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Study of Singularities in Nonsmooth Dynamical Systems via Singular Perturbation*

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Abstract. In this article we describe some qualitative and geometric aspects of nonsmooth dynamical systems theory around typical singularities. We also establish an interaction between nonsmooth systems and geometric singular perturbation theory. Such systems are represented by discontinuous vector fields on $\mathbb{R}^\ell$, $\ell \geq 2$, where their discontinuity set is a codimension one algebraic variety. By means of a regularization process proceeded by a blow-up technique we are able to bring about some results that bridge the space between discontinuous systems and singularly perturbed smooth systems. We also present an analysis of a subclass of discontinuous vector fields that present transient behavior in the 2-*dimensional* case, and we dedicate a section to providing sufficient conditions in order for our systems to have local asymptotic stability.

Key words. regularization, vector fields, singular perturbation, discontinuous vector fields, sliding vector fields

AMS subject classifications. 34C20, 34C26, 34D15, 34H05

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1. Introduction.

The purpose of this article is to present some aspects of the geometric theory of a class of nonsmooth systems. Our main concern is understanding the dynamics of such systems by means of tools in the geometric singular perturbation theory. Many similarities between such fields were observed, and a comparative study of the two categories is called for. The main task for the future is to bring the theory of nonsmooth dynamical systems to a maturity similar to that of smooth systems. Needless to say, geometric singular perturbation theory is an important tool in the field of continuous dynamical systems, of which very good surveys are available (see [6, 7, 9]). The techniques of geometric singular perturbation theory can be used to obtain information on the dynamics of the nonsmooth system, mainly in searching minimal sets.

The study of nonsmooth dynamical systems has in recent years established an important bridge between Mathematics, Physics, and Engineering. The book [5] presents motivating models that arise in the occurrence of impact motion in impact systems as well as switchings in electronic systems and hybrid dynamics in control systems. For a survey on qualitative

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aspects of such systems we refer the reader to [14] and references therein. In many applications examples of nonsmooth systems where the discontinuities are located on algebraic varieties are available (see, for example, [1, 2]). See, for instance, the system represented by $\ddot{x} + x\, sign(x) + sign(\dot{x}) = 0$. Concerning theoretical results on this subject we refer the reader to [3, 12]. This paper is mainly motivated by such issues. It is worthwhile to point out that in [4] and [10] preceding results were established in two dimensions and three dimensions, respectively, when the discontinuity set is a codimension one submanifold.

Let X be a nonsmooth vector field in $\mathbb{R}^l$, and denote by $\mathcal{S}$ its discontinuity set. First we focus our attention on the flow (determined by the orbits of X tending to $\mathcal{S}$) that is tangent to $\mathcal{S}$, which is denominated the *sliding mode*. It appears when the flow of X across $\mathcal{S}$ and points "inward" cannot leave such a surface.

Now we address the discussion to a more general context. Let $\mathcal{U} \subseteq \mathbb{R}^\ell$, $\ell \geq 2$, be an open set with $0 \in \mathcal{U}$. Let $S = \mathcal{S}_1 \bigcup \mathcal{S}_2$, where $\mathcal{S}_1$ and $\mathcal{S}_2$ are codimension one submanifolds of $\mathcal{U}$ with $0 \in \mathcal{S}_1 \cap \mathcal{S}_2$ that are in general position. Around $0 \in \mathcal{U}$ we have that $\mathcal{S}_1 \cup \mathcal{S}_2$ separates $\mathcal{U}$ into four open quadrants: $M_1, \ldots, M_4$. In our approach $\mathcal{S}$ will be the discontinuity boundary of our systems, also named the *switching variety*.

Let X_i, $i = 1, \ldots, 4$, be C^κ vector fields, with $\kappa \geqslant 1$, $\kappa = \infty$, or $\kappa = \omega$, defined on $\mathcal{U}$. We are concerned with the behavior of the sliding mode generated from a discontinuous differential system expressed by

$$\dot{z} = X(z) = X_i(z), \quad z \in M_i, \quad i = 1, \ldots, 4. \tag{1.1}$$

We will denote these systems by $X = (X_1, \ldots, X_4) \in \Omega^\kappa_{1234}(\mathcal{U})$ and the intersection $\overline{M_i} \cap \overline{M_j}$ by $\mathcal{S}_{ij}$.

Throughout the paper we consider local coordinates $(x_1, x_2, \ldots, x_\ell)$ such that $\mathcal{S}_1 = \mathcal{S}_{14} \cup \mathcal{S}_{23} = \{x_2 = 0\}$, $\mathcal{S}_2 = \mathcal{S}_{12} \cup \mathcal{S}_{34} = \{x_1 = 0\}$, and

$$M_1 = \{x_1 > 0,\, x_2 > 0\}, \quad M_2 = \{x_1 < 0,\, x_2 > 0\},$$
$$M_3 = \{x_1 < 0,\, x_2 < 0\}, \quad M_4 = \{x_1 > 0,\, x_2 < 0\}.$$

Denote by $\mathcal{S}^* = \mathcal{S} \setminus (\mathcal{S}_1 \cap \mathcal{S}_2)$ the regular part of $\mathcal{S}$. In $\mathcal{S}^*$ the definition of an orbit-solution obeys, whenever possible, the Filippov convention (see [8]). Consider M_i and M_j, $i \neq j$, having a common boundary. According to this convention there may exist generically a *sliding region* $\mathcal{S}^{sl} \subset \mathcal{S}^*$ such that any orbit which meets $\mathcal{S}^{sl}$ remains tangent to $\mathcal{S}^*$ for positive time. This region is the part of $\mathcal{S}^*$ on which X_i and X_j point inward to $\mathcal{S}^*$. Analogously there may exist generically an *escaping region* $\mathcal{S}^{es} \subset \mathcal{S}^*$ such that any orbit which meets $\mathcal{S}^{es}$ remains tangent to $\mathcal{S}^*$ for negative time.

On $\mathcal{S}^{sl} \bigcup \mathcal{S}^{es}$ the flow slides on $\mathcal{S}^*$; the flow follows a well-defined vector field $X^{\mathcal{S}}$ called the *sliding vector field*. See Figure 1. The sewing region $\mathcal{S}^{sw} \subset \mathcal{S}^*$ is the part of $\mathcal{S}^*$ where the flow crosses $\mathcal{S}^*$. The boundary between the three regions is the locus of points where the vector field is tangent to $\mathcal{S}^*$ and the flow grazes the switching surface. In our context such points together with the critical points of $X^{\mathcal{S}}$ and $\mathcal{S}_1 \cap \mathcal{S}_2$ constitute the set of singularities of X.

We remark that outside the three above-mentioned open regions (grazing regions), non-uniqueness of solutions is allowed, and the setting of local qualitative analysis has successfully

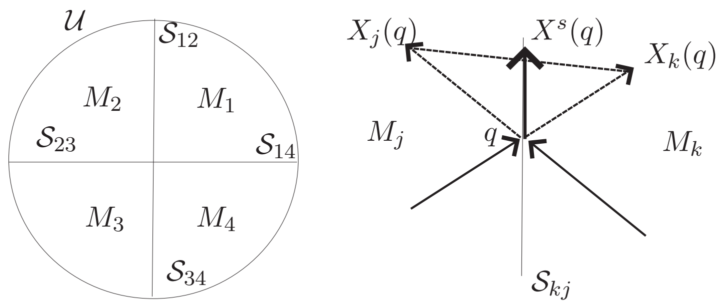


Figure 1. *Boundary of four regions and the sliding vector field.*

been the object of many studies. In the specific topic addressed in this paper the situation may be even more complicated since we want to study the dynamics of the vector field X around the origin. Alexander and Seidman in [1, 2] established a discussion of such a situation by using two mechanisms termed *blending* and *hysteresis*. Our approach is closest in spirit to the works [4] and [10], and one of our concerns is to know when a sliding flow in S^* can be continued until the intersection $\mathcal{S}_1 \cap \mathcal{S}_2$. In addition, we also present some applications of the techniques developed in [4] and [10] for analyzing the behavior of the so-called transient flow around 0.

We begin by explaining the main concepts with a concrete example before discussing the general result of this setting.

Example. Consider $\mathcal{U} \subseteq \mathbb{R}^3$, an open set with $0 \in \mathcal{U}$ and $X = (X_1, \ldots, X_4) \in \Omega^{\kappa}_{1234}(\mathcal{U})$. We denote $X_i = (f_i, g_i, h_i)$, $i = 1, \ldots, 4$. Suppose that $f_1 < 0$, $g_1 < 0$; $f_2 > 0$, $g_2 < 0$; $f_3 > 0$, $g_3 > 0$; and $f_4 < 0$, $g_4 > 0$. This means that all vector fields point inward to the intersecting switching curve $\Sigma = \mathcal{S}_1 \cap \mathcal{S}_2 = \{(0, 0, x_3)\}$. It is natural to extend the definition of the sliding vector field to this set. A possible sliding vector field is a convex combination of X_1, X_2, X_3, and X_4. We denote this combination by

$$X^{\Sigma}_{\lambda_1,\ldots,\lambda_4} = \lambda_1 X_1 + \cdots + \lambda_4 X_4, \quad \sum_{i=1}^{i=4} \lambda_i = 1.$$

If

$$\operatorname{rank} \begin{bmatrix} f_1(0) & f_2(0) & f_3(0) & f_4(0) \\ g_1(0) & g_2(0) & g_3(0) & g_4(0) \\ 1 & 1 & 1 & 1 \end{bmatrix} = 3,$$

then there exists an open set $0 \in \mathcal{U} \subseteq \mathbb{R}^3$ such that for any $(0, 0, x_3) \in \mathcal{U}$ the set

$$\left\{ \lambda \in \mathbb{R}^4;\ \sum_{i=1}^{i=4} \lambda_i = 1,\ X^{\Sigma}_{\lambda_1,\ldots,\lambda_4}(0, 0, x_3) \in \Sigma \right\}$$

is a nonempty set with dimension one in $\mathbb{R}^4$. Clearly we face here an ambiguity situation, and naturally some questions arise. For example, *What is required to avoid such ambiguity?*, or *How about the dynamics of smooth systems nearby X?*

Let $X = (X_1, \dots, X_4) \in \Omega^{\kappa}_{1234}(\mathcal{U})$. Summarizing, in what follows we give a rough overall description of the main results of this paper.

