JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **227,** 305–318 (1998)
ARTICLE NO. AY986024

Nonsmooth Continuous-Time Optimization Problems: Sufficient Conditions*

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Submitted by Leonard D. Berkovitz

Received April 9, 1997

We discuss sufficient conditions of optimality for nonsmooth continuous-time nonlinear optimization problems under generalized convexity assumptions. These include both first-order and second-order criteria.

1. INTRODUCTION

Consider the following continuous-time nonlinear programming problem:

$$\left.\begin{aligned} &\text{Minimize } \phi(x) = \int_0^T f(t, x(t))\, dt, && \\ &\text{subject to } g_i(t, x(t)) \le 0 && \text{a.e. in } [0, T], \\ &\qquad i \in I = \{1, \dots, m\}, && x \in X. \end{aligned}\right\} \quad \text{(CNP)}$$

* This work was supported by research grant 96/9705-0-FAPESP.
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0022-247X/98 $25.00

Here X is an open, nonempty convex subset of the Banach space $L^n_\infty[0, T]$ of all n-dimensional vector-valued Lebesgue measurable functions, which are essentially bounded, defined on the compact interval $[0, T] \subset \mathbb{R}$, with the norm $\|\cdot\|_\infty$ defined by

$$\|x\|_\infty = \max_{1 \le j \le n} \operatorname{ess\,sup}\left\{|x_j(t)|, 0 \le t \le T\right\},$$

where for each $t \in [0, T]$, $x_j(t)$ is the jth component of $x(t) \in \mathbb{R}^n$, ϕ is a real-valued function defined on X, $g(t, x(t)) = \gamma(x)(t)$, and $f(t, x(t)) = \Gamma(x)(t)$, where γ is a map from X into the normed space $\Lambda^m_1[0, T]$ of all Lebesgue measurable essentially bounded m-dimensional vector functions defined on $[0, T]$, with the norm $\|\cdot\|_1$ defined by

$$\|y\|_1 = \max_{1 \le j \le m} \int_0^T |y_j(t)|\, dt,$$

and Γ is a map from X into the normed space $\Lambda^1_1[0, T]$.

In a companion paper [13], we provided first-order necessary conditions of optimality of both Fritz John (FJ) and Karush−Kuhn−Tucker (KKT) types under nonsmooth assumptions.

In this paper, we aim at providing nonconvex sufficient conditions of global optimality for (CNP). We first prove the sufficiency of Fritz John and Karush−Kuhn−Tucker conditions in the Lipschitz case, using the notion of invex functions (which has been introduced by Hanson [6]; see also [5] and [12]). Later, we get more precise results under both Clarke regularity and generalized convexity hypotheses. Finally, by using a generalized Hessian, introduced by Cominetti and Correa [3], we provide second-order sufficient conditions of optimality.

Related results on the subject can be found, for example, in [7, 9, 10, 16, 17]. Among these, maybe the best results regarding sufficient optimality conditions for (CNP) have been provided by Zalmai in [17] for a problem with smooth data. The other authors also treat smooth problems.

In our case, no differentiability is required. We allow functions to be Lipschitz only in the second variable. Therefore, our results extend earlier results on sufficient conditions of optimality for (CNP).

This paper is divided into five sections. Section 2 is devoted to recalling some basic concepts. In Section 3, we present sufficient conditions of optimality for the Lipschitz case. In Section 4, we discuss sufficient conditions under the Clarke regularity assumption. Finally, in Section 5, we give sufficient conditions for data that are of class $C^{1,1}$, defined therein.

2. PRELIMINARIES

In this section we fix some basic concepts and notation adhered to in this paper.

Let Z be a Banach space and $\psi : Z \to \mathbb{R}$ be a locally Lipschitz function; i.e., for each $x \in Z$, there exist $\epsilon > 0$ and a constant $K > 0$, depending on ϵ, such that

$$|\psi(x_1) - \psi(x_2)| \leq K\|x_1 - x_2\| \qquad \forall x_1, x_2 \in x + \epsilon B,$$

where B is the open unit ball of Z.

The *Clarke generalized directional derivative* of ψ at x in the direction of a given $v \in Z$, denoted by $\psi^0(x; v)$, is defined by

$$\psi^0(x; v) := \limsup_{\substack{y \to x \\ s \to 0^+}} \frac{\psi(y + sv) - \psi(y)}{s}.$$

The *generalized gradient* of ψ at x, denoted by $\partial\psi(x)$, is defined by

$$\partial\psi(x) := \{\xi \in Z^* : \langle \xi, v\rangle \leq \psi^0(x; v) \quad \forall v \in Z\}.$$

Here, Z^* denotes the dual space of continuous linear functionals on Z, and $\langle \cdot, \cdot \rangle$: $Z^* \times Z \to \mathbb{R}$ is the duality pairing. For more details, see [2].

