Another connection between orthogonal polynomials
and L-orthogonal polynomials

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Abstract

We consider a connection that exists between orthogonal polynomials associated with positive measures on the real line and orthogonal Laurent polynomials associated with strong measures of the class $S^3[0, \beta, b]$. Examples are given to illustrate the main contribution in this paper.

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1. Introduction

Let $-\infty \leq c < d \leq \infty$ and let $\phi$ be a bounded and nondecreasing function on $[c, d]$ such that it has infinitely many points of increase in $[c, d]$ and that the moments $\mu_n^{\phi} = \int_c^d x^n \, d\phi(x)$, $n = 0, 1, 2, \ldots$, all exist. Then $\phi$ represents a positive measure on $[c, d]$ and one can uniquely define the set of polynomials $\{Q_n^{\phi}\}_{n=0}^\infty$ by

$Q_n^{\phi}$ is monic of degree $n$.

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\[ \int_{-\infty}^{\infty} x^s Q_n^\phi(x) \, d\phi(x) = 0, \quad s = 0, 1, \ldots, n - 1. \]

These are the monic orthogonal polynomials associated with the measure \( \phi \). It is well known that these polynomials satisfy the three term recurrence relation

\[ Q_{n+1}^\phi(z) = (z - b_{n+1}^\phi) Q_n^\phi(z) - a_{n+1}^\phi Q_{n-1}^\phi(z), \quad n \geq 1, \tag{1.1} \]

with \( Q_0^\phi(z) = 1 \) and \( Q_1^\phi(z) = z - b_1^\phi \). The coefficients \( a_n^\phi \) and \( b_n^\phi \), given by

\[ b_{n+1}^\phi = \frac{\int_{-\infty}^{\infty} t [Q_n^\phi(x)]^2 \, d\phi(x)}{\int_{\tau_1}^{\tau_2} [Q_0^\phi(x)]^2 \, d\phi(x)}, \quad n \geq 0, \quad \text{and} \quad a_{n+1}^\phi = \frac{\int_{-\infty}^{\infty} t [Q_n^\phi(x)]^2 \, d\phi(x)}{\int_{\tau_1}^{\tau_2} [Q_{n-1}^\phi(x)]^2 \, d\phi(x)}, \quad n \geq 1, \]

satisfy \( b_n^\phi \in \mathbb{R}, n \geq 1, \) and \( a_n^\phi > 0, n \geq 2 \). For further information on these polynomials we refer to Chihara [1] and Szegő [13].

Now let \( 0 \leq a < b \leq \infty \) and let \( \psi \) be a bounded and nondecreasing function on \([a, b]\) with infinitely many points of increase in \([a, b]\) and such that the moments \( \mu_n^\psi = \int_a^b t^n \, d\psi(t), n = 0, \pm 1, \pm 2, \ldots, \) all exist. We refer to \( \psi \) as a strong positive measure on \([a, b]\) and define the set of polynomials \( \{Q_n^\psi\}_{n=0}^\infty \) by

\[ Q_n^\psi \text{ is monic of degree } n, \]

\[ \int_a^b t^{-n+s} Q_n^\psi(t) \, d\psi(t) = 0, \quad s = 0, 1, \ldots, n - 1. \tag{1.2} \]

Such polynomials were introduced in [7] in order to solve the strong Stieltjes moment problem. The sequence of functions \( \{t^{-\lfloor(n+1)/2\rfloor} Q_n^\psi(t)\} \) is known as a sequence of orthogonal Laurent polynomials or orthogonal L-polynomials associated with the measure \( \psi \) (see [6]). Like the monic orthogonal polynomials, the polynomials \( Q_n^\psi \) are also unique and for convenience we refer to them as L-orthogonal polynomials.

It is known that the zeros of \( Q_n^\psi \) are all positive, distinct and lie within the interval \((a, b)\). The zeros of \( Q_n^\psi \) also interlace with the zeros of \( Q_{n-1}^\psi \). Moreover, these polynomials satisfy the three term recurrence relation

\[ Q_{n+1}^\psi(w) = (w - \beta_{n+1}^\psi) Q_n^\psi(w) - \alpha_{n+1}^\psi w Q_{n-1}^\psi(w), \quad n \geq 1, \tag{1.3} \]

with \( Q_0^\psi(w) = 1 \) and \( Q_1^\psi(w) = w - \beta_1^\psi \). \( \beta_1^\psi = \mu_0^\psi / \mu_{-1}^\psi = \rho_0^\psi / \sigma_0^\psi \) and, for \( n \geq 1, \)

\[ \beta_{n+1}^\psi = -\alpha_{n+1}^\psi \frac{\sigma_n^\psi}{\sigma_{n-1}^\psi} \quad \text{and} \quad \alpha_{n+1}^\psi = \frac{\rho_n^\psi}{\rho_{n-1}^\psi}, \]

where \( \rho_n^\psi = \int_a^b Q_n^\psi(t) \, d\psi(t) \) and \( \sigma_n^\psi = \int_a^b t^{-n} Q_n^\psi(t) \, d\psi(t) \). The coefficients \( \beta_n^\psi, n \geq 1, \) and \( \alpha_n^\psi, n \geq 2, \) are all positive. From (1.3) one can also note that \( Q_n^\psi(0) = (-1)^n \beta_1^\psi \beta_2^\psi \cdots \beta_n^\psi \).

Assuming different symmetric behaviors for the measure \( \psi \), properties of the L-orthogonal polynomials \( \{Q_n^\psi\} \) have been explored in several articles.

We say that the strong positive measure \( \psi \) belongs to the symmetric class \( S^3[\tau, \beta, b] \) if

\[ \frac{d\psi(t)}{t^\tau} = -\frac{d\psi(\beta^2 / t)}{(\beta^2 / t)^\tau}, \quad t \in (a, b), \]
where $0 < \beta < b$, $a = \beta^2 / b$ and $2 \tau \in \mathbb{Z}$. The classification of the symmetry is according to the value of $\tau$.

It seems that two of these symmetric classes, namely when $\tau = 1/2$ and $\tau = 0$, turn out to be more interesting than the others.

The L-orthogonal polynomials \{\(Q_n^\psi\)\}, when the measure $\psi$ belongs to the class $S^3[1/2, \beta, b]$, have been explored for example in [10]. In this case we can state the following results:

$$
\frac{w^n Q_n^\psi(\beta^2/w)}{Q_n^\psi(0)} = Q_n^\psi(w) \quad \text{and} \quad \beta_n^\psi = \beta \quad \text{for } n \geq 1.
$$

Moreover, with the transformation $x(t) = \frac{1}{\sqrt{\alpha}}(\sqrt{t} - \beta/\sqrt{t})$, where $\alpha > 0$, if the positive measure $\phi$ on the real interval $[-x(b), x(b)]$ is such that

$$
d\phi(x) = \frac{t + \beta}{2t} d\psi(t),
$$

then the orthogonal polynomials \{\(Q_n^\phi\)\} are connected to the L-orthogonal polynomials \{\(Q_n^\psi\)\} in the following way:

$$
Q_n^\phi(x(t)) = (2\sqrt{\alpha t})^{-n} Q_n^\psi(t) \quad \text{and} \quad a_{n+1}^\phi = \frac{1}{4\alpha} a_{n+1}^\psi \quad \text{for } n \geq 1.
$$

Since $\phi$ is symmetric, that is $d\phi(-x) = -d\phi(x)$, the orthogonal polynomials \{\(Q_n^\phi\)\} are such that

$$
Q_n^\phi(-z) = (-1)^n Q_n^\phi(z) \quad \text{and} \quad b_n^\phi = 0 \quad \text{for } n \geq 1.
$$

Now the other interesting class $S^3[0, \beta, b]$ of strong positive measures is characterized by the symmetry

$$
d\psi(t) = -d\psi(\beta^2/t), \quad t \in (a, b).
$$

Here $0 < \beta < b \leq \infty$ and $a = \beta^2 / b$. If $d\psi(t) = h(t) dt$ then the symmetry is given by

$$
t h(t) = (\beta^2/t) h(\beta^2/t).
$$

In this paper we look at a connection that exists between orthogonal polynomials associated with positive measures on the real line and L-orthogonal polynomials associated with strong measures of the class $S^3[0, \beta, b]$. We do this based on an idea similar to that of Szegő (see [13]) to connect orthogonal polynomials on $[-1, 1]$ to orthogonal polynomials on the unit circle. Examples given in Section 6 to illustrate the main contribution of this paper also lead to some new results.

We mention that, in [14], Vinet and Zhedanov have considered a connection between orthogonal polynomials associated with “symmetric” positive measures on the real line and Szegő polynomials on the real line using the transformation $x(t) = \lambda(\sqrt{t} + 1/\sqrt{t})$, where $\lambda > 0$. As we will observe in Section 2, the monic Szegő polynomials on the real line are the monic reciprocal polynomials of the L-orthogonal polynomials $Q_n^\psi$ defined when the measure $\psi$ is of the class $S^3[0, \beta, b]$.

