Renormalization Group in Potential Scattering

Sadhan K. Adhikari[1] and T. Frederico[2]
[1]*Instituto de Física Teórica, Universidade Estadual Paulista, 01405-900 São Paulo, São Paulo, Brasil*
[2]*Departamento de Física, Instituto Tecnológico de Aeronautica, Centro Técnico Aeroespacial, 12228–900 São José dos Campos, São Paulo, Brasil*
(Received 29 December 1994)

Renormalization group equations for the physical t matrix are written for quantum scattering with some potentials singular at the origin in two and three dimensions. The momentum-space perturbative treatment of these scattering problems exhibits ultraviolet divergences and permits renormalization leading to a scale. These equations yield the correct asymptotic behavior and some low-energy properties of the t matrix. Illustrations are made for the Dirac delta potential in two and three dimensions.

PACS numbers: 03.80.+r, 11.10.Gh, 11.10.Hi, 11.55.Bq

Renormalization group (RG) equations [1,2] are very practical for handling a host of divergent problems in physics. In particular, these equations are easily derived in quantum field theory in dealing with ultraviolet divergences in perturbative expansion, and have proved to be extremely useful in applications to quantum electrodynamics and quantum chromodynamics. The ultraviolet divergences in perturbative quantum field theory can be eliminated by renormalization to yield a scale [1–3]. Except in some simple cases the renormalized perturbative series cannot be summed and this makes it difficult to draw conclusions about the full solution. The RG equations, on the other hand, yield many general properties of this solution.

Ultraviolet divergences also appear in the nonrelativistic quantum mechanical scattering problem for potentials with certain singular behavior at short distances [4–9] in two and three dimensions. Renormalization of these potential models leads to a scale and finite physical observables [5]. In our dimension these divergences are absent.

Recently, there have been discussions on renormalization in configuration [7,8] and momentum [4,5,9] spaces for potential scattering with Dirac delta, contact, or zero-range potential. This potential is simple, local, and separable at the same time, and has been used in atomic, particle [9,10], nuclear, and surface physics [7,8].

In this Letter, RG equations are written for potential scattering with the Dirac delta potential in two and three dimensions. In both cases there are ultraviolet divergences. In the two-dimensional case the ultraviolet divergence is logarithmic in nature, whereas in the three-dimensional case it is linear in nature. In both cases, however, the renormalization can be carried out in a similar fashion and the RG equations written.

The one-dimensional delta potential has been frequently used to model many physical systems. Because of the above-mentioned ultraviolet divergences, the same is not true of two- and three-dimensional quantum scattering with delta potentials. However, the renormalization and RG equations of the present Letter for these problems will allow one to use these potentials to model actual physical systems, which are mostly two and three dimensional.

There is another interest in studying the nonrelativistic scattering with delta potential in two dimensions. This problem can be considered to be a good model of the ultraviolet structure and high energy behavior of $\lambda\phi^4$ field theory [2,7,9] in (3 + 1) dimensions. This is because both problems have ultraviolet logarithmic divergences, require regularization, are perturbatively renormalizable, collapse for attractive interaction, but are asymptotically free, etc. However, there is an important difference between (3 + 1)-dimensional quantum field theory and nonrelativistic potential scattering with delta potential. In the simplest field theoretical model, because of creation and destruction of particles one cannot go beyond the few lowest orders of perturbation theory. The nonrelativistic scattering problem with the delta potential in two dimensions, on the other hand, can be solved analytically. We shall also study the nonrelativistic scattering problem with the three-dimensional delta potential. Both these studies will allow one to understand most of the subtleties of renormalization and RG equations.

We discuss potential scattering with the delta potential. This potential exists only in S wave, as the centrifugal barrier washes it out in all other partial waves. The partial-wave Lippmann-Schwinger equation for the scattering amplitude $T(p,q,k^2)$ in $\mathcal{D}$ dimensions at c.m. energy k^2 is given by

$$T(p',p,k^2) = V(p',p) + \int d^{\mathcal{D}}q\, V(p',q) \times G(q;k^2)T(q,p,k^2)\,, \qquad (1)$$

with the free Green function $G(q;k^2) = (k^2 - q^2 + i0)^{-1}$, in units $\hbar = 2m = 1$, where m is the reduced mass. The integrals in Eq. (1) and in the following are over the relevant S-wave phase space, e.g., we take