Let $\mathbb{F}$ be the set of all feasible solutions to (CNP) (we suppose nonempty), i.e.,

$$\mathbb{F} = \{x \in X : g_i(t, x(t)) \leq 0 \quad \text{a.e. in } [0, T], i \in I\}.$$

Let V be an open convex subset of $\mathbb{R}^n$ containing the set

$$\{x(t) \in \mathbb{R}^n : x \in \mathbb{F}, t \in [0, T]\}.$$

We assume f and g_i, $i \in I$, are real functions defined on $[0, T] \times V$. The function $t \to f(t, x(t))$ is assumed to be Lebesgue measurable and integrable for all $x \in X$.

We assume that, given $a \in V$, there exist an $\epsilon > 0$ and a positive number k such that $\forall t \in [0, T]$, and $\forall x_1$, $x_2 \in a + \epsilon B$ (B denotes the unit ball of $\mathbb{R}^n$) we have

$$|f(t, x_1) - f(t, x_2)| \leq k\|x_1 - x_2\|.$$

Similar hypotheses are assumed for g_i, $i \in I$. Hence, $f(t, \cdot)$ and $g_i(t, \cdot)$, $i \in I$, are locally Lipschitz on V throughout $[0, T]$.

We can suppose the Lipschitz constant is (locally) the same for all functions involved.

Now, assume $\bar{x} \in X$ and $h \in L_\infty^n[0,T]$ are given. The *continuous Clarke generalized directional derivatives* of f and g_i's are given by

$$f^0(t, \bar{x}(t); h(t)) := \Gamma^0(\bar{x}; h)(t) := \limsup_{\substack{y \to \bar{x} \\ s \to 0^+}} \frac{\Gamma(y + sh)(t) - \Gamma(y)(t)}{s}$$

and

$$g_i^0(t, \bar{x}(t); h(t)) := \gamma_i^0(\bar{x}; h)(t) := \limsup_{\substack{y \to \bar{x} \\ s \to 0^+}} \frac{\gamma_i(y + sh)(t) - \gamma_i(y)(t)}{s}$$

a.e. in $[0,T]$.

It follows easily from the assumptions that

$$t \to f^0(t, \bar{x}(t)); h(t)),$$
$$t \to g_i^0(t, \bar{x}(t)); h(t)), \qquad i \in I,$$

are Lebesgue measurable and integrable for all $\bar{x} \in X$, and $h \in L_\infty^n[0,T]$.

Let U be a nonempty subset of Z and $\psi : U \to \mathbb{R}$ be a locally Lipschitz function on U. The function ψ is said to be *invex* at $\bar{z} \in U$ (with respect to U) if there exists a function $\eta : U \times U \to Z$ such that

$$\psi(z) - \psi(\bar{z}) \geq \psi^0(\bar{z}; \eta(z, \bar{z}))$$

for all $z \in U$. We say that ψ is *strictly invex* if the above inequality is strict for $z \neq \bar{z}$.

We also need to use an invexity notion in the continuous-time context. Let $U \subset \mathbb{R}^n$ be a nonempty subset of $\mathbb{R}^n$ and $\bar{x} \in X$. Suppose a given function ψ: $[0,T] \times U \to \mathbb{R}$ is locally Lipschitz throughout $[0,T]$. The function $\psi(t, \cdot)$ is said to be invex at $\bar{x}(t)$ (with respect to U) if there exists $\eta : U \times U \to \mathbb{R}^n$ such that the function $t \to \eta(x(t), \bar{x}(t))$ is in $L_\infty^n[0,T]$ and

$$\psi(t, x(t)) - \psi(t, \bar{x}(t)) \geq \psi^0(t, \bar{x}(t); \eta(x(t), \bar{x}(t))) \qquad \text{a.e. in } [0,T]$$

for all $x \in X$. We say that ψ is *strictly invex* if the above inequality is strict for $x(t) \neq \bar{x}(t)$ a.e. in $[0,T]$.

3. LIPSCHITZ CASE

In this section we obtain global sufficient conditions of optimality for (CNP) in the Lipschitz case without any convexity assumptions on the data. More precisely, we prove the sufficiency of both Fritz John (Theorem 3.1) and Karush–Kuhn–Tucker (Theorem 3.2) optimality conditions, under invexity assumptions on the data of (CNP).

