

Fermion localization on two-field thick branes

L. B. Castro*

Departamento de Física e Química, Universidade Estadual Paulista, 12516-410 Guaratinguetá, São Paulo, Brazil
(Received 24 August 2010; published 2 February 2011)

In a recent paper published in this journal, Almeida and collaborators [Phys. Rev. D **79**, 125022 (2009)] analyze the issue of fermion localization of fermions on a brane constructed from two scalar fields coupled with gravity (Bloch brane model). In that meritorious research the simplest Yukawa coupling $\eta\bar{\Psi}\phi\chi\Psi$ was considered. That work does not analyze the zero mode in details. In this paper, the localization of fermions on two-field thick branes is reinvestigated. It is found that the simplest Yukawa coupling does not support the localization of fermions on the brane. In addition, the problem of fermion localization for some other Yukawa couplings are analyzed. It is shown that the zero mode for left-handed and right-handed fermions can be localized on the brane depending on the values for the coupling constant η and the Bloch brane's parameter a .

DOI: 10.1103/PhysRevD.83.045002

PACS numbers: 11.10.Kk, 04.50.-h, 11.27.+d

I. INTRODUCTION

The authors of Ref. [1] analyzed the question of fermion localization on the brane constructed from two scalar fields coupled with gravity. That work considered the Bloch brane model [2,3]. It was found that the Yukawa coupling $\eta\bar{\Psi}\phi\chi\Psi$, where η is the coupling constant, allowed left-handed fermions to possess a zero mode that localizes on the brane. Fermionic resonances for both chiralities were obtained, and their appearance is related to branes with internal structure. That meritorious research did not study the zero mode in full detail. The localization problem of spin-1/2 fermions on thick branes is interesting and important [4–18]. Therefore, we believe that the conditions for obtaining normalizable zero modes on the Bloch branes model deserve to be more explored.

In Ref. [4], a model of one-scalar brane scenario was investigated and the authors addressed the problem of chiral fermion mode confinement in thick branes. The Yukawa coupling $\eta\bar{\Psi}\phi\Psi$ was considered and the conditions for the confinement for various different Bogomol'nyi-Prasad-Sommerfeld (BPS) branes, including double walls and branes interpolating between different AdS₅ spacetimes were obtained. The condition stated that for appropriate values of the coupling constant, the zero mode can be normalized and one chiral fermion mode localized. In a similar context, the authors of Ref. [5] addressed the problem of fermion localization on asymmetric branes (asymmetric Bloch brane model [3]). It was found that the usual couplings $\eta\bar{\Psi}(\phi + \chi)\Psi$ and $\eta\bar{\Psi}\phi\chi\Psi$ considered in the literature for two-scalar brane models do not support the localization of 4-dimensional massless fermions on the branes. Furthermore, the authors stated that the normalization of the zero mode is decided by the asymptotic behavior of the Yukawa coupling.

The main motivation of this paper is inspired on the results obtained in Ref. [4,5]. We reinvestigate the localization problem of fermions on two-field thick branes and we find that the usual Yukawa coupling $\eta\bar{\Psi}\phi\chi\Psi$ does not support the localization of fermions on the brane, in opposition to what was adverted in Ref. [1]. In addition, we analyze the problem of fermion localization for some other Yukawa couplings and find that the zero mode for left-handed and right-handed fermions can be localized on the branes depending on the value for the coupling constant η and the Bloch brane's parameter a .

II. THE BLOCH BRANE MODEL

The action for our system is described by [19]

$$S = \int d^4x dy \sqrt{|g|} \left[-\frac{1}{4}R + \frac{1}{2}\partial_a\phi\partial^a\phi + \frac{1}{2}\partial_a\chi\partial^a\chi - V(\phi, \chi) \right], \quad (1)$$

where $g \equiv \text{Det}(g_{ab})$ and the metric is

$$ds^2 = g_{ab}dx^a dx^b = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2, \quad (2)$$

where $y = x^4$ is the extra dimension (the Latin indices run from 0 to 4), $\eta_{\mu\nu}$ is the Minkowski metric and e^{2A} is the so-called warp factor (the Greek indices run from 0 to 3). We suppose that $A = A(y)$, $\phi = \phi(y)$ and $\chi = \chi(y)$.

