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Companion orthogonal polynomials[1]

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Abstract

We give some properties relating the recurrence relations of orthogonal polynomials associated with any two symmetric distributions $\mathrm{d}\phi_1(x)$ and $\mathrm{d}\phi_2(x)$ such that $\mathrm{d}\phi_2(x) = (1 + kx^2)\,\mathrm{d}\phi_1(x)$. As applications of these properties, recurrence relations for many interesting systems of orthogonal polynomials are obtained.

Keywords: Symmetric orthogonal polynomials; Three-term recurrence relations; Strong distributions

AMS classification: 33C45

1. Introduction

A distribution (i.e. positive measure) $\mathrm{d}\phi(x)$, defined on $(-\infty, \infty)$, is known as a symmetric distribution if $\mathrm{d}\phi(x) = -\mathrm{d}\phi(-x)$. The monic orthogonal polynomials $Q_n^{\phi}(x)$, $n \geqslant 0$, associated with $\mathrm{d}\phi(x)$ satisfy (see, for example, [1])

$$Q_{n+1}^{\phi}(x) = xQ_n^{\phi}(x) - \alpha_{n+1}^{\phi} Q_{n-1}^{\phi}(x), \quad n \geqslant 1,$$

with $Q_0^{\phi}(x) = 1$ and $Q_1^{\phi}(x) = x$, where the unique coefficients α_{n+1}^{ϕ} are all positive.

Let $\mathrm{d}\phi_1(x)$ and $\mathrm{d}\phi_2(x)$ be two symmetric distributions such that

$$\mathrm{d}\phi_2(x) = (1 + kx^2)\,\mathrm{d}\phi_1(x), \tag{1.1}$$

where k is real and positive. In this article we give some results about how the two sequences of coefficients $\{\alpha_{n+1}^{\phi_1}\}$ and $\{\alpha_{n+1}^{\phi_2}\}$ are related. In particular, we prove the following result.

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Theorem 1. *Associated with* $\mathrm{d}\phi_1(x)$ *and* $\mathrm{d}\phi_2(x)$ *there exists a sequence of positive numbers* $\{\ell_n\}$, *with* $\ell_0 = 1$ *and* $\ell_n > 1$ *for* $n \geqslant 1$, *such that*

$$(\ell_n - 1)(\ell_{n-1} + 1) = 4k\alpha_{n+1}^{\phi_1}$$

and $(n \geqslant 1)$. (1.2)

$$(\ell_n - 1)(\ell_{n+1} + 1) = 4k\alpha_{n+1}^{\phi_2}$$

We establish this result using certain properties of the monic polynomials $\widetilde{B}_n^{\psi}(t)$, $n \geqslant 0$, uniquely defined by

$$\int_0^\infty t^{-n+s}\widetilde{B}_n^{\psi}(t)\,\mathrm{d}\psi(t) = 0, \quad 0 \leqslant s \leqslant n-1, \tag{1.3}$$

where $\mathrm{d}\psi(t)$ is a strong distribution in $(0,\infty)$. A distribution $\mathrm{d}\psi(t)$, with support inside any interval (a,b) and such that the moments $\int_a^b t^m\,\mathrm{d}\psi(t)$ exist for $m=0,\pm1,\pm2,\ldots$, is called a strong distribution in (a,b).

2. Strong distributions

For any strong distribution $\mathrm{d}\psi(t)$ on $(0,\infty)$, the monic polynomials $\widetilde{B}_n^{\psi}(t)$, $n \geqslant 0$, satisfy the recurrence relation

$$\widetilde{B}_{n+1}^{\psi}(t) = (t - \widetilde{\beta}_{n+1}^{\psi})\widetilde{B}_n^{\psi}(t) - \widetilde{\alpha}_{n+1}^{\psi}t\widetilde{B}_{n-1}^{\psi}(t), \quad n \geqslant 1,$$

with $\widetilde{B}_0^{\psi}(t) = 1$ and $\widetilde{B}_1^{\psi}(t) = t - \widetilde{\beta}_1^{\psi}$, where the unique coefficients $\widetilde{\beta}_{n+1}^{\psi}$, $n \geqslant 0$ and $\widetilde{\alpha}_{n+1}^{\psi}$, $n \geqslant 1$ are all positive. For more information on these and other related results see [4, 7, 9]. For some results on polynomials defined by a variation of (1.3) see [6, 11].