THEOREM 3.1. *Let $\bar{x} \in \mathbb{F}$. Suppose that $f(t,\cdot)$ is invex at $\bar{x}(t)$ (with respect to V) throughout $[0,T]$, and that, for each $i \in I$, $g_i(t,\cdot)$ is strictly invex at $\bar{x}(t)$ (with respect to V) throughout $[0,T]$, with the same $\eta(x(t),\bar{x}(t))$ for all functions. Suppose further that there exist $\bar{\lambda}_0 \in \mathbb{R}$, $\bar{\lambda} \in L_\infty^m[0,T]$ such that*

$$0 \le \int_0^T \left[\bar{\lambda}_0 f^0(t,\bar{x}(t);h(t)) + \sum_{i=1}^m \bar{\lambda}_i(t) g_i^0(t,\bar{x}(t);h(t)) \right] dt \qquad \forall h \in L_\infty^n[0,T], \quad (1)$$

$$\bar{\lambda}_0 \ge 0, \bar{\lambda}(t) \ge 0 \qquad a.e.\ in\ [0,T], \quad (2)$$

$$\left(\bar{\lambda}_0, \bar{\lambda}(t)\right) \ne 0 \qquad a.e.\ in\ [0,T], \quad (3)$$

$$\bar{\lambda}_i(t) g_i(t,\bar{x}(t)) = 0 \qquad a.e.\ in\ [0,T], i \in I. \quad (4)$$

Then $\bar{x}$ is a global optimal solution of (CNP).

Proof. Suppose, to the contrary, that $\bar{x}$ is not optimal for (CNP). Then there exists $\tilde{x} \in \mathbb{F}$, $\tilde{x} \ne \bar{x}$, such that

$$\int_0^T f(t,\tilde{x}(t))\,dt < \int_0^T f(t,\bar{x}(t))\,dt. \quad (5)$$

Since $f(t,\cdot)$ is invex and for each $i \in I$, $g_i(t,\cdot)$ is strictly invex at $\bar{x}(t)$ throughout $[0,T]$, we have the inequalities

$$f(t,\tilde{x}(t)) - f(t,\bar{x}(t)) \ge f^0(t,\bar{x}(t);\eta(\tilde{x}(t),\bar{x}(t))) \qquad \text{a.e. in } [0,T], \quad (6)$$

$$g_i(t,\tilde{x}(t)) - g_i(t,\bar{x}(t)) > g_i^0(t,\bar{x}(t);\eta(\tilde{x}(t),\bar{x}(t))) \qquad \text{a.e. in } [0,T], \quad i \in I, \quad (7)$$

for some $\eta(\tilde{x}(t),\bar{x}(t))$. Because $\tilde{x} \in \mathbb{F}$ and $\bar{\lambda}_i(t) \ge 0$ a.e. in $[0,T]$ for each $i \in I$, it is clear that

$$\bar{\lambda}_i(t) g_i(t,\tilde{x}(t)) \le 0 \qquad \text{a.e. in } [0,T], \quad i \in I. \quad (8)$$

Now from (2)–(8) it follows that

$$0 > \int_0^T \Bigg[\bar{\lambda}_0 f^0(t,\bar{x}(t);\eta(\tilde{x}(t),\bar{x}(t))) + \sum_{i=1}^m \bar{\lambda}_i(t) g_i^0(t,\bar{x}(t);\eta(\tilde{x}(t),\bar{x}(t))) \Bigg] dt,$$

which, with $h(t) = \eta(\tilde{x}(t), \bar{x}(t))$, contradicts (1). Therefore, we conclude that $\bar{x}$ is a global optimal solution of (CNP). ∎

Remark. From the above proof it is clear that if for each $i \in I$, $g_i(t, \cdot)$ is invex, and at least one of these functions, say $g_k(t, \cdot)$, is strictly invex at $\bar{x}(t)$ throughout $[0, T]$ such that the corresponding multiplier function $\bar{\lambda}_k$ is nonzero on a subset of $[0, T]$ with positive Lebesgue measure, then the assertion of the theorem remains valid.