As other attempts to connect orthogonal polynomials on the real line with orthogonal Laurent polynomials on the real line, we also mention the paper [5] of Hagler, Jones and Thron. However, none of these previous attempts have provided a method to obtain information on the orthogonal Laurent polynomials associated with the special $S^3[0, \beta, b]$ measures that we have considered in the first three examples of Section 6. We point out that the $S^3[0, \beta, b]$ measure given by Eq. (6.2) is, in particular, very interesting in the following sense.
Let \( \{\alpha_n\} \) and \( \{\beta_n\} \) be any sequences of bounded positive numbers such that
\[
\lim_{n \to \infty} \alpha_n = \alpha > 0 \quad \text{and} \quad \lim_{n \to \infty} \beta_n = \beta > 0.
\]
Let the sequence of polynomials \( \{B_n\} \) be generated by
\[
B_{n+1}(w) = (w - \beta_{n+1})B_n(w) - \alpha_{n+1}wB_{n-1}(w), \quad n \geq 1,
\]
with \( B_1(w) = w - \beta_1 \) and \( B_0 = 1 \). Then (see for example [12]),
\[
\lim_{n \to \infty} \frac{1}{n} B_n'(w)B_n(w) = \frac{1}{2} \sqrt{\frac{a}{(b - t)(t - a)}} \int_a^b dt
\]
holds uniformly on any compact subset of \( \mathbb{C} \setminus (0, \infty) \), where \( a = \beta_2/b \) and \( b = (\sqrt{\alpha + \beta} + \sqrt{\alpha})^2 \).

2. \( S^3[0, \beta, b] \) measures and L-orthogonal polynomials

The L-orthogonal polynomials \( \{Q_{n}^{\psi}\} \), when \( \psi \) belongs to the class \( S^3[0, \beta, b] \), have first appeared in Sri Ranga, Andrade and McCabe [11]. Some aspects of these polynomials were also considered simultaneously in Common and McCabe [3]. We can state (see [11]) the following results when \( \psi \in S^3[0, \beta, b] \).

If \( f \) is integrable with respect to \( \psi \), then
\[
\int_{\beta^2/b}^{b} f(t) d\psi(t) = \int_{\beta^2/b}^{b} f\left(\frac{\beta^2}{t}\right) d\psi(t)
\]
and, in particular, \( \mu_{n}^{\psi} = \beta_{2n}^{\psi} \mu_{-n}^{\psi} \). For the associated L-orthogonal polynomials \( \{Q_{n}^{\psi}\} \) the coefficients in the three term recurrence relation (1.3) satisfy
\[
\gamma_{n+1}^{\psi} = \frac{\beta^2}{\beta_n^{\psi} \beta_{n+1}^{\psi}}, \quad n \geq 0,
\]
where \( \gamma_{n}^{\psi} = \beta_{n}^{\psi} + \alpha_{n+1}^{\psi} \), with \( \beta_{0}^{\psi} = \gamma_{0}^{\psi} = 1 \). Consequently,
\[
Q_{n}^{\psi}(0) = (-1)^n \beta_1^{\psi} \beta_2^{\psi} \cdots \beta_n^{\psi} = (-\beta)^n \sqrt{\frac{\beta_n^{\psi}}{\gamma_n^{\psi}}}, \quad n \geq 1.
\]
Moreover, if
\[
\tilde{Q}_{n}^{\psi}(w) = \frac{w^n Q_{n}^{\psi}(\beta^2/w)}{Q_{n}^{\psi}(0)}, \quad n \geq 1,
\]
then
\[
\tilde{Q}_{n}^{\psi}(w) = Q_{n}^{\psi}(w) - \alpha_{n+1}^{\psi} Q_{n-1}^{\psi}(w) = \left\{ Q_{n+1}^{\psi}(w) + \beta_{n+1}^{\psi} Q_{n}^{\psi}(w) \right\}/w, \quad n \geq 1.
\]
Now, letting \( w = \beta \), and also letting \( w = -\beta \), in (2.4) we obtain
\[
\left[ 1 - \frac{\beta^n}{Q_{n}^{\psi}(0)} \right] Q_{n}^{\psi}(\beta) = \alpha_{n+1}^{\psi} Q_{n-1}^{\psi}(\beta) \quad \text{and}
\]
\[
\left[ 1 - \frac{(-\beta)^n}{Q_{n}^{\psi}(0)} \right] Q_{n}^{\psi}(-\beta) = \alpha_{n+1}^{\psi} Q_{n-1}^{\psi}(-\beta),
\]
for \( n \geq 1 \). Thus, for example,

\[
Q_{2n}^{\psi}(\beta) = -\beta^2 n_{\psi}^{(2n-1)} \left( \sqrt{\frac{y_{2n}^{\psi}}{\beta_{2n}^{\psi}}} + 1 \right) \left( \sqrt{\frac{y_{2n-1}^{\psi}}{\beta_{2n-1}^{\psi}}} - 1 \right) Q_{2n-2}^{\psi}(\beta), \quad n \geq 1,
\]

\[
Q_{2n}^{\psi}(-\beta) = \beta^2 n_{\psi}^{(2n-1)} \left( \sqrt{\frac{y_{2n}^{\psi}}{\beta_{2n}^{\psi}}} + 1 \right) \left( \sqrt{\frac{y_{2n-1}^{\psi}}{\beta_{2n-1}^{\psi}}} + 1 \right) Q_{2n-2}^{\psi}(-\beta), \quad n \geq 1. \quad (2.5)
\]

Note that the monic polynomials \( \tilde{Q}_n^{\psi} \) satisfy the orthogonality (to be more precise the biorthogonality)

\[
\int_a^b t^{-s} \tilde{Q}_n^{\psi}(t) \, d\psi(t) = 0, \quad s = 0, 1, \ldots, n - 1. \quad (2.6)
\]

If we define the reciprocal polynomials of \( \tilde{Q}_n^{\psi} \) by \( \tilde{Q}_n^{\psi*}(w) = (w/\beta^2)^n \tilde{Q}_n^{\psi}(\beta^2/w) \), then \( \tilde{Q}_n^{\psi*}(w) = Q_n^{\psi}(w)/Q_n^{\psi}(0) \), \( n \geq 1 \), and from (2.4),

\[
\tilde{Q}_n^{\psi}(w) = w \tilde{Q}_{n-1}(w) + \tilde{\delta}_n \tilde{Q}_{n-1}^{\psi*}(w), \quad n \geq 1, \quad (2.7)
\]

where \( \tilde{\delta}_n = \frac{\beta^{2n}}{Q_n^{\psi}(0)} = (-\beta)^n \sqrt{\frac{y_n^{\psi}}{\beta_n^{\psi}}} \) for \( n \geq 1 \).

Note that, if the integration in (2.6) is done over the unit circle then \( t^{-1} \) can be replaced by the conjugate of \( t \). Hence, based on (2.6) and (2.7), the polynomials \( \tilde{Q}_n^{\psi} \) can be called the monic Szegő polynomials on the positive interval \([a, b]\). The numbers \( \tilde{\delta}_n = Q_n^{\psi}(0) \), which play the role of reflection coefficients, satisfy

\[
|\tilde{\delta}_n| > \beta^n \quad \text{for} \quad n \geq 1.
\]

Like the zeros of \( Q_n^{\psi} \) the zeros of \( \tilde{Q}_n^{\psi} \) also lie within the interval \([a, b]\), here with \( a = \beta^2/b \). Hence it is important that \( |\tilde{\delta}_n| > \beta^n \) (see Vinet and Zhedanov [14]). Otherwise, if \( |\tilde{\delta}_n| < \beta^n \), then (just as the scaled Szegő polynomials orthogonal on the circle of radius \( \beta \) the zeros of \( Q_n^{\psi} \) would lie within the disk \(|w| < \beta|\).

To be able to talk about Szegő polynomials on a positive interval \([a, b]\) it is important that the measure is of the class \( S^3[0, \beta, b] \), with \( a = \beta^2/b \). Otherwise the monic polynomials \{\( \tilde{Q}_n^{\psi} \)\} defined on \([a, b]\) by (2.6) do not satisfy (2.7). For example, if the measure \( \psi \) is of the class \( S^3[-1/2, \beta, b] \), then for the monic polynomials \{\( \tilde{Q}_n^{\psi} \)\} defined by (2.6) we would have, instead of (2.7),

\[
\tilde{Q}_n^{\psi}(w) = \tilde{\delta}_n \tilde{Q}_{n-1}^{\psi*}(w) = w \tilde{Q}_{n-1}(w) + \tilde{\delta}_n \tilde{Q}_{n-1}^{\psi*}(w) - \tilde{\alpha}_n w \tilde{Q}_{n-2}(w),
\]

for \( n \geq 2 \), where \( \tilde{\delta}_n = \tilde{Q}_n^{\psi}(0) \) and \( \tilde{\alpha}_n > 0 \).

