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The Dirac equation in a non-Riemannian manifold. I. An analysis using the complex algebra

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The Dirac wave equation is obtained in the non-Riemannian manifold of the Einstein–Schrödinger nonsymmetric theory. A new internal connection is determined in terms of complex vierbeins, which shows the coupling of the electromagnetic potential with gravity in the presence of a spin-\(\frac{1}{2}\) field.

I. INTRODUCTION

The Einstein–Schrödinger (ES) nonsymmetric theory\(^1\) was an attempt to geometrize, in an unitary way, the gravitational and the electromagnetic fields. However, structure problems in the theory have not permitted a coherent interpretation of the field equations. On the other hand, the technique of geometrizing fields introduced by Einstein continues to be a useful tool in classical field theory. Actually there has been some work\(^2\) where the electromagnetic field is made explicit; this is accomplished by first taking the skew-symmetric part of the metric as being proportional to the electromagnetic tensor and, then, adding a source-like term to the Lagrangian of the theory. This permits one to reobtain the Einstein–Maxwell equations through a correspondence principle. When we face the problem of developing a Dirac theory using the space-time manifold of the ES nonsymmetric theory, we use then complex vierbeins. This is equivalent to introducing an internal C space in the manifold of the general relativity theory. However, we have the problem of how to obtain a coherent mechanism that fits the correct rules for the transformation group. We can see, for example, that the way the problem was developed by Marques and Oliveira,\(^4\) in a study of the geometrical properties of C, Q, and O tangent spaces, does not permit the corresponding Dirac field equations to exist (we will consider the complex case presently). The reason is that the internal C connection was ignored when they introduced complex vierbeins. Instead, they generalize the tangent real connection (the one originating on the local real-tangent space) to a complex one. This induced them to generalize both the Lorentz group (to a pseudo-unitary group) and its representation \(U(L)\), for the generalized Dirac field theory. However, in spite of it being possible to show that, on the local tangent space, the trace of the symmetric part of the tangent connection should correspond to the electromagnetic field, there is no way to obtain the desirable correspondence to an (actual R) Dirac field theory. The problem can be solved by considering besides the tangent connection, the connection corresponding to an internal C space. This forces us to maintain the Lorentz group as being that of the (local) space-time transformations on the (local) tangent space. On the other hand, we must also have an “internal” transformation, corresponding to the internal C space.

In the space-time of general relativity it is possible to generalize the Dirac field equation by doing the transition

\[
\gamma^\mu\rightarrow\gamma^\mu(x) , \quad \psi_{\mu} \rightarrow \psi_{\mu} = \psi_{\mu} + \Delta_{\mu}\psi ,
\]

where \(\psi(x)\) is now the electron wave function in the curved space-time, and \(\Delta_\mu\) is the geometrical connection with relation to the internal space generated by the constant \(\gamma\) matrices \(\{\gamma_i\}\). It is easy to show that \(\Delta_\mu\) is given by

\[
\Delta_{\mu} = \{\{\gamma_{\nu}\gamma_{\xi,\mu}\} - \{\gamma_{\nu}\gamma_{\xi}\}\} ,
\]

where \(\{\gamma_{\nu}\}\) are the Christoffel symbols for the space-time connection. In terms of real vierbeins \(h_{\nu}^\alpha\) (and their inverses \(h^\alpha_{\nu}\)), (1.1) is written as

\[
\Delta_{\mu} = ( - 1/4)i(h^{-\beta}_{\nu}h_{\beta,\mu} - \{\gamma_{\nu}\}h_{\mu}^{\alpha}h^{\nu}_{\beta})\sigma_{\alpha\beta} .
\]

The function \(\psi'(x), i = 1, \ldots, 4\), above, satisfies a Dirac equation defined on the curved space-time manifold of general relativity, which now has the form

\[
\gamma^\mu(x) (\psi_{\mu} + \Delta_{\mu}\psi) - \mu\psi = 0 ,
\]

where \(\mu\) is a mass coefficient. Thus the gravitational field is present in this equation through the connection \(\Delta_{\mu}\).

