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Another connection between orthogonal polynomials and L-orthogonal polynomials ☆

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Abstract

We consider a connection that exists between orthogonal polynomials associated with positive measures on the real line and orthogonal Laurent polynomials associated with strong measures of the class $S^3[0, \beta, b]$. Examples are given to illustrate the main contribution in this paper.

Keywords: Orthogonal polynomials; Orthogonal Laurent polynomials; Szegő polynomials on the real line; Hyperbolic Szegő transformation

1. Introduction

Let $-\infty \leqslant c < d \leqslant \infty$ and let ϕ be a bounded and nondecreasing function on $[c, d]$ such that it has infinitely many points of increase in $[c, d]$ and that the moments $\mu_n^\phi = \int_c^d x^n \, d\phi(x)$, $n = 0, 1, 2, \ldots,$ all exist. Then ϕ represents a positive measure on $[c, d]$ and one can uniquely define the set of polynomials $\{Q_n^\phi\}_{n=0}^\infty$ by

Q_n^ϕ is monic of degree n,

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$$\int_c^d x^s Q_n^\phi(x)\,\mathrm{d}\phi(x) = 0, \quad s = 0, 1, \dots, n-1.$$

These are the monic orthogonal polynomials associated with the measure ϕ. It is well known that these polynomials satisfy the three term recurrence relation

$$Q_{n+1}^\phi(z) = \big(z - b_{n+1}^\phi\big) Q_n^\phi(z) - a_{n+1}^\phi Q_{n-1}^\phi(z), \quad n \geqslant 1, \tag{1.1}$$

with $Q_0^\phi(z) = 1$ and $Q_1^\phi(z) = z - b_1^\phi$. The coefficients a_n^ϕ and b_n^ϕ, given by

$$b_{n+1}^\phi = \frac{\int_c^d x[Q_n^\phi(x)]^2\,\mathrm{d}\phi(x)}{\int_c^d [Q_n^\phi(x)]^2\,\mathrm{d}\phi(x)}, \quad n \geqslant 0, \quad \text{and} \quad a_{n+1}^\phi = \frac{\int_c^d [Q_n^\phi(x)]^2\,\mathrm{d}\phi(x)}{\int_c^d [Q_{n-1}^\phi(x)]^2\,\mathrm{d}\phi(x)}, \quad n \geqslant 1,$$

satisfy $b_n^\phi \in \mathbb{R}$, $n \geqslant 1$, and $a_n^\phi > 0$, $n \geqslant 2$. For further information on these polynomials we refer to Chihara [1] and Szegő [13].

Now let $0 \leqslant a < b \leqslant \infty$ and let ψ be a bounded and nondecreasing function on $[a, b]$ with infinitely many points of increase in $[a, b]$ and such that the moments $\mu_n^\psi = \int_a^b t^n\,\mathrm{d}\psi(t)$, $n = 0, \pm 1, \pm 2, \dots$, all exist. We refer to ψ as a strong positive measure on $[a, b]$ and define the set of polynomials $\{\mathcal{Q}_n^\psi\}_{n=0}^\infty$ by

$$\begin{aligned} &\mathcal{Q}_n^\psi \text{ is monic of degree } n, \\ &\int_a^b t^{-n+s} \mathcal{Q}_n^\psi(t)\,\mathrm{d}\psi(t) = 0, \quad s = 0, 1, \dots, n-1. \end{aligned} \tag{1.2}$$

Such polynomials were introduced in [7] in order to solve the strong Stieltjes moment problem. The sequence of functions $\{t^{-\lfloor (n+1)/2 \rfloor} \mathcal{Q}_n^\psi(t)\}$ is known as a sequence of orthogonal Laurent polynomials or orthogonal L-polynomials associated with the measure ψ (see [6]). Like the monic orthogonal polynomials, the polynomials $\mathcal{Q}_n^\psi$ are also unique and for convenience we refer to them as L-orthogonal polynomials.

It is known that the zeros of $\mathcal{Q}_n^\psi$ are all positive, distinct and lie within the interval (a, b). The zeros of $\mathcal{Q}_n^\psi$ also interlace with the zeros of $\mathcal{Q}_{n-1}^\psi$. Moreover, these polynomials satisfy the three term recurrence relation

$$\mathcal{Q}_{n+1}^\psi(w) = \big(w - \beta_{n+1}^\psi\big) \mathcal{Q}_n^\psi(w) - \alpha_{n+1}^\psi w \mathcal{Q}_{n-1}^\psi(w), \quad n \geqslant 1, \tag{1.3}$$

with $\mathcal{Q}_0^\psi(w) = 1$ and $\mathcal{Q}_1^\psi(w) = w - \beta_1^\psi$, $\beta_1^\psi = \mu_0^\psi / \mu_{-1}^\psi = \rho_0^\psi / \sigma_0^\psi$ and, for $n \geqslant 1$,

$$\beta_{n+1}^\psi = -\alpha_{n+1}^\psi \frac{\sigma_{n-1}^\psi}{\sigma_n^\psi} \quad \text{and} \quad \alpha_{n+1}^\psi = \frac{\rho_n^\psi}{\rho_{n-1}^\psi},$$

where $\rho_n^\psi = \int_a^b \mathcal{Q}_n^\psi(t)\,\mathrm{d}\psi(t)$ and $\sigma_n^\psi = \int_a^b t^{-n-1} \mathcal{Q}_n^\psi(t)\,\mathrm{d}\psi(t)$. The coefficients β_n^ψ, $n \geqslant 1$, and α_n^ψ, $n \geqslant 2$, are all positive. From (1.3) one can also note that $\mathcal{Q}_n^\psi(0) = (-1)^n \beta_1^\psi \beta_2^\psi \cdots \beta_n^\psi$.

Assuming different symmetric behaviors for the measure ψ, properties of the L-orthogonal polynomials $\{\mathcal{Q}_n^\psi\}$ have been explored in several articles.

We say that the strong positive measure ψ belongs to the symmetric class $S^3[\tau, \beta, b]$ if

$$\frac{\mathrm{d}\psi(t)}{t^\tau} = -\frac{\mathrm{d}\psi(\beta^2/t)}{(\beta^2/t)^\tau}, \quad t \in (a, b),$$

where $0 < \beta < b$, $a = \beta^2/b$ and $2\tau \in \mathbb{Z}$. The classification of the symmetry is according to the value of τ.

It seems that two of these symmetric classes, namely when $\tau = 1/2$ and $\tau = 0$, turn out to be more interesting than the others.

The L-orthogonal polynomials $\{\mathcal{Q}_n^{\psi}\}$, when the measure ψ belongs to the class $S^3[1/2, \beta, b]$, have been explored for example in [10]. In this case we can state the following results:

$$\frac{w^n \mathcal{Q}_n^{\psi}(\beta^2/w)}{\mathcal{Q}_n^{\psi}(0)} = \mathcal{Q}_n^{\psi}(w) \quad \text{and} \quad \beta_n^{\psi} = \beta \quad \text{for } n \geqslant 1.$$

Moreover, with the transformation $x(t) = \frac{1}{2\sqrt{\alpha}}(\sqrt{t} - \beta/\sqrt{t})$, where $\alpha > 0$, if the positive measure ϕ on the real interval $[-x(b), x(b)]$ is such that

$$\mathrm{d}\phi(x) = \frac{t+\beta}{2t}\,\mathrm{d}\psi(t), \tag{1.4}$$

then the orthogonal polynomials $\{Q_n^{\phi}\}$ are connected to the L-orthogonal polynomials $\{\mathcal{Q}_n^{\psi}\}$ in the following way:

$$Q_n^{\phi}\big(x(t)\big) = \big(2\sqrt{\alpha t}\big)^{-n}\mathcal{Q}_n^{\psi}(t) \quad \text{and} \quad a_{n+1}^{\phi} = \frac{1}{4\alpha}\alpha_{n+1}^{\psi} \quad \text{for } n \geqslant 1.$$

Since ϕ is symmetric, that is $\mathrm{d}\phi(-x) = -\mathrm{d}\phi(x)$, the orthogonal polynomials $\{Q_n^{\phi}\}$ are such that

$$Q_n^{\phi}(-z) = (-1)^n Q_n^{\phi}(z) \quad \text{and} \quad b_n^{\phi} = 0 \quad \text{for } n \geqslant 1.$$

Now the other interesting class $S^3[0, \beta, b]$ of strong positive measures is characterized by the symmetry

$$\mathrm{d}\psi(t) = -\mathrm{d}\psi\big(\beta^2/t\big), \quad t \in (a, b).$$

Here $0 < \beta < b \leqslant \infty$ and $a = \beta^2/b$. If $\mathrm{d}\psi(t) = h(t)\,\mathrm{d}t$ then the symmetry is given by

$$t\,h(t) = \big(\beta^2/t\big)\,h\big(\beta^2/t\big). \tag{1.5}$$

In this paper we look at a connection that exists between orthogonal polynomials associated with positive measures on the real line and L-orthogonal polynomials associated with strong measures of the class $S^3[0, \beta, b]$. We do this based on an idea similar to that of Szegő (see [13]) to connect orthogonal polynomials on $[-1, 1]$ to orthogonal polynomials on the unit circle. Examples given in Section 6 to illustrate the main contribution of this paper also lead to some new results.

We mention that, in [14], Vinet and Zhedanov have considered a connection between orthogonal polynomials associated with “symmetric” positive measures on the real line and Szegő polynomials on the real line using the transformation $x(t) = \lambda(\sqrt{t} + 1/\sqrt{t})$, where $\lambda > 0$. As we will observe in Section 2, the monic Szegő polynomials on the real line are the monic reciprocal polynomials of the L-orthogonal polynomials $\mathcal{Q}_n^{\psi}$ defined when the measure ψ is of the class $S^3[0, \beta, b]$.

As other attempts to connect orthogonal polynomials on the real line with orthogonal Laurent polynomials on the real line, we also mention the paper [5] of Hagler, Jones and Thron. However, none of these previous attempts have provided a method to obtain information on the orthogonal Laurent polynomials associated with the special $S^3[0, \beta, b]$ measures that we have considered in the first three examples of Section 6. We point out that the $S^3[0, \beta, b]$ measure given by Eq. (6.2) is, in particular, very interesting in the following sense.