### 3. Hyperbolic Szegő transformation

With \( \beta > 0 \), the transformation given by

\[
z = T(w) = \frac{1}{2} \left( \frac{w}{\beta} + \frac{\beta}{w} \right),
\]

may be called a generalized Joukowski transformation. It is well known that this transformation maps the semi-circular region \( \{w = \beta e^{i\theta} : 0 \leq \theta \leq \pi\} \) onto the interval \([-1, 1]\) (likewise the
semi-circular region \( \{ w = \beta e^{i\theta} : -\pi \leq \theta \leq 0 \} \) onto the interval \([-1, 1]\). In this special case the transformation is also known as the Szegő transformation.

Here, we consider another special case of the transformation \( T(w) \) which maps the interval \([\beta, \infty)\) onto the interval \([1, \infty)\). Also the interval \((0, \beta)\) onto the interval \([1, \infty)\). We call this the Hyperbolic Szegő Transformation and write

\[
x = x(t) = \frac{1}{2} \left( \frac{t}{\beta} + \frac{\beta}{t} \right),
\]

for \( t \in [\beta^2/b, b] \) and \( x \in [1, d] \), with \( d = \frac{1}{2}(b/\beta + \beta/b) \). Interestingly, we can consider this transformation in two stages.

**Stage 1:** \( \theta = \ln(\frac{t}{\beta}) \).

This is a mapping from \([\beta^2/b, b]\) onto \([-\vartheta, \vartheta]\), where \( \vartheta = \ln(b/\beta) \). Note that \( \theta \) is a strictly increasing function of \( t \). The inverse of this mapping is \( t = \beta e^{\theta} \).

**Stage 2:** \( x = \cosh \theta = \frac{1}{2}(e^{\theta} + e^{-\theta}) \), hence the use of the word Hyperbolic.

This can be considered as a mapping from \([0, \vartheta]\) onto \([1, d]\), where \( d = \cosh \vartheta = \frac{1}{2}(b/\beta + \beta/b) \). Here \( x \) is a strictly increasing function of \( \theta \). The inverse of this mapping is \( \theta = \ln(x + \sqrt{x^2 - 1}) \), with \( \ln(b/\beta) = \vartheta = \ln(d + \sqrt{d^2 - 1}) \).

This can also be considered as a mapping from \([-\vartheta, 0]\) onto \([1, d]\). Now \( x \) is a strictly decreasing function of \( \theta \). The inverse of this mapping is \( \theta = \ln(x - \sqrt{x^2 - 1}) \), with \( \ln(\beta/b) = -\vartheta = \ln(d - \sqrt{d^2 - 1}) \).

Now if we consider the integral \( I = \int_1^d f(x) \, d\phi(x) \), where \( \phi \) is any positive measure on \([1, d]\), then we can write

\[
2I = \int_1^d f(x) \left[ -d\phi(x) \right] + \int_1^d f(x) \, d\phi(x).
\]

Here, applying Stage 2 of our transformation, we obtain

\[
2I = \int_{-\vartheta}^{\vartheta} f(\cosh \theta) \left[ -d\phi(\cosh \theta) \right] + \int_0^{\vartheta} f(\cosh \theta) \, d\phi(\cosh \theta). \tag{3.1}
\]

Applying Stage 1 of our transformation, where \( e^{\theta} = t/\beta \), then gives

\[
2I = \int_{\beta^2/b}^{\beta} f(x(t)) \left[ -d\phi(x(t)) \right] + \int_{\beta^2/b}^{\beta} f(x(t)) \, d\phi(x(t)).
\]

Hence,

\[
\int_1^d f(x) \, d\phi(x) = \int_{\beta^2/b}^{b} f(x(t)) \, d\psi(t), \tag{3.2}
\]

where \( d\psi(t) = \frac{1}{2} d\phi(x(t)) \) for \( t \in (\beta, b) \) and \( d\psi(t) = -\frac{1}{2} d\phi(x(t)) \) for \( t \in (\beta^2/b, \beta) \). Clearly, \( \mu^\psi_0 = \mu^\phi_0 \). Moreover, one can verify that \( \psi \) is a strong positive measure on \((\beta^2/b, b)\) and that \( \psi \in S^3(0, \beta, b) \).
In particular, if \( d\phi(x) = k(x) \, dx \) then (3.1) and (3.2) can be given in the form
\[
\int_1^d f(x) k(x) \, dx = \frac{1}{2} \int_{-\theta}^{\phi} f(\cosh \theta) k(\cosh \theta) \sinh \theta \, d\theta = \int_{\beta^2/b}^{b} f(x(t)) h(t) \, dt,
\]
where
\[
h(t) = \frac{d\psi(t)}{dt} = \frac{1}{4t} \left| \frac{t}{\beta} - \frac{\beta}{t} \right| k(x(t)) \quad \text{for} \quad t \in (\beta^2/b, b).
\]  

The following lemma gives information on some auxiliary polynomials used in the next two sections.

**Lemma 3.1.** Let \( x = \cosh \theta \). Let
\[
T_n(x) = \cosh n\theta \quad \text{and} \quad U_n(x) = \frac{\sinh(n+1)\theta}{\sinh \theta}, \quad n \geq 0.
\]

Then the following hold:

- \( T_0(x) = 1 \) and, for any \( n \geq 1 \), \( T_n \) is a polynomial of degree \( n \) with leading coefficient \( 2^{n-1} \). Precisely, \( T_1(x) = x \), \( T_2(x) = 2x^2 - 1 \) and, for \( n = 1, 2, \ldots \),
\[
T_{2n+1}(x) = 2^{2n} x^{2n+1} - \sum_{j=1}^{n} \binom{2n+1}{j} T_{2n+1-2j}(x),
\]
\[
T_{2n+2}(x) = 2^{2n+1} x^{2n+2} - \sum_{j=1}^{n} \binom{2n+2}{j} T_{2n+2-2j}(x) - \frac{1}{2} \binom{2n+2}{n+1}.
\]

- For any \( n \geq 0 \), \( U_n \) is a polynomial of degree \( n \) with leading coefficient \( 2^n \). Precisely, \( U_0(x) = 1 \) and, for \( n = 1, 2, \ldots \),
\[
U_{2n-1}(x) = 2 \sum_{j=0}^{n-1} T_{2n-1-2j}(x) \quad \text{and} \quad U_{2n}(x) = 1 + 2 \sum_{j=0}^{n-1} T_{2n-2j}(x).
\]

**Proof.** Using the binomial expansion
\[
(2x)^{2n} = (e^{\theta} + e^{-\theta})^{2n} = \sum_{j=0}^{2n} \binom{2n}{j} e^{(2n-2j)\theta}.
\]

Since \( \binom{2n}{j} = \binom{2n}{2n-j} \),
\[
(2x)^{2n} = \binom{2n}{n} + \sum_{j=0}^{n-1} \binom{2n}{j} [e^{(2n-2j)\theta} + e^{-(2n-2j)\theta}]
\]
\[
= \binom{2n}{n} + 2 \sum_{j=0}^{n-1} \binom{2n}{j} T_{2n-2j}(x).
\]

This completes the proof of the result for \( T_{2n} \). Similarly, considering the expansions of \((e^{\theta} + e^{-\theta})^{2n+1}\) and \((e^{(n+1)\theta} - e^{-(n+1)\theta})/(e^{\theta} - e^{-\theta})\), we obtain the results for \( T_{2n+1} \) and \( U_n \). \( \square \)
The polynomials \( \{T_n\} \) and \( \{U_n\} \) are, respectively, the Chebyshev polynomials of the first and second kind. That is, within the interval \([-1, 1]\) these polynomials can be given by \( T_n(x) = \cos n\theta \) and \( U_n(x) = \sin(n + 1)\theta / \sin \theta \) with \( x = \cos \theta \).

Note that \( T_n \) and \( U_n \) are such that

\[
T_n(x(t)) = \frac{1}{2}[(t/\beta)^n + (\beta/t)^n], \quad n \geq 0, \\
U_n(x(t)) = [(t/\beta)^{n+1} - (\beta/t)^{n+1}]/[(t/\beta) - (\beta/t)], \quad n \geq 0.
\] (3.4)

4. Orthogonal polynomials and \( L \)-orthogonal polynomials

Let \( 0 < \beta < b \) and \( \psi \in S^3(0, \beta, b) \). Let us consider the functions \( \{R_n^{(1)}\} \) given by

\[
\lambda_n^{(1)} R_n^{(1)}(t) = \left( \frac{1}{t} \right)^n Q_{2n}^\psi(t) + \left( \frac{t}{\beta^2} \right)^n Q_{2n}^\psi \left( \frac{\beta^2}{t} \right), \quad n \geq 0,
\]

where \( \lambda_n^{(1)} = (2\beta)^n \left[ 1 + \sqrt{\beta^2/\gamma_{2n}} \right] \). Using (2.4) one can also show that

\[
R_n^{(1)}(t) = -\frac{\sqrt{\beta^2 \gamma_{2n}}}{(2\beta)^n} \left[ \left( \frac{1}{t} \right)^n Q_{2n-1}^\psi(t) + \left( \frac{t}{\beta^2} \right)^n Q_{2n-1}^\psi \left( \frac{\beta^2}{t} \right) \right], \quad n \geq 1.
\]

**Theorem 4.1.** Let \( d = x(b) \) and let the positive measure \( \phi_1 \) defined on the interval \([1, d]\) be such that

\[
d\phi_1(x(t)) = 2\,d\psi(t) \quad \text{for } t \in [\beta, b].
\]