(a) There exist four curves

$$\{\gamma_i(\psi, x_1, \rho) = 0\}\,, \quad \{\alpha_i(\theta, x_2, \rho) = 0\}\,, \quad i = 1, 2,$$

with $\theta, \psi \in [0, \pi]$, $x_1, x_2 \in \mathbb{R}$, $\rho \in \mathbb{R}^{\ell-2}$, and such that the sliding region $\mathcal{S}^{sl} \bigcup \mathcal{S}^{es}$ on the regular part $\mathcal{S}^*$ is homeomorphic to one of them. Moreover, the sliding vector field $X^{\mathcal{S}}$ is topologically equivalent to one of the following reducing problems:

$$\gamma_i(\psi, x_1, \rho) = 0, \quad \dot{x_1} = \delta_i(\psi, x_1, \rho), \quad \dot{\rho} = \nu_i(\psi, x_1, \rho)$$

or

$$\alpha_i(\theta, x_2, \rho) = 0, \quad \dot{x_2} = \beta_i(\theta, x_2, \rho), \quad \dot{\rho} = \sigma_i(\theta, x_2, \rho).$$

See Theorem 2.2(a), (b).

(b) For $\ell \geq 3$ there exists a differential system

$$\zeta(\theta, \psi, \rho) = 0, \quad \xi(\theta, \psi, \rho) = 0, \quad \rho' = \phi(\theta, \psi, \rho),$$

which extends the concept of the Filippov system for the case $\mathcal{S}_1 \cap \mathcal{S}_2$. See Theorem 2.2(c).

(c) For $\ell = 2$, $X_1 = X_3$, and $X_2 = X_4$, the flow of the regular vector fields which approach $X = (X_1, X_2, X_1, X_2)$ in the transient case is described. See Proposition 5.4.

2. Preliminaries and statement of results. In this section basic concepts and the main result of the paper are presented.

The sliding vector field $X^{\mathcal{S}}$ is defined at $q \in \mathcal{S}_{kj} \cap (\mathcal{S}^{sl} \bigcup \mathcal{S}^{es})$ by $X^{\mathcal{S}}(q) = m - q$, with m being the point where the segment joining $q + X_k(q)$ and $q + X_j(q)$ cuts $\mathcal{S}_{kj}$. In previous works (see [4, 10, 11]) it was shown that if $p = (x_1, \dots, x_\ell) \in \mathcal{S}_1 \cup \mathcal{S}_2$ and $x_1^2 + x_2^2 \neq 0$, then the sliding vector field on a neighborhood $\mathcal{V}$ of p can be studied via singular perturbation theory.

Suppose that $p = (0, x_2, \dots, x_\ell) \in \mathcal{S}_{12}$, $X = X_1 = (f_1, g_1) \in \mathbb{R} \times \mathbb{R}^{\ell-1}$ if $x_1 > 0$, $X = X_2 = (f_2, g_2) \in \mathbb{R} \times \mathbb{R}^{\ell-1}$ if $x_1 < 0$, and $f_1(p) \cdot f_2(p) \neq 0$.

Definition 2.1. *A C^∞-function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is a transition function if $\varphi(s) = -1$ for $s \leqslant -1$, $\varphi(s) = 1$ if $s \geqslant 1$, and $\varphi'(s) > 0$ if $s \in (-1, 1)$.*

The φ_{x_2}*-regularization* of X is the one-parameter family given by

$$X^{12}_{\varepsilon} = (1/2)\left[(1 + \varphi(x_1/\varepsilon))\, X_1 + (1 - \varphi(x_1/\varepsilon))\, X_2\right] \tag{2.1}$$

for $x_2 > 0$, $\varepsilon > 0$.

Consider, around the point p, the surface composed by the union of $M_1 \cap \mathcal{V}$, $M_2 \cap \mathcal{V}$, and $\widehat{\mathcal{S}_{12}} = \{(\theta, \rho) : \theta \in (0, \pi), \rho = (x_2, \dots, x_\ell) \in \mathcal{S}_{12} \cap \mathcal{V}\}$. We denote $\mathcal{M} = (M_1 \cap \mathcal{V}) \cup \widehat{\mathcal{S}_{12}} \cup (M_2 \cap \mathcal{V})$ and remark that the set $\{(0, \rho) : \rho \in S_{12} \cap \mathcal{V}\}$ has two distinct copies: $\partial(M_1 \cap \mathcal{V})$ and $\partial(M_2 \cap \mathcal{V})$.

The blow-up process is the change of coordinates $x_1 = r\cos\theta$, $\varepsilon = r\sin\theta$. It induces a smooth vector field on $\mathcal{M}$ whose trajectories coincide with those of X_1 on M_1, of X_2 on M_2, and of a singular perturbation problem on $\widehat{\mathcal{S}_{12}}$ described by

$$\theta' = \alpha(r, \theta, \rho), \quad \rho' = r\beta(r, \theta, \rho), \tag{2.2}$$

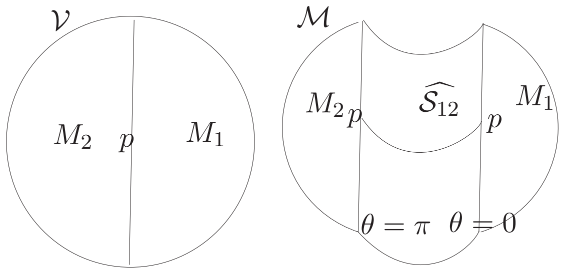


Figure 2. *The open set $\mathcal{V}$ and the set $\mathcal{M}$.*

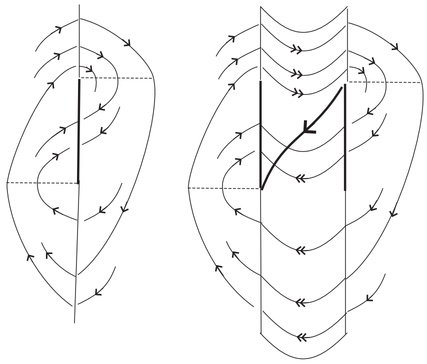


Figure 3. *Fast and slow dynamics with the slow manifold connecting two folds.*

with $r \geq 0$, $\theta \in (0, \pi)$, $\rho \in \mathcal{S}_{12} \cap \mathcal{V}$, and α and β of class C^κ. We remark that the functions $\alpha(r, \theta, \rho)$ and $\beta(r, \theta, \rho)$ are given by

$$\alpha = -\sin\theta \left[\frac{f_1 + f_2}{2} + \varphi(\cot\theta) \frac{f_1 - f_2}{2} \right], \quad \beta = \frac{g_1 + g_2}{2} + \varphi(\cot\theta) \frac{g_1 - g_2}{2},$$

with the functions f_i, g_i, $i = 1, 2$, depending on $(r\cos\theta, \rho)$. See Figures 2 and 3.

Clearly in a neighborhood of $0 \in \mathbb{R}^\ell$ we have two singular perturbation problems, SP1 and SP2, defined on the sets $r \geq 0$, $\theta \in (0, \pi)$ for $\rho \in \mathcal{S}_{12}$, and for $\rho \in \mathcal{S}_{34}$, respectively. Moreover, another two singular perturbation problems, SP3 and SP4, defined on the sets $u \geq 0$, $\psi \in (0, \pi)$ for $\eta \in \mathcal{S}_{14}$, and for $\eta \in \mathcal{S}_{23}$, respectively, still occur. Our main result is the following.

Theorem 2.2. *Consider $X = (X_1, \ldots, X_4) \in \Omega^\kappa_{1234}(\mathcal{U})$. There exist a neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of 0 and the following singular perturbation problems defined at $0 \in \mathbb{R}^\ell$:*

$$\theta' = \alpha_i(r, \theta, x_2, \rho), \quad {x_2}' = r\beta_i(r, \theta, x_2, \rho), \quad \rho' = r\sigma_i(r, \theta, x_2, \rho), \quad i = 1, 2; \tag{2.3}$$

$$\psi' = \gamma_i(u, \psi, x_1, \rho), \quad {x_1}' = u\delta_i(u, \psi, x_1, \rho), \quad \rho' = u\nu_i(u, \psi, x_1, \rho), \quad i = 1, 2; \tag{2.4}$$

and

$$u\psi' = r\zeta(r, u, \theta, \psi, \rho), \quad \theta' = \xi(r, u, \theta, \psi, \rho), \quad \rho' = r\phi(r, u, \theta, \psi, \rho) \tag{2.5}$$

with $r, u \geq 0$, $\theta, \psi \in [0, \pi]$, $(x_1, x_2, \rho) \in \mathcal{V}$, $\rho = (x_3, \ldots, x_\ell)$. *Moreover,* α_i, β_i, σ_i, γ_i, δ_i, ν_i, ζ, ξ, *and* ϕ *are of class* C^κ *for* $i = 1, 2$ *and satisfying the following:*

(a) *The sliding region* $(\mathcal{S}^{sl} \bigcup \mathcal{S}^{es}) \cap \mathcal{S}_1$ *is homeomorphic to the slow manifold* $\{\gamma_i(0, \psi, x_1, \rho) = 0\} \setminus \mathcal{Z}_\psi = \{\psi = 0, \pi\}$ *of* (2.4), *with* $i = 1$ *at* $\widehat{\mathcal{S}_{14}}$ *and with* $i = 2$ *at* $\widehat{\mathcal{S}_{23}}$. *The sliding vector field* $X^{\mathcal{S}}$ *is topologically equivalent to the reduced problem* $\gamma_i(0, \psi, x_1, \rho) = 0$, $\dot{x_1} = \delta_i(0, \psi, x_1, \rho)$, *and* $\dot{\rho} = \nu_i(0, \psi, x_1, \rho)$.

