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Exact Boundary Control for the Wave Equation in a Polyhedral Time-Dependent Domain

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Abstract—We establish exact boundary controllability for the wave equation in a polyhedral domain where a part of the boundary moves slowly with constant speed in a small interval of time. The control on the moving part of the boundary is given by the conormal derivative associated with the wave operator while in the fixed part the control is of Neuman type. For initial state $H^1 \times L^2$ we obtain controls in L^2.

Keywords—Boundary controllability, Wave equation.

SECTION 1

The controllability for the wave equation in smooth domains has been studied in the papers [1–4] by different methods. For nonsmooth or time dependent domains the literature is very restricted (see [5,6]). In the paper [6] Grisvard used the H.U.M. introduced in [2] to establish boundary controllability for the wave equation in polyhedral domains in R^2 and R^3. Some of Grivard's results were extended to higher dimensions in [7,8] using Russell's method introduced in [1].

In the present paper we study exact boundary controllability for the wave equation in a polyhedral domain $\Omega \subset R^n$, $n \geq 2$ where the entire boundary or just some of its faces move with constant speed less than one in a finite and relatively small interval of time. We use Russell's method as improved by Lagnese in [9].

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SECTION 2

Let Π be a finite collection of hyperplanes in R^n, $n \geq 2$, not all parallel and displaced in such a way that the elements of Π determine a bounded region Ω. We call Ω a polyhedral domain and denote Γ its boundary. For each hyperplane $\pi \in \Pi$, the set $\pi \cap \Gamma$ is referred as a face of Ω. We say that the face $\pi \cap \Gamma$ moves when the hyperplane π moves towards its normal direction.

Let Γ_1 and Γ_2 be two disjoint sets of faces of Ω such that $\Gamma = \Gamma_1 \cup \Gamma_2$. We assume that those faces composing Γ_2 move with constant speed less than one in an interval of time $[t_1, t_2]$ and that

$$\operatorname{diam}(\Omega) > t_2 > t_1 \geq 0. \tag{1}$$

We allow different faces in Γ_2 to move with different speed. Our method works even if the faces in Γ_2 move in different intervals of time satisfying (1). To keep the simplicity we will consider just one interval. An illustrative example is that one where Ω is a cube and one of its faces is pushed slowly, inside or outside, in a short interval of time.

Let Ω_t be the deformed domain Ω at the time $t > 0$ ($\Omega_0 = \Omega$). Assumption (1) assures $\Omega_t \neq \emptyset$ for every t.

We set

$$\tilde{Q} = \bigcup_{t>0} \Omega_t \tag{2}$$

and assume that there exists a neighborhood $\tilde{\Omega}$ of $\overline{\Omega}$ such that

$$\tilde{Q} \subset \tilde{\Omega} \times [0, \infty). \tag{3}$$

Let $\tilde{T} = \operatorname{diam} \tilde{\Omega}$ and set

$$Q_T = \bigcup_{0<t<T} \Omega_t \tag{4}$$

for some $T > \tilde{T}$. We denote Σ_T the lateral boundary of Q_T and $\nu = (\nu_x, \nu_t)$ the unit vector normal to Σ_T. Observe that Q_T is a noncylindrical polyhedral domain in R^{n+1} because the moving faces of Ω move with constant speed. The faces in Γ_1 generate faces of Q_t where the component ν_t of ν vanishes while on those faces generated by Γ_2 we have $\nu_t \neq 0$. The assumption that the speed of the faces in Γ_2 be less than one assures that the surface Σ_T is time-like. This is known to be sufficient to guarantee the well-possedness of the initial and boundary value problem studied here.

Under the assumptions above, we prove the following.

THEOREM 1. *Given an initial state $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega)$ and any $T > \tilde{T}$ there exists a control $g \in L^2(\Sigma_T)$ such that the solution $u \in H^1(Q_T)$ of the initial and boundary value problem*

$$\begin{aligned} u_{tt} - \Delta u &= 0, && \text{in } Q_T, \\ u(x,0) &= u_0(x), && \text{in } \Omega, \\ u_t(x,0) &= u_1(x), && \text{in } \Omega, \\ \nu_t u_t - \nabla u . \nu_x &= g, && \text{on } \Sigma_T, \end{aligned} \tag{5}$$

satisfies

$$u(x,T) = u_t(x,T) = 0, \qquad \text{in } \Omega_t. \qquad ∎$$

The boundary condition above arises naturally in difraction problems where the mobility of the boundary is inevitable and is obtained assuming that some elements of the reflector (boundary) acquire a velocity normal to its surface (see [10, Chapter 1]).