In [9] it was given that $P_n(t)$ is a real monic polynomial of degree $n \geqslant 2$ that satisfies

$$\int_0^\infty t^{-n+s}P_n^{\psi}(t)\,\mathrm{d}\psi(t) = 0, \quad 1 \leqslant s \leqslant n-1,$$

if and only if $P_n(t) = \widetilde{B}_n^{\psi}(\lambda, t)$ for a $\lambda \in \Re$, where

$$\widetilde{B}_n^{\psi}(\lambda, t) = \widetilde{B}_n^{\psi}(t) - \lambda\widetilde{B}_{n-1}^{\psi}(t). \tag{2.1}$$

Furthermore, it was shown that if $\mathrm{d}\psi(t)$ has its support inside $(a,b) \subseteq (0,\infty)$ then all the zeros of $\widetilde{B}_n^{\psi}(t)$ lie inside (a,b) and at least $n-1$ of the zeros of $\widetilde{B}_n^{\psi}(\lambda,t)$ lie inside (a,b).

A special type of distribution, first considered in [7], is the ScS(a,b) distribution. A strong distribution $\mathrm{d}\psi(t)$ with its support inside (a,b) is called a ScS(a,b) distribution, if

$$\frac{\mathrm{d}\psi(t)}{\sqrt{t}} = \frac{\mathrm{d}\psi(c/t)}{\sqrt{c/t}}, \quad t \in (a,b), \tag{2.2}$$

where a, b and c are such that $a = c/b$ and $0 < \sqrt{c} < b \leqslant \infty$. Taking $c = \beta^2$, we write this distribution here as a ScS$(\beta^2/b, b)$ distribution. From [7] it follows that for any ScS$(\beta^2/b, b)$ distribution

$\mathrm{d}\psi(t)$, $\widetilde{\beta}^{\psi}_{n+1} = \beta$ for $n \geqslant 0$ and

$$\widetilde{B}^{\psi}_n(t) = t^n \widetilde{B}^{\psi}_n(\beta^2/t)/(-\beta)^n, \quad n \geqslant 0. \tag{2.3}$$

Now for $\alpha > 0$ and $\beta > 0$, if

$$x(t) = \frac{1}{2\sqrt{\alpha}}(\sqrt{t} - \beta/\sqrt{t}), \quad t \in (0, \infty), \tag{2.4}$$

then from [8] the following results are obtained. For any $A > 0$, let

$$\mathrm{d}\psi(t) = A\frac{t}{t+\beta}\mathrm{d}\phi(x(t)), \tag{2.5}$$

and let $b > 0$ and $d > 0$ be such that $\sqrt{b} = \sqrt{\alpha d^2 + \beta} + \sqrt{\alpha} d$. Then $\mathrm{d}\phi(x)$ is a symmetric distribution with support inside $(-d, d)$ if and only if $\mathrm{d}\psi(t)$ is a $\mathrm{ScS}(\beta^2/b, b)$ distribution. Further, in this case,

$$\widetilde{B}^{\psi}_n(t) = (2\sqrt{\alpha t})^n Q^{\phi}_n(x), \quad n \geqslant 0$$

and (2.6)

$$\widetilde{\alpha}^{\psi}_{n+1} = 4\alpha\alpha^{\phi}_{n+1}, \quad n \geqslant 1.$$

In [8], these results are shown only when $\mathrm{d}\phi(x) = w(x)\mathrm{d}x$ and $\mathrm{d}\psi(t) = v(t)\mathrm{d}t$. When these hold then (2.5) can be written as

$$v(t) = A^* t^{-1/2} w(x(t)),$$

where $A^* > 0$. However, the extension given here is straightforward to obtain from [8].