The nonsymmetric manifold of the ES theory with the locally defined complex vierbeins referred to above will be used in this work. These vierbeins define new Fock–Ivanenko coefficients which permit the construction of the corresponding Dirac equations related to the non-Riemannian manifold of the ES theory. In Sec. II we will present briefly the properties of the complex tangent space as well as the corresponding field equations obtained in the ES nonsymmetric theory. In Sec. III the generalized Fock–Ivanenko coefficients will be determined, as well as the new Dirac equation. In Sec. IV we will proceed to their analysis. Throughout, we will use the \(\mu, \nu, \ldots\) indices as those on the non-Riemannian manifold; the \(a, b, \ldots\) indices will be those on the complex tangent space.

II. THE COMPLEX TANGENT SPACE

According to the correspondence principle there exists, at each point of the curved space-time of general relativity, a local tangent space\(^3\) with the structure of a flat space-time, with the metric given by the Minkowski tensor \(\eta_{ab}\). Therefore, we must have the line element

\[^{a}\text{On leave from UNESP, Campus de Ilha Solteira, São Paulo, Brazil.}\]
\[ ds^2 = g_{\mu \nu} \, dx^\mu \, dx^\nu = \eta_{ab} \, dx^a \, dx^b, \]  
locally, where, \( g_{\mu \nu} = g_{\nu \mu} \).

In the ES nonsymmetric theory, the metric of curved space-time has the symmetry property \( g_{\mu \nu}^* = g_{\nu \mu} \), which is the same as a gauge transformation law for an electromagnetic potential. We also have the relation

\[ R^\rho_{\mu \nu \rho} - S^a_{\mu b} b^b_{\mu} = 0, \]  
which is skew symmetric with respect to the curved-space indices and anti-Hermitian in the tangent space indices. The internal curvature can be also obtained:

\[ P_{\gamma \nu} = C_{\gamma \nu} - C_{\nu \gamma}. \]

Then, in the particular case of (2.15), the internal curvature can be considered to correspond to the Maxwell electromagnetic tensor.

One of the field equations of the ES nonsymmetric theory, obtained through a variational principle, is \( g_{\mu \nu} = 0 \), where the symbol (\( ;) \) means that the connection used in this equation is the Schrödinger connection.

\[ \theta^\rho_{\mu \nu}, \quad \partial_\alpha = \theta^\rho_{\mu \nu} = 0. \]

The notation used in this work is the same used by Marques and Oliveira in Ref. 4. It has been kept the traditional use of the “+ −” covariant derivative used in the ES nonsymmetric theory, Ref. 1. See also, M. A. Tonellat, Ref. 9.) (In general, \( \theta^\rho_{\mu \nu} \) is a nonsymmetric connection such that \( \theta^\rho_{\mu \nu} = \theta_{\alpha \nu} \).

This equation corresponds to the following vierbein equations:

\[ e_{\alpha}^{\mu} = (e_{\alpha}^{\mu})^* = e_{\mu}^\alpha - \theta^\rho_{\mu \rho} e^\alpha_\rho + \Lambda^a_{\alpha b} e^b_\rho + C_a e^\rho_\mu = 0, \]  
where (2.1) was used. Taking the inverse equation:

\[ g_{\mu \nu} = 0, \]  
and from (2.2), we have the corresponding equations for the inverse vierbeins

\[ e_{\mu}^\alpha = (e_{\mu}^\alpha)^* = e_{\alpha}^{\mu} - \theta^\rho_{\rho \mu} e_{\rho}^\alpha + \Lambda^a_{b a} e^b_\rho - C_a e^\rho_\mu = 0. \]

