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Solutions to higher Hamiltonians in the Toda hierarchies

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A method is presented for constructing the general solution to higher Hamiltonians (nonquadratic in the momenta) of the Toda hierarchies of integrable models associated with a simple Lie group G. The method is representation independent and is based on a modified version of the Lax operator. It constitutes a generalization of the method used to construct the solutions of the Toda molecule models. The SL(3) and SL(4) cases are discussed in detail.

I. INTRODUCTION

Nonlinear systems in one and two space-time dimensions have been the object of a great interest in physics and mathematics. Their structures are, in general, very rich and very useful in the understanding of nonlinear properties of physical theories. The Toda molecule model[1] (TM) is one of these systems that has been studied quite extensively using several different approaches. The equations of motion for its one-dimensional version are given by

$$\frac{d^2\phi_a}{dt^2} = -\exp\left(\sum_b K_{ab}\phi_b\right), \quad a,b = 1,2,\ldots,r. \tag{1.1}$$

For the case where the square, nonsingular matrix K is the Cartan matrix of a simple Lie group G, of rank r, these equations are completely integrable.[2,3] Their solutions have already been constructed.[4–6]

The Lie group G plays an important role in the study of the integrability properties of the TM equations. The solutions of Eqs. (1.1) can be viewed as some special geodesic motion on the symmetric space G^N/K, where G^N is the normal real form of G and K is its maximal compact subgroup.[5,7] When G^N/K is parametrized by the horospherical coordinates,[5,7] Eqs. (1.1) correspond, by a reduction procedure, to the radial part of the equations for the free motion on G^N/K. Although the TM equations are not invariant under continuous transformations associated with G, the invariance of the geodesic motion under left translations on G^N/K by G^N is, in some sense, hidden in (1.1).

Perhaps the richest structures of the TM equations associated with this hidden symmetry are contained in the so-called fundamental Poisson bracket relation (FPR) or the classical Yang–Baxter equation,[8,2,7,9,10] which relates two bracket structures. On one side is the Poisson bracket between the entries of a matrix operator A and on the other is the Lie bracket or commutator between A and a constant operator $\mathbb{P}$. In a tensor product notation, the FPR for the TM reads[2,7]

$$\{A\otimes A\}_{\rm PB} = -[\mathbb{P}, A\otimes 1 + 1\otimes A]. \tag{1.2}$$

The operator A lives in the subspace of the Lie algebra of G^N, which is the tangent space to G^N/K at unity, and it is given by

$$A = \frac{1}{2}\sum_{a,b=1}^{r} g_{ab}^{-1}p_a H_b + \frac{1}{2}\sum_{a=1}^{r}\exp\left[\frac{1}{2}\sum_{b=1}^{r}K_{ab}\phi_b\right](E_a + E_{-a}), \tag{1.3}$$

where p_a $[a = 1,2,\ldots,r$ (= rank G)] is the canonical momentum conjugate to ϕ_a; H_a are the Cartan subalgebra generators in the Chevalley basis of the group G, whose Cartan matrix is K_{ab}; E_a (E_{-a}) are the step operators for the simple roots (their negatives) of G; and g_{ab}^{-1} is the inverse of the matrix defined in (2.3a).

The operator $\mathbb{P}$ lives in the tensor product space of two copies of the Lie algebra of G, and is given by[2,7]

$$\mathbb{P} = \sum_{\alpha>0}\frac{\alpha^2}{2}(E_\alpha\otimes E_{-\alpha} - E_{-\alpha}\otimes E_\alpha), \tag{1.4}$$

where the summation is over the positive roots α of G and E_α are the corresponding step operators. The structures of the classical Yang–Baxter equations, like (1.2), are intrinsically related to the algebraic and geometric properties of symmetric spaces.[11]

From (1.2) we obtain

$$\{A,(1/N)\mathrm{Tr}\ A^N\}_{\rm PB} = [A,B_N], \tag{1.5}$$

where the operator B_N is defined as

$$B_N = \mathrm{Tr}_R[\mathbb{P}(1\otimes A^{N-1})] = -\mathrm{Tr}_L[\mathbb{P}(A^{N-1}\otimes 1)]. \tag{1.6}$$

The subindices R and L mean we are taking the trace of the right and left entries, respectively, of the tensor product.

In the case where the Hamiltonian is $\mathrm{Tr}\,A^N/N$, the relation (1.5) becomes a Lax pair equation:[12]

$$\frac{dA}{dt} = [A,B_N]. \tag{1.7}$$

For $N=2$, this is the Lax equation for the TM equation introduced in (1.1).[2,3] In this sense, the classical Yang–Baxter equation (1.4) is more fundamental than the Lax equation.

From (1.5) it follows that the charges $\mathrm{Tr}\,A^N$ are in involution, i.e.,

$$\{\mathrm{Tr}\,A^N,\mathrm{Tr}\,A^M\}_{\rm PB} = 0. \tag{1.8}$$

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The number of these charges, which are functionally independent, is equal to the rank of G.

