$ provided that $\tilde{\kappa}$ varies in the range $(-\infty, \check{\kappa}^{(\beta,\alpha,N)}]$. Hence, we obtain that $x_{n,r}^{(\alpha,\beta,\kappa,N)}$ is an increasing function of κ provided that κ varies in the range $[\hat{\kappa}^{(\alpha,\beta,N)}, \infty)$. $\square$

2. Generalized Laguerre polynomials. Let the real numbers N and κ be such that $0 \leqslant N < \infty$ and $\kappa > 0$ (or exceptionally $\kappa \geqslant 0$). We consider the monic orthogonal polynomials $L_n^{(\alpha,\kappa,N)}$ associated with the measure $d\phi^{(\alpha,\kappa,N)}$ given by

$$\int p(x)\,d\phi^{(\alpha,\kappa,N)}(x) = \frac{1}{N+1}\left\{Np(-\kappa) + \frac{\mathcal{M}_L^{(\alpha,\kappa)}[p]}{\mathcal{M}_L^{(\alpha,\kappa)}[1]}\right\}, \tag{4.2}$$

where p is any polynomial and

$$\mathcal{M}_L^{(\alpha,\kappa)}[p] = \int_0^\infty p(x)\frac{x^\alpha e^{-x}}{x+\kappa}\,dx.$$

Here, if $\kappa > 0$ then $\alpha > -1$ and if $\kappa = 0$ then $\alpha > 0$.

We can then study the zeros of $L_n^{(\alpha,\kappa,N)}$ using the Laguerre polynomials $L_n^{(\alpha)}$. For the monic Laguerre polynomials

$$L_{n+1}^{(\alpha)}(z) = \bigl(z - \beta_{n+1}^{(\alpha)}\bigr)L_n^{(\alpha)}(z) - \alpha_{n+1}^{(\alpha)}L_{n-1}^{(\alpha)}(z), \quad n \geqslant 1,$$

with $L_0^{(\alpha)} = 1$ and $L_1^{(\alpha)}(z) = z - \beta_1^{(\alpha)}$, where (see, for example, [4])

$$\beta_n^{(\alpha)} = 2n + \alpha - 1 \quad \text{and} \quad \alpha_{n+1}^{(\alpha)} = n(n+\alpha), \quad n \geqslant 1.$$

Applying Theorem 1 with κ replaced by $-\kappa$ we obtain the following result.

Corollary 1.2. *Let $n \geqslant 1$ and let $x_{n,r}^{(\alpha,\kappa,N)}$ and $x_{n,r}^{(\alpha)}$, $r = 1, 2, \ldots, n$, be the respective zeros of $L_n^{(\alpha,\kappa,N)}$ and $L_n^{(\alpha)}$ arranged in decreasing order. Then the following results hold.*

(1) $x_{n-1,r}^{(\alpha)} < x_{n,r}^{(\alpha,\kappa,N)} < x_{n,r}^{(\alpha)}$, $r = 1, \ldots, n-1$ *and* $-\kappa < x_{n,n}^{(\alpha,\kappa,N)} < x_{n,n}^{(\alpha)}$.
(2) $x_{n,r}^{(\alpha,\kappa,N)}$ *is a decreasing function of N for $N \in [0, \infty)$.*
(3) $x_{n,r}^{(\alpha,\kappa,0)}$ *is an increasing function of κ for $\kappa \in (0, \infty)$.*
(4) *If $N > 0$ then $x_{n,r}^{(\alpha,\kappa,N)}$ is a decreasing function of κ for $\kappa \in [\tilde{\kappa}^{(\alpha,N)}, \infty)$, where*

$$\tilde{\kappa}^{(\alpha,N)} = \frac{\sqrt{[N(3+\alpha)-(1+\alpha)]^2 + 4N(2+\alpha)(1+\alpha)} - [N(3+\alpha)-(1+\alpha)]}{2N}.$$

It was brought to our attention by the referee that Corollary 1.2 can also be obtain from the results of Corollary 1.1 by using the asymptotic property

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1 - 2\beta^{-1}x) = L_n^{(\alpha)}(x).$$

3. Generalized Charlier polynomials. Let the real numbers N and κ be such that $0 \leqslant N < \infty$ and $\kappa > 0$. We consider the monic orthogonal polynomials $C_n^{(\alpha,\kappa,N)}$ associated with the measure $d\psi^{(\alpha,\kappa,N)}$ given by

$$\int p(x)\,d\psi^{(\alpha,\kappa,N)}(x) = \frac{1}{N+1}\left\{Np(-\kappa) + \frac{\mathcal{M}_C^{(\alpha,\kappa)}[p]}{\mathcal{M}_C^{(\alpha,\kappa)}[1]}\right\}, \tag{4.3}$$

where p is any polynomial and

$$\mathcal{M}_C^{(\alpha,\kappa)}[p] = \int_0^\infty p(x)\frac{1}{x+\kappa}\,d\phi_C^{(\alpha)}(x) = \sum_{r=0}^\infty p(r)\frac{1}{r+\kappa}\frac{e^{-\alpha}\alpha^r}{r!}.$$

Here $\alpha > 0$. It is known that the discrete measure $d\phi_C^{(\alpha)}$ is determinate and that the associated orthogonal polynomials $C_n^{(\alpha)}$ satisfy

$$C_{n+1}^{(\alpha)}(z) = \big(z - \beta_{n+1}^{(\alpha)}\big)C_n^{(\alpha)}(z) - \alpha_{n+1}^{(\alpha)}C_{n-1}^{(\alpha)}(z), \quad n \geqslant 1,$$

with $C_0^{(\alpha)} = 1$ and $C_1^{(\alpha)}(z) = z - \beta_1^{(\alpha)}$, where (see, for example, [4])

$$\beta_n^{(\alpha)} = n + \alpha - 1 \quad \text{and} \quad \alpha_{n+1}^{(\alpha)} = n\alpha, \quad n \geqslant 1.$$

These polynomials are the Charlier polynomials. Applying Theorem 1 with κ replaced by $-\kappa$, we obtain the following result.

Corollary 1.3. *Let $n \geqslant 1$ and let $x_{n,r}^{(\alpha,\kappa,N)}$ and $x_{n,r}^{(\alpha)}$, $r = 1, 2, \ldots, n$, be the respective zeros of $C_n^{(\alpha,\kappa,N)}$ and $C_n^{(\alpha)}$ arranged in decreasing order. Then the following results hold.*

(1) $x_{n-1,r}^{(\alpha)} < x_{n,r}^{(\alpha,\kappa,N)} < x_{n,r}^{(\alpha)}$, $r = 1, \ldots, n-1$ *and* $-\kappa < x_{n,n}^{(\alpha,\kappa,N)} < x_{n,n}^{(\alpha)}$.
(2) $x_{n,r}^{(\alpha,\kappa,N)}$ *is a decreasing function of N for $N \in [0, \infty)$.*
(3) $x_{n,r}^{(\alpha,\kappa,0)}$ *is an increasing function of κ for $\kappa \in (0, \infty)$.*
(4) *If $N > 0$ then $x_{n,r}^{(\alpha,\kappa,N)}$ is a decreasing function of κ for $\kappa \in [\tilde{\kappa}^{(\alpha,N)}, \infty)$, where*

$$\tilde{\kappa}^{(\alpha,N)} = \frac{\sqrt{[N(\alpha+1)-\alpha]^2 + 4N\alpha^2} - [N(\alpha+1)-\alpha]}{2N}.$$

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