3. The $\mathrm{S\bar{c}S}(a, b)$ distributions

Another special strong distribution, which was first studied in [9], is the $\mathrm{S\bar{c}S}(a, b)$ distribution. We say that a strong distribution $\mathrm{d}\psi(t)$ with support inside $(\beta^2/b, b)$ is a $\mathrm{S\bar{c}S}(\beta^2/b, b)$ distribution if

$$\mathrm{d}\psi(t) = -\mathrm{d}\psi(\beta^2/b), \quad t \in (\beta^2/b, b).$$

As with the $\mathrm{ScS}(\beta^2/b, b)$ distribution we require $0 < \beta < b \leqslant \infty$. From [9] we have that for any $\mathrm{S\bar{c}S}(\beta^2/b, b)$ distribution $\mathrm{d}\psi(t)$, the following results hold:

If $\widetilde{\gamma}^{\psi}_n = \widetilde{\beta}^{\psi}_n + \widetilde{\alpha}^{\psi}_{n+1}$, then

$$\frac{\widetilde{\gamma}^{\psi}_{n+1}}{\widetilde{\gamma}^{\psi}_n} = \frac{\beta^2}{\widetilde{\beta}^{\psi}_n \widetilde{\beta}^{\psi}_{n+1}}, \quad n \geqslant 0,$$

with $\widetilde{\gamma}^{\psi}_0 = \widetilde{\beta}^{\psi}_0 = 1$. For any $n \geqslant 1$, if $\lambda \in \Re$ and $\eta \in \Re$ are such that

$$(\lambda + \widetilde{\beta}^{\psi}_n)(\eta + \widetilde{\beta}^{\psi}_n) = \widetilde{\beta}^{\psi}_n \, \widetilde{\gamma}^{\psi}_n,$$

then

$$\frac{t^n \widetilde{B}^{\psi}_n(\lambda, \beta^2/t)}{\widetilde{B}^{\psi}_n(\lambda, 0)} = \widetilde{B}^{\psi}_n(\eta, t),$$

where $\widetilde{B}_n^{\psi}(\lambda,t)$ are the polynomials given by (2.1). Only when λ is equal to

$$\lambda_n^{\psi,1} = \sqrt{\widetilde{\beta}_n^{\psi}}\left(+\sqrt{\widetilde{\gamma}_n^{\psi}} - \sqrt{\widetilde{\beta}_n^{\psi}}\right) \quad \text{or} \quad \lambda_n^{\psi,2} = \sqrt{\widetilde{\beta}_n^{\psi}}\left(-\sqrt{\widetilde{\gamma}_n^{\psi}} - \sqrt{\widetilde{\beta}_n^{\psi}}\right),$$

then

$$\frac{t^n \widetilde{B}_n^{\psi}(\lambda,\beta^2/t)}{\widetilde{B}_n^{\psi}(\lambda,0)} = \widetilde{B}_n^{\psi}(\lambda,t). \tag{3.1}$$

All the zeros of $\widetilde{B}_n^{\psi}(\lambda_n^{\psi,1},t)$ lie inside the ineval $(\beta^2/b,b)$. However, only $n-1$ of the zeros of $\widetilde{B}_n^{\psi}(\lambda_n^{\psi,2},t)$ lie within $(\beta^2/b,b)$ and its remaining zero is equal to $-\beta$.

4. Related ScS(a,b) and S$\bar{\text{c}}$S(a,b) distributions

Let the three strong distributions $\mathrm{d}\psi_0(t), \mathrm{d}\psi_1(t)$ and $\mathrm{d}\psi_2(t)$, all having their support inside $(\beta^2/b,b)$, be such that

$$\frac{t+\beta}{t}\mathrm{d}\psi_1(t) = \mathrm{d}\psi_0(t) = \frac{1}{t+\beta}\mathrm{d}\psi_2(t). \tag{4.1}$$

Then the following three statements are equivalent:

$\mathrm{d}\psi_1(t)$ is a ScS$(\beta^2/b,b)$ distribution,
$\mathrm{d}\psi_0(t)$ is a S$\bar{\text{c}}$S$(\beta^2/b,b)$ distribution,
$\mathrm{d}\psi_2(t)$ is a ScS$(\beta^2/b,b)$ distribution.

The proof of this follows from the definitions of the ScS$(\beta^2/b,b)$ and S$\bar{\text{c}}$S$(\beta^2/b,b)$ distributions.