 0031-9007/95/74(23)/4572(4)$06.00

$d^3q = (2/\pi)q^2dq$ and $d^2q = q\,dq$ with q varying from 0 to ∞. For the delta potential $V(p', p) = \lambda$, and

$$T(p', p, k^2) = [\lambda^{-1} - I(k)]^{-1}, \tag{2}$$

with $I(k) = \int d^{\mathcal{D}} q\, G(q; k^2)$. The integral $I(k)$ possesses ultraviolet divergence for $\mathcal{D} > 1$. For $\mathcal{D} = 3$ (2) this divergence is linear (logarithmic) in nature. Finite result for the t matrix of Eq. (2) can be obtained only if λ^{-1} also diverges in a similar fashion and cancels the divergence of $I(k)$. In Eq. (2) $\lambda I(k)$ is the trace of the kernel of the integral equation (1) and possesses ultraviolet divergence for $\mathcal{D} > 1$. The kernel of Eq. (1) is noncompact and it does not have scattering solution.

Hence some regularization is needed to give meaning to Eq. (1). This can be achieved via the following regularized Green function involving a smooth cutoff $\Lambda(>k)$:

$$G_R(q, \Lambda; k^2) = (k^2 - q^2 + i0)^{-1} + (\Lambda^2 + q^2)^{-1} = \frac{k^2 + \Lambda^2}{(k^2 - q^2 + i0)(\Lambda^2 + q^2)}. \tag{3}$$

With this Green function there is no ultraviolet divergence. The imaginary part of the Green function is unaffected by this procedure which guarantees unitarity. However, in the end, the limit $\Lambda \to \infty$ has to be taken, which will reduce the regularized Green function to the free Green function. Finite results for physical magnitudes, as $\Lambda \to \infty$, are obtained only if the coupling λ is also replaced by the so called bare coupling $\lambda(\Lambda)$, defined, for example, by

$$\lambda^{-1}(\Lambda) = -[\Lambda + \Lambda_0], \mathcal{D} = 3, \tag{4}$$

$$= -[\ln(\Lambda/\Lambda_0)], \mathcal{D} = 2, \tag{5}$$

where Λ_0 is the physical scale of the system and characterizes the strength of the interaction (compare with $\Lambda_{\rm QCD}$ of the strong interaction). The renormalized t matrix will be a function of Λ_0. The parameter Λ_0 is positive for $\mathcal{D} = 2$, but can also be negative for $\mathcal{D} = 3$; in both cases ($|\Lambda_0| < \Lambda$). Consequently, for a fixed finite Λ, the t matrix can now be written as

$$T(k, \Lambda) = [\lambda^{-1}(\Lambda) - I_R(k, \Lambda)]^{-1}, \tag{6}$$

where $I_R(k, \Lambda) = \int d^{\mathcal{D}} q\, G_R(q, \Lambda; k^2)$ is a convergent integral. As $\Lambda \to \infty$, however, this integral develops the original ultraviolet divergence. The quantity $\lambda^{-1}(\Lambda)$ of Eqs. (4) and (5) has the appropriate divergent behavior, as $\Lambda \to \infty$, and cancels the divergent part of $I_R(k, \Lambda)$. In Eq. (6) the redundant momentum labels p, p' of the t matrix have been suppressed, and the explicit dependence of the t matrix on Λ has been introduced.

Next the limit $\Lambda \to \infty$ has to be taken in Eq. (6). With this regularization procedure one has for the renormalized t matrix

$$T_R(k, \lambda_R(\mu), \mu) = [\lambda_R^{-1}(\mu) - I_R(k, \mu)]^{-1}, \tag{7}$$

where μ is the scale of the problem and emerges as a result of renormalization. The renormalization scale μ should be contrasted with the physical scale Λ_0. The renormalized t matrix will be independent of μ. In Eq. (7) the explicit dependence of the t matrix on both μ and the renormalized coupling $\lambda_R(\mu)$ has been exhibited. The limiting procedure implied by $\Lambda \to \infty$ in Eq. (6) leads to the following definition for the renormalized coupling $\lambda_R(\mu)$:

$$\lambda_R^{-1}(\mu) = \lim_{\Lambda\to\infty}[\lambda^{-1}(\Lambda) - \{I_R(k, \Lambda) - I_R(k, \mu)\}]. \tag{8}$$