Observe that on the rigid part of the boundary $\nu_t = 0$ and the boundary condition reduces to the Neuman condition.

The proof of the Theorem 1 is presented in the next sections.

SECTION 3

An important step in the proof of the Theorem 1 is the existence of L^2 trace of the conormal derivative of the solution of the wave equation on hyperplanes of R^{n+1}. In this section, we present an inequality due to Hörmander [11] to obtain such traces. We start setting some notations. Let μ be any positive integer, $D_k = i\frac{\partial}{\partial x_k}, D = (D_1, \dots, D_\mu)$ and $D^\alpha = D_1^{\alpha_1} \dots D_\mu^{\alpha_\mu}$ for every multindex $\alpha = (\alpha_1, \dots, \alpha_\mu)$. Let $P(\xi)$ and $Q(\xi)$ be two polynomials in $\xi = (\xi_1, \dots, \xi_\mu)$ with complex coefficients and $P(D)$ and $Q(D)$ the corresponding differential operators. We set

$$P^{(\alpha)} = \frac{\partial^{|\alpha|}}{\partial \xi_1^{\alpha_1} \dots \partial \xi_\mu^{\alpha_\mu}} P,$$

and

$$\tilde{P}(\xi) = \left(\Sigma \left|P^{(\alpha)}(\xi)\right|^2\right)^{1/2}.$$

THEOREM 2. *Let Ω be an open and bounded subset of R^μ and π be a linear manifold in R^μ. If*

$$\int_{\pi^1} \frac{|Q(\xi)|^2}{\tilde{P}(\xi)^2}\, d\pi^1 \le \text{Const.}$$

for every linear manifold π^1 orthogonal to π, then there exists a constant $\tilde{C} = \tilde{C}(\Omega, Q, P) > 0$ such that

$$\int_\pi |Q(D)u|^2\, d\pi \le \tilde{C} \int_{R^\mu} |P(D)u|^2\, dx$$

for every $u \in C_0^\infty(\Omega)$. ∎

The proof of theorem above is found in [11, p. 191]. Now assume that π is a hyperplane in R^μ with normal vector ν. Then

$$\pi^1 = \{\xi + t\nu; t \in R\}, \qquad \xi \in \pi.$$

If we take $Q(\xi) = \nabla P(\xi)\nu$ and set $\rho(t) = P(\xi + t\nu)$, then $\rho'(t) = Q(\xi + t\nu)$. Hence

$$\int_{\pi^1} \frac{|Q(\xi)|^2}{\tilde{P}(\xi)^2}\, d\pi^1 = \int_{-\infty}^{+\infty} \frac{|Q(\xi + t\nu)|^2}{\tilde{P}(\xi + t\nu)^2}\, dt \le \int_{-\infty}^{+\infty} \frac{|\rho'(t)|^2}{|\rho(t)|^2 + |\rho'(t)|^2}\, dt \tag{6}$$

since $\tilde{P}(\xi + t\nu)^2 = \Sigma\left|P^{(\alpha)}(\xi + t\nu)\right|^2 \ge |\rho(t)|^2 + |\rho'(t)|^2$.

Now observing that the last integral in (6) is bounded by 4 (degree $\rho)^2$ (see [11, p. 194]) we conclude that there exists $\tilde{C} = \tilde{C}(\Omega, \rho) > 0$ such that

$$\int_\pi |[\nabla P(D)\nu]\, u|^2\, d\pi \le \tilde{C} \int_{R^\mu} |P(D)u|^2\, dx \tag{7}$$

for every $u \in C_0^\infty(\Omega)$.

Let $\mu = n + 1$ and $P(\xi) = \xi_t^2 - \sum_{i=1}^n \xi_i^2$, $\xi = (\xi_1, \dots, \xi_n, \xi_t)$. Then

$$\int_\pi |\nu_t u_t - \nabla u \nu_x|\, d\pi \le C \int_{R^{n+1}} |u_{tt} - \Delta u|^2\, dx\, dt \tag{8}$$

for every $u \in C_0^\infty(\Omega)$. Here C is a constant depending only on Ω and the wave operator.

