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Chain sequences and symmetric generalized orthogonal polynomials ☆

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Abstract

In this paper, we derive an explicit expression for the parameter sequences of a chain sequence in terms of the corresponding orthogonal polynomials and their associated polynomials. We use this to study the orthogonal polynomials $K_n^{(\lambda,M,k)}$ associated with the probability measure $d\phi(\lambda,M,k;x)$, which is the Gegenbauer measure of parameter $\lambda+1$ with two additional mass points at $\pm k$. When $k=1$ we obtain information on the polynomials $K_n^{(\lambda,M)}$ which are the symmetric Koornwinder polynomials. Monotonicity properties of the zeros of $K_n^{(\lambda,M,k)}$ in relation to M and k are also given.

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1. Introduction

Koornwinder [6] considered the polynomials $P_n^{\alpha,\beta,M,N}$ defined by

$$P_n^{\alpha,\beta,M,N}(x)=\left[\frac{(s)_n}{n!}\right]^2\left\{\frac{1}{s}[(1-x)MB_n-(1+x)NA_n]\frac{d}{dx}+A_nB_n\right\}P_n^{(\alpha,\beta)}(x)$$

for $n\geqslant 0$, where $s=\alpha+\beta+1$,

$$A_n=\frac{(\alpha+1)_n n!}{(\beta+1)_n(s)_n}+\frac{n(n+s)}{s(\beta+1)}M,\quad B_n=\frac{(\beta+1)_n n!}{(\alpha+1)_n(s)_n}+\frac{n(n+s)}{s(\alpha+1)}N$$

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and $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials. He showed that these polynomials are the orthogonal polynomials associated with the measure $d\phi$ given by

$$\int_{-\infty}^{\infty} f(x)\,d\phi(x)$$
$$= Mf(-1) + Nf(1) + \frac{\Gamma(s+1)}{2^s\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1} f(x)(1-x)^\alpha(1+x)^\beta\,dx.$$

The three term recurrence relation associated with these polynomials were given by Kiesel and Wimp [4]. They also explicitly obtain the second-order differential equation satisfied by these polynomials.

In case $\alpha=\beta=\lambda-\frac{1}{2}$ and $M=N$ we get the symmetric Koornwinder polynomials $P_n^{\lambda-1/2,\lambda-1/2,M,M}$, which we write $K_n^{(\lambda,M)}$. Koekoek [5] refers to these as the symmetric generalized ultraspherical polynomials, derives all the differential equations of the form

$$\sum_{i=0}^{\infty} c_i(x)y^{(i)}(x) = 0, \quad y(x) = K_n^{(\lambda,M)}(x),$$

where $c_i(x)$ are independent of the degree n.

In this paper, we consider the orthogonal polynomials $K_n^{(\lambda,M,k)}$ associated with the probability measure $d\phi(\lambda,M,k;x)$ given by

$$\int_{-\infty}^{\infty} f(x)\,d\phi(\lambda,M,k;x)$$
$$= \frac{1}{2M+1}\left\{M[f(-k)+f(k)] + \{\tilde{S}^{(\lambda)}(k)\}^{-1}\int_{-1}^{1} f(x)\frac{(1-x^2)^{\lambda+1/2}}{k^2-x^2}\,dx\right\}, \tag{1.1}$$

where

$$\tilde{S}^{(\lambda)}(k) = \int_{-1}^{1}\frac{(1-x^2)^{\lambda+1/2}}{k^2-x^2}\,dx.$$

Here k,λ are real numbers such that $k>1$ and $\lambda > -\frac{3}{2}$ or $k=1$ and $\lambda > -\frac{1}{2}$.

Note that if $k=1$ then we have the symmetric Koornwinder polynomials $K_n^{(\lambda,M)} = K_n^{(\lambda,M,1)}$ and the probability measure given by

$$\int_{-\infty}^{\infty} f(x)\,d\phi(\lambda,M,1;x)$$
$$= \frac{1}{2M+1}\left\{M[f(-1)+f(1)] + \frac{2^{-2\lambda}\Gamma(2\lambda+1)}{[\Gamma(\lambda+1/2)]^2}\int_{-1}^{1} f(x)(1-x^2)^{\lambda-1/2}\,dx\right\}.$$

The principal tool used in this study is the chain sequences.

2. Preliminary results

Let $d\phi$ be a determinate measure defined over the real line. Let $\mathrm{Supp}(\phi)$ be the support of this measure and let I^ϕ be the smallest closed interval containing this support. We denote by

$X^\phi = (-\infty,\infty)\backslash I^\phi$ the complement of the interval I^ϕ in $(-\infty,\infty)$. Like wise $Z^\phi = \bar{\mathbb{C}}\backslash I^\phi$, where $\bar{\mathbb{C}}$ is the extended complex plane. The closure of X^ϕ within the real line will be denoted by $\tilde{X}^\phi$. Here we have assumed that the end points of I^ϕ are points of $\tilde{X}^\phi$, but $\pm\infty$ are not.