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be any sequences of bounded positive numbers such that

$$\lim_{n\to\infty} \alpha_n = \alpha > 0 \quad \text{and} \quad \lim_{n\to\infty} \beta_n = \beta > 0.$$

Let the sequence of polynomials $\{\mathcal{B}_n\}$ be generated by

$$\mathcal{B}_{n+1}(w) = (w - \beta_{n+1})\mathcal{B}_n(w) - \alpha_{n+1} w \mathcal{B}_{n-1}(w), \quad n \geqslant 1,$$

with $\mathcal{B}_1(w) = w - \beta_1$ and $\mathcal{B}_0 = 1$. Then (see for example [12]),

$$\lim_{n\to\infty} \frac{1}{n} \frac{\mathcal{B}_n'(w)}{\mathcal{B}_n(w)} = \frac{1}{2\pi} \int_a^b \frac{1}{w-t} \frac{1+\beta/t}{\sqrt{(b-t)(t-a)}}\, \mathrm{d}t$$

holds uniformly on any compact subset of $\mathbb{C}\setminus(0,\infty)$, where $a = \beta^2/b$ and $b = (\sqrt{\alpha+\beta}+\sqrt{\alpha})^2$.

2. $S^3[0, \beta, b]$ measures and L-orthogonal polynomials

The L-orthogonal polynomials $\{\mathcal{Q}_n^\psi\}$, when ψ belongs to the class $S^3[0,\beta,b]$, have first appeared in Sri Ranga, Andrade and McCabe [11]. Some aspects of these polynomials were also considered simultaneously in Common and McCabe [3]. We can state (see [11]) the following results when $\psi \in S^3[0,\beta,b]$.

If f is integrable with respect to ψ, then

$$\int_{\beta^2/b}^{b} f(t)\, \mathrm{d}\psi(t) = \int_{\beta^2/b}^{b} f\bigl(\beta^2/t\bigr)\, \mathrm{d}\psi(t) \tag{2.1}$$

and, in particular, $\mu_n^\psi = \beta^{2n}\mu_{-n}^\psi$. For the associated L-orthogonal polynomials $\{\mathcal{Q}_n^\psi\}$ the coefficients in the three term recurrence relation (1.3) satisfy

$$\frac{\gamma_{n+1}^\psi}{\gamma_n^\psi} = \frac{\beta^2}{\beta_n^\psi \beta_{n+1}^\psi}, \quad n \geqslant 0, \tag{2.2}$$

where $\gamma_n^\psi = \beta_n^\psi + \alpha_{n+1}^\psi$, with $\beta_0^\psi = \gamma_0^\psi = 1$. Consequently,

$$\mathcal{Q}_n^\psi(0) = (-1)^n \beta_1^\psi \beta_2^\psi \cdots \beta_n^\psi = (-\beta)^n \sqrt{\beta_n^\psi/\gamma_n^\psi}, \quad n \geqslant 1. \tag{2.3}$$

Moreover, if

$$\tilde{\mathcal{Q}}_n^\psi(w) = \frac{w^n \mathcal{Q}_n^\psi(\beta^2/w)}{\mathcal{Q}_n^\psi(0)}, \quad n \geqslant 1,$$

then

$$\tilde{\mathcal{Q}}_n^\psi(w) = \mathcal{Q}_n^\psi(w) - \alpha_{n+1}^\psi \mathcal{Q}_{n-1}^\psi(w) = \bigl\{\mathcal{Q}_{n+1}^\psi(w) + \beta_{n+1}^\psi \mathcal{Q}_n^\psi(w)\bigr\}/w, \quad n \geqslant 1. \tag{2.4}$$

Now, letting $w = \beta$, and also letting $w = -\beta$, in (2.4) we obtain

$$\left[1 - \frac{\beta^n}{\mathcal{Q}_n^\psi(0)}\right]\mathcal{Q}_n^\psi(\beta) = \alpha_{n+1}^\psi \mathcal{Q}_{n-1}^\psi(\beta) \quad \text{and}$$
$$\left[1 - \frac{(-\beta)^n}{\mathcal{Q}_n^\psi(0)}\right]\mathcal{Q}_n^\psi(-\beta) = \alpha_{n+1}^\psi \mathcal{Q}_{n-1}^\psi(-\beta),$$

for $n \geqslant 1$. Thus, for example,

$$\mathcal{Q}_{2n}^{\psi}(\beta) = -\beta_{2n}^{\psi}\beta_{2n-1}^{\psi}\Big(\sqrt{\gamma_{2n}^{\psi}/\beta_{2n}^{\psi}}+1\Big)\Big(\sqrt{\gamma_{2n-1}^{\psi}/\beta_{2n-1}^{\psi}}-1\Big)\mathcal{Q}_{2n-2}^{\psi}(\beta), \quad n \geqslant 1,$$

$$\mathcal{Q}_{2n}^{\psi}(-\beta) = \beta_{2n}^{\psi}\beta_{2n-1}^{\psi}\Big(\sqrt{\gamma_{2n}^{\psi}/\beta_{2n}^{\psi}}+1\Big)\Big(\sqrt{\gamma_{2n-1}^{\psi}/\beta_{2n-1}^{\psi}}+1\Big)\mathcal{Q}_{2n-2}^{\psi}(-\beta), \quad n \geqslant 1. \tag{2.5}$$

Note that the monic polynomials $\tilde{\mathcal{Q}}_n^{\psi}$ satisfy the orthogonality (to be more precise the biorthogonality)

$$\int_a^b t^{-s}\tilde{\mathcal{Q}}_n^{\psi}(t)\,\mathrm{d}\psi(t) = 0, \quad s = 0, 1, \ldots, n-1. \tag{2.6}$$

If we define the reciprocal polynomials of $\tilde{\mathcal{Q}}_n^{\psi}$ by $\tilde{\mathcal{Q}}_n^{\psi*}(w) = (w/\beta^2)^n\tilde{\mathcal{Q}}_n^{\psi}(\beta^2/w)$, then $\tilde{\mathcal{Q}}_n^{\psi*}(w) = \mathcal{Q}_n^{\psi}(w)/\mathcal{Q}_n^{\psi}(0)$, $n \geqslant 1$, and from (2.4),

$$\tilde{\mathcal{Q}}_n^{\psi}(w) = w\tilde{\mathcal{Q}}_{n-1}^{\psi}(w) + \tilde{\delta}_n\tilde{\mathcal{Q}}_{n-1}^{\psi*}(w), \quad n \geqslant 1, \tag{2.7}$$

where $\tilde{\delta}_n = \frac{\beta^{2n}}{\mathcal{Q}_n^{\psi}(0)} = (-\beta)^n\sqrt{\gamma_n^{\psi}/\beta_n^{\psi}}$ for $n \geqslant 1$.

Note that, if the integration in (2.6) is done over the unit circle then t^{-1} can be replaced by the conjugate of t. Hence, based on (2.6) and (2.7), the polynomials $\tilde{\mathcal{Q}}_n^{\psi}$ can be called the monic Szegő polynomials on the positive interval $[a, b]$. The numbers $\tilde{\delta}_n = \tilde{\mathcal{Q}}_n^{\psi}(0)$, which play the role of reflection coefficients, satisfy

$$|\tilde{\delta}_n| > \beta^n \quad \text{for } n \geqslant 1.$$

Like the zeros of $\mathcal{Q}_n^{\psi}$ the zeros of $\tilde{\mathcal{Q}}_n^{\psi}$ also lie within the interval (a, b), here with $a = \beta^2/b$. Hence it is important that $|\tilde{\delta}_n| > \beta^n$ (see Vinet and Zhedanov [14]). Otherwise, if $|\tilde{\delta}_n| < \beta^n$, then (just as the scaled Szegő polynomials orthogonal on the circle of radius β) the zeros of $\tilde{\mathcal{Q}}_n^{\psi}$ would lie within the disk $|w| < \beta$.

To be able to talk about Szegő polynomials on a positive interval $[a, b]$ it is important that the measure is of the class $S^3[0, \beta, b]$, with $a = \beta^2/b$. Otherwise the monic polynomials $\{\tilde{\mathcal{Q}}_n^{\psi}\}$ defined on $[a, b]$ by (2.6) do not satisfy (2.7). For example, if the measure ψ is of the class $S^3[-1/2, \beta, b]$, then for the monic polynomials $\{\tilde{\mathcal{Q}}_n^{\psi}\}$ defined by (2.6) we would have, instead of (2.7),

$$\tilde{\mathcal{Q}}_n^{\psi}(w) = \tilde{\delta}_n\tilde{\mathcal{Q}}_n^{\psi*}(w) = w\tilde{\mathcal{Q}}_{n-1}^{\psi}(w) + \tilde{\delta}_n\tilde{\mathcal{Q}}_{n-1}^{\psi*}(w) - \tilde{\alpha}_n w\tilde{\mathcal{Q}}_{n-2}^{\psi}(w),$$

for $n \geqslant 2$, where $\tilde{\delta}_n = \tilde{\mathcal{Q}}_n^{\psi}(0)$ and $\tilde{\alpha}_n > 0$.

3. Hyperbolic Szegő transformation

With $\beta > 0$, the transformation given by

$$z = T(w) = \frac{1}{2}\left(\frac{w}{\beta} + \frac{\beta}{w}\right),$$

may be called a generalized Joukowski transformation. It is well known that this transformation maps the semi-circular region $\{w = \beta e^{i\theta}\colon\ 0 \leqslant \theta \leqslant \pi\}$ onto the interval $[-1, 1]$ (likewise the semi-circular region $\{w = \beta e^{i\theta}\colon -\pi \leqslant \theta \leqslant 0\}$ onto the interval $[-1, 1]$). In this special case the transformation is also known as the Szegő transformation.