THEOREM 3.2. *Let $\bar{x} \in \mathbb{F}$. Suppose $f(t, \cdot)$, $g_i(t, \cdot)$, $i \in I$, are invex at $\bar{x}(t)$ (with respect to V) throughout $[0, T]$, for the same function $\eta(x(t), \bar{x}(t))$. Suppose, further, that there exists $\bar{\lambda} \in L_\infty^m[0, T]$ such that*

$$0 \leq \int_0^T \left[f^0(t, \bar{x}(t); h(t)) + \sum_{i=1}^m \bar{\lambda}_i(t) g_i^0(t, \bar{x}(t); h(t)) \right] dt \qquad \forall h \in L_\infty^n[0, T], \quad (9)$$

$$\bar{\lambda}_i(t) \geq 0 \qquad a.e.\ in\ [0, T], \quad i \in I, \quad (10)$$

$$\bar{\lambda}_i(t) g_i(t, \bar{x}(t)) = 0 \qquad a.e.\ in\ [0, T], \quad i \in I. \quad (11)$$

Then $\bar{x}$ is a global optimal solution of (CNP).

Proof. Let $x \in \mathbb{F}$ be given. It follows from (10) and (11) that

$$\bar{\lambda}_i(t) g_i(t, x(t)) \leq 0 = \bar{\lambda}_i(t) g_i(t, \bar{x}(t)) \qquad \text{a.e. in } [0, T], \quad i \in I.$$

Since for each $i \in I$, $g_i(t, \cdot)$ is invex at $\bar{x}(t)$ throughout $[0, T]$ and $\bar{\lambda}_i(t) \geq 0$ a.e. in $[0, T]$, we have that $\bar{\lambda}_i(t) g_i(t, \cdot)$ is also invex at $\bar{x}(t)$ throughout $[0, T]$ for the same function $\eta(x(t), \bar{x}(t))$. From the invexity of $\bar{\lambda}_i(t) g_i(t, \cdot)$ we obtain

$$\bar{\lambda}_i(t) g_i^0(t, \bar{x}(t); \eta(x(t), \bar{x}(t))) \leq 0 \qquad \text{a.e. in } [0, T], \quad i \in I. \quad (12)$$

Now, setting $h(t) = \eta(x(t), \bar{x}(t))$ in (9), we get

$$0 \leq \int_0^T \left[f^0(t, \bar{x}(t); \eta(x(t), \bar{x}(t))) + \sum_{i=1}^m \bar{\lambda}_i(t) g_i^0(t, \bar{x}(t); \eta(x(t), \bar{x}(t))) \right] dt. \quad (13)$$

Combining (12) and (13) we obtain

$$\int_0^T \left[f^0(t, \bar{x}(t); \eta(x(t), \bar{x}(t))) \right] dt \geq 0.$$

The invexity hypothesis on f, together with the last inequality, implies

$$\phi(\bar{x}) \leq \phi(x).$$

Hence, because $x \in \mathbb{F}$ is arbitrary, we can conclude that $\bar{x}$ is a global optimal solution of (CNP). ∎

Remark. We proved in [13, Proposition 4.1] (Theorem 5.1, respectively) that conditions (1)–(4) ((9)–(11)) are necessary for the optimality of a point $\bar{x}$, without any assumption of invexity. Therefore, because of Theorems 3.1 and 3.2, these conditions are both necessary and sufficient for optimality.

4. CLARKE REGULARITY

In this section, we obtain global sufficient conditions of optimality for (CNP) under generalized convexity and Clarke regularity assumptions. The theorems stated below generalize the smooth case by Zalmai [16].

Now, we recall the notions of Clarke regularity (*this notion is assumed to hold throughout this section*) and generalized convexities needed hereafter.

Let $U \subset Z$ be a nonempty subset of Z and ψ be a real locally Lipschitz function defined on some open subset of Z containing the set U. We say that ψ is *Clarke regular* at $x \in U$ if for all $v \in Z$, the usual one-sided directional derivative of ψ at x in the direction $v \in Z$, denoted by $\psi'(x; v)$, exists and $\psi'(x; v) = \psi^0(x; v)$.

A function ψ is said to be *pseudoconvex* at $x_1 \in U$ (with respect to U) if for all $x_2 \in U$,

$$\psi'(x_1; x_2 - x_1) \geq 0 \Rightarrow \psi(x_2) \geq \psi(x_1).$$

A function ψ is said to be *quasiconvex* at $x_1 \in U$ (with respect to U) if for all $x_2 \in U$,

$$\psi(x_2) \leq \psi(x_1) \Rightarrow \psi'(x_1; x_2 - x_1) \leq 0.$$

We define the *Lagrangean function* L: $X \times \mathbb{R} \times L_\infty^m[0, T] \to \mathbb{R}$ by

$$L(x, \lambda_0; \lambda) := \int_0^T \left[\lambda_0 f(t, x(t)) + \sum_{i=1}^m \lambda_i(t) g_i(t, x(t)) \right] dt.$$

When $\lambda_0 \neq 0$, we can assume that $\lambda_0 = 1$ by normalizing the Lagrange multipliers. In this case we denote $L(x, 1, \lambda)$ by $L(x, \lambda)$.

