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A Multidimensional Version of the Carlson Inequality

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We give a multidimensional extension of a one-dimensional integral inequality due to F. Carlson. The extension presented here involves L^P spaces with mixed norms in a very natural way.

1. INTRODUCTION

The purpose of this note is to derive a multidimensional version of a special case of the Levin generalization of the Carlson integral inequality given by formula 1.1(1) below.

PROPOSITION 1.1. *Let $h = h(t)$ be a positive measurable function on $\mathbb{R}_+$. Then, for $0 < \theta < 1$ and $1 \leqslant p_0, p_1 < \infty$, we have*

$$\int_0^\infty h(t)\frac{dt}{t} \leqslant c\left\{\int_0^\infty \{t^{-\theta}h(t)\}^{p_0}\frac{dt}{t}\right\}^{(1-\theta)/p_0}\left\{\int_0^\infty \{t^{1-\theta}h(t)\}^{p_1}\frac{dt}{t}\right\}^{\theta/p_1}. \quad (1)$$

This form was used by J. Peetre [7] in connection with the theory of interpolation spaces.

For a simple proof of the Carlson inequality, but one which does not provide the best constant, see Beckenbach and Bellman [1, p. 176]. For a derivation of the best constant see Levin [5].

The generalization we have in mind is, in the bidimensional case, as follows.

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Let $g = g(s, t)$ be a non-negative measurable function on $\mathbb{R}_+ \times \mathbb{R}_+$. For $P = (p_1, p_2)$, $1 \leqslant p_1, p_2 < \infty$, we shall set

$$\| g\|_{L_*^P} = \left\{\int_0^\infty \left\{\int_0^\infty g(s, t)^{p_1} \frac{ds}{s}\right\}^{p_2/p_1} \frac{dt}{t}\right\}^{1/p_2}.$$

PROPOSITION 1.2. *Let $h = h(s, t)$ be a positive measurable function on $\mathbb{R}_+ \times \mathbb{R}_+$. Then, for $0 < \mu$, $\nu < 1$, $1 \leqslant p_0^j$, $p_1^j < \infty$ $(j = 1, 2)$ and $P_{00} = (p_0^1, p_0^2)$, $P_{10} = (p_1^1, p_0^2)$, $P_{01} = (p_0^1, p_1^2)$ and $P_{11} = (p_1^1, p_1^2)$, we have*

$$\int_0^\infty \int_0^\infty h(s, t) \frac{ds}{s} \frac{dt}{t} \leqslant C \|s^{-\mu} t^{-\nu} h(s, t)\|_{L_*^{P_{00}}}^{(1-\mu)(1-\nu)} \|s^{1-\mu} t^{-\nu} h(s, t)\|_{L_*^{P_{10}}}^{\mu(1-\nu)}$$
$$\times \|s^{-\mu} t^{1-\nu} h(s, t)\|_{L_*^{P_{01}}}^{(1-\mu)\nu} \|s^{1-\mu} t^{1-\nu} h(s, t)\|_{L_*^{P_{11}}}^{\mu\nu}. \qquad (1)$$

The proof of the multidimensional version given below does not use Proposition 1.2.

2. THE MULTIDIMENSIONAL VERSION

The multidimensional version of the Carlson integral inequality in the line of 1.2(1) needs, to be stated and proved, some notation and preliminaries.

2.0. The set of $k = (k_1, ..., k_d) \in \mathbb{R}^d$ such that $k_j = 0$ or 1 will be denoted by $\square$.

Let $P_0 = (p_0^1, ..., p_0^d)$ and $P_1 = (p_1^1, ..., p_1^d)$ be given with $1 \leqslant p_0^j$, $p_1^j < \infty$ $(j = 1, 2, ..., d)$. We shall consider the family $\mathbb{P} = \{P_k = (p_{k_1}^1, ..., p_{k_d}^d) \mid k = (k_1, ..., k_d) \in \square\}$ associated with P_0 and P_1.