Here, we consider another special case of the transformation $T(w)$ which maps the interval $[\beta, \infty)$ onto the interval $[1, \infty)$. Also the interval $(0, \beta]$ onto the interval $[1, \infty)$. We call this the Hyperbolic Szegő Transformation and write

$$x = x(t) = \frac{1}{2}\left(\frac{t}{\beta} + \frac{\beta}{t}\right),$$

for $t \in [\beta^2/b, b]$ and $x \in [1, d]$, with $d = \frac{1}{2}(b/\beta + \beta/b)$. Interestingly, we can consider this transformation in two stages.

Stage 1: $\theta = \ln(\frac{t}{\beta})$.

This is a mapping from $[\beta^2/b, b]$ onto $[-\vartheta, \vartheta]$, where $\vartheta = \ln(b/\beta)$. Note that θ is a strictly increasing function of t. The inverse of this mapping is $t = \beta e^{\theta}$.

Stage 2: $x = \cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$, hence the use of the word Hyperbolic.

This can be considered as a mapping from $[0, \vartheta]$ onto $[1, d]$, where $d = \cosh\vartheta = \frac{1}{2}(b/\beta + \beta/b)$. Here x is a strictly increasing function of θ. The inverse of this mapping is $\theta = \ln(x + \sqrt{x^2 - 1})$, with $\ln(b/\beta) = \vartheta = \ln(d + \sqrt{d^2 - 1})$.

This can also be considered as a mapping from $[-\vartheta, 0]$ onto $[1, d]$. Now x is a strictly decreasing function of θ. The inverse of this mapping is $\theta = \ln(x - \sqrt{x^2 - 1})$, with $\ln(\beta/b) = -\vartheta = \ln(d - \sqrt{d^2 - 1})$.

Now if we consider the integral $I = \int_1^d f(x)\,\mathrm{d}\phi(x)$, where ϕ is any positive measure on $[1, d]$, then we can write

$$2I = \int_d^1 f(x)\bigl[-\mathrm{d}\phi(x)\bigr] + \int_1^d f(x)\,\mathrm{d}\phi(x).$$

Here, applying Stage 2 of our transformation, we obtain

$$2I = \int_{-\vartheta}^{0} f(\cosh\theta)\bigl[-\mathrm{d}\phi(\cosh\theta)\bigr] + \int_0^{\vartheta} f(\cosh\theta)\,\mathrm{d}\phi(\cosh\theta). \tag{3.1}$$

Applying Stage 1 of our transformation, where $e^{\theta} = t/\beta$, then gives

$$2I = \int_{\beta^2/b}^{\beta} f\bigl(x(t)\bigr)\bigl[-\mathrm{d}\phi\bigl(x(t)\bigr)\bigr] + \int_{\beta}^{b} f\bigl(x(t)\bigr)\,\mathrm{d}\phi\bigl(x(t)\bigr).$$

Hence,

$$\int_1^d f(x)\,\mathrm{d}\phi(x) = \int_{\beta^2/b}^{b} f\bigl(x(t)\bigr)\,\mathrm{d}\psi(t), \tag{3.2}$$

where $\mathrm{d}\psi(t) = \frac{1}{2}\,\mathrm{d}\phi(x(t))$ for $t \in (\beta, b)$ and $\mathrm{d}\psi(t) = -\frac{1}{2}\,\mathrm{d}\phi(x(t))$ for $t \in (\beta^2/b, \beta)$. Clearly, $\mu_0^{\psi} = \mu_0^{\phi}$. Moreover, one can verify that ψ is a strong positive measure on $(\beta^2/b, b)$ and that $\psi \in S^3(0, \beta, b)$.

In particular, if $\mathrm{d}\phi(x) = k(x)\,\mathrm{d}x$ then (3.1) and (3.2) can be given in the form

$$\int_1^d f(x)k(x)\,\mathrm{d}x = \frac{1}{2}\int_{-\vartheta}^{\vartheta} f(\cosh\theta)k(\cosh\theta)|\sinh\theta|\,\mathrm{d}\theta = \int_{\beta^2/b}^{b} f\big(x(t)\big)h(t)\,\mathrm{d}t,$$

where

$$h(t) = \frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = \frac{1}{4t}\left|\frac{t}{\beta} - \frac{\beta}{t}\right| k\big(x(t)\big) \quad \text{for } t \in \big(\beta^2/b, b\big). \tag{3.3}$$

The following lemma gives information on some auxiliary polynomials used in the next two sections.

Lemma 3.1. *Let* $x = \cosh\theta$. *Let*

$$T_n(x) = \cosh n\theta \quad \textit{and} \quad U_n(x) = \frac{\sinh(n+1)\theta}{\sinh\theta}, \quad n \geqslant 0.$$

Then the following hold:

- $T_0(x) = 1$ *and, for any* $n \geqslant 1$, T_n *is a polynomial of degree n with leading coefficient* 2^{n-1}. *Precisely,* $T_1(x) = x$, $T_2(x) = 2x^2 - 1$ *and, for* $n = 1, 2, \ldots,$

$$T_{2n+1}(x) = 2^{2n}x^{2n+1} - \sum_{j=1}^{n}\binom{2n+1}{j}T_{2n+1-2j}(x),$$

$$T_{2n+2}(x) = 2^{2n+1}x^{2n+2} - \sum_{j=1}^{n}\binom{2n+2}{j}T_{2n+2-2j}(x) - \frac{1}{2}\binom{2n+2}{n+1}.$$

- *For any* $n \geqslant 0$, U_n *is a polynomial of degree n with leading coefficient* 2^n. *Precisely,* $U_0(x) = 1$ *and, for* $n = 1, 2, \ldots,$

$$U_{2n-1}(x) = 2\sum_{j=0}^{n-1}T_{2n-1-2j}(x) \quad \textit{and} \quad U_{2n}(x) = 1 + 2\sum_{j=0}^{n-1}T_{2n-2j}(x).$$

Proof. Using the binomial expansion

$$(2x)^{2n} = \big(e^{\theta} + e^{-\theta}\big)^{2n} = \sum_{j=0}^{2n}\binom{2n}{j}e^{(2n-2j)\theta}.$$

Since $\binom{2n}{j} = \binom{2n}{2n-j}$,

$$\begin{aligned}(2x)^{2n} &= \binom{2n}{n} + \sum_{j=0}^{n-1}\binom{2n}{j}\big[e^{(2n-2j)\theta} + e^{-(2n-2j)\theta}\big]\\ &= \binom{2n}{n} + 2\sum_{j=0}^{n-1}\binom{2n}{j}T_{2n-2j}(x).\end{aligned}$$

This completes the proof of the result for T_{2n}. Similarly, considering the expansions of $(e^{\theta} + e^{-\theta})^{2n+1}$ and $(e^{(n+1)\theta} - e^{-(n+1)\theta})/(e^{\theta} - e^{-\theta})$, we obtain the results for T_{2n+1} and U_n. □

The polynomials $\{T_n\}$ and $\{U_n\}$ are, respectively, the Chebyshev polynomials of the first and second kind. That is, within the interval $[-1,1]$ these polynomials can be given by $T_n(x) = \cos n\theta$ and $U_n(x) = \sin(n+1)\theta/\sin\theta$ with $x = \cos\theta$.

Note that T_n and U_n are such that

$$\begin{aligned} T_n\big(x(t)\big) &= \frac{1}{2}\big[(t/\beta)^n + (\beta/t)^n\big], \quad n \geqslant 0, \\ U_n\big(x(t)\big) &= \big[(t/\beta)^{n+1} - (\beta/t)^{n+1}\big]\big/\big[(t/\beta) - (\beta/t)\big], \quad n \geqslant 0. \end{aligned} \tag{3.4}$$

4. Orthogonal polynomials and L-orthogonal polynomials

Let $0 < \beta < b$ and $\psi \in S^3(0,\beta,b)$. Let us consider the functions $\{R_n^{(1)}\}$ given by

$$\lambda_n^{(1)} R_n^{(1)}(t) = \left(\frac{1}{t}\right)^n \mathcal{Q}_{2n}^{\psi}(t) + \left(\frac{t}{\beta^2}\right)^n \mathcal{Q}_{2n}^{\psi}\left(\frac{\beta^2}{t}\right), \quad n \geqslant 0,$$

where $\lambda_n^{(1)} = (2\beta)^n\big[1 + \sqrt{\beta_{2n}^{\psi}/\gamma_{2n}^{\psi}}\big]$. Using (2.4) one can also show that

$$R_n^{(1)}(t) = -\frac{\sqrt{\beta_{2n}^{\psi}\gamma_{2n}^{\psi}}}{(2\beta)^n}\left[\left(\frac{1}{t}\right)^n \mathcal{Q}_{2n-1}^{\psi}(t) + \left(\frac{t}{\beta^2}\right)^n \mathcal{Q}_{2n-1}^{\psi}\left(\frac{\beta^2}{t}\right)\right], \quad n \geqslant 1.$$

Theorem 4.1. *Let $d = x(b)$ and let the positive measure ϕ_1 defined on the interval $[1,d]$ be such that*

$$\mathrm{d}\phi_1\big(x(t)\big) = 2\,\mathrm{d}\psi(t) \quad \textit{for } t \in [\beta, b].$$

Let the sequence of functions $\{P_n^{(1)}\}$ be such that

$$P_n^{(1)}\big(x(t)\big) = R_n^{(1)}(t) \quad \textit{for } n \geqslant 1.$$

Then for any $n \geqslant 1$, $P_n^{(1)}(x) = Q_n^{\phi_1}(x)$, the monic orthogonal polynomial of degree n associated with the positive measure ϕ_1 on $[1,d]$. Moreover, $\mu_0^{\phi_1} = \mu_0^{\psi}$, $b_1^{\phi_1} = \delta_1^{\psi}$ and, for $n \geqslant 1$,

$$\begin{aligned} b_{n+1}^{\phi_1} &= \frac{1}{2}\delta_{2n+1}^{\psi}\big(\delta_{2n}^{\psi} + 1\big) + \frac{1}{2}\delta_{2n-1}^{\psi}\big(\delta_{2n}^{\psi} - 1\big), \\ a_{n+1}^{\phi_1} &= \frac{1}{4}\big(\delta_{2n-2}^{\psi} + 1\big)\big(\delta_{2n-1}^{\psi} + 1\big)\big(\delta_{2n-1}^{\psi} - 1\big)\big(\delta_{2n}^{\psi} - 1\big), \end{aligned} \tag{4.1}$$

where $\delta_n^{\psi} = \sqrt{\gamma_n^{\psi}/\beta_n^{\psi}}$. Furthermore,

$$\delta_{2n-1}^{\psi} = \frac{S_n^{\phi_1}(-1) + S_n^{\phi_1}(1)}{S_n^{\phi_1}(-1) - S_n^{\phi_1}(1)} \quad \textit{and} \quad \delta_{2n}^{\psi} = S_{n+1}^{\phi_1}(1) - S_{n+1}^{\phi_1}(-1) - 1 \tag{4.2}$$

for $n \geqslant 1$, where $S_n^{\phi_1}(z) = Q_n^{\phi_1}(z)/Q_{n-1}^{\phi_1}(z)$.