In the cases where the Hamiltonian is one of those charges, the corresponding system is integrable, since Eq. (1.8) implies it has rank G independent conserved charges (including the Hamiltonian). Therefore, we have rank G integrable systems associated with each classical Yang–Baxter equation (1.2). These constitute the Toda models' hierarchies.[13] The Hamiltonian for the TM model (1.1) is $\mathrm{Tr}\, A^2/2$. It is the simplest model of the hierarchy associated with a given group G.

In this paper, we generalize the method of Refs. 4 and 5 to construct the solutions to higher Hamiltonians in the Toda hierarchy, i.e., the Hamiltonians $\mathrm{Tr}\, A^N/N$ $(N>2)$, which are not quadratic in the momenta. Although these Hamiltonians are representation dependent, our method works uniformly in any representation. The integration of the equations of motion is performed by making a suitable modification of the Lax operator A such that it becomes a polynomial in the momenta of the same degree as B_N. In fact, in a given representation D^λ, we take it to be the component of the matrix $[D^\lambda(A)]^{N-1}$ lying in the representation itself.

In Sec. II we describe the construction of the solutions to the higher Hamiltonians using the zero curvature condition (Lax equation) and the Iwasawa decomposition of the normal real form G^N of G. In Secs. III and IV we apply such a method for the Toda hierarchies associated with the groups SL(3) and SL(4), respectively.

II. THE CONSTRUCTION OF THE SOLUTIONS

Olshanetsky and Perelomov[5] have constructed the solutions of the TM equations (1.1) by projecting, on the phase space of TM, some special geodesic flows on the symmetric phase G^N/K. This space has some nice algebraic properties due to the Iwasawa decomposition[14] of G^N, the (noncompact) normal real form of the Lie group G, whose Cartan matrix appears in (1.1). According to that, the elements of the group G^N decompose as

$$g = nak, \quad g \in G^N, \tag{2.1}$$

where n is an element of the nilpotent subgroup G^N generated by the negative root step operators $E_{-\alpha}$ of G^N, a is an element of the Abelian subgroup generated by the Cartan subalgebra generators H_a $(a = 1,2,\ldots,\mathrm{rank}\ G^N)$, and k belongs to K, which is the maximal compact subgroup of G^N and is generated by $(E_\alpha - E_{-\alpha})$. These generators satisfy the commutation relations

$$[H_a, H_b] = 0, \quad a,b = 1,2,\ldots,r = \mathrm{rank}\ G^N, \tag{2.2a}$$

$$[H_a, E_{\pm\alpha}] = \pm K_{\alpha a} E_{\pm\alpha}, \tag{2.2b}$$

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha\beta} E_{\alpha+\beta}, & \text{if } \alpha+\beta \text{ is a root,} \\ 2\alpha . H/\alpha^2, & \text{if } \alpha+\beta = 0, \\ 0, & \text{otherwise,} \end{cases} \tag{2.2c}$$

where $K_{\alpha a} = 2\alpha.\, \alpha_a/\alpha_a^2$, α_a are the simple roots of G^N, and $N_{\alpha\beta}$ are some structure constants that are not important in what follows. The Killing form of G^N can be normalized as

$$\mathrm{Tr}(H_a H_b) = 4(\alpha_a . \alpha_b / \alpha_a^2 \alpha_b^2) \equiv 4 g_{ab}, \tag{2.3a}$$

$$\mathrm{Tr}(E_\alpha H_a) = 0, \tag{2.3b}$$

$$\mathrm{Tr}(E_\alpha E_{-\beta}) = (2/\alpha^2)\delta_{\alpha\beta}. \tag{2.3c}$$

Due to the Iwasawa decomposition (2.1) of G^N, the points of the symmetric space G^N/K can be put into a one-to-one correspondence with the elements of the solvable subgroup na.[7] This subgroup plays an important role in the construction of the solutions to the Hamiltonians in the Toda hierarchies, as we will now explain.

Although we are dealing with a one-dimensional theory, i.e., which depends only on time, it is much easier to consider a zero curvature condition in two space-time dimensions where the "gauge" potentials, which we shall denote by A_x and A_t, do not depend upon the extra space variable x, i.e., (see comments at the end of this section),[15]

$$[\partial_t + A_t, \partial_x + A_x] = \partial_t A_x - [A_x, A_t] = 0. \tag{2.4}$$

The potentials A_x and A_t will be chosen to be functions of the operators A and B_N, defined in (1.3) and (1.6), respectively, such that the vanishing of (2.4) is a consequence of the Lax pair equation (1.7), and therefore of the equations of motion.

It is highly desirable in our construction, the reason for which will become clear later on, to write, in the light cone variables $u = (x+t)/2$ and $v = (x-t)/2$, the component $A_v = A_x - A_t$ as a "pure gauge" potential of the solvable subgroup na, i.e.,

$$b^{-1}\frac{\partial b}{\partial v} = -b^{-1}\frac{db}{dt} = A_v, \quad b \in na, \tag{2.5}$$

where we have used the fact that the elements b of the subgroup na do not depend upon the x variable. Integrating (2.5), one obtains $b(t)$ as a path ordered integral

$$b(t) = b(0) P \exp\left[\int_0^{-v} A_v\, dv\right]. \tag{2.6}$$

Due to (2.4), the integration above is path independent and therefore we can first integrate in x from 0 to $-t$ and then in t from 0 to t. We get[15]

$$b(t) = b(0)\exp(-tA_x(0))U(t), \tag{2.7}$$

where

$$U(t) = P \exp\left[\int_0^t A_t\, dt\right]. \tag{2.8}$$

So, the unknown time dependence of b is contained in the operator $U(t)$.