One can determine the static equations of motion for the above system:

$$\phi'' + 4A'\phi' = \frac{\partial V(\phi, \chi)}{\partial \phi}, \quad (3)$$

$$\chi'' + 4A'\chi' = \frac{\partial V(\phi, \chi)}{\partial \chi}, \quad (4)$$

$$A'' = -\frac{2}{3}(\phi'^2 + \chi'^2), \quad (5)$$

*benito@feg.unesp.br

$$A'^2 = \frac{1}{6}(\phi'^2 + \chi'^2) - \frac{1}{3}V(\phi, \chi), \quad (6)$$

where prime stands for derivate with respect to y . We consider that the potential $V(\phi, \chi)$ can be written as [19]

$$V(\phi, \chi) = \frac{1}{2} \left[\left(\frac{\partial W(\phi, \chi)}{\partial \phi} \right)^2 + \left(\frac{\partial W(\phi, \chi)}{\partial \chi} \right)^2 \right] - \frac{4}{3} W(\phi, \chi)^2 \quad (7)$$

where $W(\phi, \chi)$ is the superpotential and it is given by [20]

$$W(\phi, \chi) = 2\phi - \frac{2}{3}\phi^3 - 2a\phi\chi^2 \quad (8)$$

with a being a real parameter ($0 < a < 1/2$). The above potential leads to the following first-order differential equations, which also solve the equations of motion: [2,3,19]

$$\phi' = \frac{\partial W(\phi, \chi)}{\partial \phi}, \quad \chi' = \frac{\partial W(\phi, \chi)}{\partial \chi}, \quad (9)$$

$$A' = -\frac{2}{3}W(\phi, \chi). \quad (10)$$

The classical solutions of the first-order differential Eqs. (9) and (10) are given by

$$\phi(y) = \tanh(2ay), \quad (11)$$

$$\chi(y) = \sqrt{\frac{1-2a}{a}} \operatorname{sech}(2ay), \quad (12)$$

and

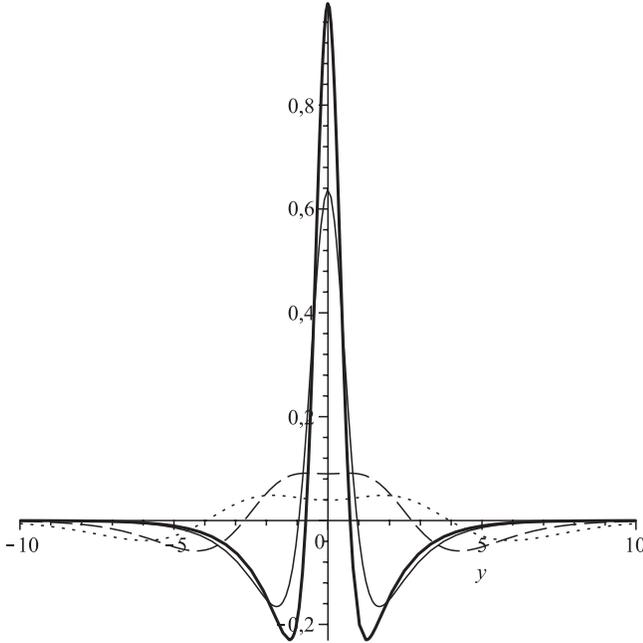


FIG. 1. The matter energy density for $a = 0.09$ (dotted line), $a = 0.15$ (dashed line), $a = 0.39$ (thin line) and $a = 0.49$ (thick line).

$$A(y) = \frac{1}{9a} \{ (1-3a)\tanh^2(2ay) - 2\ln[\cosh(2ay)] \}. \quad (13)$$

The matter energy density has the form

$$\rho(y) = e^{2A(y)} \left[\frac{1}{2} \left(\frac{d\phi(y)}{dy} \right)^2 + \frac{1}{2} \left(\frac{d\chi(y)}{dy} \right)^2 + V(\phi(y), \chi(y)) \right]. \quad (14)$$

The profiles of the energy density is shown in Fig. 1 for some values of a . Figure 1 clearly shows that the brane is localized at $y = 0$ for $0.17 < a < 0.5$ and that for $0 < a < 0.17$ the behavior for the energy density shows the appearance of two peaks (two sub-branes). More detailed discussions can be found in Ref. [2].

III. FERMION LOCALIZATION

The action for a Dirac spinor field coupled with the scalar fields by a general Yukawa coupling is

$$S = \int d^5x \sqrt{|g|} [i\bar{\Psi}\Gamma^M D_M \Psi - \eta\bar{\Psi}F(\phi, \chi)\Psi] \quad (15)$$

where η is the coupling constant between fermions and scalar fields. Here we consider the fields ϕ and χ as background fields. Making the usual change of variable

$$z = \int_0^y e^{-A(y')} dy', \quad (16)$$

we get a conformally flat metric

$$ds^2 = e^{2A}(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2). \quad (17)$$

Using the metric (17) and the representation for gamma matrices $\Gamma^M = (e^{-A}\gamma^\mu, -ie^{-A}\gamma^5)$, the equation of motion is

$$[i\gamma^\mu \partial_\mu + \gamma^5(\partial_z + 2\partial_z A) - \eta e^A F(\phi, \chi)]\Psi = 0. \quad (18)$$