Let $\{P_n^\phi\}$ be the monic orthogonal polynomials associated with $d\phi$ and $\{O_n^\phi\}$ be the monic associated polynomials given by $\mu_0^\phi O_n^\phi(z) = \int_{-\infty}^{\infty} \frac{P_n^\phi(z)-P_n^\phi(x)}{z-x}\, d\phi(x), n \geqslant 0$. Here $\mu_r^\phi = \int_{-\infty}^{\infty} x^r\, d\phi(x)$. The following results (see [2] or [8]) are well known.

$$\left.\begin{aligned} P_{n+1}^\phi(z) &= (z-\beta_{n+1}^\phi)P_n^\phi(z) - \alpha_{n+1}^\phi P_{n-1}^\phi(z),\\ O_{n+1}^\phi(z) &= (z-\beta_{n+1}^\phi)O_n^\phi(z) - \alpha_{n+1}^\phi O_{n-1}^\phi(z),\end{aligned}\right\} \quad n \geqslant 1, \tag{2.1}$$

with $P_0^\phi(z)=1$, $P_1^\phi(z) = z-\beta_1^\phi$, $O_0^\phi(z)=0$ and $O_1^\phi(z)=1$. The coefficients $\beta_n^\phi, n \geqslant 1$ are real, α_{n+1}^ϕ, $n \geqslant 1$ are all positive, and furthermore

$$\frac{O_n^\phi(z)}{P_n^\phi(z)} = \frac{1}{z-\beta_1^\phi} - \frac{\alpha_2^\phi}{z-\beta_2^\phi} - \frac{\alpha_3^\phi}{z-\beta_3^\phi} - \cdots - \frac{\alpha_n^\phi}{z-\beta_n^\phi}$$

and $\mu_0^\phi O_n^\phi(z)/P_n^\phi(z) \to \int_{-\infty}^{\infty}(z-x)^{-1}\, d\phi(x)$ uniformly on every compact subsets of Z^ϕ.

Now we let $S_n^\phi(z) = (z-\beta_1^\phi)O_n^\phi(z)/P_n^\phi(z)$, $n \geqslant 0$. Hence $S_0^\phi(z)=0$, $S_1^\phi(z)=1$ and one can write

$$S_n^\phi(z) = \frac{1}{1} - \frac{a_1^\phi(z)}{1} - \frac{a_2^\phi(z)}{1} - \cdots - \frac{a_{n-1}^\phi(z)}{1}, \quad n \geqslant 2, \tag{2.2}$$

where $a_n^\phi(z) = \alpha_{n+1}^\phi/[(z-\beta_n^\phi)(z-\beta_{n+1}^\phi)]$, $n \geqslant 1$. We also define $S^\phi(z)$ on Z^ϕ by

$$S^\phi(z) = \lim_{n\to\infty} S_n^\phi(z) = \frac{z-\beta_1^\phi}{\mu_0^\phi}\int_{-\infty}^{\infty}\frac{1}{z-x}\, d\phi(x).$$

For $S_n^\phi(z)$ the following results also hold.

$$\left.\begin{aligned} S_n^\phi(z) - S_{n-1}^\phi(z) &= \frac{\alpha_2^\phi\alpha_3^\phi\cdots\alpha_n^\phi(z-\beta_1^\phi)}{P_{n-1}^\phi(z)P_n^\phi(z)},\\ S_n^\phi(z) &= 1 + \sum_{j=2}^{n}\frac{\alpha_2^\phi\alpha_3^\phi\cdots\alpha_j^\phi(z-\beta_1^\phi)}{P_{j-1}^\phi(z)P_j^\phi(z)},\end{aligned}\right\} \quad n \geqslant 2. \tag{2.3}$$

This means that for any $z \in \tilde{X}^\phi$, $S_n^\phi(z) > 1$ for $n > 1$ and that $\{S_n^\phi(z)\}$ is an increasing sequence. That is, $1 < S_2^\phi(z) < \cdots < S_n^\phi(z) < \cdots < S^\phi(z)$, for any $z \in \tilde{X}^\phi$.

Note that if $d\phi$ is a symmetric determinate measure then $\beta_n^\phi = 0$, $n \geqslant 1$ and $S^\phi(z) = (\mu_0^\phi)^{-1}\int_{-\infty}^{\infty} z^2/(z^2-x^2)\, d\phi(x)$. Furthermore, if $d\phi_1$ and $d\phi_2$ are two symmetric measures related to each other by

$$c\, d\phi_2(x) = (k^2-x^2)\, d\phi_1(x), \tag{2.4}$$

then the relation between the polynomials $\{P_n^{\phi_1}\}$ and $\{P_n^{\phi_2}\}$ given by the Christoffel's formula is

$$(x^2-k^2)P_n^{\phi_2}(x) = P_{n+2}^{\phi_1}(x) - \frac{P_{n+2}^{\phi_1}(k)}{P_n^{\phi_1}(k)}P_n^{\phi_1}(x), \quad n \geqslant 0.$$

Other relations between these polynomials and their coefficients of the recurrence relations can be written in the following way. If (2.4) holds there exists a sequence of real numbers $\{\ell_n\}$, with $\ell_0 = 1$, such that

$$
\begin{aligned}
&P_n^{\phi_1}(z) = P_n^{\phi_2}(z) - \frac{k^2}{4}(1-\ell_n)(1-\ell_{n-1})P_{n-2}^{\phi_2}(z), \\
&\alpha_{n+1}^{\phi_1} = \frac{k^2}{4}(1-\ell_n)(1+\ell_{n-1}) \quad \text{and} \quad \alpha_{n+1}^{\phi_2} = \frac{k^2}{4}(1-\ell_n)(1+\ell_{n+1})
\end{aligned} \tag{2.5}
$$

for $n \geqslant 1$. These latter results, first observed in [7] for $k^2 < 0$ and $c < 0$, have been completely proved in [1].

It is clear that the measures $d\phi_1$ and $d\phi_2$ must have their common support in a subset of the real line inside which $(k^2 - x^2)$ does not change sign. Therefore, any choice of k such that $k^2 > 0$ is only possible if the support is contained within a region $[-d_2, -d_1] \cup [d_1, d_2]$, where if $d_1 = 0$ then $d_2 < \infty$ and if $d_2 = \infty$ then $d_1 > 0$. When $d_1 > 0$ then $0 < k < d_1$ (or sometimes $0 < k \leqslant d_1$) with $c < 0$ and when $d_2 < \infty$ then $k > d_2$ (or sometimes $k \geqslant d_2$) with $c > 0$ can be admitted.