Proof. First we show that $P_n^{(1)}$ is a monic polynomial of degree n. Letting $\mathcal{Q}_n^{\psi}(t) = \sum_{r=0}^{n} q_{n,r}t^r$, we obtain

$$\lambda_n^{(1)} R_n^{(1)}(t) = \sum_{r=0}^{2n} q_{2n,r}\beta^{-n+r}\left[\left(\frac{t}{\beta}\right)^{n-r} + \left(\frac{\beta}{t}\right)^{n-r}\right]$$
$$= 2q_{2n,n} + \sum_{r=0}^{n-1}\bigl(q_{2n,r}\beta^{-n+r} + q_{2n,2n-r}\beta^{n-r}\bigr)\left[\left(\frac{t}{\beta}\right)^{n-r} + \left(\frac{\beta}{t}\right)^{n-r}\right].$$

Hence, from (3.4),

$$\lambda_n^{(1)} P_n^{(1)}\bigl(x(t)\bigr) = 2q_{2n,n} + 2\sum_{r=0}^{n-1}\bigl(q_{2n,r}\beta^{-n+r} + q_{2n,2n-r}\beta^{n-r}\bigr)T_{n-r}\bigl(x(t)\bigr). \tag{4.3}$$

Since T_r, $r \geqslant 1$, is a polynomial of degree r in x with leading coefficient 2^{r-1}, $P_n^{(1)}$ is a polynomial of degree n in x and its leading coefficient is

$$\frac{2^n(q_{2n,0}\beta^{-n} + q_{2n,2n}\beta^n)}{\lambda_n^{(1)}} = \frac{2^n(\mathcal{Q}_{2n}^{\psi}(0)\beta^{-n} + \beta^n)}{(2\beta)^n[1 + \sqrt{\beta_{2n}^{\psi}/\gamma_{2n}^{\psi}}]}.$$

Using (2.3) one can easily verify this to be equal to 1.

Now we establish the orthogonality property of $\{P_n^{(1)}\}$. We will show that

$$I_{n,k}^{(1)} = \int_1^d T_k(x)P_n^{(1)}(x)\,\mathrm{d}\phi_1(x) = 0 \quad \text{for } k = 0, 1, \ldots, n-1,\ n \geqslant 1. \tag{4.4}$$

From (3.2),

$$I_{n,k}^{(1)} = \int_{\beta^2/b}^{b} T_k\bigl(x(t)\bigr)P_n^{(1)}\bigl(x(t)\bigr)\,\mathrm{d}\psi(t) = \frac{1}{2}\int_{\beta^2/b}^{b}\left[\left(\frac{t}{\beta}\right)^k + \left(\frac{\beta}{t}\right)^k\right]R_n^{(1)}(t)\,\mathrm{d}\psi(t).$$

We write

$$2\lambda_n^{(1)} I_{n,k}^{(1)} = \int_{\beta^2/b}^{b}\left[\left(\frac{t}{\beta}\right)^k + \left(\frac{\beta}{t}\right)^k\right]\left(\frac{1}{t}\right)^n \mathcal{Q}_{2n}^{\psi}(t)\,\mathrm{d}\psi(t)$$
$$+ \int_{\beta^2/b}^{b}\left[\left(\frac{t}{\beta}\right)^k + \left(\frac{\beta}{t}\right)^k\right]\left(\frac{t}{\beta^2}\right)^n \mathcal{Q}_{2n}^{\psi}\left(\frac{\beta^2}{t}\right)\mathrm{d}\psi(t).$$

Using (2.1) on the second integral, we obtain

$$\lambda_n^{(1)} I_{n,k}^{(1)} = \int_{\beta^2/b}^{b}\left[\left(\frac{t}{\beta}\right)^k + \left(\frac{\beta}{t}\right)^k\right]\left(\frac{1}{t}\right)^n \mathcal{Q}_{2n}^{\psi}(t)\,\mathrm{d}\psi(t). \tag{4.5}$$

Thus, from (1.2), $I_{n,k}^{(1)} = 0$ for $k = 0, 1, \ldots, n-1$. This proves the first part of the theorem.

We also have from (4.5),

$$I_{n,n}^{(1)} = \frac{\mu_0^{\psi}\alpha_2^{\psi}\alpha_3^{\psi}\cdots\alpha_{2n+1}^{\psi}}{2^n\beta^{2n}[1 + \sqrt{\beta_{2n}^{\psi}/\gamma_{2n}^{\psi}}]}, \quad n \geqslant 1.$$

On the other hand, since the leading coefficient of T_n is 2^{n-1}, we obtain from (4.4),

$$I^{(1)}_{n,n} = 2^{n-1}\mu_0^{\phi_1} a_2^{\phi_1} a_3^{\phi_1}\cdots a_{n+1}^{\phi_1}, \quad n \geqslant 1.$$

Considering the ratios $I^{(1)}_{n,n}/I^{(1)}_{n-1,n-1}$ we are led to

$$a_2^{\phi_1} = \frac{1}{1+\sqrt{\beta_2^{\psi}/\gamma_2^{\psi}}}\frac{\alpha_2^{\psi}\alpha_3^{\psi}}{2\beta^2} \quad \text{and} \quad a_{n+1}^{\phi_1} = \frac{1+\sqrt{\beta_{2n-2}^{\psi}/\gamma_{2n-2}^{\psi}}}{1+\sqrt{\beta_{2n}^{\psi}/\gamma_{2n}^{\psi}}}\frac{\alpha_{2n}^{\psi}\alpha_{2n+1}^{\psi}}{4\beta^2}, \quad n \geqslant 2.$$

Hence from (2.2), Eq. (4.1) for $a_n^{\phi_1}$ can be obtained.

Now we compare the coefficients of degree $n-1$ on both sides of the expression (4.3) and obtain

$$-\sum_{r=1}^{n} b_r^{\phi_1} = \frac{2^{n-1}(q_{2n,1}\beta^{-n+1} + q_{2n,2n-1}\beta^{n-1})}{(2\beta)^n[1+\sqrt{\beta_{2n}^{\psi}/\gamma_{2n}^{\psi}}]}, \quad n \geqslant 1.$$

The left-hand side follows from the three term recurrence relation (1.1). We need to find expressions for $q_{2n,1}$ and $q_{2n,2n-1}$. From the three term recurrence relation (1.3), $q_{n+1,n} = -\beta_{n+1}^{\psi} - \alpha_{n+1}^{\psi} + q_{n,n-1}$ for $n \geqslant 1$. Thus,

$$q_{2n,2n-1} - \alpha_{2n+1}^{\psi} = -\sum_{r=1}^{2n}\gamma_r^{\psi}, \quad n \geqslant 1.$$

From (2.4),

$$q_{2n,2n-1} - \alpha_{2n+1}^{\psi} = \frac{\beta^2 q_{2n,1}}{\mathcal{Q}_{2n}^{\psi}(0)}, \quad n \geqslant 1.$$

Therefore,

$$\sum_{r=1}^{n} b_r^{\phi_1} = \frac{1}{2\beta}\sum_{r=1}^{2n}\gamma_r^{\psi} - \frac{\alpha_{2n+1}^{\psi}}{2\beta[1+\sqrt{\beta_{2n}^{\psi}/\gamma_{2n}^{\psi}}]} = \frac{1}{2\beta}\sum_{r=1}^{2n-1}\gamma_r^{\psi} + \frac{1}{2\beta}\sqrt{\beta_{2n}^{\psi}\gamma_{2n}^{\psi}}, \quad n \geqslant 1.$$

From this and (2.2), Eq. (4.1) for $b_n^{\phi_1}$ follows.