(b) *The sliding region* $(\mathcal{S}^{sl} \bigcup \mathcal{S}^{es}) \cap \mathcal{S}_2$ *is homeomorphic to the slow manifold* $\{\alpha_i(0, \theta, x_2, \rho) = 0\} \setminus \mathcal{Z}_\theta = \{\theta = 0, \pi\}$ *of* (2.3), *with* $i = 1$ *at* $\widehat{\mathcal{S}_{12}}$ *and with* $i = 2$ *at* $\widehat{\mathcal{S}_{34}}$. *The sliding vector field* $X^{\mathcal{S}}$ *is topologically equivalent to the reduced problem* $\alpha_i(0, \theta, x_2, \rho) = 0$, $\dot{x_2} = \beta_i(0, \theta, x_2, \rho)$, *and* $\dot{\rho} = \sigma_i(0, \theta, x_2, \rho)$.

(c) *The singular perturbation* (2.5) *is the blowing up of the regularization of the systems* (2.3) *for* $i = 1, 2$. *The slow manifold is given by* $SM^1 = \{\zeta(0, 0, \theta, \psi, \rho) = 0\}$. *Furthermore, for* $\ell \geq 3$, *the slow flow is the limit, for* $r, u \downarrow 0$, *of the trajectories of another singular perturbation expressed by*

$$r\dot{\theta} = \xi(r, u, \theta, \psi, \rho), \quad \dot{\rho} = \phi(r, u, \theta, \psi, \rho). \tag{2.6}$$

The slow manifold of (2.6) *is the set on* $\mathbb{R}^\ell$ *given by*

$$SM^2 = \{\xi(0, 0, \theta, \psi, \rho) = 0, \zeta(0, 0, \theta, \psi, \rho) = 0\} \subseteq SM^1.$$

In section 3 we focus on the concept of regularization discussed above, and we prove Theorem 2.2. The sets given by $\alpha_i(0, \theta, x_2, \rho) = 0$, $\gamma_i(0, \psi, x_1, \rho) = 0$, and $\zeta(0, 0, \theta, \psi, \rho) = 0$ are called *slow manifolds*, and they will be denoted by SM. We say that a slow manifold is *nontrivial* if it is not contained in $\mathcal{Z} = \{\theta = 0, \pi\} \cup \{\psi = 0, \pi\}$.

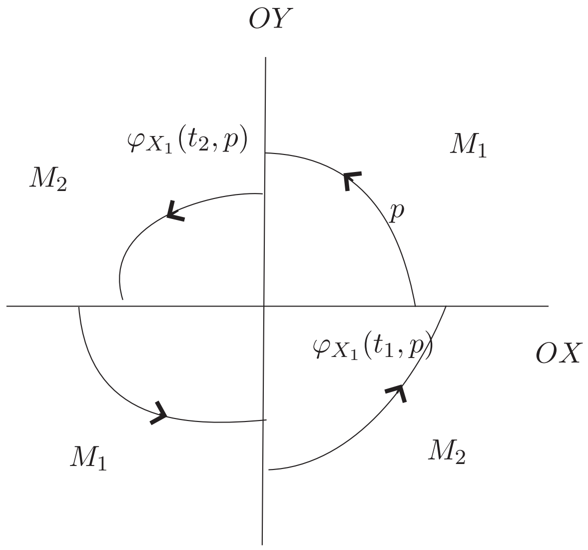


Figure 4. *Transient vector fields around* 0.

Let $X = (X_1, X_2) \in \Omega^k_{1234}(U)$ be a vector field belonging to the class $\Omega^k_{12}(U)$ of discontinuous vector fields satisfying $X_3 = X_1$ and $X_4 = X_2$. In section 5 we provide sufficient conditions for the local asymptotic stability. Finally, in section 5 we study systems in this class that present a transient behavior. See section 5 for a precise definition. For an illustration see Figure 4.

3. Proof of Theorem 2.2.

First we introduce some basic definitions. Consider a transition function φ.

(a) The φ_{x_2}-*regularization* of X is the one-parameter family given by

$$X_\varepsilon^{12} = (1/2)\left[(1+\varphi(x_1/\varepsilon))\,X_1 + (1-\varphi(x_1/\varepsilon))\,X_2\right] \tag{3.1}$$

for $x_2 > 0$, $\varepsilon > 0$, and

$$X_\varepsilon^{43} = (1/2)\left[(1+\varphi(x_1/\varepsilon))\,X_4 + (1-\varphi(x_1/\varepsilon))\,X_3\right] \tag{3.2}$$

for $x_2 < 0$, $\varepsilon > 0$.

(b) The φ_{x_1}-*regularization* of X is the one-parameter family given by

$$X_a^{14} = (1/2)\left[(1+\varphi(x_2/a))\,X_1 + (1-\varphi(x_2/a))\,X_4\right] \tag{3.3}$$

for $x_1 > 0$, $a > 0$, and

$$X_a^{23} = (1/2)\left[(1+\varphi(x_2/a))\,X_2 + (1-\varphi(x_2/a))\,X_3\right] \tag{3.4}$$

for $x_1 < 0$, $a > 0$.

Denote

$$\vartheta_{ij}^+ = \frac{\vartheta_i + \vartheta_j}{2}, \quad \vartheta_{ij}^- = \frac{\vartheta_i - \vartheta_j}{2}, \quad \lambda(z) = \varphi(\cot(z)), \quad S_\alpha = -\sin\alpha,$$

and

$$\vartheta_{abcd}^{*\star\circ} = \frac{\vartheta_a * \vartheta_b \star \vartheta_c \circ \vartheta_d}{4},$$

where $*, \star, \circ \in \{+,-\}$.

Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be a transition function. We remark that the mapping $\lambda : [0,\pi] \longrightarrow \mathbb{R}$ given by $\lambda(\theta) = \varphi(\cot\theta)$ for $0 < \theta < \pi$, $\lambda(0) = 1$, and $\lambda(\pi) = -1$ is a nonincreasing smooth function which is strictly decreasing for $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$.

Proof of Theorem 2.2. For simplicity we suppose that $\ell = 3$, and we write $X_i = (f_i, g_i, h_i)$, $i = 1, \ldots, 4$. Denote by X_r^1, X_r^2 the equations of (3.1) and (3.2), respectively, after the coordinate change $\varepsilon = r\sin\theta$, $x_1 = r\cos\theta$. Let X_1^r and X_2^r be these equations after the time rescaling $t = r\tau$. Let X_r^3, X_r^4 be the systems given by (3.3) and (3.4) now written in the coordinates $a = u\sin\psi$, $x_2 = u\cos\psi$. Represent by X_3^r, X_4^r those systems after the time rescaling $t = u\tau$. Thus X_i^r, $i = 1, 2$, are the vector fields corresponding to the differential systems (2.3), given in Theorem 2.2, on $\widehat{\mathcal{S}_{12}}$ and on $\widehat{\mathcal{S}_{34}}$, respectively. Analogously, X_i^r, $i = 3, 4$, are the vector fields corresponding to the differential systems (2.4), given in Theorem 2.2, on $\widehat{\mathcal{S}_{14}}$ and on $\widehat{\mathcal{S}_{23}}$, respectively. See the equations in Table 1 and a diagram with the local vector fields in the blowing-up locus in Figure 5.

Table 1
The equations of the regularizations.

$X_r^1 \left\{ \begin{array}{lcl} r\dot{\theta} & = & S_\theta\left(f_{12}^+ + \lambda(\theta) f_{12}^-\right) \\ \dot{x_2} & = & g_{12}^+ + \lambda(\theta) g_{12}^- \\ \dot{x_3} & = & h_{12}^+ + \lambda(\theta) h_{12}^- \end{array} \right.$	$X_r^2 \left\{ \begin{array}{lcl} r\dot{\theta} & = & S_\theta\left(f_{43}^+ + \lambda(\theta) f_{43}^-\right) \\ \dot{x_2} & = & g_{43}^+ + \lambda(\theta) g_{43}^- \\ \dot{x_3} & = & h_{43}^+ + \lambda(\theta) h_{43}^- \end{array} \right.$
$X_r^3 \left\{ \begin{array}{lcl} u\dot{\psi} & = & S_\psi\left(g_{14}^+ + \lambda(\psi) g_{14}^-\right) \\ \dot{x_1} & = & f_{14}^+ + \lambda(\psi) f_{14}^- \\ \dot{x_3} & = & h_{14}^+ + \lambda(\psi) h_{14}^- \end{array} \right.$	$X_r^4 \left\{ \begin{array}{lcl} u\dot{\psi} & = & S_\psi\left(g_{23}^+ + \lambda(\psi) g_{23}^-\right) \\ \dot{x_1} & = & f_{23}^+ + \lambda(\psi) f_{23}^- \\ \dot{x_3} & = & h_{23}^+ + \lambda(\psi) h_{23}^- \end{array} \right.$
$X_1^r \left\{ \begin{array}{lcl} \theta' & = & S_\theta\left(f_{12}^+ + \lambda(\theta) f_{12}^-\right) \\ x_2' & = & r\left(g_{12}^+ + \lambda(\theta) g_{12}^-\right) \\ x_3' & = & r\left(h_{12}^+ + \lambda(\theta) h_{12}^-\right) \end{array} \right.$	$X_2^r \left\{ \begin{array}{lcl} \theta' & = & S_\theta\left(f_{43}^+ + \lambda(\theta) f_{43}^-\right) \\ x_2' & = & r\left(g_{43}^+ + \lambda(\theta) g_{43}^-\right) \\ x_3' & = & r\left(h_{43}^+ + \lambda(\theta) h_{43}^-\right) \end{array} \right.$
$X_3^r \left\{ \begin{array}{lcl} \psi' & = & S_\psi\left(g_{14}^+ + \lambda(\psi) g_{14}^-\right) \\ x_1' & = & u\left(f_{14}^+ + \lambda(\psi) f_{14}^-\right) \\ x_3' & = & u\left(h_{14}^+ + \lambda(\psi) h_{14}^-\right) \end{array} \right.$	$X_4^r \left\{ \begin{array}{lcl} \psi' & = & S_\psi\left(g_{23}^+ + \lambda(\psi) g_{23}^-\right) \\ x_1' & = & u\left(f_{23}^+ + \lambda(\psi) f_{23}^-\right) \\ x_3' & = & u\left(h_{23}^+ + \lambda(\psi) h_{23}^-\right) \end{array} \right.$

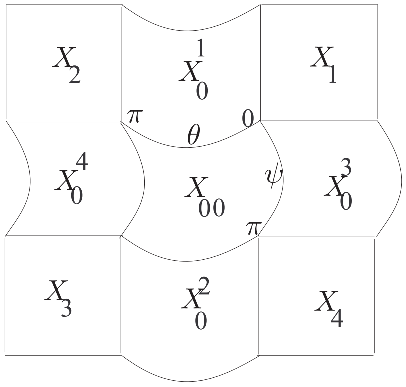


Figure 5. *Local vector fields in the blowing-up locus.*

Since the proofs of the cases (a) and (b) are similar, we will prove only (b) for $x_2 > 0$. The slow manifold for $\theta \in (0, \pi)$ is implicitly determined by the equation $f_{12}^+ + \lambda(\theta) f_{12}^- = 0$. We have that $f_{12}^- = 0$ if and only if $f_1(0, x_2, x_3) = f_2(0, x_2, x_3)$.