Finally in this section, we recall some information on chain sequences. The sequence $\{a_n\}_{n=1}^{\infty}$ is known as a chain sequence, or more precisely a positive chain sequence, if there exists a second sequence $\{g_n\}_{n=0}^{\infty}$ such that $0 \leqslant g_0 < 1, 0 < g_n < 1$ and $(1-g_{n-1})g_n = a_n$, $n \geqslant 1$. The sequence $\{g_n\}$ is called a parameter sequence of the positive chain sequence and is in general not unique. The minimal parameter sequence $\{m_n\}_{n=0}^{\infty}$ of a positive chain sequence $\{a_n\}$ is given by $m_0 = 0$, $0 < m_n < 1$ and $(1 - m_{n-1})m_n = a_n$, $n \geqslant 1$. When the parameter sequence is not unique we can talk about the maximal parameter sequence $\{M_n\}$ such that if $g_0 > M_0$ and $(1 - g_{n-1})g_n = a_n$, $n \geqslant 1$, then $\{g_n\}$ does not satisfy the condition $0 < g_n < 1$ for all $n \geqslant 1$. The maximal parameter sequence of the positive chain sequence $\{a_n\}_{n=1}^{\infty}$ can be given as

$$
M_n = 1 - \frac{a_{n+1}}{1} \, \underset{-}{} \, \frac{a_{n+2}}{1} \, \underset{-}{} \, \frac{a_{n+3}}{1} \, \underset{-}{} \, \frac{a_{n+4}}{1} \, \underset{-}{} \cdots, \qquad n \geqslant 0.
$$

One of the main contributors highlighting the importance of positive chain sequences in the theory of orthogonal polynomials is Chihara and many results on these chain sequences can be found in his book [2]. The formal theory of chain sequences (in a slightly more general form) was given earlier by Wall (see for example [9]).

A situation in the theory of orthogonal polynomials where chain sequences appear naturally is the following. From the recurrence relation (2.1) one can write (see [2, Chapter IV, Theorem 2.4])

$$
[1 - m_{n-1}^{\phi}(z)]m_n^{\phi}(z) = \frac{\alpha_{n+1}^{\phi}}{(z-\beta_n^{\phi})(z-\beta_{n+1}^{\phi})} = a_n^{\phi}(z), \quad n \geqslant 1, \tag{2.6}
$$

where $m_n^{\phi}(z) = [1 - P_{n+1}^{\phi}(z)/((z-\beta_{n+1}^{\phi})P_n^{\phi}(z))] = (P_{n-1}^{\phi}(z)/P_n^{\phi}(z))(\alpha_{n+1}^{\phi}/(z-\beta_{n+1}^{\phi}))$. Observe that $\tilde{X}^{\phi}$ does not contain $z = \pm\infty$. Hence, if $z \in \tilde{X}^{\phi}$ then $m_n^{\phi}(z)$ is the minimal parameter sequence of the positive chain sequence $\{a_n^{\phi}(z)\}$.

3. Generalized chain sequences

For the purpose of this work, we define a generalized chain sequence as follows. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that $a_n \neq 0$, $n \geqslant 1$. Then, we say that $\{a_n\}_{n=1}^{\infty}$ is a generalized chain sequence if there exists a parameter sequence $\{m_n\}_{n=0}^{\infty}$ such that $m_0=0$, m_n finite and $(1-m_{n-1})m_n=a_n$, $n\geqslant 1$. Clearly, $m_n\neq 0$ and $m_n\neq 1$ for $n\geqslant 1$. We will still call $\{m_n\}$ the minimal parameter sequence of the generalized chain sequence $\{a_n\}$.

The class of positive chain sequences is a subclass of the generalized chain sequences. For example, the chain sequence $\{a_n^{\phi}(z)\}$ given by (2.6) is a generalized chain sequence for any value of $z\in Z^{\phi}$ such that $z\neq\infty$. The following theorem gives the general expression for the elements of any of the parameter sequence of $\{a_n^{\phi}(z)\}$.

Theorem 3.1. *Given the generalized chain sequence* $\{a_n^{\phi}(z)=\alpha_{n+1}^{\phi}/((z-\beta_n^{\phi})(z-\beta_{n+1}^{\phi}))\}$ *let* $\{g_n^{\phi}(z,t)\}$ *be its parameter sequence such that the initial parameter* $g_0^{\phi}(z,t)=t$. *If* $t^{-1}\neq S_n^{\phi}(z)$ *for all* $n\geqslant 1$, *then* $\{g_n^{\phi}(z,t)\}$ *exists and can be given as*

$$g_n^{\phi}(z,t)=\frac{P_{n-1}^{\phi}(z)-t(z-\beta_1^{\phi})O_{n-1}^{\phi}(z)}{P_n^{\phi}(z)-t(z-\beta_1^{\phi})O_n^{\phi}(z)}\frac{\alpha_{n+1}^{\phi}}{(z-\beta_{n+1}^{\phi})},\quad n\geqslant 1. \tag{3.1}$$

Furthermore, if $\{a_n^{\phi}(z)\}$ *is a positive chain sequence then all its parameter sequences* $\{g_n^{\phi}(z,t)\}$, *such that* $0\leqslant g_0^{\phi}(z,t)<1$ *and* $0<g_n^{\phi}(z,t)<1$, *are obtained for those values of t given in the range* $0<t\leqslant 1/S^{\phi}(z)$.