Since $x(\beta) = 1$ and $x(-\beta) = -1$, we obtain from $Q_n^{\phi_1}(x(t)) = R_n^{(1)}(t)$,

$$Q_n^{\phi_1}(1) = \frac{2\mathcal{Q}_{2n}^{\psi}(\beta)}{\lambda_n^{(1)}\beta^n} \quad \text{and} \quad Q_n^{\phi_1}(-1) = \frac{2\mathcal{Q}_{2n}^{\psi}(-\beta)}{\lambda_n^{(1)}(-\beta)^n}, \quad n \geqslant 1.$$

Hence, with $S_n^{\phi_1}(z) = Q_n^{\phi_1}(z)/Q_{n-1}^{\phi_1}(z)$,

$$S_n^{\phi_1}(1) = -\frac{1}{2}\big(\delta_{2n-1}^{\psi} - 1\big)\big(\delta_{2n-2}^{\psi} + 1\big), \quad n \geqslant 1,$$
$$S_n^{\phi_1}(-1) = -\frac{1}{2}\big(\delta_{2n-1}^{\psi} + 1\big)\big(\delta_{2n-2}^{\psi} + 1\big), \quad n \geqslant 1.$$

Therefore we obtain (4.2) and this completes the proof of the theorem. □

We remind that $\mathcal{Q}_n^{\psi}(0) = (-\beta)^n/\delta_n^{\psi}$, $n \geqslant 1$, and

$$\beta_n^{\psi} = \beta\delta_{n-1}^{\psi}/\delta_n^{\psi}, \quad \alpha_{n+1}^{\psi} = \beta(\delta_n^{\psi} - 1)(\delta_n^{\psi} + 1)\delta_{n-1}^{\psi}/\delta_n^{\psi}, \quad n \geqslant 1, \tag{4.6}$$

with $\delta_0^{\psi} = 1$, which follow from (2.2) and (2.3). Thus, for $n \geqslant 1$,

$$\int_{\beta^2/b}^{b} \mathcal{Q}_n^{\psi}(t)\,\mathrm{d}\psi(t) = \mu_0^{\psi} \frac{\beta^n}{\delta_n^{\psi}} \prod_{j=1}^{n} \big[\big(\delta_j^{\psi} - 1\big)\big(\delta_j^{\psi} + 1\big)\big],$$

$$\int_{\beta^2/b}^{b} t^{-n-1} \mathcal{Q}_n^{\psi}(t)\,\mathrm{d}\psi(t) = (-1)^n \mu_0^{\psi} \frac{\delta_{n+1}^{\psi}}{\beta\delta_n^{\psi}} \prod_{j=1}^{n} \big[\big(\delta_j^{\psi} - 1\big)\big(\delta_j^{\psi} + 1\big)\big].$$

5. Companion polynomials of $\{Q_n^{\phi_1}\}$ and further identities

We now consider the functions $\{R_n^{(2)}\}$ given by

$$\lambda_n^{(2)} R_n^{(2)}(t) = \left[\left(\frac{1}{t}\right)^{n+1} \mathcal{Q}_{2n+2}^{\psi}(t) - \left(\frac{t}{\beta^2}\right)^{n+1} \mathcal{Q}_{2n+2}^{\psi}\left(\frac{\beta^2}{t}\right)\right] \Big/ \big[t - \beta^2/t\big], \quad n \geqslant 0,$$

where $\lambda_n^{(2)} = (2\beta)^n [1 - \sqrt{\beta_{2n+2}^{\psi}/\gamma_{2n+2}^{\psi}}]$. Again from (2.4),

$$R_n^{(2)}(t) = \frac{\sqrt{\beta_{2n+2}^{\psi}\gamma_{2n+2}^{\psi}}}{(2\beta)^n} \left[\left(\frac{1}{t}\right)^{n+1} \mathcal{Q}_{2n+1}^{\psi}(t) - \left(\frac{t}{\beta^2}\right)^{n+1} \mathcal{Q}_{2n+1}^{\psi}\left(\frac{\beta^2}{t}\right)\right] \Big/ \big[t - \beta^2/t\big],$$

for $n \geqslant 0$.

Theorem 5.1. *Let $d = x(b)$ and let the positive measure ϕ_2 defined on the interval $[1, d]$ be such that*

$$\frac{1}{x(t)^2 - 1}\,\mathrm{d}\phi_2\big(x(t)\big) = \mathrm{d}\phi_1\big(x(t)\big) = 2\,\mathrm{d}\psi(t) \quad \text{for } t \in [\beta, b].$$

Let the sequence of functions $\{P_n^{(2)}\}$ be such that

$$P_n^{(2)}\big(x(t)\big) = R_n^{(2)}(t) \quad \text{for } n \geqslant 0.$$

Then for $n \geqslant 0$, $P_n^{(2)}(x) = Q_n^{\phi_2}(x)$, the monic orthogonal polynomial of degree n associated with the positive measure ϕ_2 on $[1, d]$. Moreover, $\mu_0^{\phi_2} = \frac{1}{2}(\delta_1^{\psi} - 1)(\delta_1^{\psi} + 1)(\delta_2^{\psi} + 1)\mu_0^{\psi}$ and, for $n \geqslant 1$,

$$b_n^{\phi_2} = \frac{1}{2}\delta_{2n+1}^{\psi}\big(\delta_{2n}^{\psi} - 1\big) + \frac{1}{2}\delta_{2n-1}^{\psi}\big(\delta_{2n}^{\psi} + 1\big),$$
$$a_{n+1}^{\phi_2} = \frac{1}{4}\big(\delta_{2n}^{\psi} - 1\big)\big(\delta_{2n+1}^{\psi} - 1\big)\big(\delta_{2n+1}^{\psi} + 1\big)\big(\delta_{2n+2}^{\psi} + 1\big). \tag{5.1}$$

Proof. To prove $P_n^{(2)}(x) = Q_n^{\phi_2}(x)$, apart from establishing that $P_n^{(2)}(x(t)) = R_n^{(2)}(t)$ is a polynomial of degree n in x, one needs to prove that

$$I_{n,k}^{(2)} = \int_1^d U_k(x) P_n^{(2)}(x)\,\mathrm{d}\phi_2(x) = 0 \quad \text{for } k = 0, 1, \ldots, n-1,\ n \geqslant 1.$$

This is done in a way similar to that of Theorem 4.1.

Having proved $P_n^{(2)}(x) = Q_n^{\phi_2}(x)$, from $I_{n,n}^{(2)}$ we have

$$\int_1^d U_n(x) Q_n^{\phi_2}(x)\, d\phi_2(x) = \frac{1}{(2\beta^2)^{n+1}\big[1 - \sqrt{\beta_{2n+2}^{\psi}/\gamma_{2n+2}^{\psi}}\big]} \int_{\beta^2/b}^{b} Q_{2n+2}^{\psi}(t)\, d\psi(t),$$

for $n \geqslant 0$. From this, $\mu_0^{\phi_2} = \frac{1}{2\beta^2[1-\sqrt{\beta_2^{\psi}/\gamma_2^{\psi}}]} \mu_0^{\psi} \alpha_2^{\psi} \alpha_3^{\psi}$ and

$$2^n \mu_0^{\phi_2} a_2^{\phi_2} \cdots a_{n+1}^{\phi_2} = \frac{1}{(2\beta^2)^{n+1}\big[1 - \sqrt{\beta_{2n+2}^{\psi}/\gamma_{2n+2}^{\psi}}\big]} \mu_0^{\psi} \alpha_2^{\psi} \cdots \alpha_{2n+2}^{\psi} \alpha_{2n+3}^{\psi}, \quad n \geqslant 1.$$

Consequently the equation for $a_n^{\phi_2}$ in (5.1) follows.

Comparing the coefficients of x^{n-1} on both sides of the equation $Q_n^{\phi_2}(x(t)) = R_n^{(2)}(t)$, the equation for $b_n^{\phi_2}$ in (5.1) also follows. □

Now, from (4.1) and (5.1),

$$b_{n+1}^{\phi_1} - b_n^{\phi_2} = \delta_{2n+1}^{\psi} - \delta_{2n-1}^{\psi}, \quad n \geqslant 1,$$

with $b_1^{\phi_1} = \delta_1^{\psi}$. Thus, we can write

$$\delta_1^{\psi} = b_1^{\phi_1} \quad \text{and} \quad \delta_{2n+1}^{\psi} = \sum_{r=1}^{n+1} b_r^{\phi_1} - \sum_{r=1}^{n} b_r^{\phi_2}, \quad n \geqslant 1.$$

Also, from (4.1) and (5.1),

$$\frac{a_2^{\phi_1}}{\mu_0^{\phi_2}} = \frac{\delta_2^{\psi} + 1}{\delta_2^{\psi} - 1} \mu_0^{\psi} \quad \text{and} \quad \frac{a_{n+1}^{\phi_1}}{a_n^{\phi_2}} = \frac{\delta_{2n-2}^{\psi} + 1}{\delta_{2n-2}^{\psi} - 1} \frac{\delta_{2n}^{\psi} - 1}{\delta_{2n}^{\psi} + 1}, \quad n \geqslant 2.$$

Since $\mu_0^{\phi_1} = \mu_0^{\psi}$,

$$\frac{\delta_2^{\psi} - 1}{\delta_2^{\psi} + 1} = \frac{a_2^{\phi_1} \mu_0^{\phi_1}}{\mu_0^{\phi_2}} \quad \text{and} \quad \frac{\delta_{2n+2}^{\psi} - 1}{\delta_{2n+2}^{\psi} + 1} = \frac{a_{n+2}^{\phi_1} \cdots a_3^{\phi_1} a_2^{\phi_1} \mu_0^{\phi_1}}{a_{n+1}^{\phi_2} \cdots a_2^{\phi_2} \mu_0^{\phi_2}}, \quad n \geqslant 1.$$

Thus, $\delta_2^{\psi} = \frac{\mu_0^{\phi_2} + a_2^{\phi_1} \mu_0^{\phi_1}}{\mu_0^{\phi_2} - a_2^{\phi_1} \mu_0^{\phi_1}}$ and

$$\delta_{2n+2}^{\psi} = \frac{a_{n+1}^{\phi_2} \cdots a_2^{\phi_2} \mu_0^{\phi_2} + a_{n+2}^{\phi_1} a_{n+1}^{\phi_1} \cdots a_2^{\phi_1} \mu_0^{\phi_1}}{a_{n+1}^{\phi_2} \cdots a_2^{\phi_2} \mu_0^{\phi_2} - a_{n+2}^{\phi_1} a_{n+1}^{\phi_1} \cdots a_2^{\phi_1} \mu_0^{\phi_1}}, \quad n \geqslant 1.$$

6. Examples

We give four examples and in the first three of them we start from a known set of orthogonal polynomials and derive information on the connected set of L-orthogonal polynomials. In the last example, we consider a known set of L-orthogonal polynomials and obtain information about the connected set of orthogonal polynomials.