A study involving the right-hand equality of (4.1) has already been considered in [12] and there it was also shown that

$$\widetilde{B}_{n+1}^{\psi_0}(\lambda_{n+1}^{\psi_0,2},t) = (t+\beta)\widetilde{B}_n^{\psi_2}(t), \quad n \geqslant 0.$$

Furthermore, if $\ell_n = \sqrt{\widetilde{\gamma}_n^{\psi_0}/\widetilde{\beta}_n^{\psi_0}}$ then

$$\widetilde{\alpha}_{n+1}^{\psi_2} = \beta(\ell_n - 1)(\ell_{n+1}+1), \quad n \geqslant 1. \tag{4.2}$$

Now we consider the polynomials $\widetilde{B}_n^{\psi_1}(t)$, $n \geqslant 1$. In (1.3) for $\mathrm{d}\psi_1(t)$ substituting t by β^2/u and then using (2.2) and (2.3), we obtain

$$\int_0^{\infty} t^{-n+s}\widetilde{B}_n^{\psi_1}(u)(\beta/u)\,\mathrm{d}\psi_1(u) = 0, \quad 1 \leqslant s \leqslant n.$$

This result, with u replaced by t, added to (1.3) for $\mathrm{d}\psi_1(t)$ gives

$$\int_0^{\infty} t^{-n+s}\widetilde{B}_n^{\psi_1}(t)\,\mathrm{d}\psi_0(t) = 0, \quad 1 \leqslant s \leqslant n-1.$$

Hence, from (2.3) and (3.1), using the result preceding (2.1), gives

$$\widetilde{B}_n^{\psi_0}(\lambda_n^{\psi_{0,1}},t) = \widetilde{B}_n^{\psi_1}(t), \quad n \geqslant 1.$$

This result has already been observed in [10] when

$$d\psi_1(t) = \frac{1}{\sqrt{b-t}\sqrt{t-a}}\,dt,$$

where $a = \beta^2/b$. Since (2.1) can also be written as

$$\widetilde{B}_n^{\psi}(\lambda,t) = \frac{\lambda}{\widetilde{\alpha}_{n+1}^{\psi} t}\widetilde{B}_{n+1}^{\psi}(t) + \left\{\left(1 - \frac{\lambda}{\widetilde{\alpha}_{n+1}^{\psi}}\right) + \frac{\lambda \widetilde{\beta}_{n+1}^{\psi}}{\widetilde{\alpha}_{n+1}^{\psi} t}\right\}\widetilde{B}_n^{\psi}(t), \quad n \geqslant 1,$$

by appropriate substitutions of (2.1) and the above in the recurrence relation for $\widetilde{B}_n^{\psi_1}(t)$, we obtain (see also [10])

$$\widetilde{\alpha}_{n+1}^{\psi_1} = \beta(\ell_n - 1)(\ell_{n-1} + 1), \quad n \geqslant 1, \tag{4.3}$$

where $\ell_n = \sqrt{\widetilde{\gamma}_n^{\psi_0}/\widetilde{\beta}_n^{\psi_0}}$ as before. Since $\widetilde{\gamma}_n^{\psi} = \widetilde{\beta}_n^{\psi} + \widetilde{\alpha}_{n+1}^{\psi}$, we have $\ell_n > 1$ for $n \geqslant 1$.

5. The main results

First we prove Theorem 1. Choosing $k = \alpha/\beta$, we let

$$d\psi_1(t) = A_1\frac{t}{t+\beta}\,d\phi_1(x(t))$$

and

$$d\psi_2(t) = A_2\frac{t}{t+\beta}\,d\phi_2(x(t)),$$

where A_1 and A_2 are positive numbers and $x(t)$ is given by (2.4). Therefore, as in (2.5), $d\psi_1(t)$ is a $\mathrm{ScS}(\beta^2/b, b)$ distribution if and only if $d\phi_1(x)$ is a symmetric distribution with support inside $(-d, d)$. Similarly, $d\psi_2(t)$ is a $\mathrm{ScS}(\beta^2/b, b)$ distribution if and only if $d\phi_2(x)$ is a symmetric distribution with support inside $(-d, d)$. We recall that the relation between b and d is $\sqrt{b} = \sqrt{\alpha d^2 + \beta} + \sqrt{\alpha}d$.