We denote $f(x_1, x_2, x_3)$ and $f(r, \theta, x_2, x_3)$ before and after the blow-up, respectively. If $(0, \theta, x_2, x_3) \in (\mathcal{S}^{sl} \bigcup \mathcal{S}^{es})$, then the vector fields point inward or outward. This implies that

$$f_1(0, \theta, x_2, x_3).f_2(0, \theta, x_2, x_3) < 0.$$

Thus we have that $f_{12}^-(0, \theta, x_2, x_3) \neq 0$ for any $\theta \in (0, \pi)$, $(0, x_2, x_3) \in (\mathcal{S}^{sl} \bigcup \mathcal{S}^{es})$. Since

$$\frac{f_{12}^+}{f_{12}^-} = \pm 1 \Rightarrow f_1.f_2 = 0,$$

we have

$$-1 < -\frac{f_{12}^+(0, \theta, x_2, x_3)}{f_{12}^-(0, \theta, x_2, x_3)} < 1$$

for all $(0,\theta,x_2,x_3)\in(\mathcal{S}^{sl}\bigcup\mathcal{S}^{es})$. Moreover, the inverse of the restriction $\lambda|_{(\frac{\pi}{4},\frac{3\pi}{4})}$ is increasing on $(-1,1)$, and the equation $f_{12}^{+}+\lambda(\theta)f_{12}^{-}=0$ defines a continuous graphic contained in

$$\left\{(\theta,x_2,x_3)\mid x_2\in\mathbb{R},\,x_3\in\mathbb{R},\,\theta\in\left(\frac{\pi}{4},\frac{3\pi}{4}\right)\right\}.$$

In accordance with the definition of $X^{\mathcal{S}}$, we have that $X^{\mathcal{S}}=X_1+k(X_2-X_1)$ with $k\in\mathbb{R}$ such that $X_1(x_1,x_2,x_3)+k(X_2-X_1)(x_1,x_2,x_3)=(0,y_2,y_3)$ for some y_2,y_3. Thus it is easy to see that $X^{\mathcal{S}}$ is given by $X^{\mathcal{S}}=\left(0,\frac{f_1g_2-f_2g_1}{f_1-f_2},\frac{f_1h_2-f_2h_1}{f_1-f_2}\right)$. The reduced problem is then represented by $\dot{x}_2=g_{12}^{+}+\lambda(\theta)g_{12}^{-}$, $\dot{x}_3=h_{12}^{+}+\lambda(\theta)h_{12}^{-}$ under the restriction $\lambda(\theta)=-\frac{f_{12}^{+}}{f_{12}^{-}}$. Then we must have $\dot{x}_2=\frac{f_1g_2-f_2g_1}{f_1-f_2}$, $\dot{x}_3=\frac{f_1h_2-f_2h_1}{f_1-f_2}$. It follows immediately that the flows of $X^{\mathcal{S}}$ and the reduced problem are equivalent.

To prove case (c) we consider the φ_{x_1}-regularization of X_1^r and X_2^r:

$$X_a^{1^r,2^r}\left\{\begin{array}{lcl}\theta' &=& S_\theta\left(f_{1234}^{+++}+\lambda(\theta)f_{1243}^{-+-}+\varphi(x_2/a)f_{1234}^{+--}+\varphi(x_2/a)\lambda(\theta)f_{1243}^{--+}\right),\\ x_2' &=& r\left(g_{1234}^{+++}+\lambda(\theta)g_{1243}^{-+-}+\varphi(x_2/a)g_{1234}^{+--}+\varphi(x_2/a)\lambda(\theta)g_{1243}^{--+}\right),\\ x_3' &=& r\left(h_{1234}^{+++}+\lambda(\theta)h_{1243}^{-+-}+\varphi(x_2/a)h_{1234}^{+--}+\varphi(x_2/a)\lambda(\theta)h_{1243}^{--+}\right).\end{array}\right. \tag{3.5}$$

The equations of the equivalent system with $a=u\sin\psi$, $x_2=u\cos\psi$ are

$$\left\{\begin{array}{lcl}\theta' &=& S_\theta\left(f_{1234}^{+++}+\lambda(\theta)f_{1243}^{-+-}+\lambda(\psi)f_{1234}^{+--}+\lambda(\psi)\lambda(\theta)f_{1243}^{--+}\right),\\ u\psi' &=& rS_\psi\left(g_{1234}^{+++}+\lambda(\theta)g_{1243}^{-+-}+\lambda(\psi)g_{1234}^{+--}+\lambda(\psi)\lambda(\theta)g_{1243}^{--+}\right),\\ x_3' &=& r\left(h_{1234}^{+++}+\lambda(\theta)h_{1243}^{-+-}+\lambda(\psi)h_{1234}^{+--}+\lambda(\psi)\lambda(\theta)h_{1243}^{--+}\right).\end{array}\right. \tag{3.6}$$

By means of the time rescaling $\tau=us$ we get (keeping the notation θ',ψ',x_3' for simplicity)

$$\left\{\begin{array}{lcl}\theta' &=& uS_\theta\left(f_{1234}^{+++}+\lambda(\theta)f_{1243}^{-+-}+\lambda(\psi)f_{1234}^{+--}+\lambda(\psi)\lambda(\theta)f_{1243}^{--+}\right),\\ \psi' &=& rS_\psi\left(g_{1234}^{+++}+\lambda(\theta)g_{1243}^{-+-}+\lambda(\psi)g_{1234}^{+--}+\lambda(\psi)\lambda(\theta)g_{1243}^{--+}\right),\\ x_3' &=& r\left(h_{1234}^{+++}+\lambda(\theta)h_{1243}^{-+-}+\lambda(\psi)h_{1234}^{+--}+\lambda(\psi)\lambda(\theta)h_{1243}^{--+}\right).\end{array}\right. \tag{3.7}$$

Thus we have

$$\xi(r,u,\theta,\psi,x_3)=S_\theta\left(f_{1234}^{+++}+\lambda(\theta)f_{1243}^{-+-}+\lambda(\psi)f_{1234}^{+--}+\lambda(\psi)\lambda(\theta)f_{1243}^{--+}\right),$$

$$\zeta(r,u,\theta,\psi,x_3)=S_\psi\left(g_{1234}^{+++}+\lambda(\theta)g_{1243}^{-+-}+\lambda(\psi)g_{1234}^{+--}+\lambda(\psi)\lambda(\theta)g_{1243}^{--+}\right),$$

and

$$\phi(r,u,\theta,\psi,x_3)=h_{1234}^{+++}+\lambda(\theta)h_{1243}^{-+-}+\lambda(\psi)h_{1234}^{+--}+\lambda(\psi)\lambda(\theta)h_{1243}^{--+}.$$

We define the slow system ($u=0$ at (3.6)) and the fast system (divide by r and $u=0$ at (3.7)) at $(x_1,x_2,x_3)=(0,0,x_3)$, respectively, by

$$X_{00}^{S}\left\{\begin{array}{lcl}0 &=& S_\psi\left(g_{1234}^{+++}+\lambda(\theta)g_{1243}^{-+-}+\lambda(\psi)g_{1234}^{+--}+\lambda(\psi)\lambda(\theta)g_{1243}^{--+}\right),\\ \theta' &=& S_\theta\left(f_{1234}^{+++}+\lambda(\theta)f_{1243}^{-+-}+\lambda(\psi)f_{1234}^{+--}+\lambda(\psi)\lambda(\theta)f_{1243}^{--+}\right),\\ x_3' &=& 0\end{array}\right. \tag{3.8}$$

and

$$(3.9) \qquad X_{00}^F \begin{cases} \psi' &=& S_\psi \left(g_{1234}^{+++} + \lambda(\theta) g_{1243}^{-+-} + \lambda(\psi) g_{1234}^{+--} + \lambda(\psi)\lambda(\theta) g_{1243}^{--+} \right), \\ \theta' &=& 0, \\ x_3' &=& 0. \end{cases}$$

This finishes the proof. ■

Next the results of the theorem are illustrated.