We can rewrite Eqs. (2.13) and (2.14) as

\[ e_{\mu}^\alpha = e_{\alpha}^{\mu} - \theta^\rho_{\rho \mu} e_{\rho}^\alpha + \Lambda^a_{b a} e^b_\rho - C_a e^\rho_\mu = 0, \]

\[ e_{\mu}^\alpha = e_{\alpha}^{\mu} - \theta^\rho_{\rho \mu} e_{\rho}^\alpha + \Lambda^a_{b a} e^b_\rho = 0, \]  
where

\[ \Lambda^a_{\alpha b} = \Lambda^a_{b a} + \delta^a_{C_a}, \quad \Lambda^a_{\alpha b} = \Lambda^a_{b a} - \delta^a_{C_a}. \]

From (2.19) and (2.20) we obtain the relation

\[ \Lambda^a_{\alpha b} = e^a_{\alpha} e^{\alpha}_{b} = -e^a_{\alpha} e^{\alpha}_{b}, \]  
and from (2.23),

\[ \Lambda^a_{\alpha b} = \text{Re} \left[ e^a_{\alpha} e^{\alpha}_{b} \right] = \text{Re} \left[ -e^a_{\alpha} e^{\alpha}_{b} \right], \]

\[ \text{C}_{\alpha} = i(\text{Im} \left[ e^a_{\alpha} e^{\alpha} e^{b} \right]) = i(\text{Im} \left[ -e^a_{\alpha} e^{\alpha} e^{b} \right]). \]

Taking (2.25), we can expand it in terms of real and imaginary parts. We then obtain for \( \Lambda^a_{\alpha b} \).
The Dirac constant γ matrices satisfy the anticommutation relations
\[ \{ \gamma_a, \gamma_b \} = 2 \eta_{ab} \gamma_4 , \quad \{ \gamma^a, \gamma^b \} = 2 \eta^{ab} \gamma_4 , \]
where \( \eta_{ab} \) (and its inverse \( \eta^{ab} \)) is the Minkowski tensor with signature +2, and to the relation \( \gamma^a \gamma_a = 0 \). The set formed with combinations of γ matrices,
\[ \{ \Gamma_\gamma \} = \{ 1_4, \gamma_\mu, \sigma_{ab} = (i/2) [ \gamma_\mu, \gamma_a ] \}, \]
composes a linearly independent set in the internal space of the Dirac wavefunctions \( \psi \).

Now, multiplying (3.1) by \( e^{*\mu} a \) and \( e^\mu_b \), and using (2.1), we obtain
\[ \{ \gamma^\mu, \gamma_a \} = 2 g_{\mu a} 1_4 , \]
where \( g_{\mu a} \) is now the ES nonsymmetric metric. In (3.4) we have defined \( \gamma_\mu \) and \( \gamma_a \) by
\[ e^{*\mu} a \gamma_\mu = \gamma_\mu , \quad e^\mu_b \gamma_a = \gamma_a . \]

Analogously, multiplying (3.2) by \( e^{*a} \mu \) and \( e^a_b \), we obtain
\[ \{ \gamma^\mu, \gamma^a \} = 2 g^{*a} \mu 1_4 , \]
and the relation (2.2) was used. The covariant derivative of \( \gamma^\mu (x) \) over the non-Riemannian manifold of nonsymmetric theory, is given by
\[ \gamma^\nu_{\mu} = \gamma_\mu \nu - \partial^\nu_{\mu} \gamma_\rho + [ \Delta_{\mu} \gamma_\rho ] , \]
where \( \Delta_\mu \) is the internal connection, corresponding to the space of generalized γ matrices (or, also, of Dirac wave functions space), and \( \partial^\nu_{\mu} \) is a more general space-time affinity (that, at least in principle, includes the internal connection \( C_\mu \)). Taking the identity (3.5) and the Eq. (2.19), we have that
\[ \gamma^\nu_{\mu} = (e^{*a} \gamma_a)_{|\nu} = (e^a_b \gamma^a)_a = 0 , \]
and so \( \gamma_a \) is a constant matrix. In the same way, we obtain
\[ \gamma^\nu_{\mu} = (e^{*\mu} a)_{\gamma_a} = 0 \] .