6.1. L-orthogonal polynomials from the Chebyshev polynomials

We consider the sequence of orthogonal polynomials $\{Q_n^{\phi_1}\}$ given by

$$Q_n^{\phi_1}(x) = c_n T_n\left(\frac{2x}{d-1} - \frac{d+1}{d-1}\right), \quad n \geqslant 1,$$

where T_n are the Chebyshev polynomials (see, for example, [1,13]). The constants c_n are chosen so that $Q_n^{\phi_1}$ are monic polynomials. The following results are easily verified:

$$\mu_0^{\phi_1} = \int_1^d \frac{1}{\pi\sqrt{(d-x)(x-1)}}\,\mathrm{d}x = 1 \quad \text{and}$$

$$\int_1^d Q_n^{\phi_1}(x) Q_m^{\phi_1}(x) \frac{1}{\pi\sqrt{(d-x)(x-1)}}\,\mathrm{d}x = \delta_{n,m}\left(\frac{d-1}{4}\right)^{2n} 2, \quad n \geqslant 1,\ m \geqslant 0.$$

The three term recurrence relation for $Q_n^{\phi_1}$ is

$$Q_2^{\phi_1}(x) = \left(x - \frac{1}{2}(d+1)\right) Q_1^{\phi_1}(x) - \frac{1}{8}(d-1)^2 Q_0^{\phi_1}(x),$$
$$Q_{n+1}^{\phi_1}(x) = \left(x - \frac{1}{2}(d+1)\right) Q_n^{\phi_1}(x) - \frac{1}{16}(d-1)^2 Q_{n-1}^{\phi_1}(x), \quad n \geqslant 2,$$

with $Q_1^{\phi_1}(x) = (x - \frac{d+1}{2})$ and $Q_0^{\phi_1}(x) = 1$. Moreover, for $n \geqslant 1$,

$$2^n Q_n^{\phi_1}(x) = \left[\left(x - \frac{d+1}{2}\right) + \sqrt{(d-x)(1-x)}\right]^n + \left[\left(x - \frac{d+1}{2}\right) - \sqrt{(d-x)(1-x)}\right]^n. \tag{6.1}$$

Hence, from Theorem 4.1, with the use of (3.3) we can consider the L-orthogonal polynomials $\{\mathcal{Q}_n^{\psi}\}$ associated with the $S^3(0,\beta,b)$ measure ψ given by

$$\mathrm{d}\psi(t) = \frac{1}{2\pi} \frac{1+\beta/t}{\sqrt{(b-t)(t-a)}}\,\mathrm{d}t, \tag{6.2}$$

where $b = \beta(d + \sqrt{d^2-1})$ and $a = \beta^2/b$.

Theorem 6.1. *Let ψ be the $S^3(0,\beta,b)$ measure given by* (6.2)*. Then for the associated L-orthogonal polynomials $\{\mathcal{Q}_n^{\psi}\}$ we have that the parameters $\delta_n^{\psi} = (-\beta)^n/\mathcal{Q}_n^{\psi}(0)$ satisfy*

$$\delta_{2n-1}^{\psi} = \frac{[1+\eta]}{[1-\eta]} \frac{[1+\eta^{2n-1}]}{[1-\eta^{2n-1}]}, \qquad \delta_{2n}^{\psi} = \frac{[1+\eta]}{[1-\eta]} \frac{[1-\eta^{2n}]}{[1+\eta^{2n}]}, \quad n \geqslant 1, \tag{6.3}$$

where $\eta = [\sqrt{b} - \sqrt{\beta}]^2/[\sqrt{b} + \sqrt{\beta}]^2$. Consequently, in the three term recurrence relation (1.3) *satisfied by these polynomials, we have $\beta_1^{\psi} = \beta\frac{[1-\eta]^2}{[1+\eta]^2}$, $\alpha_2^{\psi} = \beta\frac{8\eta[1+\eta^2]}{[1-\eta^2]^2}$ and, for $n \geqslant 1$,*

$$\beta_{2n}^{\psi} = \beta \frac{[1+\eta^{2n-1}][1+\eta^{2n}]}{[1-\eta^{2n-1}][1-\eta^{2n}]}, \qquad \beta_{2n+1}^{\psi} = \beta \frac{[1-\eta^{2n}][1-\eta^{2n+1}]}{[1+\eta^{2n}][1+\eta^{2n+1}]},$$
$$\alpha_{2n+1}^{\psi} = \beta \frac{4\eta}{[1-\eta]^2} \frac{[1+\eta^{2n-1}][1-\eta^{2n+1}]}{[1+\eta^{2n}][1-\eta^{2n}]},$$
$$\alpha_{2n+2}^{\psi} = \beta \frac{4\eta}{[1-\eta]^2} \frac{[1-\eta^{2n}][1+\eta^{2n+2}]}{[1+\eta^{2n+1}][1-\eta^{2n+1}]}.$$

Proof. We obtain from (6.1),

$$Q_n^{\phi_1}(1) = 2\left(\frac{d-1}{2}\right)^n \left(\frac{-1}{2}\right)^n,$$
$$Q_n^{\phi_1}(-1) = \left[\left(\sqrt{\frac{d+1}{2}}-1\right)^{2n} + \left(\sqrt{\frac{d+1}{2}}+1\right)^{2n}\right]\left(\frac{-1}{2}\right)^n,$$

for $n \geqslant 1$. Since $d = \frac{1}{2}(b/\beta + \beta/b)$, these can be written as

$$Q_n^{\phi_1}(1) = \frac{2}{(-8b\beta)^n}\left[(\sqrt{b}+\sqrt{\beta})^{2n}(\sqrt{b}-\sqrt{\beta})^{2n}\right], \quad n \geqslant 1,$$
$$Q_n^{\phi_1}(-1) = \frac{1}{(-8b\beta)^n}\left[(\sqrt{b}+\sqrt{\beta})^{4n}+(\sqrt{b}-\sqrt{\beta})^{4n}\right], \quad n \geqslant 1.$$

With $\eta = [\sqrt{b}-\sqrt{\beta}]^2/[\sqrt{b}+\sqrt{\beta}]^2$, we can then write

$$Q_n^{\phi_1}(1) = 2\frac{(-2)^n\eta^n}{(1-\eta)^{2n}} \quad \text{and} \quad Q_n^{\phi_1}(-1) = \frac{(-2)^n(1+\eta^{2n})}{(1-\eta)^{2n}}, \quad n \geqslant 1.$$

Hence, for $S_n^{\phi_1}(z) = Q_n^{\phi_1}(z)/Q_{n-1}^{\phi_1}(z)$,

$$S_n^{\phi_1}(1) = \frac{-2\eta}{(1-\eta)^2} \quad \text{and} \quad S_n^{\phi_1}(-1) = \frac{-2}{(1-\eta)^2}\frac{1+\eta^{2n}}{1+\eta^{2n-2}}, \quad n \geqslant 1.$$

Thus, from (4.2), the results (6.3) of the theorem follow. With this, the results for $\{\beta_n^{\psi}\}$ immediately follow from (4.6).

To obtain the results for $\{\alpha_n^{\psi}\}$, first we obtain

$$\delta_{2n-1}^{\psi} - 1 = \frac{2\eta}{[1-\eta]}\frac{[1+\eta^{2n-2}]}{[1-\eta^{2n-1}]}, \qquad \delta_{2n-1}^{\psi} + 1 = \frac{2}{[1-\eta]}\frac{[1+\eta^{2n}]}{[1-\eta^{2n-1}]},$$
$$\delta_{2n}^{\psi} - 1 = \frac{2\eta}{[1-\eta]}\frac{[1-\eta^{2n-1}]}{[1+\eta^{2n}]}, \qquad \delta_{2n}^{\psi} + 1 = \frac{2}{[1-\eta]}\frac{[1-\eta^{2n+1}]}{[1+\eta^{2n}]},$$

for $n \geqslant 1$. Now the results for $\{\alpha_n^{\psi}\}$ follow from (4.6). □

Note that we can write $\eta = [\sqrt{d+\sqrt{d^2-1}}-1]^2/[\sqrt{d+\sqrt{d^2-1}}+1]^2$. Hence, the value of η is independent of the value of β.

6.2. L-orthogonal polynomials from the Laguerre polynomials

The Laguerre polynomials $\{L_n^{(\alpha)}\}$, for $\alpha > -1$, are defined by

$$\int_0^\infty L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)x^\alpha e^{-x}\,\mathrm{d}x = \Gamma(\alpha+1)\binom{n+\alpha}{n}\delta_{nm}, \quad n,m=0,1,2,\ldots,$$

where $\binom{u}{v} = \frac{\Gamma(u+1)}{\Gamma(v+1)\Gamma(u-v+1)}$. These polynomials are explicitly given by (see, for example, [1,13])

$$L_0^{(\alpha)}(x) = 1 \quad \text{and} \quad L_n^{(\alpha)}(x) = \sum_{j=0}^{n}\binom{n+\alpha}{n-j}\frac{1}{j!}(-x)^j, \quad n \geqslant 1.$$

We consider the polynomials

$$Q_0^{\phi_1}(x) = 1 \quad \text{and} \quad Q_n^{\phi_1}(x) = (-2)^n n! L_n^{(\alpha)}\left(\frac{x-1}{2}\right), \quad n \geqslant 1.$$

Then $\{Q_n^{\phi_1}(x)\}$ are the monic orthogonal polynomials given by

$$\int_1^\infty Q_n^{\phi_1}(x)Q_m^{\phi_1}(x)\,\mathrm{d}\phi_1(x) = 2^{2n}n!(\alpha+1)_n\delta_{nm}, \quad n,m=0,1,2,\ldots,$$

where $\mathrm{d}\phi_1(x) = \frac{1}{2\Gamma(\alpha+1)}[\frac{1}{2}(x-1)]^\alpha e^{-\frac{1}{2}(x-1)}\,\mathrm{d}x$. Here, $(u)_n$ is the Pochhammer-symbol defined by $(u)_0 = 1$ and $(u)_n = u(u+1)\cdots(u+n-1)$ for $n \geqslant 1$. One can easily verify that $\mu_0^{\phi_1} = 1$,

$$Q_0^{\phi_1}(x) = 1 \quad \text{and} \quad Q_n^{\phi_1}(x) = (-2)^n n! \sum_{j=0}^{n}\binom{n+\alpha}{n-j}\frac{1}{j!}\left(\frac{1-x}{2}\right)^j, \quad n \geqslant 1.$$

Moreover,

$$Q_{n+1}^{\phi_1}(x) = \left[x - (4n+3+2\alpha)\right]Q_n^{\phi_1}(x) - 4n(n+\alpha)Q_{n-1}^{\phi_1}(x), \quad n \geqslant 1,$$

with $Q_0^{\phi_1} = 1$ and $Q_1^{\phi_1}(x) = x - (3+2\alpha)$.