Example 1. Consider $X_1(x_1,x_2) = (-1,-1)$, $X_2(x_1,x_2) = (2,-1)$, $X_3(x_1,x_2) = (4,5)$, and $X_4(x_1,x_2) = (-2,3)$. The trajectories of X_0^1 are the solutions of the reduced problem $S_\theta\left(\frac{1}{2} - \frac{3}{2}\lambda(\theta)\right) = 0$, $\dot{x_2} = -1$, where $S_\theta = -\sin\theta$. The slow manifold is the curve $\theta = \theta_0$ with $\lambda(\theta_0) = 1/3$, and the slow flow points to region $x_2 < 0$. The fast vector field is $(\theta', 0)$ with $\theta' < 0$ for $\theta > \theta_0$ and $\theta' > 0$ for $\theta < \theta_0$. The trajectories of X_0^2 are the solutions of the reduced problem $S_\theta(1 - 3\lambda(\theta)) = 0$, $\dot{x_2} = 4 - \lambda(\theta)$. The slow manifold is the curve $\theta = \theta_0$ with $\lambda(\theta_0) = 1/3$, and since $4 - \lambda(\theta) > 0$ for $\theta \in (0,\pi)$, the slow flow points to region $x_2 > 0$. The fast vector field is $(\theta', 0)$ with $\theta' < 0$ for $\theta > \theta_0$ and $\theta' > 0$ for $\theta < \theta_0$. The trajectories of X_0^3 are the solutions of the reduced problem $S_\psi(1 - 2\lambda(\psi)) = 0$, $\dot{x_1} = -\frac{3}{2} + \frac{1}{2}\lambda(\psi)$. The slow manifold is the curve $\psi = \psi_0$ with $\lambda(\psi_0) = 1/2$, and since $\dot{x_1} < 0$ for $\psi \in (0,\pi)$, the slow flow points to region $x_1 < 0$. The fast vector field is $(\psi', 0)$ with $\psi' > 0$ for $\psi < \psi_0$ and $\psi' < 0$ for $\psi > \psi_0$. The trajectories of X_0^4 are the solutions of the reduced problem $S_\psi(2 - 3\lambda(\psi)) = 0$, $\dot{x_1} = 3 - \lambda(\psi)$. The slow manifold is the curve $\psi = \psi_1$ with $\lambda(\psi_1) = 2/3$, and since $\dot{x_1} > 0$ for $\psi \in (0,\pi)$, the slow flow points to region $x_1 > 0$. The fast vector field is $(\psi', 0)$ with $\psi' > 0$ for $\psi < \psi_1$ and $\psi' < 0$ for $\psi > \psi_1$. The trajectories of X_{00} are the solutions of the reduced problem $S_\psi\left(\frac{3}{2} - \frac{1}{2}\lambda(\theta) - \frac{5}{2}\lambda(\psi) + \frac{1}{2}\lambda(\psi)\lambda(\theta)\right) = 0$, $\theta' = S_\theta\left(\frac{3}{4} - \frac{9}{2}\lambda(\theta) - \frac{1}{4}\lambda(\psi) + \frac{3}{4}\lambda(\psi)\lambda(\theta)\right)$. The slow manifold is the curve $\frac{3}{2} - \frac{1}{2}\lambda(\theta) - \frac{5}{2}\lambda(\psi) + \frac{1}{2}\lambda(\psi)\lambda(\theta) = 0$, which connects the points $(\theta,\psi) = (0,\psi_0)$ and $(\theta,\psi) = (\pi,\psi_1)$ with $\lambda(\psi_0) = 1/2$ and $\lambda(\psi_1) = 2/3$. In fact, if $\theta = 0$, then $\lambda(\theta) = 1$, and $\frac{3}{2} - \frac{1}{2}\lambda(\theta) - \frac{5}{2}\lambda(\psi) + \frac{1}{2}\lambda(\psi)\lambda(\theta) = 0$ implies $\lambda(\psi) = \frac{1}{2}$, and if $\theta = \pi$, then $\lambda(\theta) = -1$, and $\frac{3}{2} - \frac{1}{2}\lambda(\theta) - \frac{5}{2}\lambda(\psi) + \frac{1}{2}\lambda(\psi)\lambda(\theta) = 0$ implies $\lambda(\psi) = \frac{2}{3}$. Observe that $\frac{3}{2} - \frac{1}{2}\lambda(\theta) - \frac{5}{2}\lambda(\psi) + \frac{1}{2}\lambda(\psi)\lambda(\theta) = 0$ implies $\theta' = \frac{S_\theta}{4(\lambda(\theta)-5)}(-15\lambda^2(\theta) + 83\lambda(\theta) - 12)$. There exists a unique $\alpha = \frac{83}{30} - \frac{1}{30}\sqrt{6169} \in (-1,1)$ such that $-15\alpha^2 + 83\alpha - 12 = 0$. Furthermore, if $\lambda < \alpha$, then $-15\lambda^2 + 83\lambda - 12 < 0$, and if $\lambda > \alpha$, then $-15\lambda^2 + 83\lambda - 12 > 0$. So there exists a unique $\theta_* \in (\frac{\pi}{4}, \frac{3\pi}{4})$ such that $\lambda(\theta_*) = \alpha$ and

$$0 < \theta < \theta_* \Rightarrow \lambda(\theta) > \lambda(\theta_*) \Rightarrow \theta' > 0$$

and

$$\theta > \theta_* \Rightarrow \lambda(\theta) < \lambda(\theta_*) \Rightarrow \theta' < 0.$$

Moreover, $\lambda(\theta_0) = 1/3$ and $\theta' > 0$ imply that $\theta_0 < \theta_*$. See Figure 6.

Example 2. Consider $X_1(x,y) = (-y,x)$ and $X_2(x,y) = (1,-1)$. The trajectories of X_0^1 are the solutions of the reduced problem

$$\frac{-y+1}{2} + \lambda(\theta)\frac{-y-1}{2} = 0, \quad \dot{y} = -\frac{1}{2}(1 - \lambda(\theta)).$$

The slow manifold is the curve $y = y(\theta)$ given by $y = \frac{1-\lambda(\theta)}{1+\lambda(\theta)}$. This function is increasing:

$$\lim_{\theta \longrightarrow \frac{\pi}{4}} y(\theta) = 0, \qquad \lim_{\theta \longrightarrow \frac{3\pi}{4}} y(\theta) = +\infty.$$

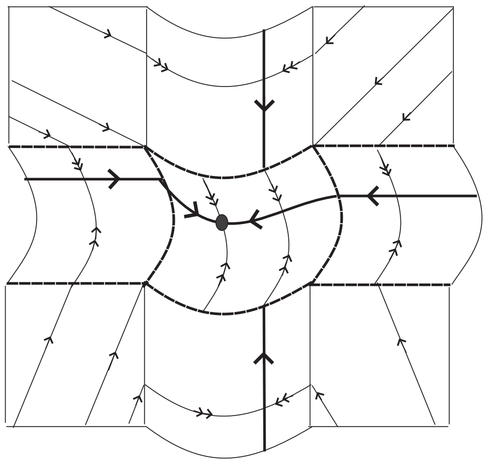


Figure 6. *Local vector fields in the blowing-up locus of Example 1.*

Since $1-\lambda(\theta)\geq 0$, we have that $\dot{y}\leq 0$. So the slow flow points to region $y<0$. Denote $\theta(y)$ the inverse function of $y(\theta)$. The fast vector field is $(\theta',0)$ with $\theta'<0$ for $\theta(y)<\theta<\pi$ and $\theta'>0$ for $0<\theta<\theta(y)$. The trajectories of X_0^2 are the solutions of the reduced problem

$$\frac{-y+1}{2}+\lambda(\theta)\frac{y+1}{2}=0,\quad \dot{y}=-\frac{1}{2}\left(1+\lambda(\theta)\right).$$

The slow manifold is the empty set. In fact, the right-hand side of $y=\frac{1+\lambda(\theta)}{-\lambda(\theta)+1}$ is positive for $\theta\in(\frac{\pi}{4},\frac{3\pi}{4})$, and here we have $y<0$. The fast vector field is $(\theta',0)$ with $\theta'<0$ for $0<\theta<\pi$.

The trajectories of X_0^3 are the solutions of the reduced problem

$$\frac{x-1}{2}+\lambda(\psi)\frac{x+1}{2}=0,\quad \dot{x}=\frac{1}{2}\left(1-\lambda(\psi)\right).$$

The slow manifold is the curve $x=x(\psi)$ given by $x=\frac{1-\lambda(\psi)}{1+\lambda(\psi)}$. This function is increasing:

$$\lim_{\psi\longrightarrow\frac{\pi}{4}}x(\psi)=0,\qquad \lim_{\psi\longrightarrow\frac{3\pi}{4}}x(\psi)=+\infty.$$

We have $\dot{x}\geq 0$. So the slow flow points to region $x>0$. Denote by $\psi(x)$ the inverse function of $x(\psi)$. The fast vector field is $(\psi',0)$ with $\psi'<0$ for $0<\psi<\psi(x)$ and $\psi'>0$ for $\psi(x)<\psi<\pi$. The trajectories of X_0^4 are the solutions of the reduced problem

$$\frac{x-1}{2}+\lambda(\psi)\frac{-x-1}{2}=0,\quad \dot{x}=\frac{1}{2}\left(1+\lambda(\psi)\right).$$

The slow manifold is the empty set. In fact, the right-hand side of $x=\frac{1+\lambda(\psi)}{-\lambda(\psi)+1}$ is positive, and here we have $x<0$. The fast vector field is $(\psi',0)$ with $\psi'<0$ for $0<\psi<\pi$. The trajectories of X_{00} are the solutions of the reduced problem

$$\frac{-1}{2}+\lambda(\psi)\lambda(\theta)\frac{1}{2}=0,\quad \theta'=\frac{S_\theta}{2}\left(1-\lambda(\psi)\lambda(\theta)\right).$$

The slow manifold is given by the equation $\lambda(\psi)\lambda(\theta) = 1$, and the slow flow is $\theta' = 0$, which is highly degenerated. The fast vector field is $(\psi', 0)$ with $\psi' \le 0$ for $0 < \psi < \pi$. Note that $\lambda(\psi)\lambda(\theta) = 1$ for all $(\theta, \psi) \in ([0, \pi/4] \times [0, \pi/4]) \cup ([3\pi/4, \pi] \times [3\pi/4, \pi])$. See Figure 7.

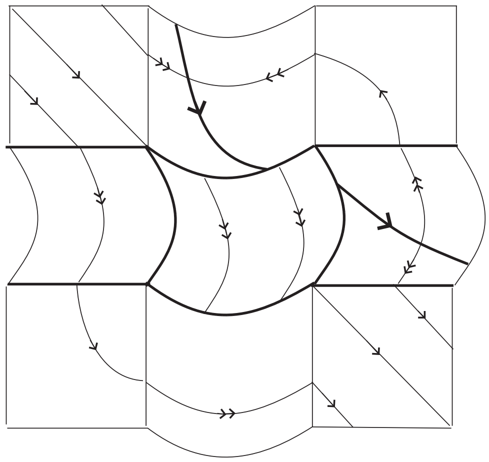


Figure 7. *Local vector fields in the blowing-up locus of Example 2.*

4. Intersecting switching surfaces on $\mathbb{R}^3$.

In this section we consider the case $\ell = 3$. We provide sufficient conditions in order for an equilibrium point of the reduced problem (2.6) to be a local attractor. See previous results in this direction in [12, 13].