Furthermore, for a suitable choice of A_1 and A_2, we also obtain that the $\mathrm{ScS}(\beta^2/b, b)$ distributions $d\psi_1(t)$ and $d\psi_2(t)$ satisfy (4.1) if and only if the symmetric distributions $d\phi_1(x)$ and $d\phi_2(x)$ satisfy (1.1). Hence, with the use of (2.6) we obtain from (4.2) and (4.3) the result (1.2). This completes the proof of Theorem 1. □

Now we see how (1.2) can be used to obtain information on the sequence $\{\alpha_{n+1}^{\phi_2}\}$ given $\{\alpha_{n+1}^{\phi_1}\}$, and vice versa. First we have from (1.2)

$$\frac{\ell_{n-1}+1}{\ell_{n+1}+1} = \frac{\alpha_{n+1}^{\phi_1}}{\alpha_{n+1}^{\phi_2}}, \quad n \geqslant 1. \tag{5.1}$$

Given $\alpha_{n+1}^{\phi_1}$, $n \geqslant 1$, we also obtain from (1.2) the continued fraction

$$\ell_n - 1 = \frac{4k\alpha_{n+1}^{\phi_1}}{2 + (\ell_{n-1} - 1)},$$
$$= \frac{4k\alpha_{n+1}^{\phi_1}}{2} \, \frac{4k\alpha_{n}^{\phi_1}}{+\ 2} \, + \cdots + \, \frac{4k\alpha_{3}^{\phi_1}}{2} \, \frac{4k\alpha_{2}^{\phi_1}}{+\ 2}, \quad n \geqslant 1.$$

This can also be written as

$$\frac{\ell_n + 1}{2} = 1 + \frac{k\alpha_{n+1}^{\phi_1}}{1} \, \frac{k\alpha_{n}^{\phi_1}}{+\ 1} \, + \cdots + \, \frac{k\alpha_{3}^{\phi_1}}{1} \, \frac{k\alpha_{2}^{\phi_1}}{+\ 1}, \quad n \geqslant 1.$$

Hence, from the theory of continued fractions, see for example [5, p. 71], that

$$\frac{\ell_n + 1}{2} = \frac{b_{n+1}^{(1)}}{b_n^{(1)}}, \quad n \geqslant 0,$$

where

$$b_{n+1}^{(1)} = b_n^{(1)} + k\alpha_{n+1}^{\phi_1} b_{n-1}^{(1)}, \quad n \geqslant 1, \tag{5.2}$$

with $b_0^{(1)} = 1$ and $b_1^{(1)} = 1$. From (5.1) then the following result is obtained.

Theorem 2. *When* (1.1) *holds then*

$$\alpha_{n+1}^{\phi_2} = \frac{b_{n-1}^{(1)} b_{n+2}^{(1)}}{b_n^{(1)} b_{n+1}^{(1)}} \alpha_{n+1}^{\phi_1}, \quad n \geqslant 1,$$

where $b_n^{(1)}, n \geqslant 0$ *are given by* (5.2).

Now, given $\alpha_{n+1}^{\phi_2}$, $n \geqslant 1$, we obtain from (1.2)

$$\frac{\ell_{n+1} + 1}{2} = \frac{k\alpha_{n+1}^{\phi_2}}{-1 + (\ell_n + 1)/2}$$
$$= \frac{k\alpha_{n+1}^{\phi_2}}{-1} \, \frac{k\alpha_{n}^{\phi_2}}{+\ -1} \, + \cdots + \, \frac{k\alpha_{2}^{\phi_2}}{-1} \, \frac{\ell_1 + 1}{+\ 2}, \quad n \geqslant 1.$$

However, since $\ell_1 - 1 = 2k\alpha_2^{\phi_1}$ we find $\ell_1 + 1 = 2\mu_0^{\phi_2}/\mu_0^{\phi_1}$, where $\mu_0^{\phi} = \int_{-\infty}^{\infty} \mathrm{d}\phi(x)$. Hence,

$$\frac{\ell_{n+1} + 1}{2} = \frac{k\alpha_{n+1}^{\phi_2}}{-1} \, \frac{k\alpha_{n}^{\phi_2}}{+\ -1} \, + \cdots + \, \frac{k\alpha_{2}^{\phi_2}}{-1} \, \frac{\mu_0^{\phi_2}/\mu_0^{\phi_1}}{+\ 1}, \quad n \geqslant 1.$$

It is easily verified that

$$\frac{\ell_{n+1} + 1}{2} = \frac{b_n^{(2)}}{b_{n+1}^{(2)}}, \quad n \geqslant 1,$$

where

$$b_{n-1}^{(2)} = b_n^{(2)} + k\alpha_{n+1}^{\phi_2} b_{n+1}^{(2)}, \quad n \geqslant 1, \tag{5.3}$$

with $b_0^{(2)} = 1$ and $b_1^{(2)} = \mu_0^{\phi_1}/\mu_0^{\phi_2}$. Hence, the following result is established.