Theorem 6.2. *Let ψ be the $S^3(0,\beta,\infty)$ measure given by*

$$\mathrm{d}\psi(t) = \frac{t^{-1}[\sqrt{t/\beta}+\sqrt{\beta/t}]}{4\Gamma(\alpha+1)}\left[\frac{1}{2}\left|\sqrt{t/\beta}-\sqrt{\beta/t}\right|\right]^{2\alpha+1} e^{-[\frac{1}{2}(\sqrt{t/\beta}-\sqrt{\beta/t})]^2}\,\mathrm{d}t.$$

Then for the associated L-orthogonal polynomials $\{\mathcal{Q}_n^\psi\}$, the parameters $\delta_n^\psi = (-\beta)^n/\mathcal{Q}_n^\psi(0)$ satisfy

$$\delta_{2n-1}^\psi = -1 + 2n\frac{\nu_n^{(\alpha)}}{\nu_{n-1}^{(\alpha+1)}} = 1 + 2(n+\alpha)\frac{\nu_{n-1}^{(\alpha)}}{\nu_{n-1}^{(\alpha+1)}}, \quad n \geqslant 1,$$

$$\delta_{2n}^\psi = -1 + 2\frac{\nu_n^{(\alpha+1)}}{\nu_n^{(\alpha)}} = 1 + 2\frac{\nu_{n-1}^{(\alpha+1)}}{\nu_n^{(\alpha)}}, \quad n \geqslant 1,$$

where $\nu_n^{(\alpha)} = L_n^{(\alpha)}(-1) = \sum_{j=0}^n \binom{n+\alpha}{n-j}\frac{1}{j!}$.

Proof. These results follow from the application of Theorem 4.1 together with (3.3). To complete the proof of this theorem we observe that

$$Q_n^{\phi_1}(1) = (-2)^n n! \binom{n+\alpha}{n} = (-2)^n n! L_n^{(\alpha)}(0), \quad n \geqslant 1,$$

$$Q_n^{\phi_1}(-1) = (-2)^n n! \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \frac{1}{j!} = (-2)^n n! L_n^{(\alpha)}(-1), \quad n \geqslant 1.$$

Hence, for $n \geqslant 1$,

$$\begin{aligned}
&Q_n^{\phi_1}(-1) Q_n^{\phi_1}(1) = 2^{2n}(n!)^2 \binom{n+\alpha}{n} \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \frac{1}{j!},\\
&Q_n^{\phi_1}(-1) Q_{n-1}^{\phi_1}(1) + Q_n^{\phi_1}(1) Q_{n-1}^{\phi_1}(-1)\\
&\quad = -2^{2n-1} \frac{(n-1)!n!}{n+\alpha} \binom{n+\alpha}{n} \sum_{j=0}^{n} (2n-j) \binom{n+\alpha}{n-j} \frac{1}{j!},\\
&Q_n^{\phi_1}(-1) Q_{n-1}^{\phi_1}(1) - Q_n^{\phi_1}(1) Q_{n-1}^{\phi_1}(-1)\\
&\quad = -2^{2n-1} \frac{(n-1)!n!}{n+\alpha} \binom{n+\alpha}{n} \sum_{j=0}^{n} j \binom{n+\alpha}{n-j} \frac{1}{j!}.
\end{aligned} \tag{6.4}$$

To obtain the last two expressions we use

$$\binom{n-1+\alpha}{n-1-j} = \frac{n-j}{n+\alpha} \binom{n+\alpha}{n-j}, \quad j = 0, 1, \ldots, n-1.$$

Using the results of (6.4) in (4.2) we conclude the proof of the theorem. □

From (4.6), we obtain that the polynomials $\{\mathcal{Q}_n^{\psi}\}$ satisfy the three term recurrence relation (1.3) with $\beta_1^{\psi} = \beta(3+2\alpha)^{-1}$, $\alpha_2^{\psi} = 4(\alpha+1)(\alpha+2)\beta_1^{\psi}$ and, for $n \geqslant 1$,

$$\beta_{2n}^{\psi} = \beta \frac{1 + 2(n+\alpha)\nu_{n-1}^{(\alpha)}/\nu_{n-1}^{(\alpha+1)}}{1 + 2\nu_{n-1}^{(\alpha+1)}/\nu_n^{(\alpha)}}, \qquad \beta_{2n+1}^{\psi} = \beta \frac{1 + 2\nu_{n-1}^{(\alpha+1)}/\nu_n^{(\alpha)}}{1 + 2(n+\alpha+1)\nu_n^{(\alpha)}/\nu_n^{(\alpha+1)}},$$

$$\alpha_{2n+1}^{\psi} = 4 \frac{\nu_{n-1}^{(\alpha+1)} \nu_n^{(\alpha+1)}}{[\nu_n^{(\alpha)}]^2} \beta_{2n}^{\psi}, \qquad \alpha_{2n+2}^{\psi} = 4(n+1)(n+\alpha+1) \frac{\nu_n^{(\alpha)} \nu_{n+1}^{(\alpha)}}{[\nu_n^{(\alpha+1)}]^2} \beta_{2n+1}^{\psi}.$$

6.3. L-orthogonal polynomials from the q-Laguerre polynomials

The q-Laguerre polynomials $\{L_n^{(\alpha)}(x;\, q)\}$, for $\alpha > -1$, can be defined by

$$\int_0^{\infty} L_n^{(\alpha)}(x;q) L_m^{(\alpha)}(x;q) \frac{x^{\alpha}}{(-x;q)_{\infty}}\, \mathrm{d}x = \frac{(q^{-\alpha};q)_{\infty}}{(q;q)_{\infty}} \frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n} \Gamma(-\alpha)\Gamma(\alpha+1)\, \delta_{nm},$$

where $(u;\, q)_0 = 1$, $(u;q)_n = (1-u)(1-uq)\cdots(1-uq^{n-1})$, $n \geqslant 1$. These polynomials, which are also called the Generalized Stieltjes–Wigert polynomials [2], have the explicit representation

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;\, q)_n}\, {}_1\Phi_1 \left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix} \,\middle|\, q; -q^{n+\alpha+1}x \right), \quad n \geqslant 1,$$

where

$$ {}_1\Phi_1\left(\begin{matrix} u \\ v \end{matrix}\,\middle|\, q; z\right) = \sum_{j=0}^{\infty} \frac{(u;q)_j}{(v;q)_j}(-1)^j q^{j(j-1)/2}\frac{z^j}{(q;q)_j}. $$

The moment problem associated with the q-Laguerre polynomials is indeterminate and therefore there are many orthogonality representations. For example, there is the following representation in terms of a discrete measure.

$$ \sum_{j=-\infty}^{\infty} \frac{q^{j(\alpha+1)}}{(-cq^j;q)_\infty} L_n^{(\alpha)}(cq^j;q) L_m^{(\alpha)}(cq^j;q) $$
$$ = \frac{(q;q)_\infty(-cq^{\alpha+1};q)_\infty(-c^{-1}q^{-\alpha};q)_\infty}{(q^{\alpha+1};q)_\infty(-c;q)_\infty(-c^{-1}q;q)_\infty}\frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n}\delta_{nm}, $$

where c is any positive constant.

Here we consider the sequence of monic polynomials

$$ Q_0^{\phi_1}(x) = 1 \quad \text{and} \quad Q_n^{\phi_1}(x) = (-2)^n \frac{(q;q)_n}{q^{n(n+\alpha)}} L_n^{(\alpha)}\big((x-1)/2;q\big), \quad n \geqslant 1. $$

From known results on $\{L_n^{(\alpha)}(q;x)\}$, the following results for the polynomials $\{Q_n^{\phi_1}\}$ can be easily written down:

$$ \int_1^{\infty} Q_n^{\phi_1}(x) Q_m^{\phi_1}(x)\,\mathrm{d}\phi_1(x) = \frac{2^{2n}(q;q)_n(q^{\alpha+1};q)_n}{q^{n(2n+2\alpha+1)}}\delta_{nm}, \quad n \geqslant 1, $$

where $\mathrm{d}\phi_1(x) = \frac{1}{2}\frac{(q;q)_\infty}{(q^{-\alpha};q)_\infty\Gamma(-\alpha)\Gamma(\alpha+1)}\frac{((x-1)/2)^\alpha}{(-(x-1)/2;q)_\infty}\,\mathrm{d}x$.