In this stage, we use the general chiral decomposition

$$\Psi(x, z) = \sum_n \psi_{L_n}(x)\alpha_{L_n}(z) + \sum_n \psi_{R_n}(x)\alpha_{R_n}(z), \quad (19)$$

with $\psi_{L_n}(x) = -\gamma^5 \psi_{L_n}(x)$ and $\psi_{R_n}(x) = \gamma^5 \psi_{R_n}(x)$. With this decomposition $\psi_{L_n}(x)$ and $\psi_{R_n}(x)$ are the left-handed and right-handed components of the four-dimensional spinor field, respectively. After applying (19) in (18), and demanding that $i\gamma^\mu \partial_\mu \psi_{L_n} = m_n \psi_{R_n}$ and $i\gamma^\mu \partial_\mu \psi_{R_n} = m_n \psi_{L_n}$, we obtain two equations for α_{L_n} and α_{R_n} :

$$[\partial_z + 2\partial_z A + \eta e^A F(\phi, \chi)]\alpha_{L_n} = m_n \alpha_{R_n}, \quad (20)$$

$$[\partial_z + 2\partial_z A - \eta e^A F(\phi, \chi)]\alpha_{R_n} = -m_n \alpha_{L_n}, \quad (21)$$

which can be reduced to the Schrödinger-like equations for $m_n \neq 0$. Inserting the general chiral decomposition (19) into the action (15), using (20) and (21) and also requiring that the result take the form of the standard four-dimensional action for the massive chiral fermions

$$S = \sum_n \int d^4x \bar{\psi}_n (\gamma^\mu \partial_\mu - m_n) \psi_n, \quad (22)$$

where $\psi_n = \psi_{L_n} + \psi_{R_n}$ and $m_n \geq 0$, the functions α_{L_n} and α_{R_n} must obey the following orthonormality conditions

$$\int_{-\infty}^{\infty} dz e^{4A} \alpha_{L_m} \alpha_{R_n} = \delta_{LR} \delta_{mn}. \quad (23)$$

Implementing the change of variables $\alpha_{L_n} = e^{-2A} L_n$ and $\alpha_{R_n} = e^{-2A} R_n$, we get

$$-L_n''(z) + V_L(z) L_n = m_n^2 L_n, \quad (24)$$

$$-R_n''(z) + V_R(z) R_n = m_n^2 R_n \quad (25)$$

where

$$V_L(z) = \eta^2 e^{2A} F^2(\phi, \chi) - \eta \partial_z (e^A F(\phi, \chi)), \quad (26)$$

$$V_R(z) = \eta^2 e^{2A} F^2(\phi, \chi) + \eta \partial_z (e^A F(\phi, \chi)). \quad (27)$$

Using the expressions $\partial_z A = e^A \partial_y A$ and $\partial_z F = e^A \partial_y F$, we can recast the potentials (26) and (27) as a function of y [5]:

$$V_L(z(y)) = \eta e^{2A} [\eta F^2 - \partial_y F - F \partial_y A(y)] \quad (28)$$

$$V_R(z(y)) = V_L(z(y)) |_{\eta \rightarrow -\eta}. \quad (29)$$

It is worthwhile to note that we can construct the Schrödinger potentials V_L and V_R from (28) and (29).

Now we focus attention on the motivation of this paper, the calculation of the zero mode. Substituting $m_n = 0$ in (20) and (21) and using $\alpha_{L_n} = e^{-2A} L_n - 2$ and $\alpha_{R_n} = e^{-2A} R_n$, respectively, we get

$$L_0 \propto \exp \left[-\eta \int_0^z dz' e^{A(z')} F(\phi, \chi) \right] \quad (30)$$

$$R_0 \propto \exp \left[\eta \int_0^z dz' e^{A(z')} F(\phi, \chi) \right]. \quad (31)$$

This fact is the same to the case of two-dimensional Dirac equation, in this context are called isolated solutions [21]. At this point is worthwhile to mention that the normalization of the zero mode and the existence of a minimum of the effective potential at the localization on the brane are essential conditions for the problem of fermion localization on the brane.

In order to check the normalization condition (23) for the left-handed fermion zero mode (30), the integral can be convergent, i.e.,

$$\int_{-\infty}^{\infty} dz \exp \left[-2\eta \int_0^z dz' e^{A(z')} F(\phi(z'), \chi(z')) \right] < \infty, \quad (32)$$

and using the expression $dz = e^{-A(y)} dy$ the integral (32) can be written as function of y and it becomes

$$\int_{-\infty}^{\infty} dy \exp \left[-A(y) - 2\eta \int_0^y dy' F(\phi(y'), \chi(y')) \right] < \infty. \quad (33)$$

This result clearly shows that the normalization of the zero mode is decided by the asymptotic behavior of $F(\phi(y), \chi(y))$. This fact has already been addressed in Ref. [4,5], for various different BPS branes and asymmetric Bloch Branes, respectively.

From (28) and (29) it can be observed that the effective potential profile depends on the choice of $F(\phi(y), \chi(y))$. This fact implies that the existence of a minimum of the effective potential $V_L(z(y))$ or $V_R(z(y))$ at the localization on the brane is decided by $F(\phi(y), \chi(y))$. This point will be more clear when it is considered a specific Yukawa coupling. Having set-up the two essential conditions for the problem of fermion localization on the brane, we are now in a position to choose some specific forms for Yukawa couplings.