We now have to express the potentials A_x and A_t in terms of the operators A and B_N such that A_v is an element of the Lie algebra of na. Using (1.4) and (1.6), we find that B_N is a linear combination of the positive and negative step operators

$$B_N = \sum_{\alpha>0} \frac{\alpha^2}{2}\{\mathrm{Tr}(E_{-\alpha}A^{N-1})E_\alpha - \mathrm{Tr}(E_\alpha A^{N-1})E_{-\alpha}\}. \tag{2.9}$$

For $N=2$, we find using (2.3) and (1.3), that the coefficients of E_α and $E_{-\alpha}$ vanish for α nonsimple, and are the same as the coefficients of $(E_\alpha + E_{-\alpha})$ in (1.3). Therefore, taking $A_x = A$ and $A_t = B_2$, we find that A_v is an element of

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the algebra of the solvable subgroup na. This is in fact the "pure gauge" potential used in the construction of the solutions of the TM equations.[4,5,15]

For $N>2$, the situation is more delicate, in part because it involves traces of more than two generators. Since B_N is a polynomial in the momenta of degree $N-1$, we need a potential of the same form to cancel the term proportional to E_α in (2.9) and get a potential in the Lie algebra of na.

In any highest weight finite-dimensional representation D^λ of the Lie algebra g^N of G^N, the generators H_a and E_α satisfy the Hermiticity property

$$D^\lambda(H_a)^+ = D^\lambda(H_a), \quad D^\lambda(E_\alpha)^+ = D^\lambda(E_{-\alpha}). \tag{2.10}$$

It then follows that the Lax operator A, defined in (1.3), is Hermitian:

$$D^\lambda(A)^+ = D^\lambda(A). \tag{2.11}$$

Therefore, in any representation D^λ (of dimension m), the matrix $[D^\lambda(A)]^{N-1}$ is Hermitian and consequently belongs to the vector space V spanned by all $m\times m$ Hermitian matrices [i.e., the defining representation of the Lie algebra of $U(m)$]. We denote by D the subspace of V spanned by $D^\lambda(H_a)$, $D^\lambda(E_\alpha+E_{-\alpha})$, and $iD^\lambda(E_\alpha-E_{-\alpha})$. Since V is a Euclidean vector space, we define M to be the orthogonal complement of D in V. Then

$$\mathrm{Tr}(MD)=0. \tag{2.12}$$

Using the fact that D is a subalgebra of V under the commutator, and the cyclic property of the trace,

$$\mathrm{Tr}\Big(([M,D]D\Big)=\mathrm{Tr}\Big(M[D,D]\Big)=0.$$

Therefore,

$$[M,D]\subset M. \tag{2.13}$$

The matrix $[D^\lambda(A)]^{N-1}$ can be written as a real linear combination of the basis of D plus some matrix M_N^λ, which belongs to M. Then, using (2.3) and (2.12),

$$\begin{aligned} d^\lambda\{&[D^\lambda(A)]^{N-1}-M_N^\lambda\}\\ &=\frac{1}{4}\sum_{a,b} g_{ab}^{-1}\,\mathrm{Tr}[D^\lambda(H_bA^{N-1})]D^\lambda(H_a)\\ &\quad+\sum_{\alpha>0}\frac{\alpha^2}{2}\{\mathrm{Tr}[D^\lambda(E_{-\alpha}A^{N-1})]D^\lambda(E_\alpha)\\ &\quad+\mathrm{Tr}[D^\lambda(E_\alpha A^{N-1})]D^\lambda(E_{-\alpha})\}, \end{aligned} \tag{2.14}$$

where d^λ is the Dynkin index[16] of the representation D^λ [the bilinear trace form in D^λ is d^λ times the Killing form (2.3)].

If the Hamiltonian $\mathscr{H}$ is taken to be a function of the charges

$$I_N^\lambda=(1/N)\mathrm{Tr}[D^\lambda(A)]^N, \tag{2.15}$$

then the time evolution of any function f of the canonical variables is given by

$$\frac{df}{dt}=\{f,\mathscr{H}\}_{\mathrm{PB}}=\sum_N\frac{\partial\mathscr{H}}{\partial I_N^\lambda}\{f,I_N^\lambda\}_{\mathrm{PB}}, \tag{2.16}$$

where the summation is over the charges I_N^λ, which are functionally independent. As a consequence of (1.8), the charges I_N^λ are conserved and the Hamiltonian $\mathscr{H}$ is integrable.