In what follows we use the notation

$$\frac{\partial(\Phi, \Psi)}{\partial(a, b)} = \begin{vmatrix} \Phi_{aa} & \Psi_{ab} \\ \Phi_{ba} & \Psi_{bb} \end{vmatrix}.$$

Proposition 4.1. *Consider $\mathcal{U} \subseteq \mathbb{R}^3$, an open set with $0 \in \mathcal{U}$ and $X = (X_1, \dots, X_4) \in \Omega^{\kappa}_{1234}(\mathcal{U})$. Denote $X_i = (f_i, g_i, h_i)$, $i = 1, \dots, 4$. Let $\overline{\zeta}$, $\overline{\psi}$ be the functions which satisfy $\zeta = S_\theta \overline{\zeta}$ and $\xi = S_\psi \overline{\xi}$ with ζ, ξ, and ϕ given by (2.5). Let $D \subseteq (0, \pi) \times (0, \pi) \times \mathbb{R}$ be an open neighborhood of $(\theta_0, \psi_0, 0)$. Suppose that $\overline{\zeta}(0, 0, \theta_0, \psi_0, 0) = \overline{\xi}(0, 0, \theta_0, \psi_0, 0) = 0$ and that for $(\theta, \psi, x_3) \in D$ we have*

$$\frac{\partial(\overline{\zeta}, \overline{\xi})}{\partial(\theta, \psi)} > 0, \quad \frac{\partial(\overline{\zeta}, \overline{\xi})}{\partial(\theta, x_3)} < 0, \quad \frac{\partial(\overline{\zeta}, \overline{\xi})}{\partial(\psi, x_3)} < 0.$$

Then the slow manifold SM_2 is a curve parameterized by $(\theta(x_3), \psi(x_3), x_3)$. Moreover, if

$$\phi(\theta_0, \psi_0, 0) = 0, \quad \frac{\partial}{\partial x_3}\phi(\theta_0, \psi_0, 0) < 0,$$

then $(\theta_0, \psi_0, 0)$ is an attracting singular point of the reduced system (2.6).

Proof. It is enough to apply the implicit function theorem. The assertions about the singular point follow the fact that the reduced problem is $x_3' = \alpha x_3 + \cdots$ with $\alpha < 0$. ■

Example. Consider $X = (X_1, \dots, X_4) \in \Omega^{\kappa}_{1234}(\mathcal{U})$ given by $X_1 = (-1, -1, -x_3)$, $X_2 = (2, -3, -x_3)$, $X_3 = (1, 2, -x_3)$, and $X_4 = (-3, 1, -x_3)$. System (2.5) takes the form

$$\begin{array}{rcl} u\psi' &=& rS_\psi(-\frac{1}{4} + \frac{1}{4}\lambda(\theta) - \frac{7}{4}\lambda(\psi) + \frac{3}{4}\lambda(\psi)\lambda(\theta)), \\ \theta' &=& S_\theta(-\frac{1}{4} - \frac{7}{4}\lambda(\theta) + \frac{3}{4}\lambda(\psi) + \frac{1}{4}\lambda(\theta)\lambda(\psi)), \\ x_3' &=& -rx_3. \end{array}$$

The slow manifold SM^1 is given by $-1 + \lambda(\theta) - 7\lambda(\psi) + 3\lambda(\theta)\lambda(\psi) = 0$. This is a surface parameterized by $(\theta, \psi(\theta), x_3)$ with $\theta \in [0, \pi]$. Moreover, the curve $(\theta, \psi(\theta))$ is such that it connects $(\theta, \psi) = (0, \pi/2)$ to (π, ψ_2) with $\lambda(\psi_2) = -1/5$. The fast vector field is such that $\psi' > 0$ on the upper side and $\psi' < 0$ otherwise. The slow manifold SM_2 is represented by

$$\left\{ \begin{array}{rcl} -1 + \lambda(\theta) - 7\lambda(\psi) + 3\lambda(\theta)\lambda(\psi) &=& 0, \\ -1 - 7\lambda(\theta) + 3\lambda(\psi) + \lambda(\theta)\lambda(\psi) &=& 0. \end{array} \right.$$

This curve is parameterized by (θ_0, ψ_0, x_3), $x_3 \in \mathbb{R}$, with $\lambda(\psi_0) = \frac{11-4\sqrt{11}}{2}$ and $\lambda(\theta_0) = \frac{3-\sqrt{11}}{2}$. Since $x_3' = -x_3$, the slow flow on SM_2 has an attracting singular point at $(\theta_0, \psi_0, 0)$. See Figure 8.

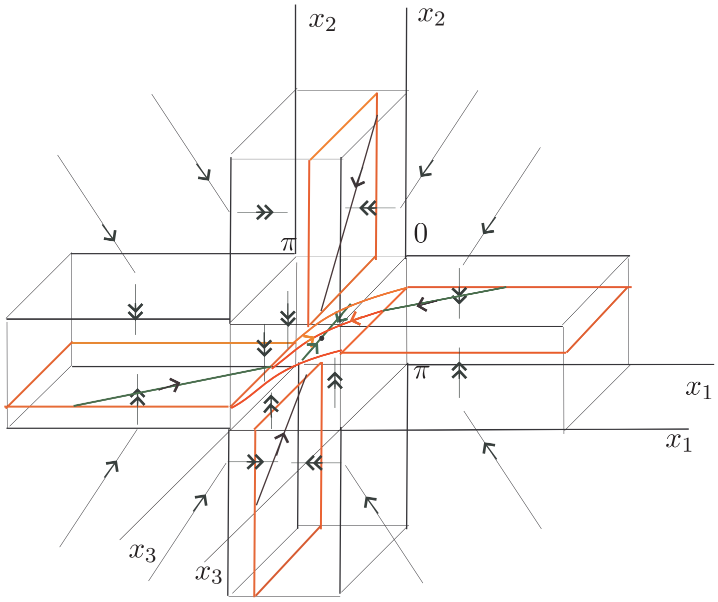


Figure 8. *A sequence of slow manifolds $SM_2 \subseteq SM_1$.*

5. Transient vector fields on $\mathbb{R}^2$. Now we give special attention to the class of discontinuous vector fields that present a transient behavior around the singularity. The following study program can be considered. Let $F : U \subseteq \mathbb{R}^2, 0 \to \mathbb{R}, 0$ be a generic Morse C^∞-function. This means that we may take a coordinate system (x_1, x_2) around $(0,0)$ such that it takes either the form (a) $F(x_1, x_2) = x_2$ or the form (b) $F(x, y) = x^2 + \varepsilon y^2$ with $\varepsilon = \pm 1$. We denote by $\mathcal{C}^\kappa(U, \mathbb{R}^2)$ the set of all vector fields of class $\mathcal{C}^\kappa$ defined on U, with $\kappa \geqslant 1$, endowed with

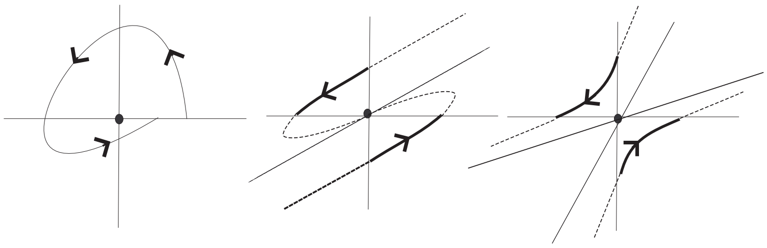


Figure 9. *T-type singular points.*

the C^κ-topology. Consider $X_1, X_2 \in \mathcal{C}^\kappa(U, \mathbb{R}^2)$. The idea is to study the following nonsmooth system in U:

$$X(q) = \begin{cases} X_1(q) = (f_1(q), g_1(q)) & \text{if} \quad F(q) > 0, \\ X_2(q) = (f_2(q), g_2(q)) & \text{if} \quad F(q) < 0. \end{cases} \tag{5.1}$$

The case (a) was treated in [11]. The case (b) can be seen as a particular case of the system studied above when $\varepsilon = -1$. In this case we will denote $X = (X_1, X_2) \in \Omega_{12}^\kappa(U)$. The case when $\varepsilon = 1$ does not deserve to be analyzed. We choose local coordinates such that $M = F^{-1}(0) = \{xy = 0\}$. Thus $M_1 = \{x.y > 0\}$ and $M_2 = \{x.y < 0\}$. In what follows we denote by $\mathcal{O}_{X_i}(p) = \{\varphi_{X_i}(t, p),\ t \in \mathbb{R}\}$ the orbit of the vector field X^i, $i = 1, 2$, through the point p.

Definition 5.1. *We say that $X = (X_1, X_2) \in \Omega_{12}^\kappa(U)$, $U \subseteq \mathbb{R}^2$, is a transient vector field around 0 if, for all $p = (x, y) \in M_i$ for some $i = 1, 2$, there exist $t_1, t_2 \in \mathbb{R}$ such that $\mathcal{O}_{X_i}(p)$ ($i = 1$ if $xy > 0$ or $i = 2$ if $xy < 0$) cuts $0X \setminus \{0\}$ and $0Y \setminus \{0\}$ transversally for $t = t_1$ and $t = t_2$, respectively. Moreover, $t_1 t_2 < 0$, and for any t between t_1 and t_2 we have that $\varphi_{X_i}(t, p) \in M_i$.*

Definition 5.2. *Let $A \in M(2)$ be a 2×2 real matrix. We say that $(0, 0)$ is a T-type singular point of A if the Jordan form of A, $J(A)$, is one of the following.*

- *(focus-type) $J(A) = \left[\begin{smallmatrix} \alpha & -\beta \\ \beta & \alpha \end{smallmatrix}\right]$, with $\alpha \cdot \beta \neq 0$;*
- *(improper node–type) $J(A) = \left[\begin{smallmatrix} \alpha & 1 \\ 0 & \alpha \end{smallmatrix}\right]$, with $\alpha \neq 0$ and with the eigenvector $\omega \in M_i$;*
- *(saddle- or node-type) $J(A) = \left[\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right]$, with $\lambda_1 \neq \lambda_2 \in \mathbb{R}$ and with both eigenvectors in the same M_i.*

See Figure 9.