Theorem 3. *When* (1.1) *holds then*

$$\alpha_2^{\phi_1} = \frac{b_2^{(2)}}{b_1^{(2)}} \alpha_2^{\phi_2}$$

and

$$\alpha_{n+2}^{\phi_1} = \frac{b_{n-1}^{(2)} b_{n+2}^{(2)}}{b_n^{(2)} b_{n+1}^{(2)}} \alpha_{n+2}^{\phi_2}, \quad n \geqslant 1,$$

where the $b_n^{(2)}$, $n \geqslant 0$, *are given by* (5.3).

6. Some special cases

In this section we consider some examples of pairs of symmetric distributions that satisfy the relation (1.1).

Example 1. Let $d = 1$ and let

$$\mathrm{d}\phi_1(x) = \frac{1}{\sqrt{1-x^2}}\mathrm{d}x, \quad \mathrm{d}\phi_2(x) = \frac{1+kx^2}{\sqrt{1-x^2}}\,\mathrm{d}x,$$

where $k > 0$. Since $\alpha_2^{\phi_1} = \frac{1}{2}$ and $\alpha_{n+1}^{\phi_1} = \frac{1}{4}$, $n \geqslant 2$, using the theory of difference equations we obtain from (5.2) that

$$b_0^{(1)} = 1, \quad b_n^{(1)} = \left\{\frac{1+\sqrt{1+k}}{2}\right\}^n + \left\{\frac{1-\sqrt{1+k}}{2}\right\}^n, \quad n \geqslant 1.$$

Hence, we can write

$$\alpha_{n+1}^{\phi_2} = \frac{1}{4}\frac{K_{n-1}K_{n+2}}{K_nK_{n+1}}, \quad n \geqslant 1,$$

where $K_n = \{1+\sqrt{1+k}\}^n + \{1-\sqrt{1+k}\}^n$, $n \geqslant 0$. This result has also been obtained in [8].

Though we have proved the above result for $k > 0$, we believe that it holds for all $k \geqslant -1$. It certainly holds for $k = -1$ and $k = 0$. When $k = 0$, we must take $K_0 = 2$.

Example 2. Let $d = 1$ and suppose that

$$\mathrm{d}\phi_1(x) = \sqrt{1-x^2}\,\mathrm{d}x, \quad \mathrm{d}\phi_2(x) = (1+kx^2)\sqrt{1-x^2}\mathrm{d}x,$$

where $k > 0$. It is known that $\alpha_{n+1}^{\phi_1} = \frac{1}{4}$, $n \geqslant 1$. Hence, from (5.2) it follows that

$$b_n^{(1)} = \frac{1}{\sqrt{1+k}}\left\{\frac{1+\sqrt{1+k}}{2}\right\}^{n+1} - \frac{1}{\sqrt{1+k}}\left\{\frac{1-\sqrt{1+k}}{2}\right\}^{n+1}, \quad n \geqslant 0.$$

Thus, we can write

$$\alpha_{n+1}^{\phi_2} = \frac{1}{4}\frac{L_{n-1}L_{n+2}}{L_nL_{n+1}}, \quad n \geqslant 1,$$

where $L_n = \{1+\sqrt{1+k}\}^{n+1} - \{1-\sqrt{1+k}\}^{n+1}$, $n \geqslant 0$. We believe that this result also holds for any $k \geqslant -1$.

Example 3. For $d = 1$ and $k > 0$, let

$$\mathrm{d}\phi_1(x) = \frac{1}{(1+kx^2)\sqrt{1-x^2}}\mathrm{d}x, \quad \mathrm{d}\phi_2(x) = \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

Here, instead of using Theorem 3 we shall consider a different approach. Since $\alpha_{n+1}^{\phi_2} = \frac{1}{4}$, $n \geqslant 2$, we obtain from (1.2)

$$\ell_n - 1 = \frac{4k\alpha_{n+1}^{\phi_2}}{2+(\ell_{n+1}-1)} = \frac{k}{2+(\ell_{n+1}-1)}, \quad n \geqslant 2.$$

