Comment on "Fun and frustration with quarkonium in a 1+1 dimension," by R. S. Bhalerao and B. Ram [Am. J. Phys. 69(7), 817–818 (2001)]

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NOTES AND DISCUSSIONS

Comment on "Fun and frustration with quarkonium in a 1+1 dimension," by R. S. Bhalerao and B. Ram [Am. J. Phys. 69(7), 817–818 (2001)]

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In a recent article in this journal, Bhalerao and Ram[1] approached the Dirac equation in a 1+1 dimension with the scalar potential

$$V(x) = g|x|. \tag{1}$$

For $x > 0$ the Dirac equation,

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{d}{dx} + (m+gx) \right] \psi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} E\psi, \tag{2}$$

reduces to

$$\left(-\frac{d^2}{d\xi^2} + \xi^2 \right)\psi_1 = \left(\frac{E^2}{g} + 1 \right)\psi_1,$$
$$\left(-\frac{d^2}{d\xi^2} + \xi^2 \right)\psi_2 = \left(\frac{E^2}{g} - 1 \right)\psi_2, \tag{3}$$

with $\xi = \sqrt{g}(m/g + x)$; ψ_1 and ψ_2 are the upper and the lower components of the bispinor ψ, respectively. For $x < 0$ the Dirac equation takes the form

$$\left(-\frac{d^2}{d\xi'^2} + \xi'^2 \right)\psi_1 = \left(\frac{E^2}{g} - 1 \right)\psi_1,$$
$$\left(-\frac{d^2}{d\xi'^2} + \xi'^2 \right)\psi_2 = \left(\frac{E^2}{g} + 1 \right)\psi_2, \tag{4}$$

with $\xi' = \sqrt{g}(m/g - x)$. The authors of Ref. 1 state that the solutions of Eqs. (3) are

$$\psi_1 = C H_{n+1}(\xi) e^{-\xi^2/2},$$
$$\psi_2 = D H_n(\xi) e^{-\xi^2/2}, \tag{5}$$

where $H_n(\xi)$ are the Hermite polynomials. Similar solutions are found for $x < 0$. By substituting Eq. (5) into the Dirac equation, they found[1]

$$E = \pm \sqrt{2(n+1)g}. \tag{6}$$

The continuity of the wave function at $x = 0$ leads to the quantization condition

$$H_{n+1}(\alpha) = \pm [2(n+1)]^{1/2} H_n(\alpha), \tag{7}$$

with

$$\alpha = \frac{m}{\sqrt{g}}. \tag{8}$$

The authors[1] found only the ground-state solution ($n = 0$) for $\alpha = 1/\sqrt{2}$. No solutions were found by the numerical evaluation for n from 1 to 250. Hence, they concluded that *it is not possible to describe the meson spectrum with Eq. (1) as the quark–antiquark potential when used as a Lorentz scalar in the* (1+1)-*dimensional Dirac equation.*[1]

Equations (3) and (4) can be cast into the form

$$\frac{d^2\psi_1}{d\eta^2} - \left(\frac{\eta^2}{4} - \nu - \frac{3}{2} \right)\psi_1 = 0,$$
$$\frac{d^2\psi_2}{d\eta^2} - \left(\frac{\eta^2}{4} - \nu - \frac{1}{2} \right)\psi_2 = 0, \tag{9}$$

$$\frac{d^2\psi_1}{d\eta'^2} - \left(\frac{\eta'^2}{4} - \nu - \frac{1}{2} \right)\psi_1 = 0,$$
$$\frac{d^2\psi_2}{d\eta'^2} - \left(\frac{\eta'^2}{4} - \nu - \frac{3}{2} \right)\psi_2 = 0, \tag{10}$$

where $\eta = \sqrt{2}\xi$, $\eta' = \sqrt{2}\xi'$, and

$$\frac{E^2}{2g} = \nu + 1. \tag{11}$$

The second-order differential equations (9)–(10) have the form

$$y''(z) - \left(\frac{z^2}{4} + a \right)y(z) = 0, \tag{12}$$

whose solution is a parabolic cylinder function.[2] The solutions $D_{-a-1/2}(z)$ and $D_{-a-1/2}(-z)$ are linearly independent unless $n = -a - 1/2$ is a non-negative integer. In this special circumstance $D_n(z)$ has the peculiar property that $D_n(-z) = (-1)^n D_n(z)$, and $D_n(z)$ is proportional to $\exp(-z^2/4)H_n(z/\sqrt{2})$. In general, the solutions do not exhibit this parity property and can instead be expressed as