In Eq. (8), if integrals I_R are evaluated and the trivial limit $\Lambda \to \infty$ taken, we get for both $\mathcal{D} = 3$ and 2

$$\lambda_R(\mu) = \lambda(\Lambda = \mu). \tag{9}$$

This relation between the renormalized coupling and bare coupling depends on the regularization scheme used. Equations (4), (5), and (9) lead to the following expressions for the renormalized couplings:

$$\lambda_R(\mu) = -[\mu + \Lambda_0]^{-1}, \mathcal{D} = 3, \tag{10}$$

$$= -[\ln(\mu/\Lambda_0)]^{-1}, \mathcal{D} = 2. \tag{11}$$

The renormalized coupling for two scales μ and μ_0 are related by

$$\lambda_R^{-1}(\mu) + \mu = \lambda_R^{-1}(\mu_0) + \mu_0, \mathcal{D} = 3, \tag{12}$$

$$\lambda_R^{-1}(\mu) + \ln\mu = \lambda_R^{-1}(\mu_0) + \ln\mu_0, \mathcal{D} = 2. \tag{13}$$

Equations (12) and (13) relate the renormalized coupling for two different scales μ and μ_0 and are the flow equations. The flow equations are independent of the regularization scheme.

The present scattering model permits analytic solutions for both $\mathcal{D} = 3$ and 2. In these cases the exact renormalized t matrices of Eq. (7) are given by

$$T_R(k, \lambda_R(\mu), \mu) = [\lambda_R^{-1}(\mu) + \mu + ik]^{-1}, \mathcal{D} = 3 \tag{14}$$

$$= [\lambda_R^{-1}(\mu) - \ln(-ik/\mu)]^{-1}, \mathcal{D} = 2, \tag{15}$$

respectively. Explicitly, using definitions (10) and (11) for the renormalized coupling, these renormalized t matrices can be written as

$$T_R(k, \lambda_R(\mu), \mu) = [ik - \Lambda_0]^{-1}, \mathcal{D} = 3 \tag{16}$$

$$= [-\ln(-ik/\Lambda_0)]^{-1}, \mathcal{D} = 2. \tag{17}$$

These t matrices depend on the renormalized coupling $\lambda_R(\mu)$, but not on μ, that is, the explicit and implicit [through $\lambda_R(\mu)$] dependences of the t matrix on μ cancel. Physics is determined by the value of $\lambda_R(\mu)$ at an arbitrary value of μ [7], or by the following μ independent quantities:

$$\lambda_R^{-1}(\mu) + \mu = -\Lambda_0, \mathcal{D} = 3, \tag{18}$$

$$\lambda_R^{-1}(\mu) + \ln\mu = \ln\Lambda_0, \mathcal{D} = 2, \tag{19}$$

as can be seen from Eqs. (14)–(17). This μ independence of Eqs. (18) and (19) is consistent with flow equations (12) and (13). One could have arrived at the renormalized t matrices of Eqs. (16) and (17) from Eqs. (14) and (15) using the μ independence of each side of the flow equations (12) and (13). The functional relationship of T_R on the physical scale Λ_0 depends on the regularization scheme. In the end the physical scale will be identified with a physical observable and once this is done T_R will be independent of the regularization scheme [5]. Other regularization schemes can be used in place of Eq. (3) [7,8]. Perturbative renormalization has been carried out in two dimensions for the delta potential [7].

The renormalized t matrix is independent of μ, so is invariant under the group of transformations $\mu \to \exp(s)\mu$, which form the RG. In the present case, as in the $\lambda\phi^4$ model, it is convenient to work in terms of the dimensionless coupling, $g_R(\mu)$, defined by

$$g_R(\mu) \equiv \mu\lambda_R(\mu), \mathcal{D} = 3, \qquad (20)$$

$$\equiv \lambda_R(\mu), \mathcal{D} = 2. \qquad (21)$$

The renormalization condition is given by

$$\mu\frac{d}{d\mu}T_R(k, g_R(\mu), \mu) = 0, \qquad (22)$$

or

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g_R)\frac{\partial}{\partial g_R}\right]T_R(k, g_R(\mu), \mu) = 0, \qquad (23)$$

where

$$\beta(g_R) = \mu\frac{\partial g_R(\mu)}{\partial\mu}. \qquad (24)$$

Equation (23) is the RG equation.