From (1.5) and (2.13), we find that the potentials

$$A_x^\lambda(\mathscr{H})=d^\lambda\sum_N\frac{\partial\mathscr{H}}{\partial I_N^\lambda}\{[D^\lambda(A)]^{N-1}-M_N^\lambda\}, \tag{2.17a}$$

$$A_t^\lambda(\mathscr{H})=\sum_N\frac{\partial\mathscr{H}}{\partial I_N^\lambda}D^\lambda(B_N) \tag{2.17b}$$

satisfy the Lax pair equation

$$\frac{dA_x^\lambda(\mathscr{H})}{dt}=[A_x^\lambda(\mathscr{H}),A_t^\lambda(\mathscr{H})] \tag{2.18}$$

and, in addition, the matrix M_N^λ has to satisfy

$$\frac{dM_N^\lambda}{dt}-[M_N^\lambda,A_t^\lambda]=0. \tag{2.19}$$

Therefore, traces of powers of M_N^λ are constants of motion.

The advantages of the Lax equation (2.18) are that the potentials A_x^λ and A_t^λ are both polynomials in the momenta of the same degree and, most important, that the potential $A_v^\lambda=A_x^\lambda-A_t^\lambda$ is an element of the Lie algebra of the subgroup na. Therefore, we can take the potentials appearing in (2.4) to be those given by (2.17). From (2.16) and (1.3), we have

$$\frac{d\phi_a}{dt}=\frac{1}{2}\sum_N\frac{\partial\mathscr{H}}{\partial I_N^\lambda}\sum_b g_{ab}^{-1}\,\mathrm{Tr}[D^\lambda(H_bA^{N-1})]. \tag{2.20}$$

Then, using, (2.5), (2.17), (2.14), and (2.9), we have (with b evaluated in the representation D^λ)

$$\begin{aligned} A_v^\lambda(\mathscr{H})&=-b^{-1}\frac{db}{dt}\\ &=\frac{1}{2}\sum_a\frac{d\phi_a}{dt}D^\lambda(H_a)+2\sum_N\frac{\partial\mathscr{H}}{\partial I_N^\lambda}\\ &\quad\times\sum_{\alpha>0}\frac{\alpha^2}{2}\mathrm{Tr}[D^\lambda(E_\alpha A^{N-1})]D^\lambda(E_{-\alpha}). \end{aligned} \tag{2.21}$$

Therefore, the coefficients of the Cartan subalgebra generators in A_v^λ are just the velocities. This fact makes the integration of the equations of motion easier. Indeed, since $b=na$, we have

$$b^{-1}\frac{db}{dt}=a^{-1}\frac{da}{dt}+a^{-1}n^{-1}\frac{dn}{dt}a. \tag{2.22}$$

The first term on the rhs is a linear combination of the Cartan subalgebra generators, and the second term is a linear combination of negative step operators. Therefore, comparing (2.21) with (2.22), we get

$$a^{-1}\frac{da}{dt}=-\frac{1}{2}\sum_a\frac{d\phi_a}{dt}D^\lambda(H_a) \tag{2.23}$$

and, consequently,

$$a=\exp(-\tfrac{1}{2}\phi^\lambda(t)), \tag{2.24}$$

where we have defined

$$\phi^\lambda(t)=\sum_a\phi_a(t)D^\lambda(H_a). \tag{2.25}$$

Using (2.10) and (2.11), we have

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$$\mathrm{Tr}\, D^{\lambda}(H_a A^{N-1})^* = \mathrm{Tr}\, D^{\lambda}(H_a A^{N-1}),$$
$$\mathrm{Tr}\, D^{\lambda}(E_a A^{N-1})^* = \mathrm{Tr}\, D^{\lambda}(E_{-a} A^{N-1}). \tag{2.26}$$

From (2.9), we then see that the operator B_N is anti-Hermitian, and therefore the potentials introduced in (2.17) satisfy

$$[A_x^{\lambda}(\mathscr{H})]^+ = A_x^{\lambda}(\mathscr{H}), \quad [A_t^{\lambda}(\mathscr{H})]^+ = -A_t^{\lambda}(\mathscr{H}). \tag{2.27}$$

Working with these potentials, we find that the operator $U(t)$, defined in (2.8), is unitary and consequently we have, from (2.7),

$$b(t)b(t)^+ = b(0)\exp\{-2tA_x^{\lambda}(\mathscr{H})|_{t=0}\}b(0)^+. \tag{2.28}$$

Using (2.24) and the fact that $\phi^{\lambda}(t)$ is Hermitian, we get

$$e^{-\phi^{\lambda}(t)} = n(t)^{-1}n(0)e^{-\phi^{\lambda}(0)/2} \times\exp\{-2tA_x^{\lambda}(\mathscr{H})|_{t=0}\} \times e^{-\phi^{\lambda}(0)/2}n(0)^+[n(t)^+]^{-1}. \tag{2.29}$$

Therefore, the unknown time dependence of the coordinates ϕ_a is contained in the elements of the nilpotent subgroup n, whose generators are the negative step operators. However, by taking the expectation value of the expression (2.29) in the highest weight state $|\lambda\rangle$ of the representation D^{λ}, we eliminate this unknown time dependence. Indeed, the state $|\lambda\rangle$ is annihilated by all positive step operators, and consequently

$$n^+|\lambda\rangle = |\lambda\rangle. \tag{2.30}$$

The fundamental weights λ_a ($a = 1,2,...,\mathrm{rank}\, G^N$) satisfy

$$2\alpha_a.\lambda_b/\alpha_a^2 = \delta_{ab}. \tag{2.31}$$

Therefore, working with the rank G^N fundamental representations of G^N, we get

$$D^{\lambda_a}(H_b)|\lambda_a\rangle = \delta_{ab}|\lambda_a\rangle. \tag{2.32}$$