A. Case 1: $F(\phi, \chi) = \phi\chi$

This Yukawa coupling has been analyzed in Ref. [1] for $a = 1/3$. In this case, from Eqs. (11)–(13) the integrand in (33) can be expressed as

$$I = \exp \left\{ -\frac{(1-3a)}{9a} \tanh^2(2ay) + \frac{2}{9a} \ln[\cosh(2ay)] - \frac{\eta}{a} \sqrt{\frac{1-2a}{a}} [1 - \operatorname{sech}(2ay)] \right\}. \quad (34)$$

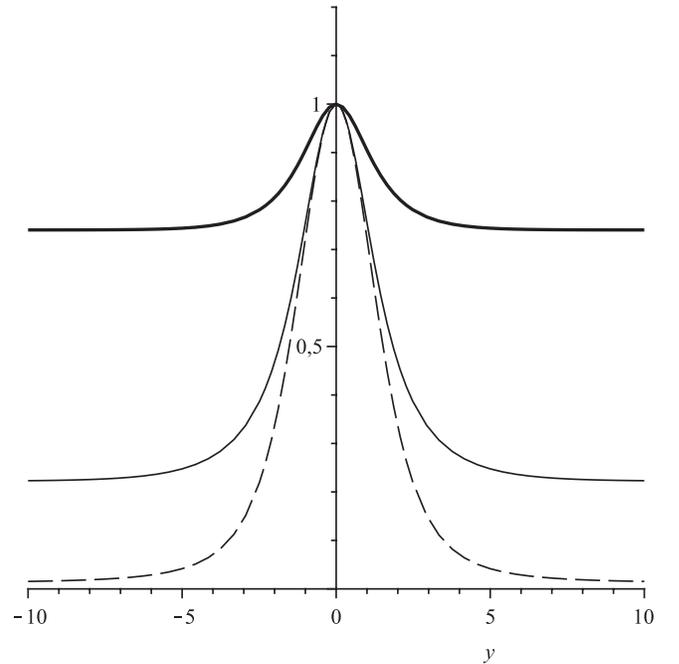


FIG. 2. The zero mode of left-handed fermion for the case $F(\phi, \chi) = \phi\chi$, with $\eta = 1$, $a = 0.20$ (dashed line), $a = 0.33$ (thin line) and $a = 0.48$ (thick line).

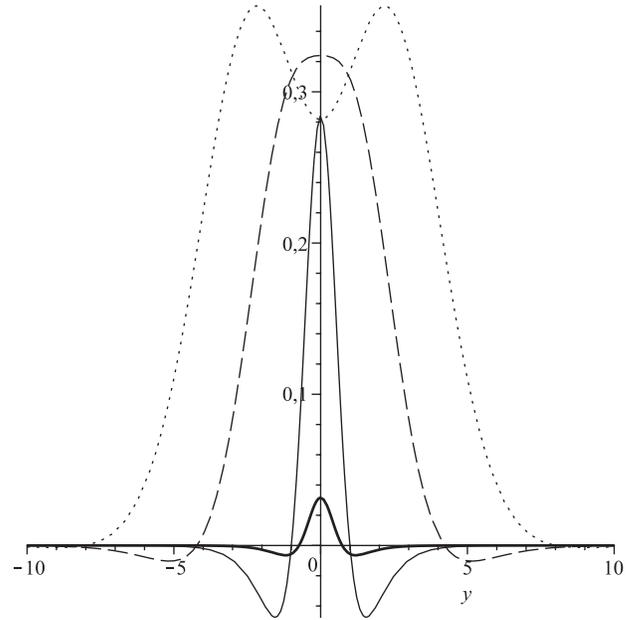
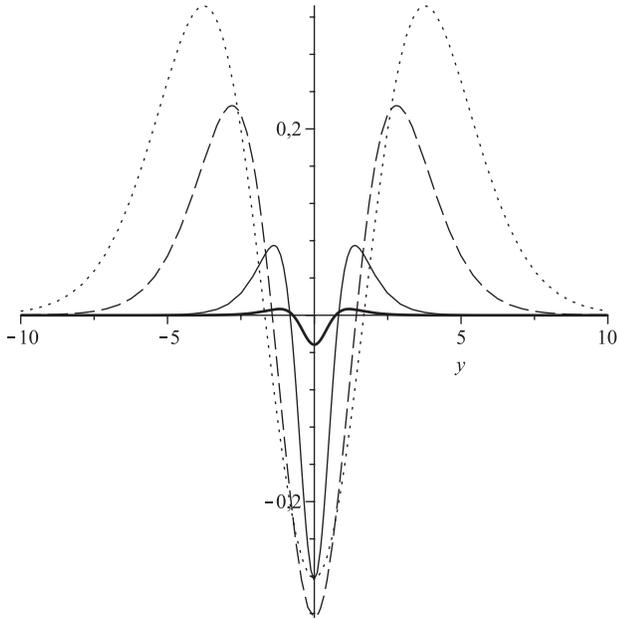


FIG. 3. Potential profiles: (a) $V_L(y)$ (left) and (b) $V_R(y)$ (right) for $\eta = 0.5$, $a = 0.09$ (dotted line), $a = 0.15$ (dashed line), $a = 0.39$ (thin line) and $a = 0.49$ (thick line).