$$\psi(x>0) = \begin{pmatrix} C D_{\nu+1}(\eta) \\ D D_\nu(\eta) \end{pmatrix},$$

$$\psi(x<0) = \begin{pmatrix} C' D_\nu(\eta') \\ D' D_{\nu+1}(\eta') \end{pmatrix}, \tag{13}$$

where the physically acceptable solutions $D_\nu(z)$ must vanish in the asymptotic region $|z| \to \infty$. When these solutions are inserted into the Dirac equation, one obtains

$$\frac{D}{C}=\frac{C'}{D'}=\frac{E}{\sqrt{2g}}. \tag{14}$$

In the last step the recurrence formula,

$$D'_\nu(z)-z/2D_\nu(z)+D_{\nu+1}(z)=0, \tag{15}$$

where the prime denotes differentiation with respect to z, has been used to obtain $D'_\nu(z)$. On the other hand, the matching condition at $x=0$ leads to the quantization condition

$$D_{\nu+1}(\sqrt{2}\alpha)=\pm\sqrt{\nu+1}D_\nu(\sqrt{2}\alpha), \tag{16}$$

where α is given by Eq. (8). This condition would be equivalent to Eq. (15) of Ref. 1 if ν were to be restricted to a non-negative integer. However, our derivation of the quantization condition does not involve such a restriction which should not be imposed because the differential equations for ψ_1 and ψ_2, Eq. (9) for $x>0$ and Eq. (10) for $x<0$, are not invariant for $x\rightarrow-x$.

By solving the quantization condition (16) for ν with the requirement that the solutions of Eq. (12) vanish in $z\rightarrow+\infty$ limit, one obtains the allowed energy levels by inserting the possible values of ν in Eq. (11). In passing, one should note that those energy levels are symmetrical about

$E=0$. The numerical solution of Eq. (16) is substantially simpler when $D_{\nu+1}(\sqrt{2}\alpha)$ is written in terms of $D'_\nu(\sqrt{2}\alpha)$:

$$D'_\nu(\sqrt{2}\alpha)=\left[\frac{\alpha}{\sqrt{2}}\mp\sqrt{\nu+1}\right]D_\nu(\sqrt{2}\alpha). \tag{17}$$

Because the normalization of the spinor is not important for the calculation of the spectrum, one can arbitrarily choose $D_\nu(\sqrt{2}\alpha)=1$. By using a Fehlberg fourth–fifth order Runge–Kutta method[3] to solve in Eq. (17), an infinite sequence of possible values of ν are found. For $\alpha=1/\sqrt{2}$, the minus sign gives 1.580×10^{-4}, 2.681, 5.038,..., whereas the plus sign gives 1.525, 3.915, 6.210,... .

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[1] R. S. Bhalerao and B. Ram, "Fun and frustration with quarkonium in a 1+1 dimension," Am. J. Phys. **69**, 817–818 (2001).
[2] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, Toronto, 1965).
[3] G. E. Forsythe, M. A. Malcolm, and C. B. Moler, *Computer Methods for Mathematical Computation* (Prentice–Hall, Englewood Cliffs, NJ, 1977).

Comment on "Fun and frustration with quarkonium in a 1+1 dimension," by R. S. Bhalerao and B. Ram [Am. J. Phys. 69 (7), 817–818 (2001)]

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In a recent paper, Bhalerao and Ram[1] solved the Dirac equation in $1+1$ dimensions with a Lorentz scalar potential given by

$$V(x)=g|x| \quad (g>0) \tag{1}$$

and came to the conclusion that the energy levels are given by

$$E=\pm\sqrt{2(n+1)g} \quad (n\in\mathcal{I}), \tag{2}$$

where $\mathcal{I}$ is the set of non-negative integers that satisfy one of the conditions

$$H_{n+1}(m/\sqrt{g})=\pm\sqrt{2(n+1)}H_n(m/\sqrt{g}). \tag{3}$$

$H_n(z)$ is the Hermite polynomial of degree n and m is the fermion mass. The purpose of this Comment is to point out that the energy spectrum obtained in Ref. 1 is not correct, and to derive the correct one.

