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Comment on "Some results from a Mellin transform expansion for the heat kernel" [J. Math. Phys. 30, 1226 (1989)]

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In this note the inaccuracies contained in the above mentioned paper are corrected and some of the results therein are modified and extended. In particular, formulae are given for directly calculating the coefficients of the heat kernel expansion, not making appeal to recurrence in the heat equation. The nonexistence of a class of anomalies in odd dimension is proved.

I. INTRODUCTION

Our starting point is the Mellin transform expansion for the solution of the "heat equation"—the so-called *heat kernel* associated to an elliptic operator H of order m, acting on a D-dimensional compact manifold without boundary, which has been obtained in a previous paper.[1] We consider in this note only differential operators of order *m even*. The diagonal part of the heat kernel has, for $t \ll 1$, the asymptotic expansion,

$$F(t;x,x) = -(4\pi t)^{-D/m} \times \left\{ \sum_{l=0}^{\infty} (4\pi)^{D/m} \frac{(-1)^l}{l!} K(l;x,x) t^{l+(D/m)} + \sum_j^{\infty} \Gamma\left(\frac{D-j}{m}\right) R_j(x)\, (4\pi)^{D/m} t^{j/m} \right\}, \quad (1)$$

where $R_j(x)$ is the residue of $K(1;x,x)$ at the pole situated at $s=(j-D)/m$, $K(s;x,x)$ is the diagonal part of the Seeley's analytic extension to the complex s plane of the kernel $K(s;x,y)$ of the power operator H^s (see Ref. 2), and the sum over j *excludes* the values of j such that $(j-D)/m = 0,1,2,\ldots$.

Concerning our paper in the title (henceforth refered to as Paper I) we first observe that we have not taken into account the vanishing of the residues of the poles of $K(s;x,x)$ at *all* integers s values $\geqslant 0$ in the best way, in the derivation of our asymptotic expansion. When this is done, we need neither introduce the function $\phi(l)$ in formula (1.4) of Paper I, nor calculate its derivatives at the points $s=l$, so the coefficients of the sum over l in (1) are obtained in a much simpler way. The simplified derivation of the asymptotic expansion (1) is given in Ref. 1. It follows that the formula (4.3) in Paper I is unnecessary [by the way, there is a misprint in that formula—the factor (-2) must be dropped out] and the expression for the anomaly in dimension D is given directly by

$$A = q\, \mathrm{Tr}\{(X+Y)\, [K(0;x,x) + P_o(x)]\}. \quad (2)$$

Equation (2) replaces Eqs. (4.2) and (4.5) of Paper I.

As it was remarked in Ref. 1 the series (1) looks different from the de Witt ansatz[3] currently used for even D and operators of order $m=2$,

$$F(t;x,x) = (4\pi t)^{-D/2} \sum_{n=0}^{\infty} c_n(x)\, t^n. \quad (3)$$

For our expansion (1) to coincide formally with (3) it would be necessary, for example in even D and $m=2$, that all the residues $R_j(x)$ vanish for odd values of j and $(j-D)/m \neq 0,1,2,\ldots$, which is not explicitly stated in Seeley's paper.[2]

II. THE VANISHING OF SOME RESIDUES OF THE POLES OF $K(s;x,x)$

To further investigate this point we remember that $R_j(x)$ is given by

$$R_j(x) = \frac{1}{\mathrm{im}(2\pi)^{D+1}} \int_{|\xi|=1} d\xi \times \int_\Gamma d\lambda\, \lambda^{(j-D)/m} b_{-m-j}(x,\xi,\lambda), \quad (4)$$

where Γ is a curve coming from ∞ along a ray of minimal growth, clockwise on a small circle around the origin and then going back to ∞. The quantities b_{-m-j} are obtained from the coefficients a_{m-k} of the symbol of H, $\sigma(H)(x,\xi) = \Sigma_0^m a_{m-k}(x,\xi)$, by the set of equations

$$b_{-m}(a_m - \lambda) = 1, \quad l=0, \quad (5a)$$

$$b_{-m-l} = -(a_m-\lambda)^{-1} \sum \left(\frac{\partial}{\partial \xi}\right)^\alpha b_{-m-j} \times D_x^\alpha \frac{1}{\alpha!} a_{m-k}, \quad l>0, \quad (5b)$$

where multi-index notation for α is used and the sum is taken for

$$j<l, \quad j+k+|\alpha| = l, \quad |\alpha| \equiv \alpha_1 + \cdots + \alpha_D.$$

Now, from the definition of the a_{m-k}'s,

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$$a_{m-k}(x,\xi) = \sum_{|\alpha| = m-k} H_\alpha \xi^\alpha,$$

where the H_α are the coefficients of the operator H in Seeley's notation, we see that the parity in the ξ-variables of a_{m-k} is $(-1)^k$, and from (5a) that the parity in ξ of b_{-m} is always equal to 1. Then by induction (the recurrence hypothesis is easily verified directly for the two first steps) we obtain from (5b) that the parity in ξ of $b_{-m-l}(x,\xi,\lambda)$ equals $(-1)^l$ for any $l \geqslant 0$. Thus from Eq. (4) since the integration over ξ is constrained to the unit sphere (in the cotangent space) $|\xi| = 1$, we see that $R_j(x)$ *vanishes for j odd.*

To see the implications of the vanishing of the residues of these poles of $K(s;x,x,)$, we consider separately the two situations of the manifold dimension D being even and being odd.