We follow the same procedure of Ref. [5]. As $y \rightarrow \infty$,

$$I \rightarrow \exp\left(\frac{4}{9}y\right) \rightarrow \infty \quad (35)$$

and as $y \rightarrow -\infty$,

$$I \rightarrow \exp\left(-\frac{4}{9}y\right) \rightarrow \infty \quad (36)$$

which leads to a non-normalizable zero mode. This fact implies that the zero mode of the left-handed fermions can not be localized on the brane, in opposition what was adverted in Ref. [1]. It is instructive to note that the asymptotic behavior of the integrand is independent of a . On the other side, changing η by $-\eta$ ($L_0 \rightarrow R_0$) we obtain that the right-handed fermions cannot be localized on the

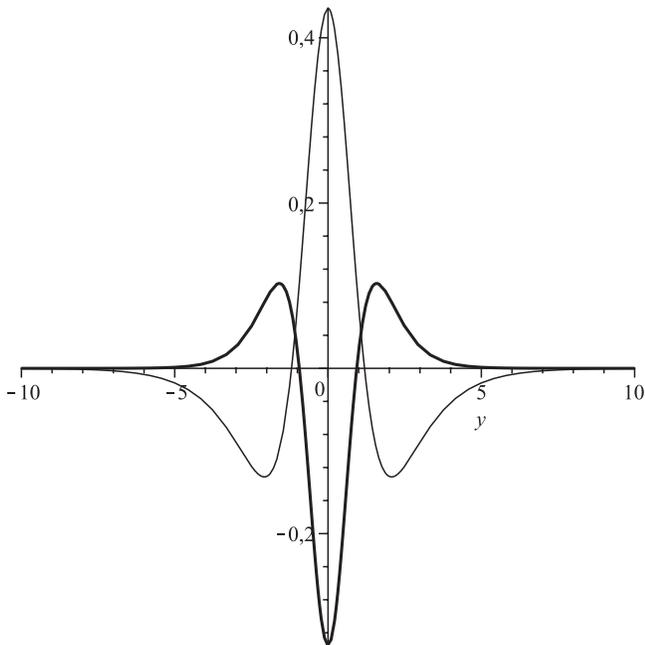


FIG. 4. The profiles of the matter energy density (thin line) and $V(y)$ (thick line) for $\eta = 0.5$ and $a = 1/3$.

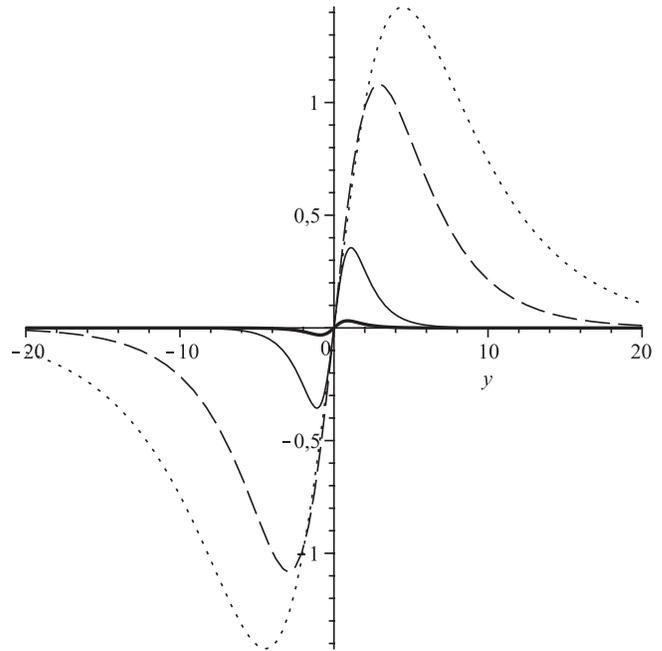


FIG. 5. $F(\phi, \chi) = \phi\chi$ as a function of y with $\eta = 0.5$, $a = 0.09$ (dotted line), $a = 0.15$ (dashed line), $a = 0.39$ (thin line) and $a = 0.49$ (thick line).

brane too. Figure 2 clearly shows that the zero mode of left-handed fermions is not normalizable. For this case the behavior of the V_L and V_R potentials are shown in the Fig. 3 for some values of a . Figure 3(a) shows that the potential of left-handed fermions, V_L , is indeed a volcano-like potential. The shapes of the energy density and $V_L(y)$ potential for this case are shown in the Fig. 4 for $a = 1/3$, as used in Ref. [1]. Figure 4 clearly shows that the effective potential $V_L(y)$ has a minimum at the localization on the brane, but this fact does not guarantee the existence of a normalized zero mode. The behavior of $F(\phi, \chi) = \phi\chi$ as a function of y for some values of a is shown in Fig. 5. From Fig. 5 one can see that $F(\phi, \chi) \rightarrow 0$ as $y \rightarrow \pm\infty$.