In the sequel $L'_x(\bar{x}, \lambda_0, \lambda; h)$ denotes the usual directional derivative of $L(\cdot, \lambda_0, \lambda)$ at $\bar{x}$ in the direction $h \in L_\infty^n[0, T]$, and $\partial_x L(\bar{x}, \lambda_0, \lambda)$ means the generalized gradient of $L(\cdot, \lambda_0, \lambda)$.

We point out that conditions (1)–(4) ((9)–(11)) in Theorem 3.1 (Theorem 3.2) cannot be written in terms of the Clarke generalized gradient of the Lagrangean function, in general. In this section, we show that under the Clarke regularity assumption, it is possible. In fact, if f and g_i's are Clarke regular, then condition (1) is equivalent to $L'_x(\bar{x}, \bar{\lambda}_0, \bar{\lambda}; h) \geq 0$ for all $h \in L^n_\infty[0,T]$ and, therefore, $0 \in \partial_x L(\bar{x}, \bar{\lambda}_0, \bar{\lambda})$. Formally, we have the following corollaries:

COROLLARY 4.1. *Let $\bar{x} \in \mathbb{F}$. Suppose that $f(t,\cdot)$ is invex at $\bar{x}(t)$ (with respect to V) throughout $[0,T]$, and that, for each $i \in I$, $g_i(t,\cdot)$ is strictly invex at $\bar{x}(t)$ (with respect to V) throughout $[0,T]$, with respect to the same $\eta(x(t), \bar{x}(t))$ for all functions. Suppose, further, that there exist $\bar{\lambda}_0 \in \mathbb{R}$, $\bar{\lambda} \in L^m_\infty[0,T]$ such that*

$$0 \in \partial_x L\left(\bar{x}, \bar{\lambda}_0, \bar{\lambda}\right), \tag{14}$$

$$\bar{\lambda}_0 \geq 0, \bar{\lambda}(t) \geq 0 \qquad a.e.\ in\ [0,T], \tag{15}$$

$$\left(\bar{\lambda}_0, \bar{\lambda}(t)\right) = \left(\bar{\lambda}_0, \bar{\lambda}_1(t), \dots, \bar{\lambda}_m(t)\right) \neq 0, \qquad a.e.\ in\ [0,T], \tag{16}$$

$$\bar{\lambda}_i(t) g_i(t, \bar{x}(t)) = 0 \qquad a.e.\ in\ [0,T], \quad i \in I. \tag{17}$$

Then $\bar{x}$ is a global optimal solution of (CNP).

COROLLARY 4.2. *Let $\bar{x} \in \mathbb{F}$. Suppose $f(t,\cdot)$, $g_i(t,\cdot)$, $i \in I$, are invex at $\bar{x}(t)$ (with respect to V) throughout $[0,T]$, for the same function $\eta(x(t), \bar{x}(t))$. Suppose, further, that there exists $\bar{\lambda} \in L^m_\infty[0,T]$ such that*

$$0 \in \partial_x L(\bar{x}, \bar{\lambda}), \tag{18}$$

$$\bar{\lambda}_i(t) \geq 0 \qquad a.e.\ in\ [0,T], \quad i \in I, \tag{19}$$

$$\bar{\lambda}_i(t) g_i(t, \bar{x}(t)) = 0 \qquad a.e.\ in\ [0,T], \quad i \in I. \tag{20}$$

Then $\bar{x}$ is a global optimal solution of (CNP).

The next result provides a global minimal criterion, in which we only assume the Lagrangean function is invex in the first variable.

PROPOSITION 4.3. *Let $\bar{x} \in \mathbb{F}$. If there exists $\bar{\lambda} \in L^m_\infty[0,T]$ such that $(\bar{x}, \bar{\lambda})$ satisfies (1)–(4), and if the Lagrangean function $L(x; \bar{\lambda})$ is invex at $\bar{x}$ (with respect to $\mathbb{F}$), then $\bar{x}$ is a global optimal solution of (CNP).*

Proof. Condition (1) implies $0 \leq L'_x(\bar{x}, \bar{\lambda}; \eta(x, \bar{x}))$ $\forall x \in \mathbb{F}$. From the invexity assumption on the Lagrangean, we obtain

$$L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda}) \qquad \forall x \in \mathbb{F}.$$

This inequality implies that $\phi(\bar{x}) \leq \phi(x)$ $\forall x \in \mathbb{F}$, which finishes the proof. ∎

Next, we provide two results on the sufficiency of the Karush–Kuhn–Tucker conditions. The first is obtained under the hypotheses of pseudoconvexity on the objective function and quasiconvexity on the constraint functions. The second result shows that global optimality is maintained if we impose quasiconvexity on a certain function defined in terms of the constraint functions g_i, instead of assuming quasiconvexity on the g_i's individually.