Let the sequence of functions \( \{P_n^{(1)}\} \) be such that

\[
P_n^{(1)}(x(t)) = R_n^{(1)}(t) \quad \text{for } n \geq 1.
\]

Then for any \( n \geq 1 \), \( P_n^{(1)}(x) = Q_n^{\phi_1}(x) \), the monic orthogonal polynomial of degree \( n \) associated with the positive measure \( \phi_1 \) on \([1, d]\). Moreover, \( \mu_0^{\phi_1} = \mu_0^\psi \), \( b_1^{\phi_1} = \delta^{\psi} \) and, for \( n \geq 1 \),

\[
\begin{align*}
\delta_n^{\psi} &= \sqrt{\gamma_n^\psi / \beta_n^\psi} \quad \text{Furthermore,} \\
\delta_n^{\psi} &= \frac{S_n^{\phi_1}(-1) + S_n^{\phi_1}(1)}{S_n^{\phi_1}(-1) - S_n^{\phi_1}(1)} \quad \text{and} \\
\delta_n^{\psi} &= S_{n+1}^{\phi_1}(1) - S_{n+1}^{\phi_1}(-1) - 1
\end{align*}
\]

(4.1) for \( n \geq 1 \), where \( S_n^{\phi_1}(z) = Q_n^{\phi_1}(z)/Q_n^{\phi_1}(-1) \).

**Proof.** First we show that \( P_n^{(1)} \) is a monic polynomial of degree \( n \). Letting \( Q_n^\psi(t) = \sum_{r=0}^n q_n,r \, t^r \), we obtain
\[
\lambda_n^{(1)} R_n^{(1)}(t) = \sum_{r=0}^{2n} q_{2n,r} \beta^{-n+r} \left[ \left( \frac{t}{\beta} \right)^{n-r} + \left( \frac{\beta}{t} \right)^{n-r} \right]
\]

\[
= 2q_{2n,n} + \sum_{r=0}^{n-1} \left[ q_{2n,r} \beta^{-n+r} + q_{2n,2n-r} \beta^{n-r} \right] \left[ \left( \frac{t}{\beta} \right)^{n-r} + \left( \frac{\beta}{t} \right)^{n-r} \right].
\]

Hence, from (3.4),

\[
\lambda_n^{(1)} P_n^{(1)}(x(t)) = 2q_{2n,n} + \sum_{r=0}^{n-1} \left[ q_{2n,r} \beta^{-n+r} + q_{2n,2n-r} \beta^{n-r} \right] T_{n-r}(x(t)).
\] (4.3)

Since \( T_r, r \geq 1 \), is a polynomial of degree \( r \) in \( x \) with leading coefficient \( 2^{r-1} \), \( P_n^{(1)} \) is a polynomial of degree \( n \) in \( x \) and its leading coefficient is

\[
\frac{2^n(q_{2n,0} \beta^{-n} + q_{2n,2n} \beta^n)}{\lambda_n^{(1)}} = \frac{2^n(Q^{\psi}_{2n}(0) \beta^{-n} + \beta^n)}{(2\beta)^n[1 + \sqrt{\beta^{2n}/\gamma_{2n}^{\psi}}]}. \]

Using (2.3) one can easily verify this to be equal to 1.

Now we establish the orthogonality property of \( \{P_n^{(1)}\} \). We will show that

\[
I_{n,k}^{(1)} = \int_1^d T_k(x) P_n^{(1)}(x) \, d\phi_1(x) = 0 \quad \text{for } k = 0, 1, \ldots, n-1, \ n \geq 1.
\] (4.4)

From (3.2),

\[
I_{n,k}^{(1)} = \int_{\beta^2/b}^{b} T_k(x(t)) P_n^{(1)}(x(t)) \, d\psi(t) = \frac{1}{2} \int_{\beta^2/b}^{b} \left[ \left( \frac{t}{\beta} \right)^k + \left( \frac{\beta}{t} \right)^k \right] R_n^{(1)}(t) \, d\psi(t).
\]

We write

\[
2\lambda_n^{(1)} I_{n,k}^{(1)} = \int_{\beta^2/b}^{b} \left[ \left( \frac{t}{\beta} \right)^k + \left( \frac{\beta}{t} \right)^k \right] \left( \frac{1}{t} \right)^n Q^{\psi}_{2n}(t) \, d\psi(t)
\]

\[
+ \int_{\beta^2/b}^{b} \left[ \left( \frac{t}{\beta} \right)^k + \left( \frac{\beta}{t} \right)^k \right] \left( \frac{t}{\beta^2} \right)^n Q^{\psi}_{2n} \left( \frac{\beta^2}{t} \right) \, d\psi(t).
\]

Using (2.1) on the second integral, we obtain

\[
\lambda_n^{(1)} I_{n,k}^{(1)} = \int_{\beta^2/b}^{b} \left[ \left( \frac{t}{\beta} \right)^k + \left( \frac{\beta}{t} \right)^k \right] \left( \frac{1}{t} \right)^n Q^{\psi}_{2n}(t) \, d\psi(t).
\] (4.5)

Thus, from (1.2), \( I_{n,k}^{(1)} = 0 \) for \( k = 0, 1, \ldots, n-1 \). This proves the first part of the theorem.

We also have from (4.5),

\[
I_{n,n}^{(1)} = \frac{\mu_0^{\psi} \alpha_2^{\psi} \alpha_3^{\psi} \cdots \alpha_{2n+1}^{\psi}}{2^n \beta^{2n}[1 + \sqrt{\beta^{2n}/\gamma_{2n}^{\psi}}]}, \ n \geq 1.
\]
On the other hand, since the leading coefficient of $T_n$ is $2^{n-1}$, we obtain from (4.4),
\[ I_{n,n}^{(1)} = 2^{n-1} \mu_0 \phi_1 \phi_2 \phi_3 \cdots \phi_{n+1}, \quad n \geq 1. \]

Considering the ratios $I_{n,n}^{(1)}/I_{n-1,n-1}^{(1)}$ we are led to
\[ a_2^{\phi_1} = \frac{1}{1 + \sqrt{\beta_2^\psi / \gamma_2^\psi}} \frac{\alpha_2^\psi \beta_2^\psi}{2 \beta^2} \quad \text{and} \quad a_n^{\phi_1} = \frac{1 + \sqrt{\beta_{2n-2}^\psi / \gamma_{2n-2}^\psi}}{1 + \sqrt{\beta_{2n}^\psi / \gamma_{2n}^\psi}} \frac{\alpha_{2n}^\psi \alpha_{2n+1}^\psi}{4 \beta^2}, \quad n \geq 2. \]

Hence from (2.2), Eq. (4.1) for $a_n^{\phi_1}$ can be obtained.

Now we compare the coefficients of degree $n-1$ on both sides of the expression (4.3) and obtain
\[ - \sum_{r=1}^{n} b_r^{\phi_1} = \frac{2^{n-1}(q_{2n,1} \beta^{-n+1} + q_{2n,2n-1} \beta^{n-1})}{(2 \beta)^n[1 + \sqrt{\beta_{2n}^\psi / \gamma_{2n}^\psi}]}, \quad n \geq 1. \]

The left-hand side follows from the three term recurrence relation (1.1). We need to find expressions for $q_{2n,1}$ and $q_{2n,2n-1}$. From the three term recurrence relation (1.3), $q_{n+1,n} = -\beta_{n+1}^\psi - \alpha_n^\psi + q_{n,n}$ for $n \geq 1$. Thus,
\[ q_{2n,2n-1} - \alpha_{2n+1}^\psi = - \sum_{r=1}^{2n} \gamma_r^\psi, \quad n \geq 1. \]

From (2.4),
\[ q_{2n,2n-1} - \alpha_{2n+1}^\psi = \frac{\beta^2 q_{2n,1}}{Q_{2n}^\psi (0)}, \quad n \geq 1. \]

Therefore,
\[ \sum_{r=1}^{n} b_r^{\phi_1} = \frac{1}{2 \beta} \sum_{r=1}^{2n} \gamma_r^\psi - \frac{\alpha_{2n+1}^\psi}{2 \beta[1 + \sqrt{\beta_{2n}^\psi / \gamma_{2n}^\psi}]} = \frac{1}{2 \beta} \sum_{r=1}^{2n-1} \gamma_r^\psi + \frac{1}{2 \beta} \sqrt{\beta_{2n}^\psi / \gamma_{2n}^\psi}, \quad n \geq 1. \]

From this and (2.2), Eq. (4.1) for $b_n^{\phi_1}$ follows.