B. Case 2: $F(\phi, \chi)_\pm = \phi \pm \chi$

For this case the integrand in (33) can be expressed as

$$I = \exp\left\{-\frac{1}{a}\left(\eta - \frac{2}{9}\right)\ln[\cosh(2ay)] - \frac{(1-3a)}{9a}\tanh^2(2ay) \mp \frac{\eta}{a}\sqrt{\frac{1-2a}{a}}\arctan[\sinh(2ay)]\right\}. \quad (37)$$

The behavior of (37) as $y \rightarrow \infty$ is given by

$$I \rightarrow \exp\left[-2\left(\eta - \frac{2}{9}\right)y\right] \rightarrow 0, \quad \text{for } \eta > 2/9 \quad (38)$$

and as $y \rightarrow -\infty$, is

$$I \rightarrow \exp\left[2\left(\eta - \frac{2}{9}\right)y\right] \rightarrow 0, \quad \text{for } \eta > 2/9. \quad (39)$$

This result clearly shows that the zero mode of the left-handed fermions is normalized only for $\eta > 2/9$. Now,

under the change $\eta \rightarrow -\eta$ ($L_0 \rightarrow R_0$) we obtain that the right-handed fermions can not be a normalizable zero mode. The shape of the potentials for $F(\phi, \chi)_+$ are shown in Fig. 6 for some values of a . Figure 6(a) shows a well structure that grows and tend to around $y = 0$ as $a \rightarrow 1/2$. Figure 6(b) shows that the potential $(V_R(y))_+$ is always positive, therefore, the potential can not trap any bound fermions with right chirality. These kind of potentials have not appeared in previous studies. The shapes of the energy density, $(V_L(y))_+$ potential and zero mode for this case are shown in Fig. 7. The Fig. 7(a) ($a = 0.25$) shows that the effective potential $(V_L(y))_+$ has a minimum outside the localization of the brane, as a consequence the normalizable zero mode is not localized on the brane. On the other hand, Fig. 7(b) ($a = 0.48$) clearly shows that the effective potential $(V_L(y))_+$ has a minimum at the localization of the brane. Therefore, this result clearly shows that the zero mode of the left-handed fermions is localized on the brane only as $a \rightarrow 1/2$.

The behavior of the potentials for $F(\phi, \chi)_-$ can be written out easily by replacing $(V_L)_- = (V_L(-y))_+$ and $(V_R)_- = (V_R(-y))_+$. The behavior of $F(\phi(y), \chi(y))_+$ is shown in Fig. 8.

C. Case 3: $F(\phi, \chi) = \chi - \phi$

For this case the integrand in (33) can be expressed as

$$I = \exp\left\{\frac{1}{a}\left(\eta + \frac{2}{9}\right)\ln[\cosh(2ay)] - \frac{(1-3a)}{9a}\tanh^2(2ay) - \frac{\eta}{a}\sqrt{\frac{1-2a}{a}}\arctan[\sinh(2ay)]\right\}. \quad (40)$$