PROPOSITION 4.4. *Let $\bar{x} \in \mathbb{F}$. Suppose $\phi(\cdot)$ is pseudoconvex at $\bar{x}$ (with respect to $\mathbb{F}$) and that, for each $i \in I$, $\bar{\lambda}_i(t) g_i(t, \cdot)$ is quasiconvex at $\bar{x}(t)$ (with respect to V) throughout $[0, T]$. If there exists $\bar{\lambda} \in L_\infty^m[0, T]$ such that*

$$0 \in \partial_x L(\bar{x}, \bar{\lambda}), \tag{21}$$

$$\bar{\lambda}_i(t) \geq 0 \qquad a.e. \text{ in } [0, T], \quad i \in I, \tag{22}$$

and

$$\bar{\lambda}_i(t) g_i(t, \bar{x}(t)) = 0 \qquad a.e. \text{ in } [0, T], \quad i \in I, \tag{23}$$

then $\bar{x}$ is a global optimal solution to (CNP).

Proof. Since, for each $x \in \mathbb{F}$,

$$\bar{\lambda}_i(t) g_i(t, x(t)) \leq 0 = \bar{\lambda}_i(t) g_i(t, \bar{x}(t)) \qquad \text{a.e. in } [0, T], \quad i \in I,$$

the quasiconvexity hypothesis implies

$$\bar{\lambda}_i(t) g_i'(t, \bar{x}(t); x(t) - \bar{x}(t)) \leq 0 \qquad \text{a.e. in } [0, T], \quad i \in I,$$

and, hence,

$$\int_0^T \sum_{i \in I} \bar{\lambda}_i(t) g_i'(t, \bar{x}(t); x(t) - \bar{x}(t))\, dt \leq 0 \qquad \forall x \in \mathbb{F}. \tag{24}$$

Now, $0 \in \partial_x L(\bar{x}, \bar{\lambda})$ implies

$$0 \leq \int_0^T \left[f'(t, \bar{x}(t); x(t) - \bar{x}(t)) + \sum_{i \in I} \bar{\lambda}_i(t) g_i'(t, \bar{x}(t); x(t) - \bar{x}(t)) \right] dt \qquad \forall x \in \mathbb{F}. \tag{25}$$

From (24) and (25) we conclude that

$$\int_0^T f'(t, \bar{x}(t); x(t) - \bar{x}(t))\, dt \geq 0 \qquad \forall x \in \mathbb{F}.$$

Since ϕ is pseudoconvex at $\bar{x}$, the last inequality implies

$$\phi(\bar{x}) \le \phi(x) \qquad \forall x \in \mathbb{F}.$$

Therefore, $\bar{x}$ is a global optimal solution to (CNP). ∎

PROPOSITION 4.5. *Let $\bar{x} \in \mathbb{F}$. Suppose $\phi(\cdot)$ is pseudoconvex at $\bar{x}$ (with respect to $\mathbb{F}$). If there exists $\overline{\lambda} \in L_\infty^m[0,T]$ such that $(\bar{x}, \overline{\lambda})$ satisfies* (21)–(23), *and if the function $G(\cdot; \overline{\lambda}) : L_\infty^n[0,T] \to \mathbb{R}$ given by*

$$G(x, \overline{\lambda}) = \int_0^T \sum_{i=1}^m \overline{\lambda}_i(t) g_i(t, x(t))\, dt,$$

is quasiconvex at $\bar{x}$ (with respect to $\mathbb{F}$), then $\bar{x}$ is a global optimal solution of (CNP).