Expanding (3.8) and (3.9) we have
\[ \gamma^\nu_{\mu} = \gamma^\nu_{\mu} - \partial^\nu_{\mu} \gamma_\rho + [ \Delta_\nu \gamma_\rho ] + C_\nu \gamma_\rho = 0 , \]
and
\[ \gamma^\nu_{\mu} = 0 . \]

We can observe then, from (3.10) and (3.11), that we obtain a relation similar of that of general relativity, i.e.,
\[ \Delta_\nu = (1/4i) \Lambda_\nu \sigma_{ab} = (1/4i) Re [ e^{*a} \gamma_a ] \sigma_{ab} , \]
where it was used (2.21) for \( \Lambda_\mu \).

If we now consider \( \psi (x) \) as the wave function of a spin-1/2 particle of mass \( m \), placed in a non-Riemannian manifold of ES nonsymmetric theory, \( \psi (x) = \psi_{\gamma_0} \) will be the wave
function of its antiparticle, and the corresponding Dirac wave equations are, respectively,

\[ \gamma^\mu (\partial_\mu + \Delta_\mu + C_\mu) \psi - \mu \psi = 0, \quad (3.14) \]

\[ -\bar{\psi} (\partial_\mu + \Delta_\mu - C_\mu) \gamma^\mu - \mu \bar{\psi} = 0, \quad (3.15) \]

where \( \mu = mc^2/h \).

The new operator \((\partial_\mu + \Delta_\mu + C_\mu)\) comes from the covariant derivation of the function \( \psi(x) \), which, besides being an object that transforms, locally, under the representation of Lorentz group \([U(L)]\), also transforms under the (internal) \( U(1) \) group. Equations (3.14) and (3.15) describe particles in a curved non-Riemannian space-time of the ES nonsymmetric theory, since the connections \( \Delta_\mu \) and \( C_\mu \) are now related to complex vierbeins, as well as the complex space-time connection.

Another way to obtain Eqs. (3.14) and (3.15) is through a minimal action principle. In this case the action is

\[ A = \int \mathcal{L} d^4x, \]

where the Lagrangian is given by

\[ \mathcal{L} = \sqrt{-g} \left[ \bar{\psi} \gamma^\mu (\partial_\mu + \Delta_\mu + C_\mu) \psi + \bar{\psi} (\partial_\mu + \Delta_\mu - C_\mu) \gamma^\mu - \mu \bar{\psi} \psi \right]. \quad (3.16) \]

From (3.15), the wave equation for the charge conjugate function, \( \psi^c \), is

\[ \gamma^\mu (\partial_\mu + \Delta_\mu + C_\mu) \psi^c - \mu \psi^c = 0, \quad (3.17) \]

where \( \psi^c = C \bar{\psi}^* \), and \( C \) is the charge conjugate matrix. Therefore, if the wave equation of a particle is constructed with the set \( \gamma^\mu \) and \( (\Delta_\mu + C_\mu) \), the wave equation for its "charge conjugate" will be constructed with the set \( \gamma^\mu \) and \( (\Delta_\mu - C_\mu) \).

Let us write now the internal connection \( C_\mu \) as

\[ C_\mu = i e A_\mu (x). \quad (3.18) \]

Then, after (2.15), we can interpret \( e \) as the electric charge for the electron, \( A_\mu (x) \) as the electromagnetic potential, and \( I \) will be a constant such that it balances units. Equations (3.14) and (3.17) can be written now as

\[ \gamma^\mu (\partial_\mu + \Delta_\mu + i e A_\mu) \psi - \mu \psi = 0, \quad (3.19) \]

\[ \gamma^\mu (\partial_\mu + \Delta_\mu - i e A_\mu) \psi^c - \mu \psi^c = 0. \quad (3.20) \]