This leads to the convergent periodic continued fraction

$$\ell_n - 1 = \frac{k}{2}\,\frac{k}{+2}\,\frac{k}{+2}+\cdots,$$

the limit of which is $-1+\sqrt{1+k}$. Hence,

$$\ell_n + 1 = 1 + \sqrt{1+k}, \quad n \geqslant 2.$$

Since $\ell_1 - 1 = 2k/(\ell_2+1)$, it also follows that $\ell_1 + 1 = 2\sqrt{1+k}$. Thus, from (5.1),

$$\alpha_2^{\phi_1} = \frac{1}{1+\sqrt{1+k}}, \quad \alpha_3^{\phi_1} = \frac{\sqrt{1+k}}{2(1+\sqrt{1+k})}$$

and

$$\alpha_{n+1}^{\phi_1} = \tfrac{1}{4}, \quad n \geqslant 3.$$

This result has already been obtained in [3] for all $k \geqslant -1$.

Example 4. With $d = 1$ and $k > 0$ let

$$\mathrm{d}\phi_1(x) = \frac{\sqrt{1-x^2}}{(1+kx^2)}\mathrm{d}x, \quad \mathrm{d}\phi_2(x) = \sqrt{1-x^2}\,\mathrm{d}x.$$

Here, since $\alpha_{n+1}^{\phi_2} = \frac{1}{4}$, $n \geqslant 1$, we obtain from (1.2) that $\ell_n + 1 = 1 + \sqrt{1+k}$, $n \geqslant 1$. Hence, from (5.1)

$$\alpha_2^{\phi_1} = \frac{1}{2(1+\sqrt{1+k})}$$

and

$$\alpha_{n+1}^{\phi_1} = \tfrac{1}{4}, \quad n \geqslant 2.$$

This result follows from [2] for all $k \geqslant -1$. For information on the results of the last two examples, see also [1, p. 205].

Example 5. Let $d = 1$ and that for $k > 0$ and $k_1 > 0$, let

$$\mathrm{d}\phi_1(x) = \frac{\sqrt{1-x^2}}{(1+kx^2)(1+k_1x^2)}\,\mathrm{d}x, \quad \mathrm{d}\phi_2(x) = \frac{\sqrt{1-x^2}}{(1+k_1x^2)}\,\mathrm{d}x.$$

From the previous example we note that $\alpha_2^{\phi_2} = \{2(1+\sqrt{1+k_1})\}^{-1}$ and $\alpha_{n+1}^{\phi_2} = \frac{1}{4}$, $n \geqslant 2$. Thus, we obtain from (1.2) that $\ell_n + 1 = 1 + \sqrt{1+k}$, $n \geqslant 2$. Furthermore, also from (1.2)

$$\ell_1 - 1 = \frac{2k}{(1+\sqrt{1+k})(1+\sqrt{1+k_1})},$$

and hence,

$$\ell_1 + 1 = \frac{2[\sqrt{1+k} + \sqrt{1+k_1}]}{(1+\sqrt{1+k_1})}.$$

Consequently, from (5.1)

$$\alpha_2^{\phi_1} = \frac{1}{(1+\sqrt{1+k})(1+\sqrt{1+k_1})}, \quad \alpha_3^{\phi_1} = \frac{(\sqrt{1+k} + \sqrt{1+k_1})}{2(1+\sqrt{1+k})(1+\sqrt{1+k_1})}$$

and

$$\alpha_{n+1}^{\phi_1} = \tfrac{1}{4}, \quad n \geqslant 3.$$

Example 6. Here, with $d = 1$, $k > 0$ and $k_1 > 0$ we take

$$\mathrm{d}\phi_1(x) = \frac{\sqrt{1-x^2}}{(1+k_1x^2)}\,\mathrm{d}x, \quad \mathrm{d}\phi_2(x) = \frac{(1+kx^2)}{(1+k_1x^2)}\sqrt{1-x^2}\,\mathrm{d}x.$$

Since $\alpha_2^{\phi_1} = \{2(1+\sqrt{1+k_1})\}^{-1}$ and $\alpha_{n+1}^{\phi_1} = \frac{1}{4}$, $n \geqslant 2$, we obtain from Theorem 2 that

$$\alpha_2^{\phi_2} = \frac{1}{2(1+\sqrt{1+k_1})}\frac{b_3^{(1)}}{b_2^{(1)}} \quad \text{and} \quad \alpha_{n+1}^{\phi_2} = \frac{1}{4}\frac{b_{n-1}^{(1)}b_{n+2}^{(1)}}{b_n^{(1)}b_{n+1}^{(1)}}, \quad n \geqslant 2.$$