Proposition 5.3. *Let $X = (X_1, X_2) \in \Omega_{12}^\kappa(U)$, $U \subseteq \mathbb{R}^2$, be a transient vector field. If $(0, 0)$ is a hyperbolic singular point of X_i, $i = 1, 2$, then $(0, 0)$ is a singular point of the linear vector field $DX_i(0, 0)$ of T type.*

Proof. We have that $(0, 0)$ is a saddle or a node or a focus. The transient behavior of X_i implies that we cannot have orbits on $M_i \cup \{(0, 0)\}$ with α-limit or ω-limit equal to $\{(0, 0)\}$. Moreover, if $(0, 0)$ is a saddle or a node, then the stable and unstable manifolds are on $M_i \cup \{(0, 0)\}$. ■

Now we introduce some notation. $Q1 = \{(x, y);\ x > 0,\ y > 0\}$.

- We say that X_i for $i = 1, 2$ is of F_{ij} type in a neighborhood of $(0, 0)$ if $(0, 0)$ is a focus or an improper node of X_i. Moreover, the trajectories are positively oriented if $j = 1$ and negatively oriented if $j = 2$.

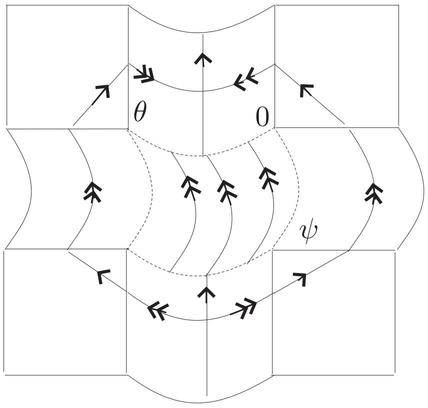


Figure 10. *Transient vector field of* (R_{11}, R_{21}) *type.*

- We say that X_i for $i = 1, 2$ is of S_{ij} type in a neighborhood of $(0, 0)$ if $(0, 0)$ is a saddle or a node of X_i and the trajectories on M_i are positively oriented if $j = 1$ and negatively oriented if $j = 2$.
- We say that X_1 is of R_{1j} type in a neighborhood of $(0, 0)$ if $(0, 0)$ is a regular point of X_1 and $X_1(0, 0) \in Q2$ if $j = 1$ and $X_1(0, 0) \in Q4$ if $j = 2$.
- We say that X_2 is of R_{2j} type in a neighborhood of $(0, 0)$ if $(0, 0)$ is a regular point of X_2 and $X_2(0, 0) \in Q1$ if $j = 1$ and $X_1(0, 0) \in Q3$ if $j = 2$.
- We say that (X_1, X_2) is of (A, B) type if X_1 is of A type and X_2 is of B type.

Remark. We have 36 possible combinations for X_1 and X_2. However, we can reduce this number, observing that some cases can be identified by means of a rotation or a change of sign. For example, roughly speaking, (F_{12}, F_{21}) can be identified with $-(F_{11}, F_{22})$, and (S_{11}, F_{21}) can be identified with $R_{\pi/2}(F_{11}, S_{21})$. Thus we have the following:

(a) If $X_1(0, 0) \in M_2$, $X_2(0, 0) \in M_1$, then, by means of a rotation, $X = (X_1, X_2)$ is such that its phase portrait is of (R_{11}, R_{21}) type. See Figure 10.

(b) If $|X_1(0, 0)|.|X_2(0, 0)| = 0$ and $|X_1(0, 0)|^2 + |X_2(0, 0)|^2 \neq 0$, then $X = (X_1, X_2)$ is such that its phase portrait is of (F_{11}, R_{21}), (F_{11}, R_{22}), (S_{11}, R_{21}), or (S_{11}, R_{22}) type. See Figures 11 and 12.

(c) If $|X_1(0, 0)|^2 + |X_2(0, 0)|^2 = 0$, then $X = (X_1, X_2)$ is such that its phase portrait is of (F_{11}, F_{21}), (F_{11}, F_{22}), (F_{11}, S_{21}), (F_{11}, S_{22}), (S_{11}, S_{21}), or (S_{11}, S_{22}) type. See Figures 13, 14, and 15.

Example. Let $X = (X_1, X_2) \in \Omega^k_{12}(U)$ be a transient vector field defined on a neighborhood of $0 \in U \subseteq \mathbb{R}^2$. Suppose that $X_1(0, 0) = (a, b)$, $ab < 0$, and $X_2(0, 0) = (c, d)$, with $cd > 0$. The slow manifold on the blowing-up locus corresponding to $x_1 > 0$, $x_2 = 0$ is $S_\psi(\frac{b+d}{2} + \frac{b-d}{2}\lambda(\psi)) = 0$ and on the blowing-up locus corresponding to $x_1 < 0$, $x_2 = 0$ is $S_\psi(\frac{b+d}{2} + \frac{d-b}{2}\lambda(\psi)) = 0$. If we consider $bd < 0$, we have that $-\frac{b+d}{b-d} \in (-1, 1)$, which implies that there exist $\psi_1, \psi_2 \in (\frac{\pi}{4}, \frac{3\pi}{4})$ such that the slow manifolds are $\psi = \psi_1$ and $\psi = \psi_2$, respectively. Finally, the equation of the slow manifold in the intersection is $S_\psi(\frac{b+d}{2} + \frac{b-d}{2}\lambda(\theta)\lambda(\psi)) = 0$. We have that this manifold connects the points $(\theta, \psi) = (0, \psi_1)$ and $(\theta, \psi) = (\pi, \psi_2)$.

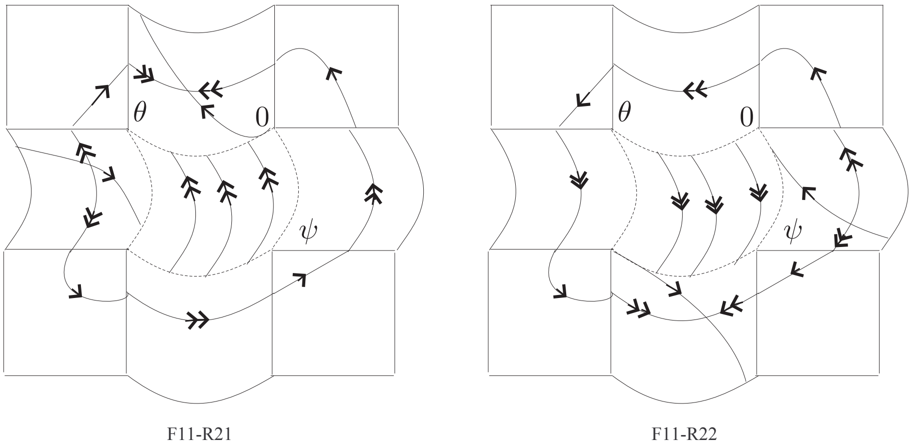


Figure 11. *Transient vector fields of* (F_{11}, R_{21}) *type and* (F_{11}, R_{22}) *type.*

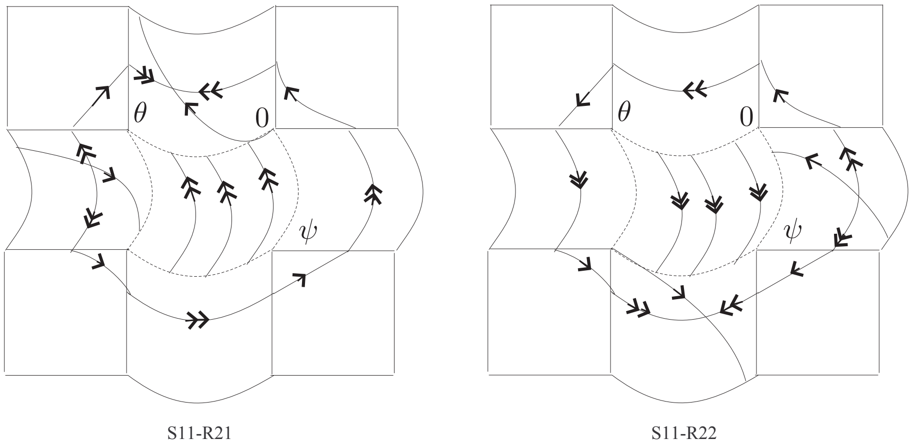


Figure 12. *Transient vector fields of* (S_{11}, R_{21}) *type and* (S_{11}, R_{22}) *type.*

Proposition 5.4. *Let* $X = (X_1, X_2) \in \Omega^k_{12}(U)$ *be a transient vector field defined on a neighborhood of* $0 \in U \subseteq \mathbb{R}^2$*. Denote*

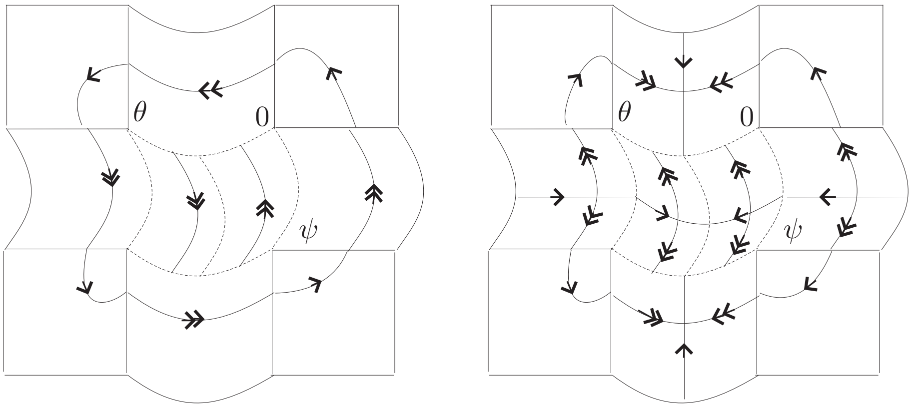
$$A_i = \begin{bmatrix} a_i & -b_i \\ b_i & a_i \end{bmatrix}, \quad i = 1, 2,$$

$$B_\lambda = \begin{bmatrix} \sin(\lambda_1 + \lambda_2) & -2\cos\lambda_1 \cos\lambda_2 \\ 2\sin\lambda_1 \sin\lambda_2 & -\sin(\lambda_1 + \lambda_2) \end{bmatrix},$$

and

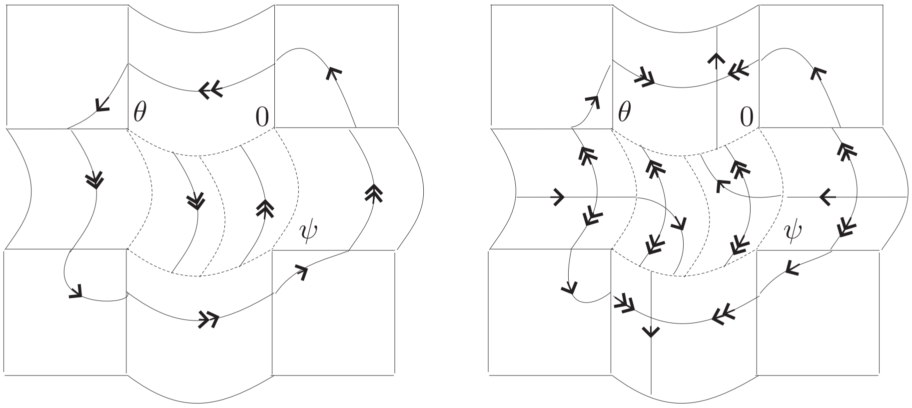
$$C_\lambda = \begin{bmatrix} -2\sin\lambda_1 \sin\lambda_2 & \sin(\lambda_1 + \lambda_2) \\ \sin(\lambda_1 + \lambda_2) & -2\cos\lambda_1 \cos\lambda_2 \end{bmatrix}.$$