Since $x(\beta) = 1$ and $x(-\beta) = -1$, we obtain from $Q_n^{\phi_1}(x(t)) = R_n^{(1)}(t)$,
\[ Q_n^{\phi_1}(1) = 2\frac{Q_{2n}^\psi (\beta)}{\lambda_n^{(1)}(1) \beta^n} \quad \text{and} \quad Q_n^{\phi_1}(-1) = 2\frac{Q_{2n}^\psi (-\beta)}{\lambda_n^{(1)}(-\beta)^n}, \quad n \geq 1. \]

Hence, with $S_n^{\phi_1}(z) = Q_n^{\phi_1}(z)/Q_{n-1}^{\phi_1}(z)$,
\[ S_n^{\phi_1}(1) = -\frac{1}{2}(\delta_{2n-1}^\psi - 1)(\delta_{2n-2}^\psi + 1), \quad n \geq 1, \]
\[ S_n^{\phi_1}(-1) = -\frac{1}{2}(\delta_{2n-1}^\psi + 1)(\delta_{2n-2}^\psi + 1), \quad n \geq 1. \]

Therefore we obtain (4.2) and this completes the proof of the theorem. \( \square \)

We remind that $Q_n^\psi(0) = (-\beta)^n / \delta_n^\psi$, $n \geq 1$, and
\[ \beta_n^\psi = \beta \delta_{n-1}^\psi / \delta_n^\psi, \quad \alpha_n^{\psi+1} = \beta(\delta_n^\psi - 1)(\delta_n^\psi + 1)\delta_{n-1}^\psi / \delta_n^\psi, \quad n \geq 1, \quad (4.6) \]
with $\delta_0^{\psi} = 1$, which follow from (2.2) and (2.3). Thus, for $n \geq 1$,
\[
\int_{\beta^n}^{b} Q_n^{\psi}(t) \, d\psi(t) = \mu_0^{\psi} \prod_{j=1}^{n} [(\delta_j^{\psi} - 1)(\delta_j^{\psi} + 1)],
\]
\[
\int_{\beta^n}^{b} t^{-n-1} Q_n^{\psi}(t) \, d\psi(t) = (-1)^n \mu_0^{\psi} \prod_{j=1}^{n} [(\delta_j^{\psi} - 1)(\delta_j^{\psi} + 1)].
\]

5. Companion polynomials of $Q_n^{\phi_1}$ and further identities

We now consider the functions $\{R_n^{(2)}\}$ given by
\[
\lambda_n^{(2)} R_n^{(2)}(t) = \left[ \left( \frac{1}{t} \right)^{n+1} Q_2^{\psi}(t) - \left( \frac{t}{\beta^2} \right)^{n+1} Q_2^{\psi}(t) \right] / [t - \beta^2 / t], \quad n \geq 0,
\]
where $\lambda_n^{(2)} = (2\beta)^n [1 - \sqrt{\beta^{2n}_2 / \gamma^{2n}_2}].$ Again from (2.4),
\[
R_n^{(2)}(t) = \frac{\sqrt{\beta^{2n}_2 \gamma^{2n}_2}}{(2\beta)^n} \left[ \left( \frac{1}{t} \right)^{n+1} Q_2^{\psi}(t) - \left( \frac{t}{\beta^2} \right)^{n+1} Q_2^{\psi}(t) \right] / [t - \beta^2 / t],
\]
for $n \geq 0$.

**Theorem 5.1.** Let $d = x(b)$ and let the positive measure $\phi_2$ defined on the interval $[1, d]$ be such that
\[
\frac{1}{x(t)^2 - 1} \, d\phi_2(x(t)) = d\phi_1(x(t)) = 2 \, d\psi(t) \quad \text{for} \ t \in [\beta, b].
\]
Let the sequence of functions $\{P_n^{(2)}\}$ be such that
\[
P_n^{(2)}(x(t)) = R_n^{(2)}(t) \quad \text{for} \ n \geq 0.
\]
Then for $n \geq 0$, $P_n^{(2)}(x) = Q_n^{\phi_2}(x)$, the monic orthogonal polynomial of degree $n$ associated with the positive measure $\phi_2$ on $[1, d]$. Moreover, $\mu_0^{\phi_2} = \frac{1}{2} (\delta_1^{\psi} - 1)(\delta_1^{\psi} + 1)(\delta_2^{\psi} + 1) \mu_0^{\psi}$ and, for $n \geq 1$,
\[
b_n^{\phi_2} = \frac{1}{2} \delta_2^{\psi}(\delta_2^{\psi} - 1) + \frac{1}{2} \delta_2^{-1}(\delta_2^{\psi} + 1),
\]
\[
da_n^{\phi_2} = \frac{1}{4} (\delta_2^{\psi} - 1)(\delta_2^{\psi} + 1)(\delta_2^{\psi} + 1)\delta_2^{\psi} + 1).
\]

**Proof.** To prove $P_n^{(2)}(x) = Q_n^{\phi_2}(x)$, apart from establishing that $P_n^{(2)}(x(t)) = R_n^{(2)}(t)$ is a polynomial of degree $n$ in $x$, one needs to prove that
\[
I_n^{(2)} = \int_{1}^{d} U_k(x) P_n^{(2)}(x) \, d\phi_2(x) = 0 \quad \text{for} \ k = 0, 1, \ldots, n - 1, \ n \geq 1.
\]
This is done in a way similar to that of Theorem 4.1.
Having proved $P_n^{(2)}(x) = Q_n^{\phi_2}(x)$, from $I_{n,n}^{(2)}$ we have

$$
\int_1^d U_n(x)Q_n^{\phi_2}(x)\,d\phi_2(x) = \frac{1}{(2\beta^2)^{n+1}[1 - \sqrt{\beta_2^{\psi^{n+2}} / \gamma_2^{n+2}}]} \int_b^\infty Q_2^{\psi^{n+2}}(t)\,d\psi(t),
$$

for $n \geq 0$. From this, $\mu_0^{\phi_2} = \frac{1}{2\beta^2[1 - \sqrt{\beta_2^{\psi^{n+2}} / \gamma_2^{n+2}}]} \mu_0^{\psi^{n+2}} \alpha_2^{\psi^{n+2}} \alpha_3^{\psi^{n+2}}$ and

$$
2^n \mu_0^{\phi_2} \alpha_2^{\phi_2} \cdots \alpha_{n+1}^{\phi_2} = \frac{1}{(2\beta^2)^{n+1}[1 - \sqrt{\beta_2^{\psi^{n+2}} / \gamma_2^{n+2}}]} \mu_0^{\psi^{n+2}} \alpha_2^{\psi^{n+2}} \cdots \alpha_{2n+2}^{\psi^{n+2}} \alpha_{2n+3}^{\psi^{n+2}}, \quad n \geq 1.
$$

Consequently the equation for $a^{\phi_2}$ in (5.1) follows.

Comparing the coefficients of $x^n$ on both sides of the equation $Q_n^{\phi_2}(x(t)) = R_n^{(2)}(t)$, the equation for $b^{\phi_2}$ in (5.1) also follows.

Now, from (4.1) and (5.1),

$$
b_{n+1}^{\phi_1} - b_n^{\phi_2} = \delta_2^{n+1} - \delta_2^n, \quad n \geq 1,
$$

with $b_1^{\phi_1} = \delta_1$. Thus, we can write

$$
\delta_1^{\psi} = b_1^{\phi_1} \quad \text{and} \quad \delta_2^{n+1} = \sum_{r=1}^{n+1} b_r^{\phi_1} - \sum_{r=1}^n b_r^{\phi_2}, \quad n \geq 1.
$$

Also, from (4.1) and (5.1),

$$
a^{\phi_1} = \frac{\delta_2^{\psi} + 1}{\delta_2^{\psi} - 1} \mu_0^{\psi} \quad \text{and} \quad a_n^{\phi_1} = \frac{\delta_2^n - 1}{\delta_2^{n-1} - 1} \delta_2^{\psi} - 1, \quad n \geq 2.
$$

Since $\mu_0^{\phi_1} = \mu_0^{\psi}$,

$$
\frac{\delta_2^n - 1}{\delta_2^{n-1} + 1} = \frac{\mu_0^{\phi_1}}{\mu_0^{\phi_2}} \quad \text{and} \quad \frac{\delta_2^{n+2} - 1}{\delta_2^{n+1} + 1} = \frac{a_{n+2}^{\phi_1} \cdots a_2^{\phi_1} \mu_0^{\phi_1}}{a_{n+2}^{\psi} \cdots a_2^{\psi} \mu_0^{\psi}}, \quad n \geq 1.
$$

Thus, $\delta_2^{\psi} = \frac{\mu_0^{\psi} + a_2^{\phi_1} \mu_0^{\phi_1}}{\mu_0^{\phi_2} - a_2^{\phi_1} \mu_0^{\psi}}$ and

$$
\delta_2^{n+2} = \frac{a_{n+1}^{\phi_1} \cdots a_2^{\phi_1} \mu_0^{\phi_1} + a_{n+2}^{\phi_1} a_{n+1}^{\phi_1} \cdots a_2^{\phi_1} \mu_0^{\phi_1}}{a_{n+1}^{\phi_1} \cdots a_2^{\phi_1} \mu_0^{\phi_2} - a_{n+2}^{\phi_1} a_{n+1}^{\phi_1} \cdots a_2^{\phi_1} \mu_0^{\phi_2}}, \quad n \geq 1.
$$