For $a = (a_1, ..., a_d)$, $b = (b_1, ..., b_d) \in \mathbb{R}_+^d$ we set $a^b = a_1^{b_1} \cdots a_d^{b_2}$.

For $a = (a_1, ..., a_d, a_{d+1}) \in \mathbb{R}^{d+1}$ we shall write $a = (\hat{a}, a_{d+1})$, where $\hat{a} = (a_1, ..., a_d) \in \mathbb{R}^d$.

2.1. We shall work in the context of the L^P spaces with mixed norms of Benedek–Panzone [2]. For $d \geqslant 1$, we shall denote by $L_*^1(\mathbb{R}_+^d)$ the linear space of all functions $h: \mathbb{R}_+^d \to \mathbb{R}$, measurable with respect to the measure $d_*t = d_*t_1 \cdots d_*t_d = dt_1/t_1 \cdots dt_d/t_d$ such that

$$\|h\|_{L_*^1} = \|h\|_{L_*^1(\mathbb{R}_+^d)} = \int_0^\infty \cdots \int_0^\infty |h(t_1, ..., t_d)| \frac{dt_1}{t_1} \cdots \frac{dt_d}{t_d} < \infty. \qquad (1)$$

Let $P = (p_1, ... p_d)$ be given with $1 \leqslant p_j < \infty$, $j = 1, 2, ..., d$. We shall define the linear space $L_*^P = L_*^P(\mathbb{R}_+^d)$ as follows. If $d = 1$ and $P = p_1$, L_*^P is the linear space of all measurable functions $h: \mathbb{R}_+ \to \mathbb{R}$ such that

$$\|h\|_{L_*^P} = \|h\|_{L_*^P(\mathbb{R}_+)} = \left\{ \int_0^\infty |h(t_1)|^{p_1} \frac{dt_1}{t_1} \right\}^{1/p_1} < \infty. \tag{2}$$

Now, if $d > 1$ and $P = (p_1, ..., p_d, p_{d+1}) = (\hat{P}, p_{d+1})$ we suppose $L_*^{\hat{P}} = L_*^{\hat{P}}(\mathbb{R}_+^d)$ already defined and we shall define $L_*^P = L_*^P(\mathbb{R}_+^{d+1})$ as the linear space of all measurable $h: \mathbb{R}_+^{d+1} \to \mathbb{R}$ such that

$$\|h\|_{L_*^P} = \|h\|_{L_*^P(\mathbb{R}_+^{d+1})} = \left\{ \int_0^\infty \{\|h(\hat{t}, t_{d+1})\|_{L_*^{\hat{P}}(\mathbb{R}_+^d)}\}^{p_{d+1}} \frac{dt_{d+1}}{t_{d+1}} \right\}^{1/p_{d+1}}. \tag{3}$$

If we agree that $\|h\|_{L_*^P(\mathbb{R}_+^0)} = |h(t)|$, we see that (2) is a particular case of (3).

LEMMA 2.2. *Let $k = (k_1, ..., k_d) \in \square$ and let $\Theta = (\theta_1, ..., \theta_d)$ be given with $0 < \theta_j < 1$, $j = 1, 2, ..., d$. If we define $\Theta(k) = \prod_{j=1}^d \theta(k_j)$, where $\theta(k_j) = 1 - k_j + (-1)^{k_j+1}\theta_j$, $j = 1, 2, ..., d$, we shall have*

$$\sum_{k \in \square} \Theta(k) = 1. \tag{1}$$

(*Observe that, when $d = 1$, we have $\Theta(0) = 1 - \theta$ and $\Theta(1) = \theta$, and when $d = 2$, we have $\Theta(0,0) = (1 - \theta_1)(1 - \theta_2)$, $\Theta(1,0) = \theta_1(1 - \theta_2)$, $\Theta(0,1) = (1 - \theta_1)\,\theta_2$ and $\Theta(1,1) = \theta_1\theta_2$.*)