Consequently, from (2.29) and (2.30), it follows that

$$e^{-\phi_a(t)} = e^{-\phi_a(0)}\langle\lambda_a|\exp[-2tA_x^{\lambda_a}(\mathscr{H})|_{t=0}]|\lambda_a\rangle. \tag{2.33}$$

This is the general solution of the equations of motion for a model described by any Hamiltonian $\mathscr{H}$, which is a function of the charges $\mathrm{Tr}\, A^N$. We notice that the solutions have the same formal expression for any $\mathscr{H}$, including the solutions to the Toda molecule models[4,5] corresponding to $\mathscr{H} = (1/2)\mathrm{Tr}\, A^2$. The actual different time evolutions are encoded into the operator $A_x^{\lambda_a}(\mathscr{H})|_{t=0}$ containing the initial values of coordinates and momenta.

The introduction of the extra space varaiable x is an artifice of the construction. It enables us to treat the Lax operators as the components of a "gauge potential" in two dimensions. Then we can play with the path-independent integration of (2.5) to select a component of the gauge potential, which points to a suitable direction in the algebra g^N. In fact, the choice of A_x^{λ} is made in such a way that A_v^{λ} lies in the subalgebra na. This choice implies, in addition, that the coefficients of the Cartan subalgebra generators in A_v^{λ} are just the velocities. This makes the integration possible. We point out, however, that the physics of the problem is unchanged by the introduction of the extra space variable x since the gauge potentials are x independent.

In the next two sections, we show how to use the procedure described above to find the general solutions to the Hamiltonians in the hierarchies of the Toda models associated with the groups SL(3) and SL(4).

III. THE SL(3) TODA HIERARCHY

For the case of SL(3), there are only two charges $\mathrm{Tr}\, A^N$ that are functionally independent, namely, $\mathrm{Tr}\, A^2$ and $\mathrm{Tr}\, A^3$. In the case where the Hamiltonian is $\mathrm{Tr}\, A^2$, we get the Toda molecule equations for SL(3), whose solutions are known.[4,5] We now want to discuss the model defined by the Hamiltonian $\mathrm{Tr}\, A^3$.

The equations become simpler if we perform the canonical transformation

$$\rho_a = 2\sum_b g_{ab}\phi_b = \sum_b K_{ab}\frac{\phi_b}{\alpha_a^2}, \tag{3.1a}$$

$$\pi_a = \frac{1}{2}\sum_b g_{ab}^{-1}p_b. \tag{3.1b}$$

The Lax operator A, defined in (1.3), then becomes

$$A = \sum_a \pi_a H_a + \frac{1}{2}\sum_a \exp\left(\rho_a\frac{\alpha_a^2}{2}\right)(E_a + E_{-a}). \tag{3.2}$$

All roots of SL(3) have the same length and so we can set $\alpha_a^2 = 2$. We shall denote by λ_1 and λ_2 the fundamental weights of SL(3), which are associated to the triplet and antitriplet representations, respectively. By taking the Hamiltonian $\mathscr{H}$ of our system to be $\mathrm{Tr}\, A^3/3$, evaluated in the triplet representation, we get

$$\mathscr{H} = \tfrac{1}{3}\mathrm{Tr}[D^{\lambda_1}(A)]^3 = \pi_1\pi_2(\pi_1 - \pi_2) + \tfrac{1}{4}[\pi_2 e^{2\rho_1} - \pi_1 e^{2\rho_2}]. \tag{3.3}$$

When evaluated in the antitriplet representation, the quantity $\mathrm{Tr}\, A^3$ gets a factor (-1) with respect to its value in the triplet representation. So we have

$$\mathscr{H} = -\tfrac{1}{3}\mathrm{Tr}[D^{\lambda_2}(A)]^3. \tag{3.4}$$

This Hamiltonian is not positive definite and it is singular whenever

$$\pi_1 = e^{\pm i\pi/3}\pi_2. \tag{3.5}$$

However, it describes an integrable system since, from (1.8), we get that the quantity

$$I_2 = \tfrac{1}{2}\mathrm{Tr}[D^{\lambda_1}(A)]^2 = \tfrac{1}{2}\mathrm{Tr}[D^{\lambda_2}(A)]^2 = \pi_1^2 + \pi_2^2 - \pi_1\pi_2 + \tfrac{1}{4}[e^{2\rho_1} + e^{2\rho_2}] \tag{3.6}$$

is conserved.