6. Examples

We give four examples and in the first three of them we start from a known set of orthogonal polynomials and derive information on the connected set of L-orthogonal polynomials. In the last example, we consider a known set of L-orthogonal polynomials and obtain information about the connected set of orthogonal polynomials.
6.1. L-orthogonal polynomials from the Chebyshev polynomials

We consider the sequence of orthogonal polynomials \( \{Q_\phi^n\} \) given by

\[
Q_\phi^n(x) = c_n T_n \left( \frac{2x}{d - 1} - \frac{d + 1}{d - 1} \right), \quad n \geq 1,
\]

where \( T_n \) are the Chebyshev polynomials (see, for example, [1,13]). The constants \( c_n \) are chosen so that \( Q_\phi^n \) are monic polynomials. The following results are easily verified:

\[
\mu_{\phi_0} = \int_1^d \frac{1}{\sqrt{(d - x)(x - 1)}} \, dx = 1 \quad \text{and} \quad \\
\int_1^d Q_\phi^n(x) Q_\phi^m(x) \frac{1}{\pi \sqrt{(d - x)(x - 1)}} \, dx = \delta_{n,m} \left( \frac{d - 1}{4} \right)^{2n}, \quad n \geq 1, \ m \geq 0.
\]

The three term recurrence relation for \( Q_\phi^1 \) is

\[
Q_\phi^2(x) = \left( x - \frac{1}{2}(d + 1) \right) Q_\phi^1(x) - \frac{1}{8} (d - 1)^2 Q_\phi^0(x),
\]

\[
Q_\phi^{n+1}(x) = \left( x - \frac{1}{2}(d + 1) \right) Q_\phi^n(x) - \frac{1}{16} (d - 1)^2 Q_\phi^{n-1}(x), \quad n \geq 2,
\]

with \( Q_\phi^1(x) = (x - \frac{d + 1}{2}) \) and \( Q_\phi^0(x) = 1 \). Moreover, for \( n \geq 1 \),

\[
2^n Q_\phi^n(x) = \left[ \left( x - \frac{d + 1}{2} \right) + \sqrt{(d - x)(1 - x)} \right]^n + \left[ \left( x - \frac{d + 1}{2} \right) - \sqrt{(d - x)(1 - x)} \right]^n.
\]

Hence, from Theorem 4.1, with the use of (3.3) we can consider the L-orthogonal polynomials \( \{Q_\psi^n\} \) associated with the \( S_3(0, \beta, b) \) measure \( \psi \) given by

\[
d\psi(t) = \frac{1}{2\pi} \frac{1 + \beta/t}{\sqrt{(b - t)(t - a)}} \, dt,
\]

where \( b = \beta(d + \sqrt{d^2 - 1}) \) and \( a = \beta^2/b \).

**Theorem 6.1.** Let \( \psi \) be the \( S_3(0, \beta, b) \) measure given by (6.2). Then for the associated L-orthogonal polynomials \( \{Q_\psi^n\} \) we have that the parameters \( \delta_\psi^n = (-\beta)^n / Q_\psi^n(0) \) satisfy

\[
\delta_{2n-1} = \frac{[1 + \eta][1 + \eta^{2n-1}]}{[1 - \eta][1 - \eta^{2n-1}]}, \quad \delta_{2n} = \frac{[1 + \eta][1 - \eta^{2n}]}{[1 - \eta][1 + \eta^{2n}]}, \quad n \geq 1,
\]

where \( \eta = [\sqrt{\beta} - \sqrt{\beta^2}] / [\sqrt{\beta} + \sqrt{\beta^2}] \). Consequently, in the three term recurrence relation (1.3) satisfied by these polynomials, we have \( \beta_1^\psi = \beta \frac{[1 - \eta]^2}{[1 + \eta]^2} \), \( \alpha_2^\psi = \beta \frac{8\eta[1 + \eta^2]}{[1 - \eta^2]^2} \) and, for \( n \geq 1 \),
\[
\beta_{2n}^\psi = \beta \left[ 1 + \eta^{2n-1} \right] \left[ 1 + \eta^{2n} \right] \left[ 1 - \eta^{2n-1} \right] \left[ 1 - \eta^{2n} \right],
\]
\[
\beta_{2n+1}^\psi = \beta \left[ 1 - \eta^{2n} \right] \left[ 1 - \eta^{2n+1} \right] \left[ 1 + \eta^{2n} \right] \left[ 1 + \eta^{2n+1} \right].
\]
\[
\alpha_{2n+1}^\psi = \beta \frac{4\eta}{\left[ 1 - \eta \right]^2} \left[ 1 + \eta^{2n-1} \right] \left[ 1 - \eta^{2n+1} \right] \left[ 1 + \eta^{2n} \right] \left[ 1 - \eta^{2n} \right],
\]
\[
\alpha_{2n+2}^\psi = \beta \frac{4\eta}{\left[ 1 - \eta \right]^2} \left[ 1 + \eta^{2n+1} \right] \left[ 1 + \eta^{2n+2} \right] \left[ 1 - \eta^{2n+1} \right] \left[ 1 - \eta^{2n+2} \right].
\]

**Proof.** We obtain from (6.1),

\[
Q_n^{\phi_1}(1) = 2 \left( \frac{d - 1}{2} \right)^n \left( \frac{-1}{2} \right)^n,
\]
\[
Q_n^{\phi_1}(-1) = \left[ \left( \sqrt{\frac{d + 1}{2}} - 1 \right)^{2n} + \left( \sqrt{\frac{d + 1}{2}} + 1 \right)^{2n} \right] \left( \frac{-1}{2} \right)^n,
\]
for \( n \geq 1 \). Since \( d = \frac{1}{2}(b/\beta + \beta/b) \), these can be written as

\[
Q_n^{\phi_1}(1) = \frac{2}{(8b\beta)^n} \left[ (\sqrt{b} + \sqrt{\beta})^{2n} (\sqrt{b} - \sqrt{\beta})^{2n} \right], \quad n \geq 1,
\]
\[
Q_n^{\phi_1}(-1) = \frac{1}{(8b\beta)^n} \left[ (\sqrt{b} + \sqrt{\beta})^{4n} + (\sqrt{b} - \sqrt{\beta})^{4n} \right], \quad n \geq 1.
\]

With \( \eta = [\sqrt{b} - \sqrt{\beta}]^2/[\sqrt{b} + \sqrt{\beta}]^2 \), we can then write

\[
Q_n^{\phi_1}(1) = 2 \frac{(-2)^n \eta^n}{(1 - \eta)^{2n}} \quad \text{and} \quad Q_n^{\phi_1}(-1) = \frac{(-2)^n (1 + \eta^{2n})}{(1 - \eta)^{2n}}, \quad n \geq 1.
\]

Hence, for \( S_n^{\phi_1}(z) = Q_n^{\phi_1}(z)/Q_{n-1}^{\phi_1}(z) \),

\[
S_n^{\phi_1}(1) = \frac{-2\eta}{(1 - \eta)^2} \quad \text{and} \quad S_n^{\phi_1}(-1) = \frac{-2}{(1 - \eta)^2} \frac{1 + \eta^{2n}}{1 + \eta^{2n-2}}, \quad n \geq 1.
\]

Thus, from (4.2), the results (6.3) of the theorem follow. With this, the results for \( \beta_n^\psi \) immediately follow from (4.6).

To obtain the results for \( \alpha_n^\psi \), first we obtain

\[
\delta_{2n-1}^\psi - 1 = \frac{2\eta}{[1 - \eta][1 - \eta^{2n-1}]} \left[ 1 + \eta^{2n-2} \right], \quad \delta_{2n-1}^\psi + 1 = \frac{2}{[1 - \eta][1 - \eta^{2n-1}]} \left[ 1 + \eta^{2n} \right],
\]
\[
\delta_{2n}^\psi - 1 = \frac{2\eta}{[1 - \eta][1 + \eta^{2n}]} \left[ 1 - \eta^{2n-1} \right], \quad \delta_{2n}^\psi + 1 = \frac{2}{[1 - \eta][1 + \eta^{2n}]} \left[ 1 - \eta^{2n+1} \right],
\]
for \( n \geq 1 \). Now the results for \( \alpha_n^\psi \) follow from (4.6). \( \Box \)

Note that we can write \( \eta = [\sqrt{d + \sqrt{d^2 - 1}} - 1]/[\sqrt{d + \sqrt{d^2 - 1}} + 1] \). Hence, the value of \( \eta \) is independent of the value of \( \beta \).
6.2. L-orthogonal polynomials from the Laguerre polynomials

The Laguerre polynomials \( \{L_n^{(\alpha)}\} \), for \( \alpha > -1 \), are defined by
\[
\int_0^\infty L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) x^\alpha e^{-x} \, dx = \Gamma(\alpha + 1) \binom{n + \alpha}{n} \delta_{nm}, \quad n, m = 0, 1, 2, \ldots,
\]
where \( \binom{u}{v} = \frac{\Gamma(u+1)}{\Gamma(v+1)\Gamma(u-v+1)} \). These polynomials are explicitly given by (see, for example, [1,13])
\[
L_0^{(\alpha)}(x) = 1 \quad \text{and} \quad L_n^{(\alpha)}(x) = \sum_{j=0}^{n} \binom{n + \alpha}{n-j} \frac{1}{j!} (-x)^j, \quad n \geq 1.
\]