Proof. From (3.1) and the recurrence relation (2.1) one can easily establish that

$$1-g_n^{\phi}(z,t)=\frac{P_{n+1}^{\phi}(z)-t(z-\beta_1^{\phi})O_{n+1}^{\phi}(z)}{P_n^{\phi}(z)-t(z-\beta_1^{\phi})O_n^{\phi}(z)}\frac{1}{(z-\beta_{n+1}^{\phi})},\quad n\geqslant 0.$$

This provides immediately $[1-g_{n-1}^{\phi}(z,t)]g_n^{\phi}(z,t)=a_n^{\phi}(z)$. It is also clear that $g_n^{\phi}(z,t)$ is bounded if and only if $t^{-1}\neq S_n^{\phi}(z)=(z-\beta_1^{\phi})(O_n(z)/P_n(z))$. This proves the first part of the theorem. Now $g_n^{\phi}(z,0)=m_n^{\phi}(z)$ and from (3.1),

$$t^{-1}=(z-\beta_1^{\phi})\frac{O_n^{\phi}(z)-(\alpha_{n+1}^{\phi}/((z-\beta_{n+1}^{\phi})g_n^{\phi}(z,t)))O_{n-1}^{\phi}(z)}{P_n^{\phi}(z)-(\alpha_{n+1}^{\phi}/((z-\beta_{n+1}^{\phi})g_n^{\phi}(z,t)))P_{n-1}^{\phi}(z)}.$$

From the theory of continued fractions, this can be written as

$$t^{-1}=\frac{z-\beta_1^{\phi}}{z-\beta_1^{\phi}}\,\frac{\alpha_2^{\phi}}{-z-\beta_2^{\phi}}\,\frac{\alpha_3^{\phi}}{-z-\beta_3^{\phi}}-\cdots\frac{\alpha_n^{\phi}}{-z-\beta_n^{\phi}}\,\frac{\alpha_{n+1}^{\phi}}{-[z-\beta_{n+1}^{\phi}]g_n^{\phi}(z,t)}.$$

Hence from the continued fraction (2.2), $t^{-1}=S^{\phi}(z)$ if and only if

$$g_n^{\phi}(z,t)=1-\frac{a_{n+1}^{\phi}(z)}{1}\,\frac{a_{n+2}^{\phi}(z)}{-\quad 1}\,\frac{a_{n+3}^{\phi}(z)}{-\quad 1}\,\frac{a_{n+4}^{\phi}(z)}{-\quad 1}-\cdots,\quad n\geqslant 1.$$

When $\{a_n^{\phi}(z)\}$ is a positive chain sequence (i.e., when $z\in\tilde{X}^{\phi}$), these are the elements of its maximal parameter sequence $\{M_n^{\phi}(z)\}$. Furthermore, it follows from [2, Chapter III, Theorem 5.1] that if

$0 < t_1 < t_2 < 1/S^{\phi}(z)$ then

$$m_n^{\phi}(z) = g_n^{\phi}(z,0) < g_n^{\phi}(z,t_1) < g_n^{\phi}(z,t_2) < M_n^{\phi}(z) = g_n^{\phi}(z,1/S^{\phi}(z)).$$

Since $g_n^{\phi}(z,t)$ is a continuous function in $0 \leqslant t \leqslant 1/S^{\phi}(z)$, this completes the proof of the theorem. □

This theorem can also be obtained in the following manner. First we note from (2.6) and the continued fraction (2.2) that $S_0^{\phi}(z) = 0$, $S_1^{\phi}(z) = 1$ and

$$S_{n+1}^{\phi}(z) = 1 + \sum_{j=1}^{n} \frac{m_1^{\phi}(z)\, m_2^{\phi}(z)\, \cdots\, m_j^{\phi}(z)}{[1 - m_1^{\phi}(z)][1 - m_2^{\phi}(z)] \cdots [1 - m_j^{\phi}(z)]}.$$

(See Chihara [2, Lemmas 3.1 and 3.2]). Furthermore, from (3.1),

$$g_n^{\phi}(z,t) = \frac{1 - tS_{n-1}^{\phi}(z)}{1 - tS_n^{\phi}(z)} m_n^{\phi}(z), \quad n \geqslant 1.$$

Thus, we can use the following result given by Chihara in [3].

If $\{m_n\}$ is the minimal parameter sequence of the positive chain sequence $\{a_n\}$ then any other parameters $\{g_n\}$, where $0 < g_n < 1$, $n \geqslant 0$, of this chain sequence can be given by

$$g_0 = t \qquad \text{and} \qquad g_n = \frac{1 - tS_{n-1}}{1 - tS_n} m_n, \quad n \geqslant 1,$$

where $S_0 = 0$, $S_1 = 1$,

$$S_{n+1} = 1 + \sum_{j=1}^{n} \frac{m_1\, m_2\, \cdots\, m_j}{(1 - m_1)(1 - m_2) \cdots (1 - m_j)}, \quad n \geqslant 1$$

and $\lim_{n\to\infty} S_n \leqslant t^{-1} < \infty$. The proof of this follows from the observation that $S_n - m_n S_{n-1} = (1 - m_n) S_{n+1}$ for $n \geqslant 1$.

If we are interested also in other parameter sequences then we must consider other values of t subject to the restriction $1 - tS_n \neq 0$, $n \geqslant 0$. Since $m_n \neq 1$, with the same restriction these results are valid also for obtaining the parameter sequences of the generalized chain sequence. Hence the results of the theorem follow.