Proof. For each $x \in \mathbb{F}$, conditions (22) and (23) imply $G(x, \overline{\lambda}) \le 0 = G(\bar{x}, \overline{\lambda})$. In view of the quasiconvexity of G at $\bar{x}$, we deduce that

$$G'_x(\bar{x}, \overline{\lambda}; x - \bar{x}) \le 0 \qquad \forall x \in \mathbb{F},$$

that is,

$$\int_0^T \sum_{i=1}^m \overline{\lambda}_i(t) g'_i(t, \bar{x}(t); x(t) - \bar{x}(t))\, dt \le 0 \qquad \forall x \in \mathbb{F}.$$

The rest of the proof follows by using the same arguments as in the proof of the previous proposition. ∎

5. SECOND-ORDER CONDITIONS

Considering a problem similar to (CNP) with twice continuously differentiable data, Zalmai [16] proved sufficient conditions of optimality in terms of the usual Hessian of the associated Lagrangean function. We aim at extending his results for the case where the functions involved are of class $C^{1,1}$, defined below. For this matter, we use the notion of second-order generalized derivative due to Cominetti and Correa [3]. However, the results we present here remain true if, instead of using the above notion of generalized Hessian, we use any other notion available in the literature with the same features. For different notions of generalized Hessians see, for instance, [1, 11, 14, 15]. The connections among several known second-order directional derivatives are discussed in [11].

Before engaging in the optimality conditions, we recall some basic notation and results from [3].

Let Z be a Banach space. Let $\psi : Z \to \mathbb{R}$ and $x \in Z$ be given. The *generalized second-order directional derivative* of ψ at x in the direction $(u, v) \in Z \times Z$ is given by

$$\psi^{oo}(x; u, v) = \limsup_{\substack{y \to x \\ s, t \to 0}} \frac{\psi(y + su + tv) - \psi(y + su) - \psi(y + tv) + \psi(y)}{st}.$$

ψ^{oo} is allowed to eventually assume the values $+\infty$ and $-\infty$.

The *generalized Hessian* of ψ at x is defined to be the multifunction $\partial^2\psi(x)$: $Z \to Z^*$ given by

$$\partial^2\psi(x)(u) = \{x^* \in Z^*: \langle x^*, v\rangle \le \psi^{oo}(x; u, v) \quad \forall v \in Z\}.$$

THEOREM 5.1 ([3]).

(a) *The function* $(u, v) \to \psi^{oo}(x; u, v)$ *is symmetric* $(\psi^{oo}(x, u, v) = \psi^{oo}(x, v, u))$ *and bisublinear* (*sublinear in each variable separately*).

(b) *The map* $x \to \psi^{oo}(x; u, v)$ *is upper semicontinuous* (*u.s.c.*) *at* x *for every* $(u, v) \in Z \times Z$, *and the multifunction* $x \to \partial^2\psi(x)(u)$ *is closed at* x *for each fixed* $u \in Z$.

(c) $\partial^2\psi(x)(u)$ *is convex and w*-closed.*

(d) $\psi^{oo}(x; -u, v) = \psi^{oo}(x; u, -v) = (-\psi)^{oo}(x; u, v)$.

A function ψ is said to be of class $C^{1,1}$ on Z if for each $x \in Z$, ψ is Gâteaux differentiable at x with a locally Lipschitz gradient.

THEOREM 5.2. ([3]). *If* ψ *is of class* $C^{1,1}$ *at* $x \in Z$, *then*

(a) $\partial^2\psi(x)(u)$ *is nonempty.*

(b) $\partial^2\psi(x)(u)$ *is w*-compact and*

$$\psi^{oo}(x; u, v) = \max_{x^* \in \partial^2\psi(x)(u)} \langle x^*, v\rangle;$$

(c) $(x, u) \to \partial^2\psi(x)(u)$ *is upper semicontinuous.*

LEMMA 5.3 ([3]). *Suppose* ψ *is of class* $C^{1,1}$ *on the closed line segment* $[x,y] \subset Z$. *Then there exists* ξ *in the open line segment* $]x, y[$ *such that*

$$\psi(y) \in \psi(x) + \langle \nabla\psi(x), y - x\rangle + \tfrac{1}{2}\langle \partial^2\psi(\xi)(y - x), y - x\rangle.$$

We say that $\partial^2\psi(x)$ is *positive definite* (p.d.) if $-\psi^{oo}(x, u, -u) \ge 0$ $\forall u \in Z$. If the above inequality is strict for $u \ne 0$, then $\partial^2\psi(x)$ is said to be *strictly positive definite* (s.p.d.).

For the rest of this section, we assume that the functions f and g_i, $i \in I$, in (CNP) are Gâteaux differentiable with respect to the second variable, and their partial derivatives, denoted by $\nabla f(t, x)$ and $\nabla g_i(t, x)$, $i \in I$, respectively, are locally Lipschitz. Actually, we assume that there exists a function $k \in L_1[0, T]$ such that

$$\|\nabla f(t, y) - \nabla f(t, z)\| \leq k(t)\|y - z\| \qquad \text{a.e. in } [0, T],$$
$$\|\nabla g_i(t, y) - \nabla g_i(t, z)\| \leq k(t)\|y - z\| \qquad \text{a.e. in } [0, T], \quad i \in I,$$

for all y, z in a neighborhood of x.