The Hamilton's equations of motion for such systems are given by

$$\dot\rho_1 = \pi_2(2\pi_1 - \pi_2) - \tfrac{1}{4}e^{2\rho_2}, \tag{3.7a}$$

$$\dot\rho_2 = -\pi_1(2\pi_2 - \pi_1) + \tfrac{1}{4}e^{2\rho_1}, \tag{3.7b}$$

$$\dot\pi_1 = -\tfrac{1}{2}\pi_2 e^{2\rho_1}, \tag{3.7c}$$

$$\dot\pi_2 = \tfrac{1}{2}\pi_1 e^{2\rho_2}. \tag{3.7d}$$

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Notice that if the momenta vanish at a given time, then they will vanish for all times. In this case the energy of the system is zero. According to (3.5) this corresponds to a singular point of the Hamiltonian. The dynamics of the system at this singular point become very simple since Eqs. (3.7) reduce to

$$\ddot{\rho}_- = 0, \tag{3.8a}$$

$$\ddot{\rho}_+ = -\tfrac{1}{4}e^{2\rho_+}, \tag{3.8b}$$

where $\rho_\pm = \rho_1 \pm \rho_2$. Therefore, in this case the system is composed of a particle satisfying the Liouville equation plus a free particle. However, due to the conservation of I_2, they are subjected to the constraint

$$\tfrac{1}{2}\dot{\rho}_-^2 = \tfrac{1}{2}\dot{\rho}_+^2 + \tfrac{1}{8}e^{2\rho_+}. \tag{3.9}$$

The equations of motion (3.7) can be written in a more interesting form by eliminating the momenta using the conserved quantity I_2. From (3.7) we have

$$2\ddot{\rho}_1 + \ddot{\rho}_2 = -(\pi_2^2 + \pi_1\pi_2 + \tfrac{1}{4}e^{2\rho_2} - \tfrac{1}{2}\dot{\rho}_1)e^{2\rho_1}, \tag{3.10a}$$

$$2\ddot{\rho}_2 + \ddot{\rho}_1 = -(\pi_1^2 + \pi_1\pi_2 + \tfrac{1}{4}e^{2\rho_1} + \tfrac{1}{2}\dot{\rho}_2)e^{2\rho_2}. \tag{3.10b}$$

Using (3.1), (3.6), (3.7a), (3.7b), and the fact that, for SL(3), $K_{11} = K_{22} = 2$, $K_{12} = K_{21} = -1$, we can write (3.10) as

$$\ddot{\phi}_a = [\tfrac{1}{2}\epsilon_{ac}\dot{\phi}_c - \tfrac{2}{3}I_2]e^{K_{ab}\phi_b}, \quad a = 1,2, \tag{3.11}$$

where $\epsilon_{11} = \epsilon_{22} = 0$, $\epsilon_{12} = -\epsilon_{21} = 1$.

We now construct the solutions to this equation using the procedure of Sec. II. In the l-dimensional fundamental representations of SL(l), the matrix M_N^λ is proportional to the $l \times l$ unity matrix. The reason is that any $l \times l$ real matrix can be written as a linear combination of the matrices representing the $(l^2 - 1)$ generators of SL(l), in one of these representations, and the unity matrix. The trace of a generator of a semisimple Lie algebra vanishes in any finite-dimensional representation. Then, by taking the trace on both sides of (2.14) and using (3.6), we get

$$M_3^{\lambda_1} = M_3^{\lambda_2} = \tfrac{2}{3}I_2\mathbb{1}_{3\times 3}, \tag{3.12}$$

which obviously satisfies (2.19).

Therefore, the potentials (2.17), in the triplet representation, are given by

$$A_x^{\lambda_1} = [D^{\lambda_1}(A)]^2 - \tfrac{2}{3}I_2\mathbb{1}_{3\times 3}, \tag{3.13a}$$

$$A_t^{\lambda_1} = D^{\lambda_1}(B_3), \tag{3.13b}$$

and in the antitriplet representation by

$$A_x^{\lambda_2} = -[D^{\lambda_2}(A)]^2 + \tfrac{2}{3}I_2\mathbb{1}_{3\times 3}, \tag{3.14a}$$

$$A_t^{\lambda_2} = -D^{\lambda_2}(B_3), \tag{3.14b}$$

where we have used the fact that $d^{\lambda_1} = d^{\lambda_2} = 1$.

Since λ_1 and λ_2 are the fundamental weights of SL(3), the expression for the solutions $\phi_1(t)$ and $\phi_2(t)$ are given by (2.33) for the potentials (3.13a) and (3.14a), respectively.

IV. THE SL(4) TODA HIERARCHY

We will denote by λ_1, λ_2, and λ_3 the fundamental weights of SL(4). They are, respectively, the highest weights of the 4, 6, and $\bar{4}$ fundamental representations of SL(4). In this case there are three charges $\mathrm{Tr}\,A^N$ that are functionally independent. Using the notation introduced in (2.15), we find that in the D^{λ_1} fundamental representation these three charges are

$$I_2^{\lambda_1} = \pi_1^2 + \pi_2^2 + \pi_3^2 - \pi_1\pi_2 - \pi_2\pi_3 + \tfrac{1}{4}(e^{2\rho_1} + e^{2\rho_2} + e^{2\rho_3}), \tag{4.1a}$$