Moreover,

$$ Q_n^{\phi_1}(1) = (-2)^n \frac{(q;q)_n}{q^{n(n+\alpha)}} L_n^{(\alpha)}(0;q) = (-2)^n \frac{(q^{\alpha+1};q)_n}{q^{n(n+\alpha)}}, $$
$$ Q_n^{\phi_1}(-1) = (-2)^n \frac{(q;q)_n}{q^{n(n+\alpha)}} L_n^{(\alpha)}(-1;q) = (-2)^n \frac{1}{q^{n(n+\alpha)}}, $$

for $n \geqslant 1$. The last expression follows, for example, from the summation formula (see [8, p. 15])

$$ {}_1\Phi_1\left(\begin{matrix} u \\ v \end{matrix}\,\middle|\, q; \frac{v}{u}\right) = \frac{(u^{-1}v;q)_\infty}{(v;q)_\infty}. $$

Hence, from Theorem 4.1,

Theorem 6.3. *Let ψ be the $S^3(0,\beta,\infty)$ measure given by*

$$ \mathrm{d}\psi(t) = \frac{(q;q)_\infty t^{-1}[\sqrt{t/\beta}+\sqrt{\beta/t}\,]}{4(q^{-\alpha};q)_\infty\Gamma(-\alpha)\Gamma(\alpha+1)}\frac{[\frac{1}{2}|\sqrt{t/\beta}-\sqrt{\beta/t}\,|]^{2\alpha+1}}{(-[\frac{1}{2}(\sqrt{t/\beta}-\sqrt{\beta/t})]^2;q)_\infty}\,\mathrm{d}t. $$

Then for the associated L-orthogonal polynomials $\{\mathcal{Q}_n^{\psi}\}$, the parameters $\delta_n^{\psi} = (-\beta)^n/\mathcal{Q}_n^{\psi}(0)$ satisfy

$$ \delta_{2n-1}^{\psi} = 2q^{-n-\alpha} - 1 \quad \textit{and} \quad \delta_{2n}^{\psi} = 2q^{-n} - 1, \quad n \geqslant 1. $$

6.4. Orthogonal polynomials from the L-orthogonal polynomials associated with the log-normal distribution

In [9], Pastro gave the orthogonal Laurent polynomials associated with the log-normal distribution. Here we consider the L-orthogonal polynomials $\{\mathcal{Q}_n^{\psi}\}$ defined by (1.2) in relation to the strong measure ψ given by the shifted log-normal distribution

$$\mathrm{d}\psi(t) = \frac{1}{2\kappa\sqrt{\pi}}\, t^{-1} e^{-[\ln(t)/(2\kappa)]^2}\,\mathrm{d}t.$$

It is easily verified from (1.5) that this measure belongs to the class $S^3[0,\beta,b]$, with $\beta = 1$ and $b = \infty$.

From Pastro's results, and also results given in [3,4], it follows that

$$\mathcal{Q}_n^{\psi}(w) = \sum_{r=0}^{n} (-1)^r q^{-r(n-r)} \begin{Bmatrix} n \\ r \end{Bmatrix}_q q^{r/2} w^{n-r}, \quad n \geqslant 1,$$

and

$$\mathcal{Q}_{n+1}^{\psi}(w) = \big(w - q^{1/2}\big)\mathcal{Q}_n^{\psi}(w) - q^{1/2}\big(q^{-n} - 1\big) w\,\mathcal{Q}_{n-1}^{\psi}(w), \quad n \geqslant 1,$$

with $\mathcal{Q}_1^{\psi}(w) = w - q^{1/2}$. Here, $q = e^{-2\kappa^2}$ and $\left\{\begin{smallmatrix} n \\ r \end{smallmatrix}\right\}_q$ are the q-binomial coefficients given by

$$\begin{Bmatrix} n \\ r \end{Bmatrix}_q = \frac{(q;q)_n}{(q;q)_r (q;q)_{n-r}}, \quad r = 0, 1, \ldots, n.$$

The above results can also be found in [11].

Theorem 6.4. *Let the positive measures ϕ_A and ϕ_B, both having their common support on the entire positive real axis, be given by*

$$\mathrm{d}\phi_B(x) = x(x+2)\,\mathrm{d}\phi_A(x) = \frac{1}{\kappa\sqrt{\pi}}\sqrt{x(x+2)}\,e^{-[\ln(x+1+\sqrt{x(x+2)})^{1/(2\kappa)}]^2}\,\mathrm{d}x.$$

Then $\mu_0^{\phi_A} = 1$, $\mu_0^{\phi_B} = q^{-2} - 1$ and the associated monic orthogonal polynomials $\{Q_n^{\phi_A}\}$ and $\{Q_n^{\phi_B}\}$ satisfy $Q_1^{\phi_A}(z) = z + 1 - q^{-1/2}$, $Q_1^{\phi_B}(z) = z + 1 - \frac{1}{2}q^{-1/2}(q^{-2}+1)$ and

$$Q_{n+1}^{\phi_A}(z) = \big(z - b_{n+1}^{\phi_A}\big)Q_n^{\phi_A}(z) - a_{n+1}^{\phi_A} Q_{n-1}^{\phi_A}(z), \quad n \geqslant 1,$$
$$Q_{n+1}^{\phi_B}(z) = \big(z - b_{n+1}^{\phi_B}\big)Q_n^{\phi_B}(z) - a_{n+1}^{\phi_B} Q_{n-1}^{\phi_B}(z), \quad n \geqslant 1,$$

where

$$b_{n+1}^{\phi_A} = \frac{1}{2}\big[q^{-(2n+1)/2}\big(q^{-n}+1\big) + q^{-(2n-1)/2}\big(q^{-n}-1\big)\big] - 1,$$
$$a_{n+1}^{\phi_A} = \frac{1}{4}\big(q^{-n+1}+1\big)\big(q^{-n}-1\big)\big(q^{-2n+1}-1\big),$$
$$b_{n+1}^{\phi_B} = \frac{1}{2}\big[q^{-(2n+3)/2}\big(q^{-n-1}-1\big) + q^{-(2n+1)/2}\big(q^{-n-1}+1\big)\big] - 1,$$
$$a_{n+1}^{\phi_B} = \frac{1}{4}\big(q^{-n}-1\big)\big(q^{-n-1}+1\big)\big(q^{-2n-1}-1\big),$$

for $n \geqslant 1$.

Proof. We have $\delta_n^{\psi} = q^{-n/2}$, $n \geqslant 1$. Hence, applying Theorems 4.1 and 5.1, we obtain for the measures ϕ_1 and ϕ_2, both defined on $[1, \infty)$,

$$\mathrm{d}\phi_2(x) = \left(x^2 - 1\right) \mathrm{d}\phi_1(x) = \frac{1}{\kappa\sqrt{\pi}}\sqrt{x^2-1}e^{-[\ln(x+\sqrt{x^2-1})^{1/(2\kappa)}]^2}\,\mathrm{d}x,$$

$\mu_0^{\phi_1} = 1$, $\mu_0^{\phi_2} = q^{-2} - 1$. Moreover, the associated sequences of monic orthogonal polynomials $\{Q_n^{\phi_1}\}$ and $\{Q_n^{\phi_2}\}$ satisfy $Q_1^{\phi_1}(x) = x - q^{-1/2}$, $Q_1^{\phi_2}(x) = x - \frac{1}{2}q^{-1/2}(q^{-2}+1)$ and

$$Q_{n+1}^{\phi_1}(x) = \left(x - b_{n+1}^{\phi_1}\right)Q_n^{\phi_1}(x) - a_{n+1}^{\phi_1}Q_{n-1}^{\phi_1}(x), \quad n \geqslant 1,$$
$$Q_{n+1}^{\phi_2}(x) = \left(x - b_{n+1}^{\phi_2}\right)Q_n^{\phi_2}(x) - a_{n+1}^{\phi_2}Q_{n-1}^{\phi_2}(x), \quad n \geqslant 1,$$

where

$$b_{n+1}^{\phi_1} = \frac{1}{2}\left[q^{-(2n+1)/2}\left(q^{-n}+1\right) + q^{-(2n-1)/2}\left(q^{-n}-1\right)\right],$$
$$a_{n+1}^{\phi_1} = \frac{1}{4}\left(q^{-n+1}+1\right)\left(q^{-n}-1\right)\left(q^{-2n+1}-1\right),$$
$$b_{n+1}^{\phi_2} = \frac{1}{2}\left[q^{-(2n+3)/2}\left(q^{-n-1}-1\right) + q^{-(2n+1)/2}\left(q^{-n-1}+1\right)\right],$$
$$a_{n+1}^{\phi_2} = \frac{1}{4}\left(q^{-n}-1\right)\left(q^{-n-1}+1\right)\left(q^{-2n-1}-1\right),$$

for $n \geqslant 1$. Hence the substitution $x = y + 1$ gives the results of the theorem. □

References

[1] T.S. Chihara, An Introduction to Orthogonal Polynomials, Math. Appl. Ser., Gordon and Breach, New York, 1978.

[2] T.S. Chihara, On generalized Stieltjes–Wigert and related orthogonal polynomials, J. Comput. Appl. Math. 5 (1979) 291–297.

[3] A.K. Common, J.H. McCabe, The symmetric strong moment problem, J. Comput. Appl. Math. 67 (1996) 327–341.

[4] S.C. Cooper, W.B. Jones, W.J. Thron, Asymptotics of orthogonal L-polynomials for log-normal distributions, Construct. Approx. 8 (1992) 59–67.

[5] B.A. Hagler, W.B. Jones, W.J. Thron, Orthogonal Laurent polynomials of Jacobi, Hermite, and Laguerre types, in: W.B. Jones, A. Sri Ranga (Eds.), Orthogonal Functions, Moment Theory, and Continued Fractions: Theory and Applications, in: Lecture Notes in Pure and Appl. Math., vol. 199, Marcel Dekker, New York, 1998, pp. 187–208.

[6] W.B. Jones, O. Njåstad, W.J. Thron, Two point Padé expansions for a family of analytic functions, J. Comput. Appl. Math. 9 (1983) 105–123.

[7] W.B. Jones, W.J. Thron, H. Waadeland, The strong Stieltjes moment problem, Trans. Amer. Math. Soc. 261 (1980) 503–528.

[8] R. Koekoek, R.F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Delft University of Technology, Faculty of Information Technology and Systems, Department of Technical Mathematics and Informatics, Report no. 98-17, 1998.

[9] P.I. Pastro, Orthogonal polynomials and some q-beta integrals of Ramanujan, J. Math. Anal. Appl. 112 (1985) 517–540.

[10] A. Sri Ranga, Symmetric orthogonal polynomials and the associated orthogonal L-polynomials, Proc. Amer. Math. Soc. 123 (1995) 3135–3141.

[11] A. Sri Ranga, E.X.L. de Andrade, J.H. McCabe, Some consequences of symmetry in strong distributions, J. Math. Anal. Appl. 193 (1995) 158–168.

[12] A. Sri Ranga, W. Van Assche, Blumenthal's theorem for Laurent orthogonal polynomials, J. Approx. Theory 117 (2002) 255–278.

[13] G. Szegő, Orthogonal Polynomials, fourth ed., Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, RI, 1975.

[14] L. Vinet, A. Zhedanov, Szegő polynomials on the real axis, Integral Transforms Spec. Funct. 8 (1999) 149–164.