As the present problem permits analytic solution, the constant $\beta(g_R)$ of Eq. (24) can be exactly calculated. For both $\mathcal{D} = 3$, and 2, $\beta(g_R)$ is a finite quantity independent of μ, and so are the different terms of the RG equation (23). For $\mathcal{D} = 3$, from Eqs. (20) and (24) we have

$$\beta(g_R) = g_R(\mu) + \mu^2\frac{\partial\lambda_R(\mu)}{\partial\mu}. \qquad (25)$$

With $\lambda_R(\mu)$ defined by Eq. (10), we have

$$\mu^2\frac{\partial\lambda_R(\mu)}{\partial\mu} = \mu^2\lambda_R^2(\mu). \qquad (26)$$

Then from Eqs. (20), (25), and (26) we have

$$\beta(g_R) = g_R + g_R^2, \mathcal{D} = 3. \qquad (27)$$

Similarly, for $\mathcal{D} = 2$, from Eq. (11) we have

$$\mu\frac{\partial\lambda_R(\mu)}{\partial\mu} = \lambda_R^2(\mu). \qquad (28)$$

Then Eqs. (21), (24), and (28) lead to

$$\beta(g_R) = g_R^2, \mathcal{D} = 2. \qquad (29)$$

Next an equation can be written down expressing the invariance of the t matrix $T_R(k, g_R(\mu), \mu)$ under a change of scale:

$$T_R(\gamma k, g_R(\mu), \mu) = \gamma^{(2-\mathcal{D})}T_R(k, g_R(\mu), \mu\gamma^{-1}), \qquad (30)$$

valid for both $\mathcal{D} = 2$ and 3, so that

$$\left[\gamma\frac{\partial}{\partial\gamma} + \mu\frac{\partial}{\partial\mu} + (\mathcal{D} - 2)\right]T_R(\gamma k, g_R(\mu), \mu) = 0. \qquad (31)$$

Eliminating the partial derivative $\mu(\partial T_R/\partial\mu)$ between Eqs. (23) and (31) we have

$$\left[\gamma\frac{\partial}{\partial\gamma} - \beta(g_R)\frac{\partial}{\partial g_R} + (\mathcal{D} - 2)\right]T_R(\gamma k, g_R(\mu), \mu) = 0, \qquad (32)$$

with $\beta(g_R)$ given by Eqs. (27) and (29), for $\mathcal{D} = 3$ and 2, respectively. RG equation (32) expresses the effect on the t matrix of scaling up momentum by a factor γ.

We now wish to find a condition for the solution to Eq. (32). This equation expresses the fact that a change in γ can be compensated for by a change in g_R. So the following functional form can be postulated for the t matrix [2]:

$$T_R(\gamma k, g_R(\mu), \mu) = f(\gamma)T_R(k, g_R(\gamma), \mu), \qquad (33)$$

so that

$$\left[\gamma\frac{\partial}{\partial\gamma} - \frac{\gamma}{f(\gamma)}\frac{df(\gamma)}{d\gamma} + \gamma\frac{\partial g_R(\gamma)}{\partial\gamma}\frac{\partial}{\partial g_R(\gamma)}\right] \times T_R(\gamma k, g_R, \mu) = 0. \qquad (34)$$

Comparing Eqs. (32) and (34), the coefficients of $\partial/\partial g_R$ lead to

$$\gamma\frac{\partial g_R(\gamma)}{\partial\gamma} = \beta(g_R), \qquad (35)$$

where $g_R(\gamma)$ is the so called running coupling constant.

Equation (35) can be solved for g_R for both $\mathcal{D} = 3$, and 2. For $\mathcal{D} = 3$, we have from Eqs. (27) and (35)

$$\gamma\frac{\partial g_R(\gamma)}{\partial\gamma} = g_R(\gamma) + g_R^2(\gamma). \qquad (36)$$

Integrating Eq. (36) between $\gamma = \mu_0$ and $\gamma = \mu$ and using Eq. (20), we obtain

$$\lambda_R(\mu) = \frac{\lambda_R(\mu_0)}{1 - (\mu - \mu_0)\lambda_R(\mu_0)}. \qquad (37)$$

For $\mathcal{D} = 2$, from Eqs. (29) and (35), we have

$$\gamma\frac{\partial g_R(\gamma)}{\partial\gamma} = g_R^2(\gamma). \qquad (38)$$

Integrating Eq. (38) between $\gamma = \mu_0$ and $\gamma = \mu$, and using Eq. (21), we obtain

$$\lambda_R(\mu) = \frac{\lambda_R(\mu_0)}{1 - \lambda_R(\mu_0)\ln(\mu/\mu_0)}. \qquad (39)$$

Equations (37) and (39) are interesting consequences of the RG equation (32). Equations (37) and (39) are the previously derived flow equations (12) and (13). The present consideration of the RG equation yields identical results, as the RG equation contains the information about the renormalization condition in a subtle fashion. In both two and three dimensions $\lambda_R(\mu)$ increases with μ. Thus if we start with a small $\lambda_R(\mu_0)$ at a given scale μ_0, the effective coupling constant increases with μ as in the $\lambda\phi^4$ model [2]. With the increase of μ one can reach a large enough $\lambda_R(\mu)$ where perturbative treatment is not valid. So by using the RG equation one can go from the case of weak coupling to the case of strong coupling.