We consider the polynomials
\[
Q_{\phi_1}^0(x) = 1 \quad \text{and} \quad Q_{\phi_1}^n(x) = (-2)^n n! L_n^{(\alpha)} \left( \frac{x - 1/2}{2} \right), \quad n \geq 1.
\]
Then \( \{Q_{\phi_1}^n(x)\} \) are the monic orthogonal polynomials given by
\[
\int_1^\infty Q_{\phi_1}^n(x) Q_{\phi_1}^m(x) d\phi_1(x) = 2^{2n} n! (\alpha + 1) \delta_{nm}, \quad n, m = 0, 1, 2, \ldots,
\]
where \( d\phi_1(x) = \frac{1}{2\Gamma(\alpha + 1)} \left( \frac{1}{2} (x - 1) \right)^\alpha e^{-\frac{1}{2} (x-1)} \, dx \). Here, \( (u)_n \) is the Pochhammer-symbol defined by \( (u)_0 = 1 \) and \( (u)_n = u(u+1) \cdots (u+n-1) \) for \( n \geq 1 \). One can easily verify that \( \mu_{\phi_1}^1 = 1 \),
\[
Q_{\phi_1}^0(x) = 1 \quad \text{and} \quad Q_{\phi_1}^n(x) = (-2)^n n! \sum_{j=0}^{n} \binom{n + \alpha}{n-j} \frac{1}{j!} \left( \frac{1 - x}{2} \right)^j, \quad n \geq 1.
\]
Moreover,
\[
Q_{\phi_1}^n(x) = [x - (4n + 3 + 2\alpha)] Q_{\phi_1}^n(x) - 4n(n + \alpha) Q_{\phi_1}^{n-1}(x), \quad n \geq 1,
\]
with \( Q_{\phi_1}^0 = 1 \) and \( Q_{\phi_1}^1(x) = x - (3 + 2\alpha) \).

**Theorem 6.2.** Let \( \psi \) be the \( S^3(0, \beta, \infty) \) measure given by
\[
d\psi(t) = \frac{1}{4\Gamma(\alpha+1)} \left[ \frac{1}{2} \sqrt{t/\beta} + \sqrt{\beta/t} \right] e^{-\frac{1}{2} (\sqrt{t/\beta} - \sqrt{\beta/t})^2} \, dt.
\]
Then for the associated L-orthogonal polynomials \( \{Q_{\psi}^n\} \), the parameters \( \delta_{\psi}^n = (-\beta)^n / Q_{\psi}^n(0) \) satisfy
\[
\delta_{2n-1}^\psi = -1 + 2 \frac{v_n^{(\alpha)}}{v_{n-1}^{(\alpha+1)}} = 1 + 2(n + \alpha) \frac{v_n^{(\alpha)}}{v_{n-1}^{(\alpha+1)}}, \quad n \geq 1,
\]
\[
\delta_{2n}^\psi = -1 + 2 \frac{v_n^{(\alpha+1)}}{v_n^{(\alpha)}} = 1 + 2 \frac{v_{n-1}^{(\alpha+1)}}{v_n^{(\alpha)}}, \quad n \geq 1,
\]
where \( v_n^{(\alpha)} = L_n^{(\alpha)}(-1) = \sum_{j=0}^{n} \binom{n + \alpha}{n-j} \frac{1}{j!} \).
Proof. These results follow from the application of Theorem 4.1 together with (3.3). To complete the proof of this theorem we observe that

\[ Q_n^{\phi_1}(1) = (-2)^n n! \left(\frac{n + \alpha}{n}\right) = (-2)^n n! L_n^{(\alpha)}(0), \quad n \geq 1, \]

\[ Q_n^{\phi_1}(-1) = (-2)^n n! \sum_{j=0}^{n} \left(\frac{n + \alpha}{n - j}\right) \frac{1}{j!} = (-2)^n n! L_n^{(\alpha)}(-1), \quad n \geq 1. \]

Hence, for \( n \geq 1 \),

\[ Q_n^{\phi_1}(-1) Q_n^{\phi_1}(1) = 2^{2n} (n!)^2 \left(\frac{n + \alpha}{n}\right) \sum_{j=0}^{n} \left(\frac{n + \alpha}{n - j}\right) \frac{1}{j!}, \]

\[ Q_n^{\phi_1}(-1) Q_n^{\phi_1}(1) + Q_n^{\phi_1}(1) Q_n^{\phi_1}(-1) \]

\[ = -2^{2n-1} (n-1)! n! \left(\frac{n + \alpha}{n}\right) \sum_{j=0}^{n} (2n - j) \left(\frac{n + \alpha}{n - j}\right) \frac{1}{j!}, \]

\[ Q_n^{\phi_1}(-1) Q_n^{\phi_1}(1) - Q_n^{\phi_1}(1) Q_n^{\phi_1}(-1) \]

\[ = -2^{2n-1} (n-1)! n! \left(\frac{n + \alpha}{n}\right) \sum_{j=0}^{n} j \left(\frac{n + \alpha}{n - j}\right) \frac{1}{j!}. \] (6.4)

To obtain the last two expressions we use

\[ \left( n - 1 + \alpha \right) \left( n - j - 1 \right) = \frac{n - j}{n - 1} \left(\frac{n + \alpha}{n - j}\right), \quad j = 0, 1, \ldots, n - 1. \]

Using the results of (6.4) in (4.2) we conclude the proof of the theorem. \( \square \)

From (4.6), we obtain that the polynomials \( \{ Q_n^{\psi} \} \) satisfy the three term recurrence relation (1.3) with \( \beta_1^{\psi} = \beta(3 + 2\alpha)^{-1} \), \( \alpha_2^{\psi} = 4(\alpha + 1)(\alpha + 2)\beta_1^{\psi} \) and, for \( n \geq 1 \),

\[ \beta_{2n}^{\psi} = \beta \frac{1 + 2(n + \alpha)\nu_{n-1}^{(\alpha)+1}}{1 + 2\nu_{n-1}^{(\alpha)+1}/\nu_{n-1}^{(\alpha)}}, \quad \beta_{2n+1}^{\psi} = \beta \frac{1 + 2\nu_{n+1}^{(\alpha)+1}/\nu_{n}^{(\alpha)}}{1 + 2(n + \alpha + 1)\nu_{n}^{(\alpha)+1}/\nu_{n}^{(\alpha)+1}}, \]

\[ \alpha_{2n+1}^{\psi} = 4 \nu_{n-1}^{(\alpha)+1} \nu_{n}^{(\alpha)+1} \beta_{2n}^{\psi}, \quad \alpha_{2n+2}^{\psi} = 4(n + 1)(n + \alpha + 1) \nu_{n}^{(\alpha)+1} \nu_{n+1}^{(\alpha)+1} \beta_{2n+1}^{\psi}. \]

6.3. L-orthogonal polynomials from the q-Laguerre polynomials

The q-Laguerre polynomials \( \{ L_n^{(\alpha)}(x; q) \} \), for \( \alpha > -1 \), can be defined by

\[ \int_{0}^{\infty} L_n^{(\alpha)}(x; q) L_m^{(\alpha)}(x; q) \frac{x^\alpha}{(-x; q)_{\infty}} dx = \frac{(q^{-\alpha}; q)_{\infty}}{(q; q)_{\infty}} \frac{(q^{\alpha+1}; q)_n}{(q; q)_n q^n} \Gamma(-\alpha) \Gamma(\alpha + 1) \delta_{nm}, \]

where \( (u; q)_0 = 1, (u; q)_n = (1 - u)(1 - uq) \cdots (1 - uq^{n-1}), n \geq 1 \). These polynomials, which are also called the Generalized Stieltjes–Wigert polynomials [2], have the explicit representation

\[ L_n^{(\alpha)}(x; q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \Phi_1 \left( \frac{q^{-n}}{q^{\alpha+1}} \middle| q; -q^{\alpha+1}x \right), \quad n \geq 1, \]
where
\[
\Phi_1 \left( \frac{u}{v} \mid q; z \right) = \sum_{j=0}^{\infty} \frac{(u; q)_j}{(v; q)_j} (-1)^j q^{j(j-1)/2} \frac{z^j}{(q; q)_j}.
\]

The moment problem associated with the q-Laguerre polynomials is indeterminate and therefore there are many orthogonality representations. For example, there is the following representation in terms of a discrete measure.
\[
\sum_{j=0}^{\infty} \frac{q^{j(a+1)}}{(-cq^j; q)_\infty} L_n^{(a)}(cq^j; q)L_m^{(a)}(cq^j; q) = \frac{(q; q)_\infty}{(q^{a+1}; q)_\infty} \frac{(cq^j; q)_\infty}{(-c^{-1}q^{-a}; q)_\infty} \frac{(q^{a+1}; q)_n}{(q; q)_n q^n} \delta_{nm},
\]
where c is any positive constant.