4. Chain sequences and related measures

Now we consider the chain sequences that arise when considering the relation $c\,\mathrm{d}\phi_2(x) = (k^2 - x^2)\,\mathrm{d}\phi_1(x)$. Substituting $g_n = (1 - \ell_n)/2$, $n \geqslant 0$ in (2.5) and, since $g_0 = 0$, comparing g_n with the parameters in (2.6), we find that $g_n = m_n^{\phi_1}(k)$. That is, the equations in (2.5) can be written as

$$\begin{gathered} P_n^{\phi_1}(z) = P_n^{\phi_2}(k) - k^2 m_n^{\phi_1}(k) m_{n-1}^{\phi_1}(k) P_{n-2}^{\phi_2}(z), \\ [1 - m_{n-1}^{\phi_1}(k)] m_n^{\phi_1}(k) = \frac{\alpha_{n+1}^{\phi_1}}{k^2} \quad \text{and} \quad [1 - m_{n+1}^{\phi_1}(k)] m_n^{\phi_1}(k) = \frac{\alpha_{n+1}^{\phi_2}}{k^2} \end{gathered} \tag{4.1}$$

for $n \geqslant 1$, where $\{\alpha_{n+1}^{\phi_1}/k^2\}$ is a generalized chain sequence. It is a positive chain sequence if k is real and lies outside the support of the measures. The associated Christoffel's formula can be written as

$$(x^2 - k^2)P_n^{\phi_2}(x) = P_{n+2}^{\phi_1}(x) - k^2[1 - m_{n+1}^{\phi_1}(k)][1 - m_n^{\phi_1}(k)]P_n^{\phi_1}(x), \quad n \geqslant 0.$$

The above relations permit one to obtain information about the polynomials $P_n^{\phi_2}$ from those of $P_n^{\phi_1}$.

Now we see how one can obtain information about $P_n^{\phi_1}$ from those of $P_n^{\phi_2}$. Substitution of $g_n = (1 + \ell_{n+1})/2$, $n \geqslant -1$, in (2.5) leads to

$$\begin{aligned} &P_n^{\phi_1}(z) = P_n^{\phi_2}(z) - k^2[1 - g_{n-1}][1 - g_{n-2}]P_{n-2}^{\phi_2}(z), \\ &[1 - g_{n-1}]g_{n-2} = \alpha_{n+1}^{\phi_1}/k^2 \quad \text{and} \quad [1 - g_{n-1}]g_n = \alpha_{n+1}^{\phi_2}/k^2 \end{aligned} \tag{4.2}$$

for $n \geqslant 1$. Here $g_{-1} = -1$. We have a situation where $\{\alpha_{n+1}^{\phi_2}/k^2\}$ is the generalized chain sequence and $\{g_n\}$ is it's parameter sequence that starts with the initial parameter $g_0 = 1 - \alpha_2^{\phi_1}/k^2$. Application of Theorem 1 gives

$$g_n = g_n^{\phi_2}(k, t), \quad n \geqslant 0, \tag{4.3}$$

where $g_0^{\phi_2}(k, t) = t = 1 - \alpha_2^{\phi_1}/k^2$.

Note that if $0 < g_0^{\phi_2}(k, t) < 1$, then clearly k must be real and the relations

$$g_n^{\phi_2}(k, t) = \frac{\alpha_{n+1}^{\phi_2}}{k^2[1 - g_{n-1}^{\phi_2}(k, t)]} \quad \text{and} \quad 1 - g_n^{\phi_2}(k, t) = \frac{\alpha_{n+2}^{\phi_1}}{k^2 g_{n-1}^{\phi_2}(k, t)}, \quad n \geqslant 1,$$

obtained from (4.2) and (4.3), give $0 < g_n^{\phi_2}(k, t) < 1$ for all $n \geqslant 0$.

We now use the above results to obtain information about the orthogonal polynomials $K_n^{(\lambda,M,k)}$ associated with the measure $\mathrm{d}\phi(\lambda, M, k; x)$ given by (1.1). As we have mentioned, when $k = 1$ then the polynomials are the symmetric Koornwinder polynomials.

Observe that if we take $\mathrm{d}\phi_1(x) = \mathrm{d}\phi(\lambda, M, k; x)$ then the measure $\mathrm{d}\phi_2$ given by $\mathrm{d}\phi_2(x) = (k^2 - x^2)\,\mathrm{d}\phi_1(x)$ is the measure associated with the Gegenbauer polynomials $P_n^{(\lambda+1)}$. To be precise,

$$\mathrm{d}\phi_2(x) = \{(2M + 1)\tilde{S}^{(\lambda)}(k)\}^{-1}(1 - x^2)^{\lambda+1/2}\,\mathrm{d}x.$$

The polynomials $P_n^{\phi_2} = P_n^{(\lambda+1)}$ can be given explicitly as $P_0^{\phi_2}(x) = 1$ and

$$P_n^{\phi_2}(x) = (2)^{-n}\frac{(2 + 2\lambda)_n}{(1 + \lambda)_n}\,{}_2F_1\left(-n, n + 2\lambda + 2; \lambda + \frac{3}{2}; \frac{1 - x}{2}\right), \quad n \geqslant 1.$$

The coefficients of the recurrence relation (2.1) associated with $\mathrm{d}\phi_2$ is

$$\alpha_{n+1}^{\phi_2} = \alpha_{n+1}^{(\lambda+1)} = \frac{1}{4}\frac{n(n + 2\lambda + 1)}{(n + \lambda)(n + \lambda + 1)}, \quad n \geqslant 1.$$