Our starting point is Eq. (6a) of Ref. 1. Its general solution is given by[2]

$$\psi_1(\xi)=CH_\nu(\xi)e^{-\xi^2/2}+DH_{-\nu-1}(i\xi)e^{\xi^2/2}, \tag{4}$$

where $\nu\equiv E^2/2g$, $\xi\equiv(m+gx)/\sqrt{g}$, and $H_\nu(z)$ is the Hermite function,[3] defined as

$$H_\nu(z)=\frac{2^\nu\Gamma(\frac{1}{2})}{\Gamma\left(\frac{1-\nu}{2}\right)}\Phi\left(-\frac{\nu}{2},\frac{1}{2};z^2\right)$$

$$+\frac{2^\nu\Gamma(-\frac{1}{2})}{\Gamma\left(-\frac{\nu}{2}\right)}z\Phi\left(\frac{1-\nu}{2},\frac{3}{2};z^2\right). \tag{5}$$

The confluent hypergeometric function $\Phi(a,b;z)$ is expressed as

$$\Phi(a,b;z)=\sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k}\frac{z^k}{k!} \tag{6}$$

with $(a)_0=1$, $(a)_k=\Gamma(a+k)/\Gamma(a)=a(a+1)\cdots(a+k-1)$, $k=1,2,\dots$.

For large z and fixed ν, $H_\nu(z)$ behaves asymptotically as

$$H_\nu(z)\sim(2z)^\nu \quad (|z|\rightarrow\infty) \tag{7}$$

provided that $|\arg z|\leq 3\pi/4-\delta$. Therefore, in order that $\psi_1(x)$ be square integrable at $x\rightarrow\infty$, it is necessary and sufficient to take $D=0$ in Eq. (4), *regardless of the value of ν*. From Eq. (5a) of Ref. 1 and the identity $H'_\nu(\xi)$

ν	$\alpha=0$	$\alpha=1$	$\alpha=2$
ν_0	0.345 459	1.396 274	3.338 595
ν_1	1.548 571	3.056 760	5.452 161
ν_2	2.468 573	4.306 277	7.006 087
ν_3	3.522 295	5.615 211	8.568 946
ν_4	4.482 395	6.804 771	9.978 608

$=2\nu H_{\nu-1}(\xi)$, it then follows that the second component of the Dirac bispinor ψ is given by

$$\psi_2(\xi) = C\frac{E}{\sqrt{g}}H_{\nu-1}(\xi)e^{-\xi^2/2}. \tag{8}$$

Equations (4) (with $D=0$) and (8) are valid for $x \geq 0$; for $x \leq 0$ they must be replaced by

$$\psi_1(\xi') = C'\frac{E}{\sqrt{g}}H_{\nu-1}(\xi')e^{-\xi'^2/2}, \tag{9a}$$

$$\psi_2(\xi') = C'H_\nu(\xi')e^{-\xi'^2/2}, \tag{9b}$$

where $\xi' \equiv (m-gx)/\sqrt{g}$. The continuity of ψ_1 and ψ_2 at $x=0$ leads to the condition

$$H_\nu^2(m/\sqrt{g}) - 2\nu H_{\nu-1}^2(m/\sqrt{g}) = 0. \tag{10}$$

This equation is equivalent to Eq. (15) of Ref. 1 [Eq. (3) of this Comment] if n is not required to be a non-negative integer. In Ref. 1, however, this equation is a supplementary condition imposed on the previously determined energy levels (2). Here, in contrast, the energy levels are determined solely by Eq. (10).

It remains to be seen whether Eq. (10) has any solutions. In the Appendix, I prove that it has an infinite number of solutions if $m=0$. Unfortunately, I have not been able to generalize the proof to the $m \neq 0$ case. However, a numerical attack to the problem reveals the existence of solutions to Eq. (10) for $m \neq 0$. Table I lists the first five solutions to that equation for several values of m.

APPENDIX

Let me show that Eq. (10) has an infinite number of solutions if $m=0$. Indeed, in this case Eq. (10) reduces to

$$f(\nu) \equiv H_\nu^2(0) - 2\nu H_{\nu-1}^2(0) = 0. \tag{A1}$$

It is clear that ν cannot be negative, for then $f(\nu)$ is strictly positive. If $\nu=0$ or $\nu=2k$, k a positive integer, then $2\nu H_{\nu-1}^2(0)$ vanishes [in the latter case because $H_{2k-1}(x)$ is an odd function of x] and $H_\nu^2(0) > 0$. Therefore, $f(\nu) > 0$ if ν is a non-negative even integer. On the other hand, $f(\nu) < 0$ if ν is a positive odd integer, for then $H_\nu(0) = 0$ and $2\nu H_{\nu-1}^2(0) > 0$. Because $f(\nu)$ is a continuous function of ν, one is led to the conclusion that $f(\nu)$ vanishes at least once in each interval $(n, n+1)$, $n=0,1,2,...$. A more refined analysis, in which one uses the identity [cf. Eq. (5)]