(1) D *even*: $D = 2p$; $m = 2q$ (remember we consider m even). In this case, since the sum over j in (1) is such that $(j-D)/m \neq 0,1,2,...,$ we must have $j \neq 2(qk+p)$, $k = 0,1,2,...,$ i.e., the only allowed even values of j are those smaller than $2p-2$; otherwise j is odd. But if j is odd, $R_j(x)$ vanishes, therefore for even D the asymptotic expansion is given by the first sum (sum over l) in (1) and a piece of the sum over j, corresponding to $j = 0,2,...,2p-2$.

(2) D *odd*: $D = 2p+1$; $m = 2q$. All the $K(l;x,x)$ vanish: they are given by the integral[2]

$$K(l;x,x) = \frac{1}{(-1)^l \cdot l (2\pi)^D} \int_{|\xi|=1} d\xi \times \int_\Gamma d\lambda\, \lambda^l b_{-m-lm-D}(x,\xi,\lambda), \quad (6)$$

which is zero by the parity argument above, since $lm + D = 2(lq+p)+1$ is odd. The asymptotic expansion is given by the second sum (sum over j) in Eq. (1), if their coefficients do not vanish. This is indeed the case, for we must have $(j-D)/m \neq k$, $k = 0,1,2,...;$ so, $j \neq 2(qk+p)+1$, i.e., j must be *even*. But, if j is even, the $R_j(x)$'s do not vanish by the same parity argument above.

III. CONCLUSIONS

In the case of D *even*, the expansion coincides formally exactly with the de Witt ansatz, in the sense that the series remaining after factorization of $(4\pi t)^{-D/m}$ is a series of integer powers of t, if p is a multiple of q ($D/2$ is a multiple of $m/2$). Otherwise the sum over l in (1) contains only fractionary powers of t, although of course the series as a whole is made up of integer powers of t.

For *odd D*, the series has, after factorization of $(4\pi t)^{-D/m}$, integer or fractionary powers of t if, respectively, $j/2$ is or is not a multiple of $q = m/2$. The global powers of t are in any case fractionary.

In the special but important case of a differential operator H of order $m = 2$, the asymptotic expansion (1) has the form of the de Witt ansatz,

$$F(t;x,x) = (4\pi t)^{-D/2} \sum_{l=0}^{\infty} \alpha_l(x)\, t^l, \quad (7)$$

where the coefficients $\alpha_l(x)$ are:

for *even D*,

$$\alpha_1(x) = -\Gamma\left(\frac{D-21}{2}\right) R_{21}(x)(4\pi)^{D/2};$$

$$1 = 0,1,2,..., \frac{D}{2} - 1,$$

$$\alpha_l(x) = -\frac{(-1)^l}{1!} K(l;x,x)(4\pi)^{D/2};$$

$$l = \frac{D}{2}, \frac{D}{2}+1,..., \quad (8)$$

and for *odd D*,

$$\alpha_l(x) = -\Gamma\left(\frac{D-21}{2}\right) R_{21}(x)(4\pi)^{D/2}. \quad (9)$$

Another implication of the vanishing of $R_j(x)$ for odd j is that the residues of the new poles of the Hawking's zeta function $\xi(s)$, which we had claimed to exist in Paper **I**, vanish in even dimension D. On the other side, those poles are already present with nonvanishing residues for *odd D* and situated at the values $s = (D-j)/m$ for $j = 0,2,4,...$.

Also, replacing (6) in (2) we see that all the class of anomalies described by Cognola and Zerbini[4] *cannot exist in odd dimension*, due to the vanishing of $K(0;x,x)$. This is to be compared with previous results from both the mathematical and physical literature, such as those in the papers by Greiner,[5] Gilkey,[6] and Romanov and Schwartz,[7] which explicitly exhibit the connection between the Seeley's kernel and the coefficients of the heat kernel expansion, in the case of a differential operator. As a consequence, an elliptic differential operator acting on an odd-dimensional compact manifold without boundary has zero index. From a physical point of view this may be interpreted as the absence of anomalies. Here we recover this fact.

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