Hence from (4.2) and (4.3), by writing $g_n^{\phi_2}(k, 1 - \alpha_2^{(\lambda,M,k)}/k^2) = h_n^{\phi_2}(M, k)$, one obtains for $K_n^{(\lambda,M,k)} = P_n^{\phi_1}$ and $\alpha_n^{(\lambda,M,k)} = \alpha_n^{\phi_1}$ that

$$K_n^{(\lambda,M,k)}(z) = P_n^{\phi_2}(z) - k^2[1 - h_{n-1}^{\phi_2}(M, k)][1 - h_{n-2}^{\phi_2}(M, k)]P_{n-2}^{\phi_2}(z), \tag{4.4}$$

$$\alpha_2^{(\lambda,M,k)} = k^2[1 - h_0^{\phi_2}(M, k)] \quad \text{and} \quad \alpha_{n+1}^{(\lambda,M,k)} = k^2[1 - h_{n-1}^{\phi_2}(M, k)]h_{n-2}^{\phi_2}(M, k)$$

for $n \geqslant 2$, where

$$h_0^{\phi_2}(M,k) = 1 - \frac{\mu_2^{\phi_1}}{k^2\mu_0^{\phi_1}} = \frac{\int_{-1}^{1}(1-x^2)^{\lambda+1/2}\,\mathrm{d}x}{(2M+1)k^2\tilde{S}^{(\lambda)}(k)} = \{(2M+1)S^{\phi_2}(k)\}^{-1} \tag{4.5}$$

and $h_n^{\phi_2}(M,k) = \alpha_{n+1}^{\phi_2}/[k^2(1-h_{n-1}^{\phi_2}(M,k))]$ for $n \geqslant 1$. Written explicitly,

$$h_n^{\phi_2}(M,k) = \frac{P_{n-1}^{\phi_2}(k) - tkO_{n-1}^{\phi_2}(k)}{P_n^{\phi_2}(k) - tkO_n^{\phi_2}(k)}\frac{\alpha_{n+1}^{\phi_2}}{k} = \frac{1-tS_{n-1}^{\phi_2}(k)}{1-tS_n^{\phi_2}(k)}\frac{P_{n-1}^{\phi_2}(k)\alpha_{n+1}^{\phi_2}}{kP_n^{\phi_2}(k)}$$

for $n \geqslant 1$, with $t = h_0^{\phi_2}(M,k) = 1 - \alpha_2^{(\lambda,M,k)}/k^2$. From (2.3) we see that the functions $S_{n+1}^{\phi_2}(k)$ satisfy the recurrence relation

$$S_{n+1}^{\phi_2}(k) = S_n^{\phi_2}(k) + \frac{n!}{4^n}\frac{(1+2\lambda)_n}{(1+\lambda)_n(2+\lambda)_n}\frac{k}{P_n^{\phi_2}(k)P_{n+1}^{\phi_2}(k)}, \quad n \geqslant 1, \tag{4.6}$$

with $S_1^{\phi_2}(k) = 1$. Since $S_0^{\phi_2}(k) = 0$, the above result also holds for $n = 0$.

When $k = 1$ these results turn out to be even simpler as one has

$$P_n^{\phi_2}(1) = 2^{-n}\frac{(2+2\lambda)_n}{(1+\lambda)_n}, \quad \text{and} \quad S_{n+1}^{\phi_2}(1) = S_n^{\phi_2}(1) + \frac{n!}{(3+2\lambda)_n}, \quad n \geqslant 1.$$

Note that $S_{n+1}^{\phi_2}(1)$ is the nth partial sum of ${}_2F_1(1,1;3+2\lambda;1)$.

For the special case $\lambda = 0$, since $\alpha_{n+1}^{\phi_2} = \frac{1}{4}$ and $O_n^{\phi_2} = P_{n-1}^{\phi_2}$ (Chebyshev polynomial of degree $n-1$ of the second kind), we can write

$$h_n^{\phi_2}(M,k) = \frac{P_{n-1}^{\phi_2}(k) - tkP_{n-2}^{\phi_2}(k)}{P_{n-1}^{\phi_2}(k) - (1/(4k(1-t)))P_{n-2}^{\phi_2}(k)}\frac{1}{4k^2(1-t)}, \quad n \geqslant 1.$$

Hence, if $tk = [4k(1-t)]^{-1}$ then $h_n^{\phi_2}(M,k)$ is simply a constant. This is achieved when $t = [k \pm \sqrt{k^2-1}]/(2k)$ with the value for the parameters $h_n^{\phi_2}(M,k) = 1/\{2k[k \mp \sqrt{k^2-1}]\}$, $n \geqslant 1$. Therefore, when $2M+1 = 2k[k \mp \sqrt{k^2-1}]/S^{\phi_2}(k)$, the coefficients of the recurrence relation satisfied by the polynomials $K_n^{(0,M,k)}$ are

$$\alpha_2^{(0,M,k)} = \frac{k}{2}[k \mp \sqrt{k^2-1}], \quad \text{and} \quad \alpha_{n+2}^{(0,M,k)} = \frac{1}{4}, \quad n \geqslant 1.$$

Since $S^{\phi_2}(k) = 2k[k - \sqrt{k^2-1}]$, the minus sign corresponds to $M = 0$ and the plus sign corresponds to $M = \sqrt{k^2-1}/[k - \sqrt{k^2-1}]$.

5. Monotonicity properties of the zeros of $K_n^{(\lambda,M,k)}$

In this section, we study the behaviour of the zeros of $K_n^{(\lambda,M,k)}$ as a function of the parameters. Since both $K_n^{(\lambda,M,k)}$ and $P_n^{\phi_2} = P_n^{(\lambda+1)}$ are orthogonal with respect to even measures, they are even (odd) polynomials when n is even (odd). Then their zeros are symmetric with respect to the origin and it suffices to consider only their positive zeros. In what follows we denote by $x_{n,j}^{(\lambda,M,k)}$ and $x_{n,j}^{(\lambda+1)}$, $j = 1,2,\ldots,[n/2]$, the positive zeros of $K_n^{(\lambda,M,k)}$ and $P_n^{(\lambda+1)}$ arranged in decreasing order.