$$H_\nu(0) = \frac{2^\nu \Gamma(\frac{1}{2})}{\Gamma\left(\frac{1-\nu}{2}\right)}, \tag{A2}$$

shows that $f(\nu)$ in fact vanishes only once in these intervals. It follows that the energy levels in the massless case are given by

$$E = \pm \sqrt{2\nu_n g}, \tag{A3}$$

with ν_n satisfying $n < \nu_n < n+1$, $n=0,1,2,...$.

[1]R. S. Bhalerao and B. Ram, "Fun and frustration with quarkonium in a 1+1 dimension," Am. J. Phys. **69** (7), 817–818 (2001).
[2]N. N. Lebedev, *Special Functions and Their Applications* (Dover, New York, 1972), Chap. 10.
[3]The Hermite function $H_\nu(z)$ should not be confused with the Hermite polynomials. The latter are a special case of the former, when ν is a non-negative integer.

Note on "Comment on 'A check of Prigogine's theorem of minimum entropy production in a rod in a nonequilibrium stationary state,' by Irena Danielewicz-Ferchmin and A. Ryszard Ferchmin" [Am. J. Phys. 68 (10), 962–965 (2000)], by Peter Palffy-Muhoray [Am. J. Phys. 69 (7), 825–826 (2001)]

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Peter Palffy-Muhoray has made a very interesting comment on the entropy production associated with a temperature profile $T(x)$ in a heat conducting rod.[1] With constant thermal conductivity, he shows that an *exponential* temperature profile, fitted to the hot and cold boundary temperatures of the rod, results in lower entropy production than does the *linear* profile observed by Irena Danielewicz-Ferchmin and A. Ryszard Ferchmin.[2] Palffy-Muhoray correctly points out

that despite its *lower* entropy production, the exponential profile has never been observed. Thus Prigogine's theorem (minimum entropy production for observed nonequilibrium stationary states)[3] is violated in this case.