(a) *If* $X_1(0,0) = X_2(0,0) = (0,0)$*,* $DX_1(0,0) = A_1$*, and* $DX_2(0,0) = A_2$*, then for* $b_1 > 0$*,* $b_2 > 0$ *the slow manifold is nontrivial and the reduced flow of system* (2.5) *is singular;*



F11-F21 F11-F22

Figure 13. *Transient vector fields of* (F_{11}, F_{21}) *type and* (F_{11}, F_{22}) *type.*



F11-S21 F11-S22

Figure 14. *Transient vector fields of* (F_{11}, S_{21}) *type and* (F_{11}, S_{22}) *type.*

and for $b_1 > 0$*,* $b_2 < 0$*, and* $a_1 \neq a_2$ *the slow manifold is nontrivial and the slow flow has a singular point.*

(b) *If* $X_1(0,0) = X_2(0,0) = (0,0)$*,* $DX_1(0,0) = A_1$*, and* $b_1 > 0$*,* $0 < \lambda_1 < \lambda_2 < \pi/2$*, then for* $DX_2(0,0) = B_\lambda$ *the slow manifold is nontrivial and the reduced flow of system* (2.5) *is singular; and for* $DX_2(0,0) = -B_\lambda$ *the slow manifold is nontrivial and the slow flow has not only singular points. Moreover, if* $2a_1 \sin(\lambda_1)\sin(\lambda_2) > b_1 \sin(\lambda_1 + \lambda_2)$*, then the slow flow has a singular point.*

(c) *If* $X_1(0,0) = X_2(0,0) = (0,0)$*,* $DX_1(0,0) = C_\lambda$*,* $0 < \lambda_1 < \lambda_2 < \pi/2$*, and* $0 < \mu_1 < \mu_2 < \pi/2$*, then for* $DX_2(0,0) = B_\mu$ *the slow manifold is nontrivial and the reduced flow of system* (2.5) *is singular; and for* $DX_2(0,0) = -B_\mu$ *the slow manifold is nontrivial and the slow flow has a part composed only of singular points and another part where there exists a singular point.*

We remark that A_i is the matrix of a linear vector field which has a singular point of

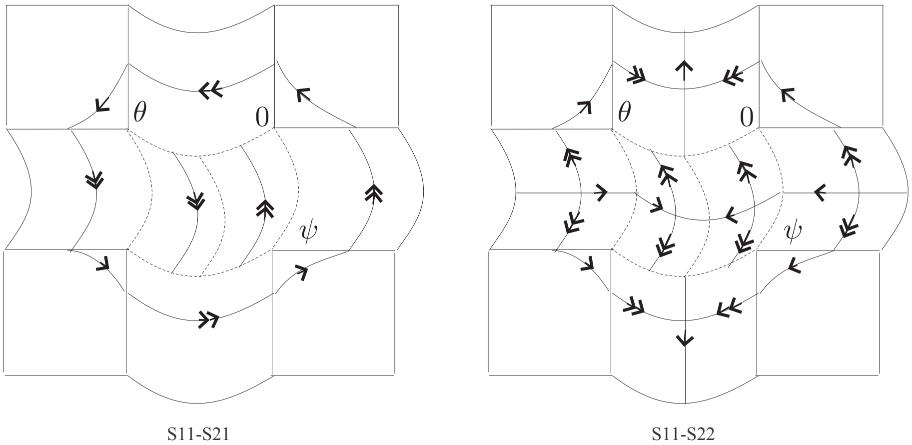


Figure 15. *Transient vector fields of* (S_{11}, S_{21}) *type and* (S_{11}, S_{22}) *type.*

focus type at $(0,0)$, and B_λ and C_λ are the matrices of linear vector fields which have singular points of saddle type at $(0,0)$ with eigenvectors $(\cos\lambda_1, \sin\lambda_1)$ and $(\cos\lambda_2, \sin\lambda_2)$ in the first and second quadrants, respectively.

Proof of Proposition 5.4. Suppose that X_1 and X_2 satisfy the hypothesis of (a). The equation of the slow manifold is $\frac{S_\psi}{2}\cos(\theta)((b_1+b_2)+(b_1-b_2)\lambda(\theta)\lambda(\psi)) = 0$. If $b_1 > 0$ and $b_2 > 0$, then $|\frac{b_1+b_2}{b_1-b_2}| > 1$. This implies that the nontrivial part of the slow manifold is given by $\theta = \pi/2$. Moreover, the slow flow is determined by θ', which is zero for $\theta = \pi/2$. If $b_1 > 0$ and $b_2 < 0$, then $|\frac{b_1+b_2}{b_1-b_2}| < 1$, and thus $(b_1+b_2)+(b_1-b_2)\lambda(\theta)\lambda(\psi) = 0$ defines two smooth curves. The slow flow has a singular point because $\theta' = S_\theta\cos(\theta)(a_2-a_1)$ has a sign change at $\theta = \frac{\pi}{2}$. So (a) is proved.

Suppose now that both X_1 and X_2 satisfy the hypothesis of (b). The equation of the slow manifold is $\frac{S_\psi}{2}\cos(\theta)[(b_1+2S_{\lambda_1}S_{\lambda_2})+(b_1-2S_{\lambda_1}S_{\lambda_2})\lambda(\theta)\lambda(\psi)] = 0$. If $b_1 > 0$ and $0 < \lambda_1 < \lambda_2 < \pi/2$, then $S_{\lambda_1}S_{\lambda_2} > 0$. This implies that the nontrivial part of the slow manifold is given by $\theta = \pi/2$. Moreover, the slow flow is determined by θ', which is zero for $\theta = \pi/2$. If $DX_2(0,0) = -B_\lambda$, then the slow manifold is $\frac{S_\psi}{2}\cos(\theta)[(b_1-2S_{\lambda_1}S_{\lambda_2})+(b_1+2S_{\lambda_1}S_{\lambda_2})\lambda(\theta)\lambda(\psi)] = 0$. Since $\frac{b_1-2S_{\lambda_1}S_{\lambda_2}}{b_1+2S_{\lambda_1}S_{\lambda_2}} \in (-1,1)$, the equation defines at least two curves. The slow flow has a singular point because $\theta' = S_\theta\cos(\theta)\left(\frac{2a_1S_{\lambda_1}S_{\lambda_2}+b_1S_{\lambda_1+\lambda_2}}{b_1+2S_{\lambda_1}S_{\lambda_2}}\right)$ has a sign change at $\theta = \frac{\pi}{2}$. So (b) is proved.

Suppose now that X_1 and X_2 satisfy the hypothesis of (c). The equation of the slow manifold is $\frac{S_\psi}{2}\cos(\theta)[(-S_{\lambda_1+\lambda_2}+2S_{\mu_1}S_{\mu_2})+(-S_{\lambda_1+\lambda_2}-2S_{\mu_1}S_{\mu_2})\lambda(\theta)\lambda(\psi)] = 0$. If $0 < \lambda_1 < \lambda_2 < \pi/2$ and $0 < \mu_1 < \mu_2 < \pi/2$, then $-S_{\lambda_1+\lambda_2} > 0$ and $S_{\mu_1}S_{\mu_2} > 0$. This implies that the nontrivial part of the slow manifold is given by $\theta = \pi/2$. Moreover, the slow flow is determined by θ', which is zero for $\theta = \pi/2$. If $DX_2(0,0) = -B_\mu$, then the slow manifold is $\frac{S_\psi}{2}\cos(\theta)[(-S_{\lambda_1+\lambda_2}-2S_{\mu_1}S_{\mu_2})+(-S_{\lambda_1+\lambda_2}+2S_{\mu_1}S_{\mu_2})\lambda(\theta)\lambda(\psi)] = 0$. Since $\frac{-S_{\lambda_1+\lambda_2}-2S_{\mu_1}S_{\mu_2}}{-S_{\lambda_1+\lambda_2}+2S_{\mu_1}S_{\mu_2}} \in (-1,1)$, the equation defines at least two curves. The slow flow has a singular point because $\theta' = S_\theta\cos(\theta)\left(\frac{-8S_{\lambda_1}S_{\lambda_2}S_{\mu_1}S_{\mu_2}-2S_{\lambda_1+\lambda_2}S_{\mu_1+\mu_2}}{-S_{\lambda_1+\lambda_2}+2S_{\mu_1}S_{\mu_2}}\right)$ has a sign change at $\theta = \frac{\pi}{2}$. In fact, observe that if

$$\delta = \frac{-8S_{\lambda_1}S_{\lambda_2}S_{\mu_1}S_{\mu_2} - 2S_{\lambda_1+\lambda_2}S_{\mu_1+\mu_2}}{-S_{\lambda_1+\lambda_2} + 2S_{\mu_1}S_{\mu_2}} \Rightarrow sgn(\delta) = -1,$$

then (c) is proved. ■

6. Conclusions. In this paper we propose a method of studying the dynamics around the intersection of codimension one discontinuity submanifolds. Using a regularization process proceeded by a blow-up, we get a singular perturbation problem. We also consider questions like transient behavior and asymptotic stability.

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