Here we consider the sequence of monic polynomials
\[
Q_{\phi_1}^0(x) = 1 \quad \text{and} \quad Q_{\phi_1}^n(x) = (-2)^n \frac{(q; q)_n}{q^n(n+\alpha)} L_n^{(\alpha)}((x - 1)/2; q), \quad n \geq 1.
\]

From known results on \(\{L_n^{(\alpha)}(q; x)\}\), the following results for the polynomials \(\{Q_{\phi_1}^n\}\) can be easily written down:
\[
\int_1^\infty Q_{\phi_1}^n(x) Q_{\phi_1}^m(x) d\phi_1(x) = \frac{2^{2n}(q; q)_n (q^{a+1}; q)_n}{q^{n(2n+2\alpha+1)}} \delta_{nm}, \quad n \geq 1,
\]
where \(d\phi_1(x) = \frac{1}{2(q^{-\alpha}; q)_\infty \Gamma(-\alpha) \Gamma(\alpha+1)} \frac{((x-1)/2)^{\alpha}}{(-(x-1)/2; q)_\infty} \) dx.

Moreover,
\[
Q_{\phi_1}^n(1) = (-2)^n \frac{(q; q)_n}{q^n(n+\alpha)} L_n^{(\alpha)}(0; q) = (-2)^n \frac{(q^{a+1}; q)_n}{q^{n(n+\alpha)}},
\]
\[
Q_{\phi_1}^n(-1) = (-2)^n \frac{(q; q)_n}{q^n(n+\alpha)} L_n^{(\alpha)}(-1; q) = (-2)^n \frac{1}{q^n(n+\alpha)},
\]
for \(n \geq 1\). The last expression follows, for example, from the summation formula (see [8, p. 15])
\[
\Phi_1 \left( \frac{u}{v} \mid q; z \right) = \frac{(u^{-1}v; q)_\infty}{(v; q)_\infty}.
\]

Hence, from Theorem 4.1,

**Theorem 6.3.** Let \(\psi\) be the \(S^3(0, \beta, \infty)\) measure given by
\[
d\psi(t) = \frac{(q; q)_\infty t^{-[\sqrt{t}\beta + \sqrt{t}\beta]} \left[ \frac{1}{2} [\sqrt{t}\beta - \sqrt{t}\beta] \right]^{2\alpha+1}}{4(q^{-\alpha}; q)_\infty \Gamma(-\alpha) \Gamma(\alpha+1) (-[\frac{1}{2} (\sqrt{t}\beta - \sqrt{t}\beta)]^2; q)_\infty} \) dr.

Then for the associated L-orthogonal polynomials \(\{Q_{\psi}^n\}\), the parameters \(\delta_{\psi}^n = (-\beta)^n / Q_{\psi}^n(0)\) satisfy
\[
\delta_{2n-1}^\psi = 2q^{-n-\alpha} - 1 \quad \text{and} \quad \delta_{2n}^\psi = 2q^{-n} - 1, \quad n \geq 1.
\]
6.4. Orthogonal polynomials from the L-orthogonal polynomials associated with the log-normal distribution

In [9], Pastro gave the orthogonal Laurent polynomials associated with the log-normal distribution. Here we consider the L-orthogonal polynomials \( \{Q_n^\psi\} \) defined by (1.2) in relation to the strong measure \( \psi \) given by the shifted log-normal distribution

\[
d\psi(t) = \frac{1}{2\kappa \sqrt{\pi}} t^{-1} e^{-[\ln(t)/(2\kappa)]^2} \, dt.
\]

It is easily verified from (1.5) that this measure belongs to the class \( S^3[0, \beta, b] \), with \( \beta = 1 \) and \( b = \infty \).

From Pastro’s results, and also results given in [3,4], it follows that

\[
Q_n^\psi(w) = \sum_{r=0}^{n} (-1)^r q^{-r(n-r)} \binom{n}{r}_q q^{r/2} w^{n-r}, \quad n \geq 1,
\]

and

\[
Q_{n+1}^\psi(w) = (w - q^{1/2}) Q_n^\psi(w) - q^{1/2} (q^{-n-1} w Q_n^\psi(w), \quad n \geq 1,
\]

with \( Q_0^\psi(w) = w - q^{1/2} \). Here, \( q = e^{-2\kappa} \) and \( \binom{n}{r}_q \) are the q-binomial coefficients given by

\[
\binom{n}{r}_q = \frac{(q; q)_n}{(q; q)_r (q; q)_{n-r}}, \quad r = 0, 1, \ldots, n.
\]

The above results can also be found in [11].

Theorem 6.4. Let the positive measures \( \phi_A \) and \( \phi_B \), both having their common support on the entire positive real axis, be given by

\[
d\phi_B(x) = x(x+2) \, d\phi_A(x) = \frac{1}{\kappa \sqrt{\pi}} \sqrt{x(x+2)} e^{-[\ln(x+1+\sqrt{x(x+2)})/(2\kappa)]^2} \, dx.
\]

Then \( \mu_0^\phi_A = 1 \), \( \mu_0^\phi_B = q^{-2} - 1 \) and the associated monic orthogonal polynomials \( \{Q_n^{\phi_A}\} \) and \( \{Q_n^{\phi_B}\} \) satisfy \( Q_1^{\phi_A}(z) = z + 1 - q^{-1/2} \), \( Q_1^{\phi_B}(z) = z + 1 - \frac{1}{2} q^{-1/2}(q^{-2} + 1) \) and

\[
Q_{n+1}^{\phi_A}(z) = (z - b_{n+1}^{\phi_A}) Q_n^{\phi_A}(z) - a_{n+1}^{\phi_A} Q_{n+1}^{\phi_A}(z), \quad n \geq 1,
\]

\[
Q_{n+1}^{\phi_B}(z) = (z - b_{n+1}^{\phi_B}) Q_n^{\phi_B}(z) - a_{n+1}^{\phi_B} Q_{n+1}^{\phi_B}(z), \quad n \geq 1,
\]

where

\[
b_{n+1}^{\phi_A} = \frac{1}{2} [q^{-(2n+1)/2} (q^{-n} + 1) + q^{-(2n-1)/2} (q^{-n-1}) - 1],
\]

\[
a_{n+1}^{\phi_A} = \frac{1}{2} (q^{-n+1} + 1)(q^{-n} - 1)(q^{-2n+1} - 1),
\]

\[
b_{n+1}^{\phi_B} = \frac{1}{2} [q^{-(2n+3)/2} (q^{-n-1} - 1) + q^{-(2n+1)/2} (q^{-n-1} + 1)] - 1,
\]

\[
a_{n+1}^{\phi_B} = \frac{1}{4} (q^{-n} - 1)(q^{-n-1} + 1)(q^{-2n-1} - 1),
\]

for \( n \geq 1 \).
**Proof.** We have $\delta_n^\psi = q^{-n/2}$, $n \geq 1$. Hence, applying Theorems 4.1 and 5.1, we obtain for the measures $\phi_1$ and $\phi_2$, both defined on $[1, \infty)$,

$$d\phi_2(x) = (x^2 - 1) d\phi_1(x) = \frac{1}{\kappa \sqrt{\pi}} \sqrt{x^2 - 1} e^{-(\ln(x + \sqrt{x^2 - 1})^{1/2}x)^2} dx,$$

$$\mu_{\phi_1} = 1, \mu_{\phi_2} = q^{-2} - 1.$$ Moreover, the associated sequences of monic orthogonal polynomials \{Q_{n}^{\phi_1}\} and \{Q_{n}^{\phi_2}\} satisfy $Q_{n}^{\phi_1}(x) = x - q^{-1/2}$, $Q_{1}^{\phi_1}(x) = x - \frac{1}{2}q^{-1/2}(q^{-2} + 1)$ and

$$Q_{n+1}^{\phi_1}(x) = (x - b_{n+1}^{\phi_1}) Q_{n}^{\phi_1}(x) - a_{n+1}^{\phi_1} Q_{n-1}^{\phi_1}(x), \quad n \geq 1,$$

$$Q_{n+1}^{\phi_2}(x) = (x - b_{n+1}^{\phi_2}) Q_{n}^{\phi_2}(x) - a_{n+1}^{\phi_2} Q_{n-1}^{\phi_2}(x), \quad n \geq 1,$$

where

$$b_{n+1}^{\phi_1} = \frac{1}{2} \left[ q^{-2(n+1)/2} \left( q^n + 1 \right) + q^{-2(n-1)/2} \left( q^n - 1 \right) \right],$$

$$a_{n+1}^{\phi_1} = \frac{1}{4} \left[ q^{-n+1} + 1 \right] \left[ q^n - 1 \right] \left[ q^{2n+1} - 1 \right],$$

$$b_{n+1}^{\phi_2} = \frac{1}{2} \left[ q^{-2(n+3)/2} \left( q^n - 1 \right) + q^{-2(n+1)/2} \left( q^{n-1} + 1 \right) \right],$$

$$a_{n+1}^{\phi_2} = \frac{1}{4} \left[ q^n - 1 \right] \left[ q^{n-1} + 1 \right] \left[ q^{2n-1} - 1 \right],$$

for $n \geq 1$. Hence the substitution $x = y + 1$ gives the results of the theorem. \(\square\)

**References**