Prigogine's theorem states that "in the linear regime, the total entropy production in a system subject to flow of energy and matter...reaches a minimum value at the nonequilibrium steady state."[3] Alternatively, the theorem can be stated as "...the steady state of a system in which an irreversible pro-cess occurs is that state for which the rate of entropy production has the minimum value consistent with the constraints which prevent the system from reaching equilibrium."[4] Reference 4 continues: "This attractively simple criterion for the steady state is, unfortunately, not always valid." Klein illustrates this point by considering the evolution of a simple nonequilibrium problem, for which the minimum entropy production solution is close to, but different from the actual solution.[4]
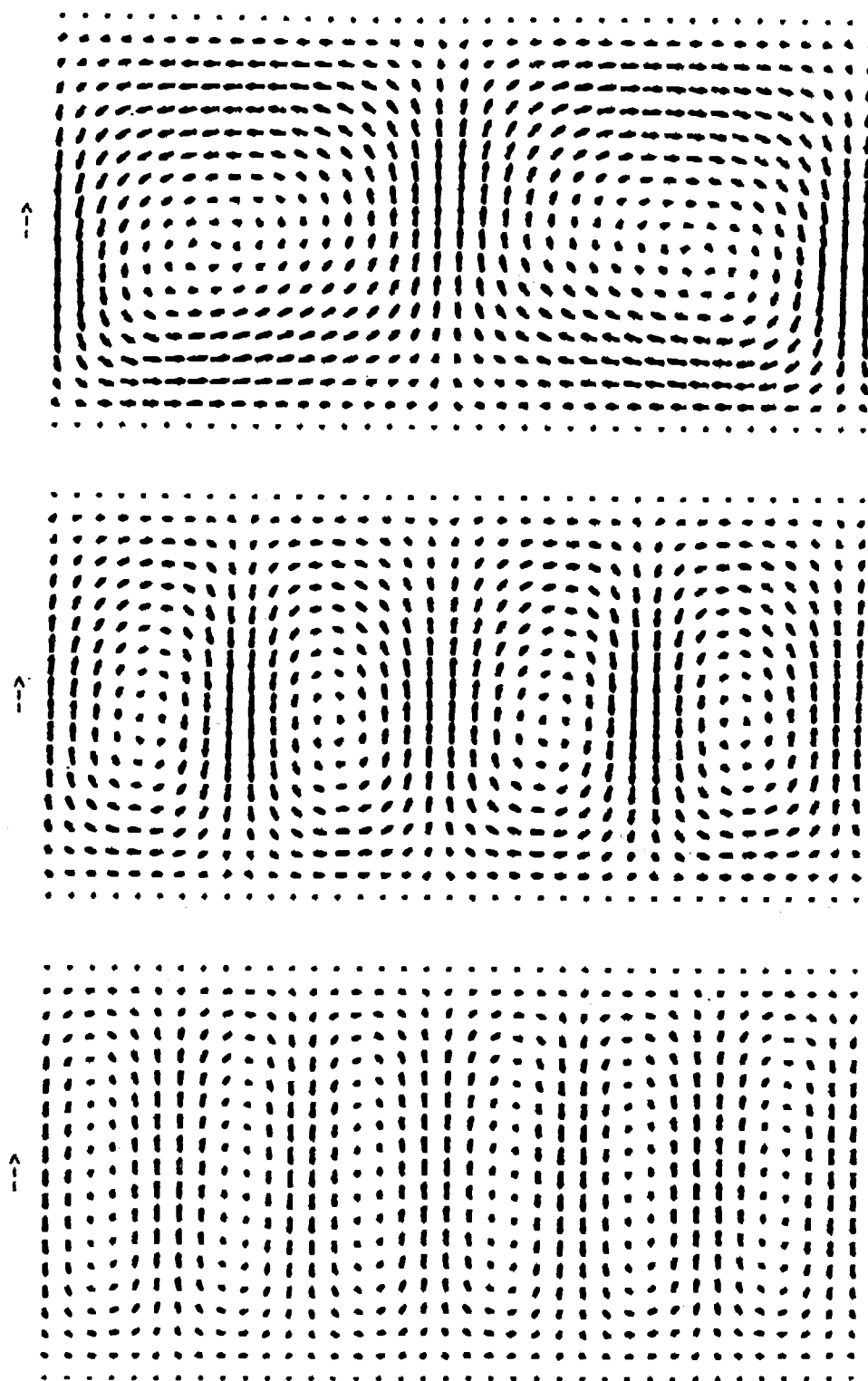

Fig. 1. Two-roll, four-roll, and six-roll stationary stable flows for a two-dimensional ideal gas with Rayleigh number of 40 000 and Prandtl number of unity.

Because the exponential profile is not stationary (because $\partial T/\partial t \propto \partial^2 T/\partial x^2$ is nonzero), one might argue that this profile does not constitute a violation of the theorem. A different type of failure of the theorem can result in nonequilibrium problems that have several stable stationary solutions, not just one (for fixed boundary conditions). Two-dimensional Rayleigh–Bénard flow is an example.[5] In this case heat flows upward, against gravity, in a heat-conducting viscous compressible fluid. The fluid is confined between reservoirs maintained at different boundary temperatures. There can be several different stationary solutions, with the observed one depending (sensitively) on the initial conditions of the fluid. Figure 1 shows three different Rayleigh–Bénard flows for an ideal gas law fluid with constant viscosity and conductivity. The dimensionless ratio of the kinematic viscosity and the thermal diffusivity is unity. The three stationary flows are stable to small perturbations, match the boundary conditions, vanishing flow velocity and fixed temperature at the top and bottom boundaries, and satisfy all three conservation laws. The six-roll solution has the least energy of the three. The rate of growth from a motionless initial condition is fastest for the four-roll solution. The entropy productions for the three solutions are not at all equal:[5]

$$\dot{S}_6 < \dot{S}_4 < \dot{S}_2.$$

Nevertheless, by choosing appropriate initial conditions any one of the three solutions can result.[5] The existence of the three solutions shows that—*even in the restricted case of stationary states*—Prigogine's theorem does not hold.

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[1]Peter Palffy-Muhoray, "Comment on 'A check of Prigogine's theorem of minimum entropy production in a rod in a nonequilibrium stationary state,' " Am. J. Phys. **69** (7), 825–826 (2001).

[2]Irena Danielewicz-Ferchmin and A. Ryszard Ferchmin, "A check of Prigogine's theorem of minimum entropy production in a rod in a nonequilibrium stationary state," Am. J. Phys. **68** (10), 962–965 (2000).

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