Now let $K \subset \pi$ be a compact set and $\Omega \subset R^{n+1}$ be a bounded neighborhood of K. Let $u \in H^1(\Omega)$ be a solution of $u_{tt} - \Delta u = 0$ in Ω. Let $\varphi \in C_0^\infty(\Omega)$ be a function such that $\varphi \equiv 1$ near K. Then the function $v \equiv \varphi u$ satisfies $v_{tt} - \Delta v = f$ for a convenient $f \in L^2(\Omega)$. If $\{\rho_n\}$ is a mollifier in Ω, then $v * \rho_n \in C_0^\infty(\Omega)$ and satisfies

$$(v * \rho_n)_{tt} - \Delta(v * \rho_n) = f * \rho_n.$$

The inequality (8) applied to $v * \rho_n$ shows that v has L^2 conormal derivative in K. Since $\varphi \equiv 1$ near K the same is true for u. Hence u has conormal derivative in $L^2_{\text{loc}}(\pi)$.

SECTION 4

In this section, we prove the Theorem 1. Let $\widetilde{u_0}$ and $\widetilde{u_1}$ be extensions of u_0 and $u_1$such that

$$\widetilde{u_0} \in H^1(R^n), \qquad \text{supp } \widetilde{u_0} \subset \tilde{\Omega}, \tag{9}$$
$$\widetilde{u_1} \in L^2(R^n), \qquad \text{supp } \widetilde{u_1} \subset \tilde{\Omega}, \tag{10}$$

and let $\widetilde{u}$ be the solution of the Cauchy problem

$$\begin{aligned} \widetilde{u}_{tt} - \Delta\widetilde{u} &= 0, && \text{in } R^{n+1}, \\ \widetilde{u}(x,0) &= \widetilde{u}_0(x), && \text{in } R^n, \\ \widetilde{u}_t(x,0) &= \widetilde{u}_1(x), && \text{in } R^n. \end{aligned} \tag{11}$$

From well-known results on the propagation of singularities and the assumptions (9), (10), and the choice of $T > \tilde{T}$ it follows that $\widetilde{u}(.,T), \widetilde{u}_t(.,T) \in C_0^\infty(\tilde{\Omega})$. Now consider the reverse control problem for the wave equation with initial state $(\widetilde{u}(.,T), \widetilde{u}_t(.,T))$ at the time $t = T$ in the smooth domain $\tilde{\Omega}$. Since controllability for this problem is already established in [1] and [9], we may assert that there exists a smooth function h on the lateral boundary $\partial\tilde{\Omega} \times [0,T]$ such that the solution of

$$\begin{aligned} w_{tt} - \Delta w &= 0, && \text{in } \tilde{\Omega} \times [0,T], \\ w(x,T) &= \widetilde{u}(x,T), && \text{in } \tilde{\Omega}, \\ w_t(x,T) &= \widetilde{u}_t(x,T), && \text{in } \tilde{\Omega}, \\ w(x,t) &= h(x,t), && \text{in } \partial\tilde{\Omega} \times [0,T] \end{aligned}$$

satisfies

$$w(x,0) = w_t(x,0) = 0, \qquad \text{in } \tilde{\Omega}.$$

Now we define $u = \widetilde{u} - w$ and observe that the restriction of u to Q_T solves

$$\begin{aligned} u_{tt} - \Delta u &= 0, && \text{in } Q_T, \\ u(x,0) &= u_0(x), && \text{in } \Omega, \\ u_t(x,0) &= u_1(x), && \text{in } \Omega, \end{aligned}$$

and

$$u(x,T) = u_t(x,T) = 0, \qquad \text{in } \Omega_T.$$

Now, to conclude the proof, all we need is to read of the trace of the conormal derivative of u on the surface Σ_T. We observe that the component w of u is smooth then its conormal derivative on Σ_T is smooth. For the component $\widetilde{u}$ of u, we apply the Theorem 2 as discussed in Section 3 and read the conormal derivative of $\widetilde{u}$ on the faces of Σ_T as an L^2 function. The desired control is then given by taking

$$\nu_t u_t - \nabla u.\nu_x = g, \qquad \text{on } \Sigma_T$$

which completes the proof.

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