Theorem 5.1. *For any $k \geqslant 1$ and $M \geqslant 0$ the zeros of the polynomials $K_n^{(\lambda,M,k)}$ and $P_n^{(\lambda+1)}$, $n \geqslant 2$, satisfy*

$$x_{n,1}^{(\lambda+1)} < x_{n,1}^{(\lambda,M,k)}$$

and

$$x_{n,r}^{(\lambda+1)} < x_{n,r}^{(\lambda,M,k)} < x_{n-2,r-1}^{(\lambda+1)}, \quad r = 2, \ldots, [n/2].$$

Moreover,

(a) *for any n, r such that $1 \leqslant r \leqslant [n/2]$ and $n \geqslant 2$, $x_{n,r}^{(\lambda,M,k)}$ is an increasing function of M when M varies within the range $[0, \infty)$;*

(b) *for any n, r such that $1 \leqslant r \leqslant [n/2]$ and $n \geqslant 3$, $x_{n,r}^{(\lambda,M,k)}$ is an increasing function of k when k varies within the range $[\hat{k}(\lambda, M), \infty)$, where $\hat{k}(\lambda, M) = \sqrt{1 + (2\lambda+3)/(4(\lambda+2)M)}$.*

Proof. The following simple lemmas provide the basic tools for the proof of the above theorem.

Lemma 1. *Let $q_n(x) = (x - x_1)\cdots(x - x_n)$ and $q_{n-1}(x) = (x - y_1)\cdots(x - y_{n-1})$ be polynomials with real and interlacing zeros,*

$$x_n < y_{n-1} < x_{n-1} < \cdots < y_1 < x_1.$$

Then, for any real constant c, the polynomial

$$Q(x) = q_n(x) - cq_{n-1}(x)$$

has n real zeros $\xi_n < \xi_{n-1} < \cdots < \xi_2 < \xi_1$ which interlace with both the zeros of $q_n(x)$ and $q_{n-1}(x)$. More precisely,

(i) *if $c > 0$ then*

$$x_1 < \xi_1, \tag{5.1}$$

$$x_r < \xi_r < y_{r-1}, \quad r = 2, \ldots, n \tag{5.2}$$

(ii) *and if $c < 0$ then*

$$y_r < \xi_r < x_r, \quad r = 1, \ldots, n-1,$$

$$\xi_n < x_n.$$

Moreover, each ξ_r is an increasing function of c.

Proof. Obviously, $\xi_r = x_r$, $r = 1, \ldots, n$ when $c = 0$. The proofs of the statements (i) and (ii) are similar. Suppose that $c > 0$. Then

$$\operatorname{sign} Q(x_r) = -\operatorname{sign} q_{n-1}(x_r) = (-1)^r$$

and

$$\operatorname{sign} Q(y_r) = \operatorname{sign} q_n(y_r) = (-1)^r.$$

Hence there exist zeros $\xi_r, r = 2, \ldots, n$ of $Q(x)$ satisfying (5.2). The existence of ξ_1 with (5.1) follows from $Q(x_1) < 0$ and from the fact that $q_n(x)$ goes to infinity faster than $q_{n-1}(x)$ as x goes to infinity.

We remark that this result is equivalent to that regarding the separation properties of the zeros of quasi-orthogonal polynomials due to M. Riesz (see for example [2, Page 65]). In fact, according to a result of Wendroff [10], one can find a measure on the real line for which $q_n(x)$ and $q_{n-1}(x)$ are consecutive orthogonal polynomials. Consequently $Q(x)$ is a quasi-orthogonal polynomial associated with this measure with the required interlacing property.

In order to prove the monotonicity of the zeros of $Q(x)$ with respect to c, suppose that the zeros of

$$Q_\varepsilon(x) = q_n(x) - (c+\varepsilon)q_{n-1}(x),$$

where $\varepsilon \geqslant 0$, are $\xi_n(\varepsilon) < \cdots < \xi_1(\varepsilon)$. It is clear that $\xi_r = \xi_r(0)$ and

$$Q_\varepsilon(x) = Q(x) - \varepsilon q_{n-1}(x).$$

Thus $Q_\varepsilon(\xi_r) = -\varepsilon q_{n-1}(\xi_r)$ and then, for $\varepsilon > 0$,

$$\operatorname{sign} Q_\varepsilon(\xi_r) = -\operatorname{sign} q_{n-1}(\xi_r) = (-1)^r \tag{5.3}$$

because of the interlacing properties of y_r and ξ_r that we have established. Since the zeros of $Q_\varepsilon(x)$ are all real, (5.3) implies that $\xi_r < \xi_r(\varepsilon)$. Completing the proof of Lemma 1. □

The above lemma can be easily modified to even or odd polynomials.

Lemma 2. *Let both $Q_n(x)$ and $Q_{n-2}(x)$ be even (odd) polynomials with positive leading coefficients and with only real zeros whose positive zeros $x_1 > \cdots > x_{[n/2]}$ and $y_1 > \cdots > y_{[(n-2)/2]}$ interlace. That is,*

$$x_{[n/2]} < y_{[(n-2)/2]} < \cdots < y_1 < x_1.$$

Then, for any real constant c, the polynomial

$$Q(x) = Q_n(x) - cQ_{n-2}(x)$$

has n zeros and its positive zeros $\xi_{[n/2]} < \cdots < \xi_2 < \xi_1$ interlace with the positive zeros of $Q_n(x)$ and of $Q_{n-2}(x)$. More precisely,
(i) *if $c > 0$ then*

$$x_1 < \xi_1,$$

$$x_r < \xi_r < y_{r-1}, \quad r = 2,\ldots,[n/2]$$

(ii) *and if $c < 0$ then*

$$y_r < \xi_r < x_r, \quad r = 1,\ldots,[n/2]-1,$$

$$\xi_{[n/2]} < x_{[n/2]}.$$

Moreover, every positive zero ξ_r, for $r = 1,\ldots,[n/2]$, is an increasing function of c.