$$I_3^{\lambda_1} = \pi_1\pi_2(\pi_1 - \pi_2) + \pi_2\pi_3(\pi_2 - \pi_3) + \tfrac{1}{4}[\pi_2 e^{2\rho_1} + (\pi_3 - \pi_1)e^{2\rho_2} - \pi_2 e^{2\rho_3}], \tag{4.1b}$$

$$\begin{aligned} I_4^{\lambda_1} = \tfrac{1}{2}\{&\pi_1^4 + \pi_2^4 + \pi_3^4 - 2\pi_2(\pi_1^3 + \pi_3^3) + 3\pi_2^2(\pi_1^2 + \pi_3^2) \\ &- 2\pi_2^3(\pi_1 + \pi_3) + \tfrac{1}{2}(\pi_1^2 - \pi_1\pi_2 + \pi_2^2)e^{2\rho_1} \\ &+ \tfrac{1}{2}(\pi_3^2 - \pi_2\pi_3 + \pi_2^2)e^{2\rho_3} \\ &+ \tfrac{1}{2}[\pi_1^2 + \pi_2^2 + \pi_3^2 - \pi_2(\pi_1 + \pi_3) - \pi_1\pi_3]e^{2\rho_2} \\ &+ \tfrac{1}{16}(e^{4\rho_1} + e^{4\rho_2} + e^{4\rho_3} + 2e^{2(\rho_1+\rho_2)} + 2e^{2(\rho_2+\rho_3)}\}, \end{aligned} \tag{4.1c}$$

where we have used the canonical variables introduced in (3.1).

In the D^{λ_3} fundamental representation we have

$$I_2^{\lambda_3} = I_2^{\lambda_1}, \quad I_3^{\lambda_3} = -I_3^{\lambda_1}, \quad I_4^{\lambda_3} = I_4^{\lambda_1}. \tag{4.2}$$

In the D^{λ_2} fundamental representation the charges $I_3^{\lambda_2}$ and $I_5^{\lambda_2}$ vanish. Then the functionally independent charges in this representation are

$$I_2^{\lambda_2} = 2I_2^{\lambda_1}, \tag{4.3a}$$

$$I_4^{\lambda_2} = 3(I_2^{\lambda_1})^2 - 4I_4^{\lambda_1}, \tag{4.3b}$$

$$I_6^{\lambda_2} = \tfrac{14}{3}(I_2^{\lambda_1})^3 - 8I_2^{\lambda_1}I_4^{\lambda_1} + (I_3^{\lambda_1})^2. \tag{4.3c}$$

The relations (4.1)–(4.3) can be easily checked using a REDUCE program.

The solutions to any Hamiltonian that is a function of the charges $\mathrm{Tr}\,A^N$ can be written in terms of the solutions of three Hamiltonians that are independent functions of these charges. The SL(4) Toda molecule model itself is described by the Hamiltonian $I_2^{\lambda_1}$. Therefore, we will take the Hamiltonians of the other two models in this hierarchy to be

$$\mathscr{H}_4 = I_4^{\lambda_2}, \tag{4.4}$$

$$\mathscr{H}_6 = I_6^{\lambda_2}. \tag{4.5}$$

We have

$$M_2^{\lambda_1} = 0, \quad M_3^{\lambda_1} = \tfrac{1}{2}I_2^{\lambda_1}\mathbb{1}_{4\times 4}, \quad M_4^{\lambda_1} = \tfrac{3}{4}I_3^{\lambda_1}\mathbb{1}_{4\times 4}, \tag{4.6}$$

$$M_2^{\lambda_2} = M_4^{\lambda_2} = M_6^{\lambda_2} = 0, \tag{4.7}$$

$$M_2^{\lambda_3} = 0, \quad M_3^{\lambda_3} = \tfrac{1}{2}I_2^{\lambda_3}\mathbb{1}_{4\times 4}, \quad M_4^{\lambda_3} = \tfrac{3}{4}I_3^{\lambda_3}\mathbb{1}_{4\times 4}. \tag{4.8}$$

Equations (4.6) and (4.8) are obtained using the same arguments leading to (3.12). Equation (4.7) can be checked using a REDUCE program. The Dynkin indices for the fundamental representations of SL(4) are $d^{\lambda_1} = d^{\lambda_3} = 1$ and $d^{\lambda_2} = 2$. Therefore, from (2.17), (4.2), and (4.3), we find that the Lax pair operators for the Hamiltonian $\mathscr{H}_4$, in the fundamental representations, are

$$A_x^{\lambda_1}(\mathscr{H}_4) = 6I_2^{\lambda_1}D^{\lambda_1}(A) - 4\{[D^{\lambda_1}(A)]^3 - \tfrac{3}{4}I_3^{\lambda_1}\mathbb{1}_{4\times 4}\}, \tag{4.9a}$$

$$A_t^{\lambda_1}(\mathscr{H}_4) = 6I_2^{\lambda_1}D^{\lambda_1}(B_2) - 4D^{\lambda_1}(B_4), \tag{4.9b}$$

$$A_x^{\lambda_2}(\mathscr{H}_4) = 2[D^{\lambda_2}(A)]^3, \tag{4.10a}$$

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$$A_t^{\lambda_2}(\mathscr{H}_4) = D^{\lambda_2}(B_4), \quad (4.10b)$$

$$A_x^{\lambda_3}(\mathscr{H}_4) = 6I_2^{\lambda_3}D^{\lambda_3}(A) - 4\{[D^{\lambda_3}(A)]^3 - \tfrac{3}{4}I_3^{\lambda_3}\mathbf{1}_{4\times4}\}, \quad (4.11a)$$

$$A_t^{\lambda_3}(\mathscr{H}_4) = 6I_2^{\lambda_3}D^{\lambda_3}(B_2) - 4D^{\lambda_3}(B_4). \quad (4.11b)$$