### IV. CONCLUSION

We have learned that complexifying the space-time manifold of general relativity is equivalent to attaching it an internal C space. The new metric is no longer symmetric and its antisymmetric part should be proportional to the electromagnetic tensor [see Einstein, Ref. 1, Eqs. (11)–(17) in Sec. III]. Through complex vierbeins, it is possible to obtain a (complex) tangent space local to the nonsymmetric curved space of the ES type. Using these concepts we obtained here Dirac field equations for a spin-\( 1/2 \) particle. The internal complex connection corresponds to the electromagnetic potential.

We observe that it is possible to define the complex vierbeins as

\[ e^\mu_\mu = e^\mu_\mu + i e \lambda n^\mu_\mu, \quad (4.1) \]

where \( \kappa \) is considered now as a parameter, and \( \lambda \) is a constant. We use here, as in Ref. 2,

\[ \lambda = \frac{e^2}{L^2} \approx \frac{c^3 e^2}{\hbar G} = 1.82 \times 10^{56} \text{ statvolt}/\text{cm}. \]

In a limit where the parameter \( \kappa \to 0 \), we obtain from (2.27) and (2.28) the real connection \( \Lambda \), from general relativity. Using the above expression for the vierbeins, we can display the interesting behavior of the new Dirac equations that appears when we split up \( \gamma^\mu (x) \) in terms of its real and imaginary parts, and also suppose a complex mass term: \( \mu = \mu_R + i \mu_I \), where we again can take \( \mu_I = \kappa \lambda m \). Then, from (3.19),

\[ [e^\mu_\mu \gamma^\mu (\partial_\mu + \Delta_\mu + i e A_\mu) \psi + \nu \mu_\mu \psi] + i \kappa \left[ n^\mu_\mu \gamma^\mu (\partial_\mu + \Delta_\mu + i e A_\mu) \psi + m \psi \right] = 0. \quad (4.2) \]

In the limit of the parameter \( \kappa \to 0 \), we should get the normal Dirac equations in the presence of gravitational and electromagnetic fields. Therefore, it means that we can get another identical set of Dirac equations if we take \( n^\mu_\mu = e^\mu_\mu - h^\mu_\mu \), and \( m \equiv \mu_R \), where \( h^\mu_\mu \) and \( \mu_R \) will be vierbeins and the mass term of general relativity theory.

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### APPENDIX: COMMENTS ON A MORE GENERAL TRANSFORMATION LAW IN A TANGENT SPACE ASSOCIATED TO A COMPLEX INTERNAL SPACE

Let us consider, instead of (2.10), a more general transformation law for objects in the complex tangent space. Considering, for instance, the vectors \( e^\mu_\mu \), it can be defined as

\[ e^\mu_\mu (x) = L^s \psi_s (x) e^\mu_\mu (x), \quad (A1) \]

\[ e^{*\mu}_\mu (x) = L^{*s} \psi^*_s \mu_\mu (x). \quad (A2) \]

The complex matrix \( L^s \mu_\mu \) now is a kind of pseudo-Lorentz matrix that follows the relation

\[ L^s \eta L = \eta. \quad (A3) \]

The covariant derivative of \( e^\mu_\mu \) on this complex tangent space is then defined as

\[ e^\mu_\mu |_L = e^\mu_\mu + A^\mu_\mu e^\mu_\mu, \quad (A4) \]

\[ e^{*\mu}_\mu |_L = e^{*\mu}_\mu + A^{*\mu}_\mu e^{*\mu}_\mu, \quad (A5) \]

where the affinity is complex. Its transformation law is

\[ A'_\mu = L\Lambda_\mu L^{-1} - L \mu_\mu L^{-1}, \quad A'^*_\mu = L^* \Lambda^* L^* - L^* \mu_\mu L^* \mu^* L^{-1}. \quad (A6) \]