Under these assumptions one can easily prove the following lemma.

LEMMA 5.4. *Let $\lambda \in L_\infty^m[0, T]$ be given. The Lagrangean function $L(\cdot, \lambda)$: $L_\infty^n[0, T] \to \mathbb{R}$ is of class $C^{1,1}$.*

Let $\nabla_x L(\bar{x}, \lambda)$ denote the Gâteaux derivative of $L(\cdot, \lambda)$ at $\bar{x}$, and $\partial_x^2 L(\bar{x}, \lambda)$ the generalized Hessian of $L(\cdot, \lambda)$ at $\bar{x}$.

We are now in a position to state and prove the main result of this section.

THEOREM 5.5. *Let $(\bar{x}, \lambda) \in \mathbb{F} \times L_\infty^m[0, T]$. Assume that $(\bar{x}, \lambda)$ satisfies the first-order necessary optimality conditions,*

(i) $\nabla_x L(\bar{x}, \lambda) = 0$.

(ii) $\lambda_i(t) \geq 0$ *a.e. in* $[0, T]$, $i \in I$.

(iii) $\lambda_i g_i(t, \bar{x}(t)) = 0$ *a.e. in* $[0, T]$, $i \in I$.

Assume also that the generalized Hessian $\partial_x^2 L(\bar{x}, \lambda)$ is s.p.d. Then $\bar{x}$ is a strict local minimum for (CNP). Furthermore, if $L(\cdot, \lambda)$ is quasiconvex at $\bar{x}$ (with respect to F), then $\bar{x}$ is a strict global minimum for (CNP).

Proof. Suppose $\bar{x}$ is not a strict local minimum for the Lagrangean function $L(\cdot, \lambda)$ on $\mathbb{F}$. Then there exists $(x_n) \subseteq \mathbb{F}$, $x_n \to \bar{x}$, $x_n \neq \bar{x}$, such that $L(x_n, \lambda) \leq L(\bar{x}, \lambda)$. Let

$$u_n = \frac{x_n - \bar{x}}{\|x_n - \bar{x}\|} \to u.$$

It follows from Lemma 5.4 that $L(\cdot, \lambda)$ is of class $C^{1,1}$. Hence, from Lemma 5.3, for each n, there exists $\xi_n \in]\bar{x}, x_n[$ such that

$$L(x_n, \lambda) - L(\bar{x}, \lambda) \in \langle \nabla_x L(\bar{x}, \lambda), x_n - \bar{x} \rangle$$
$$+ \tfrac{1}{2}\langle \partial_x^2 L(\xi_n, \lambda)(x_n - \bar{x}), x_n - \bar{x} \rangle.$$

Since assumption (i) is in force, the above inclusion is equivalent to

$$a_n = \frac{2\big(L(x_n, \lambda) - L(\bar{x}, \lambda)\big)}{\|x_n - \bar{x}\|^2} \in \langle \partial_x^2 L(\xi_n, \lambda) u_n, u_n \rangle.$$

We have built sequences $\xi_n \to \bar{x}$, $u_n \to u$, and $x_n^* \in \partial_x^2 L(\xi_n, \lambda)(u_n)$, such that $a_n = \langle x_n^*, u_n \rangle \leq 0$. By Theorem 5.2, we can suppose (by taking subsequences, if necessary) that (x_n^*) converges to some $x^* \in \partial_x^2 L(\bar{x}, \lambda)(u)$. It follows that $\langle x^*, u \rangle = \lim \langle x_n^*, u_n \rangle \leq 0$, which implies $-L^{oo}(\bar{x}, \lambda; u, -u) \leq 0$. This is a contradiction of the fact that $\partial_x^2 L(\bar{x}, \lambda)$ is s.p.d. Hence, $\bar{x}$ is a strict local minimum of $L(x, \lambda)$ on $\mathbb{F}$, which, under assumptions (ii) and (iii), readily implies that $\bar{x}$ is a strict local minimum of (CNP). The quasiconvexity hypothesis on the Lagrangean function easily implies the strict global optimality of $\bar{x}$ for (CNP). ∎

ACKNOWLEDGMENT

We are very grateful to the referee for his helpful comments and suggestions.

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