Here, from (5.2)

$$b_n^{(1)} = c_1\left\{\frac{1+\sqrt{1+k}}{2}\right\}^n + c_2\left\{\frac{1-\sqrt{1+k}}{2}\right\}^n, \quad n \geqslant 1,$$

where

$$c_1 = \frac{\sqrt{1+k}+\sqrt{1+k_1}}{\sqrt{1+k}(1+\sqrt{1+k_1})}, \quad c_2 = \frac{\sqrt{1+k}-\sqrt{1+k_1}}{\sqrt{1+k}(1+\sqrt{1+k_1})}.$$

Example 7. For $d=1$, $k>0$ and $k_1>0$ let

$$\mathrm{d}\phi_1(x) = \frac{1}{(1+kx^2)(1+k_1x^2)\sqrt{1-x^2}}\,\mathrm{d}x, \quad \mathrm{d}\phi_2(x) = \frac{1}{(1+k_1x^2)\sqrt{1-x^2}}\,\mathrm{d}x.$$

Using the results of Example 3, we obtain from (1.2) that $\ell_n+1 = 1+\sqrt{1+k}$, $n \geqslant 3$,

$$\ell_2+1 = \frac{2(1+\sqrt{1+k}\sqrt{1+k_1})}{1+\sqrt{1+k_1}}$$

and

$$\ell_1+1 = \frac{2\sqrt{1+k}(\sqrt{1+k}+\sqrt{1+k_1})}{1+\sqrt{1+k}\sqrt{1+k_1}}.$$

Thus, the use of (5.1) gives

$$\alpha_2^{\phi_1} = \frac{1}{1+\sqrt{1+k}\sqrt{1+k_1}},$$

$$\alpha_3^{\phi_1} = \frac{\sqrt{1+k}\sqrt{1+k_1}(\sqrt{1+k}+\sqrt{1+k_1})}{(1+\sqrt{1+k})(1+\sqrt{1+k_1})(1+\sqrt{1+k}\sqrt{1+k_1})},$$

$$\alpha_4^{\phi_1} = \frac{(1+\sqrt{1+k}\sqrt{1+k_1})}{2(1+\sqrt{1+k})(1+\sqrt{1+k_1})}$$

and

$$\alpha_{n+1}^{\phi_1} = \tfrac{1}{4}, \quad n \geqslant 4.$$

Example 8. Finally, for $d=1$, $k>0$ and $k_1>0$ let

$$\mathrm{d}\phi_1(x) = \frac{1}{(1+k_1x^2)\sqrt{1-x^2}}\,\mathrm{d}x, \quad \mathrm{d}\phi_2(x) = \frac{(1+kx^2)}{(1+k_1x^2)\sqrt{1-x^2}}\,\mathrm{d}x.$$

Then from Theorem 2

$$\alpha_2^{\phi_2} = \frac{1}{1+\sqrt{1+k_1}}\frac{b_3^{(1)}}{b_2^{(1)}}, \quad \alpha_3^{\phi_2} = \frac{\sqrt{1+k_1}}{2(1+\sqrt{1+k_1})}\frac{b_4^{(1)}}{b_2^{(1)}b_3^{(1)}}$$

and

$$\alpha_{n+1}^{\phi_2} = \frac{1}{4}\frac{b_{n-1}^{(1)}b_{n+2}^{(1)}}{b_n^{(1)}b_{n+1}^{(1)}}, \quad n \geqslant 3.$$

Here, from (5.2) it follows that

$$b_n^{(1)} = c_1 \left\{ \frac{1+\sqrt{1+k}}{2} \right\}^{n-1} - c_2 \left\{ \frac{1-\sqrt{1+k}}{2} \right\}^{n-1}, \quad n \geqslant 2,$$

where

$$c_1 = \frac{\sqrt{1+k}+\sqrt{1+k_1}}{(1+\sqrt{1+k_1})}, \quad c_2 = \frac{\sqrt{1+k}-\sqrt{1+k_1}}{(1+\sqrt{1+k_1})}.$$

In all four of the above examples we believe that the results hold for $k > -1$ and $k_1 > -1$.

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