Proof. We proceed by induction. For $d = 1$ we have

$$\sum_{k \in \square} \Theta(k) = \Theta(0) + \Theta(1) = 1 - \theta + \theta = 1.$$

Now, we suppose that (1) holds for $d > 1$ and we shall prove that it also holds for $d + 1$. Indeed, we have

$$\begin{aligned} \sum_{k \in \square} \Theta(k) &= \sum_{k \in \square} \Theta(k_1, ..., k_d, k_{d+1}) = \sum_{k \in \square} \Theta(\hat{k}, k_{d+1}) \\ &= \sum_{\hat{k} \in \hat{\square}} \Theta(\hat{k}, 0) + \sum_{\hat{k} \in \hat{\square}} \Theta(\hat{k}, 1) \\ &= \sum_{\hat{k} \in \hat{\square}} \Theta(\hat{k})(1 - \theta_{d+1}) + \sum_{\hat{k} \in \hat{\square}} \Theta(\hat{k})\,\theta_{d+1} = 1, \end{aligned}$$

as desired.

We are now ready to state and prove the multidimensional version of the Carlson inequality.

THEOREM 2.3. *Let $\Theta = (\theta_1, ..., \theta_d)$ be given with $0 < \theta_j < 1$ and let $P_0 = (p_0^1, ..., p_0^d)$ and $P_1 = (p_1^1, ..., p_1^d)$ be given with $1 \leqslant p_0^j$, $p_1^j < \infty$, $j = 1, 2, ..., d$. Let us set $P_k = (p_{k_1}^1, ..., p_{k_d}^d)$ for each $k = (k_1, ..., k_d) \in \square$. Then, for all positive measurable functions $h = h(t)$ on $\mathbb{R}_+^d$, we have*

$$\|h\|_{L_*^1(\mathbb{R}_+^d)} \leqslant C \prod_{k \in \square} \|t^{k-\Theta} h(t)\|_{L_*^{P_k}(\mathbb{R}_+^d)}^{\Theta(k)}, \tag{1}$$

where $\Theta(k)$ is given as in Lemma 2.2, $C = \prod_{j=1}^d C(p_0^j, p_1^j, \theta_j)$ and for each $j = 1, ..., d$, $C(p_0^j, p_1^j, \theta_j)$ is any valid constant for 1.1(1).

Proof. If $d = 1$ the inequality (1) is the Carlson inequality 1.1(1). Now, let us suppose that (1) holds for $d \geqslant 1$ and let us prove that it also holds for $d + 1$. Then, if we apply the case $d = 1$ to $\|h(\hat{t}, t_{d+1})\|_{L_*^1(\mathbb{R}_+^d)}$ with $\theta = \theta_{d+1} = \theta'$, $p_0 = p_0^{d+1} = p_0'$ and $p_1 = p_1^{d+1} = p_1'$, $k' = k_{d+1}$ and $t_{d+1} = s$, we get

$$\|h\|_{L_*^1(\mathbb{R}_+^{d+1})} = \|h(\hat{t}, t_{d+1})\|_{L_*^1(\mathbb{R}_+^{d+1})} = \int_0^\infty \|h(\hat{t}, s)\|_{L_*^1(\mathbb{R}_+^d)} \frac{ds}{s}$$

$$\leqslant C_{d+1} \prod_{k'=0,1} \left\{ \int_0^\infty \left\{ s^{k'-\theta'} \|h(\hat{t}, s)\|_{L_*^1(\mathbb{R}_+^d)} \right\}^{p'_{k'}} \frac{ds}{s} \right\}^{\theta(k')/p'_{k'}}$$