In the present problem one can write alternative RG equations equivalent to Eq. (32). Next, we do this for $\mathcal{D} = 3$. The $\mathcal{D} = 2$ case can be worked out similarly. We introduce a new physical scale a by

$$a = -1/\Lambda_0 \tag{40}$$

so that Eq. (16) reduces to

$$T_R(k,a) = [ik + 1/a]^{-1}. \tag{41}$$

The physical scale a is now recognized to be the scattering length. The name scale is justified as scattering length is a measure of low energy scattering. This t matrix has a bound-state pole at $k = i/a = -i\Lambda_0$ and, for positive (negative) a, $1/a^2$ is the bound (virtual) state energy in this model. With definitions (20) and (40), one has the identity

$$\beta(g_R)\frac{\partial}{\partial g_R} \equiv (g_R + g_R^2)\frac{\partial}{\partial g_R} = a\frac{\partial}{\partial a}, \tag{42}$$

so that the RG equation (32) becomes

$$\left[\gamma\frac{\partial}{\partial\gamma} - a\frac{\partial}{\partial a} + 1\right]T_R(\gamma k, a) = 0. \tag{43}$$

Equations (32) and (43) express the fact that the effect of a change in γ on T_R can be compensated for by the effect of a change in $g_R(\mu)$ or a, respectively. In principle, the RG equation (32) can be solved to yield the exact renormalized t matrix. However, it is illustrative to obtain the asymptotic high energy behavior of this t matrix from RG equations (32) or (43). At high energies Eqs. (32) or (43) reduce to

$$\gamma\frac{\partial T_R(\gamma k)}{\partial\gamma} + T_R(\gamma k) = 0. \tag{44}$$

This has the simple solution $\lim_{\gamma\to\infty} T_R(\gamma k) \sim 1/\gamma$ again consistent with the t matrices of Eq. (41).

In summary, we have derived and studied RG equations for potential scattering with delta potential in two and three dimensions. The RG equations yield certain general scaling properties of the renormalized t matrix. Similar RG equations should be valid in general for potentials with certain renormalizable singular behavior at short distances. The RG equations are expected to be very useful in situations where the analytic solution is not known, for example, in other few- and many-body problems. The study of RG equations in these cases will be an interesting topic for future investigation.

We thank Dr. C. E. I. Carneiro, Dr. D. Linhares, and Dr. L. Tomio for discussions and the Conselho Nacional de Desenvolvimento Científico e Tecnológico and Financiadora de Estudos e Projetos of Brazil for partial financial support.

[1] K. G. Wilson, Phys. Rev. B **4**, 3184 (1971); K. G. Wilson and J. Kogut, Phys. Rep. **12C**, 78 (1974).
[2] L. H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1985); P. Ramond, *Field Theory: A Modern Primer* (Benjamin/Cummings, Reading, 1981).
[3] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965); C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).
[4] S. Weinberg, Nucl. Phys. **B363**, 2 (1991).
[5] S. K. Adhikari, T. Frederico, and I. D. Goldman, Phys. Rev. Lett. **74**, 487 (1995).
[6] R. Jackiw, in *M. A. B. Beg Memorial Volume,* edited by A. Ali and P. Hoodbhoy (World Scientific, Singapore, 1991).
[7] C. Manuel and R. Tarrach, Phys. Lett. B **328**, 113 (1994); R. Tarrach, University of Barcelona Report No. UB-ECM-PF-38-94, 1994 (unpublished).
[8] P. Gosdzinsky and R. Tarrach, Am. J. Phys. **59**, 70 (1991); L. R. Mead and J. Godines, *ibid.* **59**, 935 (1991).
[9] M. A. B. Beg and R. C. Furlong, Phys. Rev. D **31**, 1370 (1985).
[10] C. Thorn, Phys. Rev. D **19**, 639 (1979).