It is directly shown (see Ref. 4 that, through the Einstein field equations for the nonsymmetric theory (a complex theory), \( g_{\mu\nu} = 0 \), we obtain the same corresponding field equations for the vierbeins described in Eqs. (2.21) and...
(2.22). However, we must also have \( \eta_{\mu \nu} = 0 \), where the "minus" sign corresponds to the complex conjugate of the affinity \( \Lambda_{\mu} \):

\[
\eta_{\mu \nu} = \eta_{\alpha \beta} = -\Lambda_{\mu}^{\alpha} \eta_{\alpha \beta} - \Lambda^{* \beta}_{\mu} \eta_{\mu \beta} = 0 .
\]  
(A7)

As \( \eta_{\mu \nu} \) lowers indices, we have that \( \Lambda_{\mu} \) is anti-Hermitian with respect to the index of the tangent space. Then, we have

\[
\Lambda_{\mu \nu} = \Lambda_{\nu \mu}^* + i \Lambda_{\mu \nu}.
\]  
(A8)

The expansion of \( \Lambda \) in first order is, from (A.1)–(A.3),

\[
\Lambda \approx 1 + \epsilon + i \eta, \quad \Lambda^{-1} \approx 1 - \epsilon - i \eta ,
\]  
(A9)

where \( \epsilon = \epsilon(x) \) are infinitesimal rotation matrices as before and \( \mu = \mu(x) \) are symmetric infinitesimal matrices. We can write the latter as

\[
\mu_{ab} = (\alpha + 1) \text{Tr} \mu_{ab} ,
\]  
(A10)

where \( \alpha \) is a symmetric trace-free matrix. Considering then, a particular transformation such that

\[
\Lambda \approx 1 + i K, \quad K = \text{Tr} \mu ,
\]  
(A11)

the affinity \( \Lambda_{\alpha} \) of this complex theory transforms as

\[
\Lambda'_{\alpha} = \Lambda_{\alpha} - (i/4) K_{\alpha} , \quad \text{Tr} \Lambda'_{\alpha} = \text{Tr} \Lambda_{\alpha} - i K_{\alpha} ,
\]  
(A12)

which is similar to the gauge transformation of an electromagnetic potential. (In the same way, we can show that the complex part of a nonsymmetric tangent curvature obtained with the above \( \Lambda_{\alpha} \) will be related to the Maxwell electromagnetic tensor.)

Now, from (A7), (3.8), and (3.9), we can easily obtain a relation between \( \Lambda_{\alpha} \) and the connection \( \Delta_{\alpha} \):

\[
\Lambda_{\alpha}^{\beta \gamma \delta} \gamma_{\delta} = \eta^{\alpha \beta \gamma} \Delta_{\alpha}^{(1)} \gamma_{\delta} ,
\]

\[
\Lambda^{* \alpha \beta \gamma \delta} \gamma_{\delta} = \eta^{* \alpha \beta \gamma} \Delta_{\alpha}^{(2)} \gamma_{\delta} ,
\]  
(A13)

where \( \Delta_{\alpha}^{(1)} \) and \( \Delta_{\alpha}^{(2)} \) are now (general) Dirac connections corresponding to the Fock–Ivanenko coefficients. However, expanding \( \Delta_{\alpha}^{(1)} \) and \( \Delta_{\alpha}^{(2)} \) in terms of the set \( \{ \Gamma_{\alpha} \} \), we can see that the real part of \( \Lambda_{\alpha} \) is of the same form as in general relativity, but that there is no way to relate the complex symmetrical part of \( \Lambda_{\alpha} \) in terms of that expansion, since the only symmetrical term there, which is proportional to the unit element of the set (the unit \( 4 \times 4 \) matrix), is eliminated through the commutator in (A13). This shows that this is not the correct choice for the transformation matrix \( \Lambda \). As we saw the correct one is the product expressed in (2.10).

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