(recall that $\theta(k') = \theta(k_{d+1}) = 1 - k_{d+1} + (-1)^{k_{d+1}+1} \theta_{d+1} = 1 - \theta_{d+1}$ or $\theta(k') = \theta_{d+1}$ when $k' = k_{d+1} = 0$ or 1, respectively), where $C_{d+1} = C(p_0^{d+1}, p_1^{d+1}, \theta_{d+1})$. Now, by the induction assumption we have

$$\|h\|_{L_*^1(\mathbb{R}_+^{d+1})} \leqslant C_{d+1} \hat{C} \prod_{k'=0,1} \left\{ \int_0^\infty \left\{ s^{k'-\theta'} \prod_{\hat{k} \in \hat{\square}} \|\hat{t}^{\hat{k}-\hat{\Theta}} h(\hat{t}, s)\|_{L_*^{P_{\hat{k}}}(\mathbb{R}_+^d)}^{\Theta(\hat{k})} \right\}^{p'_{k'}} \frac{ds}{s} \right\}^{\theta(k')/p'_{k'}},$$

where $\hat{C} = \prod_{j=1}^d C(p_0^j, p_1^j, \theta_j)$, and, since $\sum_{\hat{k} \in \hat{\square}} \Theta(\hat{k}) = 1$ we get

$$\|h\|_{L_*^1(\mathbb{R}_+^{d+1})} \leqslant C \prod_{k'=0,1} \left\{ \int_0^\infty \left\{ \prod_{\hat{k} \in \hat{\square}} \|\hat{t}^{\hat{k}-\hat{\Theta}} s^{k'-\theta'} h(\hat{t}, s)\|_{L_*^{P_{\hat{k}}}(\mathbb{R}_+^d)}^{\Theta(\hat{k})} \right\}^{p'_{k'}} \frac{ds}{s} \right\}^{\theta(k')/p'_{k'}},$$

where $C = \prod_{j=1}^{d+1} C(p_0^j, p_1^j, \theta_j)$. Finally, applying the general form of Hölder's inequality [4, p. 139], we get

$$\|h\|_{L_*^1(\mathbb{R}_+^{d+1})} \leqslant C \prod_{k'=0,1} \prod_{\hat{k} \in \hat{\square}} \left\{ \int_0^\infty \|\hat{t}^{\hat{k}-\hat{\Theta}} s^{k'-\theta'} h(\hat{t}, s)\|_{L_*^{P_{\hat{k}}}(\mathbb{R}_+^d)}^{p'_{k'}} \frac{ds}{s} \right\}^{\Theta(\hat{k})\theta(k')/p'_{k'}}$$

$$= C \prod_{k'=0,1} \prod_{\hat{k} \in \hat{\square}} \|(\hat{t}, s)^{(\hat{k}-\hat{\Theta}, k'-\theta')} h(\hat{t}, s)\|_{L_*^{P(\hat{k}, k')}(\mathbb{R}_+^{d+1})}^{\Theta(\hat{k}, k')}$$

$$= C \prod_{k \in \square} \|t^{k-\Theta} h(t)\|_{L_*^{P_k}(\mathbb{R}_+^{d+1})}^{\Theta(k)}.$$

The theorem is proved.

3. Remarks

3.1. The best possible value for C_1, when $p_0 \geqslant 1$ and $p_1 \geqslant 1$ (with strict inequality for at least one) and $0 < \theta < 1$, is

$$C_1(p_0, p_1, \theta)$$
$$= (1-\theta)^{-s}\theta^{-t}\left\{B\left(\frac{s}{1-s-t}, \frac{t}{1-s-t}\right)\Big/[p_0\theta + p_1(1-\theta)]\right\}^{1-s-t},$$

where $s = (1-\theta)/p_0$, $t = \theta/p_1$, and B denotes the beta function. This is the special case of the constant obtained by Levin (see Mitrinović [6, p. 371]).

3.2. The authors have used the inequality 2.3(1) in connection with the multidimensional Mellin transform (see Bertolo and Fernandez [3]).

3.3. Another multidimensional version of the Carlson inequality was obtained by Pigolkin [8].

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