Now we apply the above lemma to the relation

$$K_n^{(\lambda,M,k)}(z) = P_n^{\phi_2}(z) - c_n(\lambda,M,k)P_{n-2}^{\phi_2}(z),$$

where, from (4.4),

$$c_n(\lambda, M, k) = k^2(1 - h^{\phi_2}_{n-1}(M,k))(1 - h^{\phi_2}_{n-2}(M,k)).$$

Observe that we can also write $c_n(\lambda, M, k) = [P^{\phi_2}_n(k) - kh^{\phi_2}_0(k)O^{\phi_2}_n(k)]/[P^{\phi_2}_{n-2}(k) - kh^{\phi_2}_0(k)O^{\phi_2}_{n-2}(k)]$.

From (4.5) since $S^{\phi_2}(k) > 1$, it follows that $0 < h^{\phi_2}_0(M,k) < 1$. Hence from (4.4) and from $\alpha^{\phi_2}_{n+1} = k^2[1 - h^{\phi_2}_{n-1}(M,k)]h^{\phi_2}_n(M,k)$ we have $h^{\phi_2}_n(M,k)$ are such that $0 < h^{\phi_2}_n(M,k) < 1, n \geqslant 1$. Thus $c_n(\lambda, M, k) > 0$ and we obtain from Lemma 2 that

$$x^{(\lambda+1)}_{n,1} < x^{(\lambda,M,k)}_{n,1}$$

and

$$x^{(\lambda+1)}_{n,r} < x^{(\lambda,M,k)}_{n,r} < x^{(\lambda+1)}_{n-2,r-1}, \quad r = 2, \ldots, [n/2].$$

First we analyse the behaviour of $c_n(\lambda, M, k)$ in relation to M. Note that from (4.5) that $\{h^{\phi_2}_0(M,k)\}$ is a decreasing function of M. In fact, $\{h^{\phi_2}_n(0,k)\}$ is the maximal parameter sequence of the positive chain sequence $\{\alpha^{\phi_2}_{n+1}/k^2\}$. As $M \to \infty$ we also obtain $h^{\phi_2}_n(\infty,k) \to m^{\phi_2}_n(k)$, the minimal parameter sequence. That is, as M increases from 0 to ∞ in (1.1), $h^{\phi_2}_n(M,k)$ decreases from the maximal parameter to the minimal of the chain sequence $\{\alpha^{\phi_2}_{n+1}/k^2\}$. Consequently, $c_n(\lambda, M, k)$ is an increasing function of M. Thus from Lemma 2 any positive zero $x^{(\lambda,M,k)}_{n,j}$ of $K^{(\lambda,M,k)}_n(z)$ is an increasing function of M.

To analyse the behaviour of $c_n(\lambda, M, k)$ in relation to k, we consider the relation $h^{\phi_2}_n(M,k) = \alpha^{\phi_2}_{n+1}/(k^2[1 - h^{\phi_2}_{n-1}(M,k)]), n \geqslant 1$. It follows that for any given $r \geqslant 0$ if $h^{\phi_2}_r(M,k)$ is a decreasing function of k then $h^{\phi_2}_n(M,k)$ are decreasing functions of k for $n = r+1, r+2, \ldots$ .Consequently, $c_n(\lambda, M, k)$ are increasing functions of k for $n = r+2, r+3, \ldots$. We note that $h^{\phi_2}_0(M,k)$ is an increasing function of k for $k \in [1, \infty)$. However, we show that $h^{\phi_2}_1(M,k)$ is a decreasing function of k for $k \in [\hat{k}(\lambda, M), \infty)$.

Since,

$$h^{\phi_2}_1(M,k) = \frac{\alpha^{\phi_2}_2}{k^2[1 - h^{\phi_2}_0(M,k)]} = \frac{\alpha^{\phi_2}_2}{k^2[1 - 1/((2M+1)S^{\phi_2}(k))]},$$

we look for the values of k for which the derivative of $f(k) = k^2[1 - 1/((2M+1)S^{\phi_2}(k))]$ is positive. By considering the expression

$$S^{\phi_2}(k) = (\mu^{\phi_2}_0)^{-1} \int_0^1 \frac{2k^2}{k^2 - x^2}\, \mathrm{d}\phi_2(x) = \frac{k^2}{\int_0^1 (1-x^2)^{\lambda+1/2}\,\mathrm{d}x} \int_0^1 \frac{(1-x^2)^{\lambda+1/2}}{k^2 - x^2}\, \mathrm{d}x,$$

we obtain

$$\frac{\partial f(k)}{\partial k} = 2k\left[1 - \frac{2k^4}{\mu^{\phi_2}_0(2M+1)\{S^{\phi_2}(k)\}^2} \int_0^1 \frac{1}{(k^2 - x^2)^2}\, \mathrm{d}\phi_2(x)\right].$$

Thus for $k > 1$, since $\int_0^1 1/((k^2 - x^2)^2)\,\mathrm{d}\phi_2 < \int_0^1 1/((k^2-1)(k^2 - x^2))\,\mathrm{d}\phi_2$, we obtain the inequality

$$\frac{\partial f(k)}{\partial k} > 2k\left[1 - \frac{k^2}{(k^2-1)(2M+1)S^{\phi_2}(k)}\right].$$

Again for $k > 1$, since $S_2^{\phi_2}(k) < S^{\phi_2}(k)$, we obtain

$$\frac{\partial f(k)}{\partial k} > 2k\left[1 - \frac{k^2 - \alpha_2^{\phi_2}}{(k^2-1)(2M+1)}\right].$$

Therefore, if $k^2 \geqslant 1+(2\lambda+3)/[4(\lambda+2)M]$ then $\partial f(k)/\partial k > 0$. This completes the proof of Theorem 5.1. $\square$

Observe that $\hat{k}(\lambda, 0) = \infty$. This is expected as one can prove using the Markoff Theorem (see [8, Theorem 6.12.1]) that all the zeros $x_{n,r}^{(\lambda,0,k)}$ are decreasing functions of k.

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