Analogously, the Lax pair operators for the Hamiltonian $\mathscr{H}_6$ in the fundamental representations are

$$A_x^{\lambda_1}(\mathscr{H}_6) = [14(I_2^{\lambda_1})^2 - 8I_4^{\lambda_1}]D^{\lambda_1}(A) + 2I_3^{\lambda_1}\{[D^{\lambda_1}(A)]^2 - \tfrac{1}{2}I_2^{\lambda_1}\mathbf{1}_{4\times4}\} - 8I_2^{\lambda_1}\{[D^{\lambda_1}(A)]^3 - \tfrac{3}{4}I_3^{\lambda_1}\mathbf{1}_{4\times4}\}, \quad (4.12a)$$

$$A_t^{\lambda_1}(\mathscr{H}_6) = [14(I_2^{\lambda_1})^2 - 8I_4^{\lambda_1}]D^{\lambda_1}(B_2) + 2I_3^{\lambda_1}D^{\lambda_1}(B_3) - 8I_2^{\lambda_1}D^{\lambda_1}(B_4), \quad (4.12b)$$

$$A_x^{\lambda_2}(\mathscr{H}_6) = 2[D^{\lambda_2}(A)]^5, \quad (4.13a)$$

$$A_t^{\lambda_2}(\mathscr{H}_6) = D^{\lambda_2}(B_6), \quad (4.13b)$$

$$A_x^{\lambda_3}(\mathscr{H}_6) = [14(I_2^{\lambda_3})^2 - 8I_4^{\lambda_1}]D^{\lambda_3}(A) + 2I_3^{\lambda_3}\{[D^{\lambda_3}(A)]^2 - \tfrac{1}{2}I_2^{\lambda_3}\mathbf{1}_{4\times4}\} - 8I_2^{\lambda_1}\{[D^{\lambda_3}(A)]^3 - \tfrac{3}{4}I_3^{\lambda_3}\mathbf{1}_{4\times4}\}, \quad (4.14a)$$

$$A_t^{\lambda_3}(\mathscr{H}_6) = [14(I_2^{\lambda_3})^2 - 8I_4^{\lambda_1}]D^{\lambda_3}(B_2) + 2I_3^{\lambda_3}D^{\lambda_3}(B_3) - 8I_2^{\lambda_3}D^{\lambda_3}(B_4). \quad (4.14b)$$

The general solutions to the Hamiltonians $\mathscr{H}_4$ and $\mathscr{H}_6$ are obtained from (2.33) using the "gauge potentials" given above.

V. CONCLUSIONS

We have presented a method for constructing the general solution to higher Hamiltonians in the Toda hierarchies associated to any simple Lie group G. The method is representation independent and it is a generalization of the work of Olshanetsky and Perelomov,[5] and Leznov and Saveliev.[4] The key point of the construction is the modification of the Lax operator. For the Toda molecule models the operators A and B_2, defined in (1.3) and (1.6), respectively, are such that their difference is an element of the solvable subalgebra *na*. This fact plays an important role in the construction of the solutions of the TM[4,5] since it makes the connection with the geodesics on the symmetric space G^N/K.[5,7] For the higher Hamiltonians, $\mathrm{Tr}\, A^N$ ($N > 2$), in the hierarchies the operator B_N depends upon the momenta and, unlike A, it contains nonsimple root step operators. The operator A is Hermitian in any representation D^λ, and therefore powers of the matrix $D^\lambda(A)$ (of dimension m) belong to the vector space of the $m\times m$ Hermitian matrices. Since this is a Euclidean space, the matrices belonging to the representation D^λ can be split from the rest in a way that relations (2.12) and (2.13) hold. Then, by replacing the operator A by the component of the matrix $[D^\lambda(A)]^{N-1}$ lying in the representation D^λ, we obtain from (1.7) two decoupled Lax equations, namely (2.18) and (2.19). The first one leads us to the solution in a quite simple way. Analogously to the TM case, the difference between the new Lax operators (2.17) is an element of the subalgebra *na*. In addition, the integration is made easy by the fact the component of the Lax operator (2.21) lying in the Cartan subalgebra is linear in the velocities. This is very similar to what happens in Refs. 4 and 5.

It would be very interesting to investigate further the possibility of understanding this construction in terms of the universal enveloping algebra of G^N. We believe that such an investigation could shed some light on the quantum integrability properties of these models and their relation with quantum groups. Natural extensions of our work would be the construction of the solutions of the two-dimensional version of these higher Hamiltonians. One could also try to make a connection between the solution presented here and special motions on the symmetric spaces G^N/K.

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