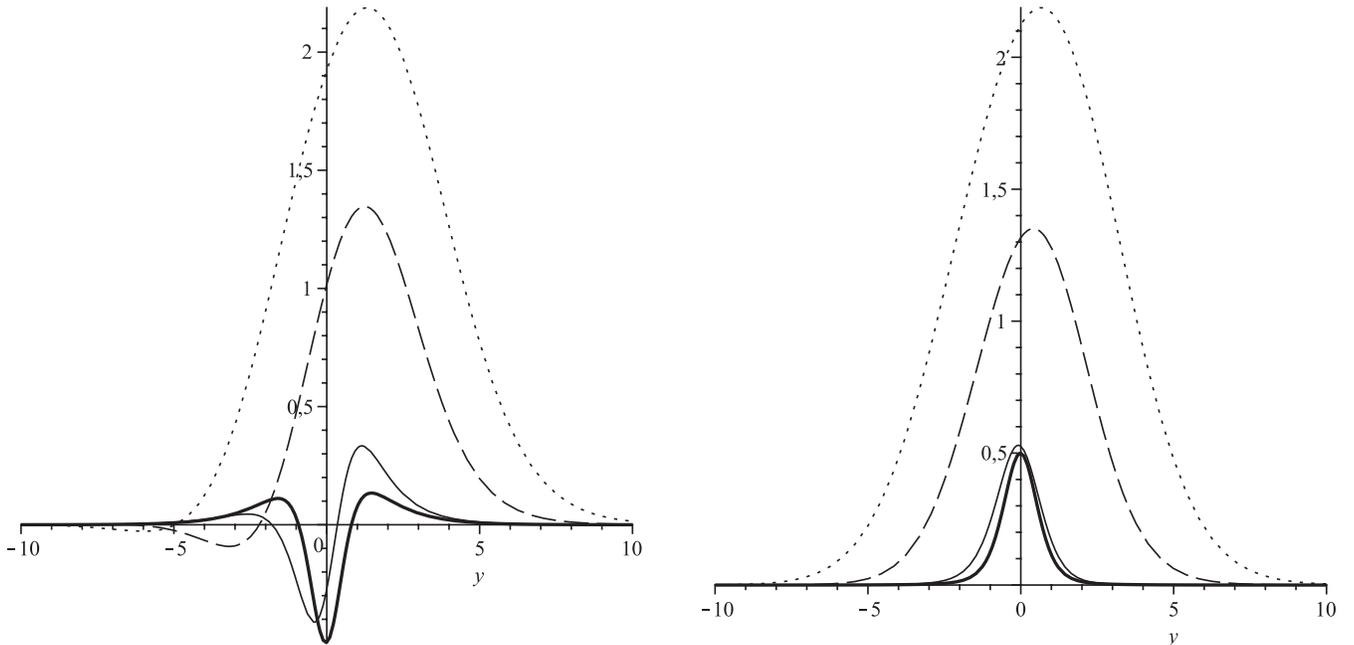


FIG. 6. Potential profile: (a) $(V_L(y))_+$ (left) and (b) $(V_R(y))_+$ (right) for $\eta = 0.5$, $a = 0.09$ (dotted line), $a = 0.15$ (dashed line), $a = 0.39$ (thin line) and $a = 0.49$ (thick line).

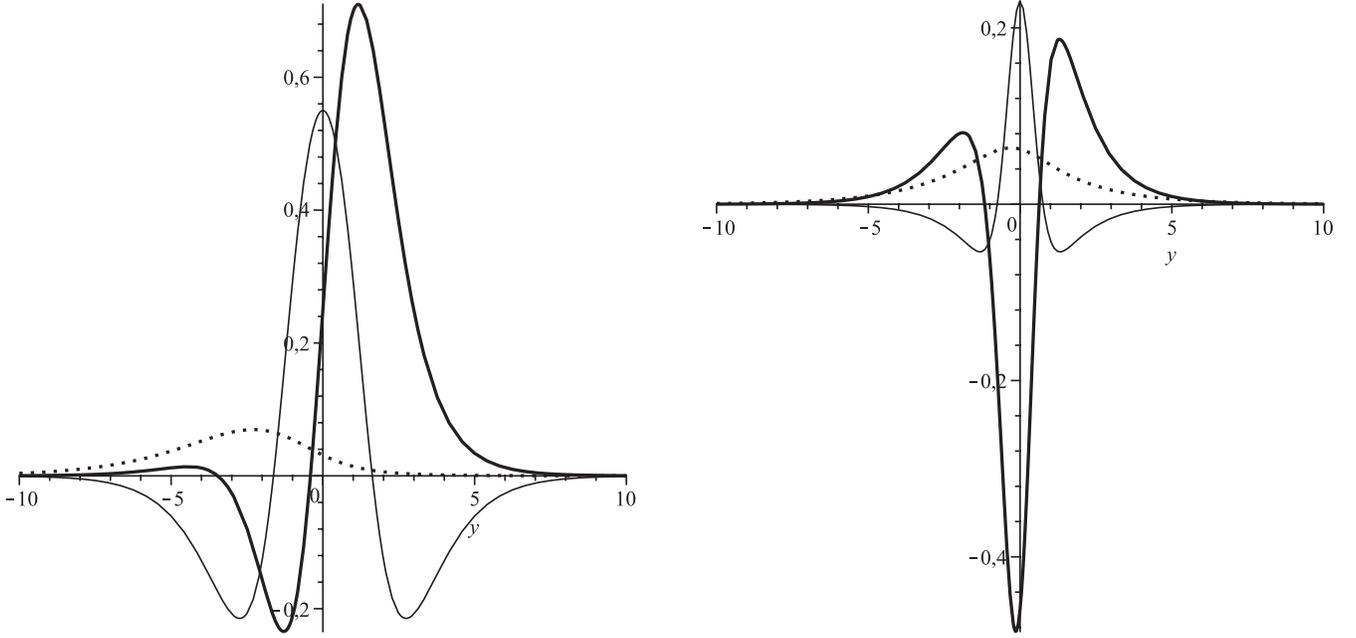


FIG. 7. The profiles of the energy density (thin line), $(V_L(y))_+$ (thick line) and zero mode (dotted line) for $\eta = 0.5$; (a) $a = 0.25$ (left) and (b) $a = 0.48$ (right).

The behavior of (40) as $y \rightarrow \infty$ is

$$I \rightarrow \exp\left[2\left(\eta + \frac{2}{9}\right)y\right] \rightarrow \infty \quad (41)$$

and as $y \rightarrow -\infty$, is given by

$$I \rightarrow \exp\left[-2\left(\eta + \frac{2}{9}\right)y\right] \rightarrow \infty, \quad (42)$$

which leads to a non-normalizable zero mode. The zero mode of the left-handed fermions cannot be localized on the brane. Otherwise, the change $\eta \rightarrow -\eta$ ($L_0 \rightarrow R_0$) allowed us to conclude that the right-handed fermions can be localized on the brane on the condition that $\eta > 2/9$ and $a \rightarrow 1/2$. The behavior of the potentials for this $F(\phi, \chi)$ can be written out easily by replacing $V_L = (V_R(-y))_+$ and $V_R = (V_L(-y))_+$.

IV. CONCLUSIONS

We have reinvestigated the localization problem of fermions on two-field thick branes (Bloch brane model). We showed that the simplest Yukawa coupling $F(\phi, \chi) = \phi\chi$ does not support the localization of fermions on the brane, as incompletely argued in Ref. [1]. This fact is a consequence of the absence of a normalized zero mode. We showed that the zero mode of left-handed fermions for the Yukawa coupling $F(\phi, \chi)_\pm = \phi \pm \chi$ is normalizable under the condition $\eta > 2/9$ and it can be trapped on the brane only for $a \rightarrow 1/2$, because the effective potential have a minimum at the localization of the brane. On the other hand, the zero mode of right-handed is not normalizable, this result can also be validated from the behavior of the potential $(V_R)_\pm$ [Fig. 6(b)]. In the same way, we also showed that the zero mode right-handed fermions can be localized on the brane for $\eta > 2/9$ and $a \rightarrow 1/2$ for $F(\phi, \chi) = \chi - \phi$. At this point, it is worthwhile to

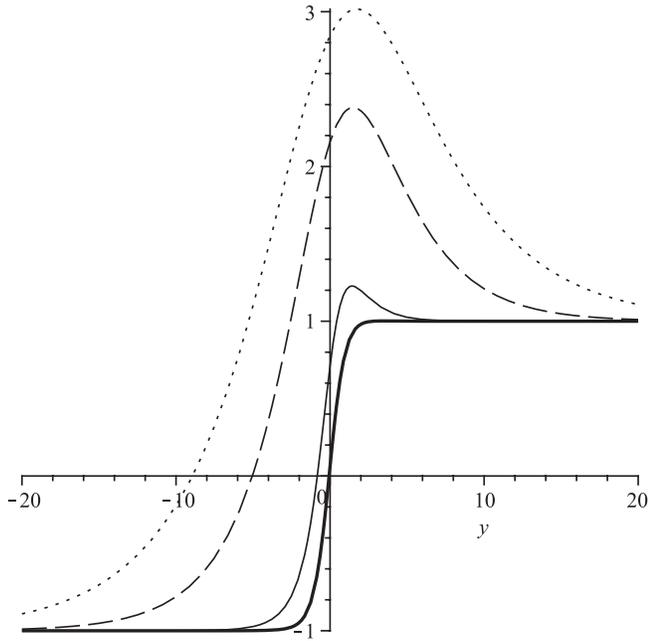


FIG. 8. $F(\phi, \chi)_+ = \phi + \chi$ as a function of y with $\eta = 0.5$, $a = 0.09$ (dotted line), $a = 0.15$ (dashed line), $a = 0.39$ (thin line) and $a = 0.49$ (thick line).

mention that the limit $a \rightarrow 1/2$ changes the two-field solution to the one-field solution [2]. In this context, we believe that an interesting line of investigation could follow [22], which construct effective models with only one scalar field from models with two interacting scalar fields.

We can conclude that the normalization of the zero mode and the existence of a minimum of the effective potential at the localization on the brane are essential conditions for the problem of fermion localization on the brane. The normalization of the zero mode is decided by the asymptotic behavior of $F(\phi, \chi)$ and also the presence of $F(\phi, \chi)$ is an essential ingredient for the effective potential profile. Therefore, the behavior of $F(\phi, \chi)$ plays a leading role for the fermion localization on the brane. This work completes the analyzing of the research in Ref. [1], because in that

work it does not analyze the zero mode in full detail. An interesting issue concerns the natural extension of the present work in the case of massive modes using the other kinds of Yukawa couplings, which support the normalizable zero mode presented in this paper in order to investigate possible effects on the resonances modes and bear out the main conclusions of Ref. [1].

ACKNOWLEDGMENTS

L. B. Castro would like to thank Professor Dr. Antonio S. de Castro and Professor Marcelo B. Hott. Thanks also go to the referee for useful comments and suggestions. This work was supported by means of funds provided by CAPES.

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