so(2,1) Lie algebra and the Green's functions for the conditionally exactly solvable potentials

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The Green's functions of the recently discovered conditionally exactly solvable potentials are computed. This is done through the use of a second-order differential realization of the so(2,1) Lie algebra. So we present the dynamical symmetry underlying the solvability of such potentials and show that they belong to a general class of solvable and partially solvable potentials.

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I. INTRODUCTION

The so(2,1) Lie algebra [isomorphic to su(1,1) and sl(2,R)] is known to generate a family of exactly solvable (ES) potentials, among them the harmonic oscillator, the Coulomb and Morse potentials [1]. This was presented in a unified way using a direct construction of the so(2,1) generators [2] or with the Milshtein-Strakhovenko method [3,4]. So it seemed that, at least, the one-dimensional so(2,1) class of potentials was exhausted. In fact, the so(2,1) Lie algebra can also be used to describe two, three, and higher dimensional systems [5].

Recently, however, it was pointed out that there is a special class of exactly solvable potentials which can be mapped into a harmonic oscillator although with a restriction in some of its parameters. They were called conditionally exactly solvable (CES) potentials. They were discussed independently by Stillinger [6] and de Souza Dutra [7]. Probably due to historical reasons, Stillinger has not perceived that the potentials he was studying were only the first representatives of a broaden class. In fact, two-dimensional examples were recently added to this family [8] and also some other one-dimensional ones are under investigation [9]. Furthermore, a supersymmetric origin for these potentials was discussed in Ref. [10].

What distinguishes the potentials belonging to the CES class, is that it is only possible to get exact solutions when specific conditions are provided. Namely, the exact solvability is only possible when some of the potential parameters hold fixed to a very special value [6–10].

Until 1979, as far as we know, only two classes of quantum-mechanical potentials were known, the exact and the nonexact ones. But in that year, Flessas [11] presented examples of potentials that could be solved analytically only when some constraints among the potential parameters were obeyed. Besides only part of their spectra could be obtained exactly, the remaining should be obtained by standard numerical calculations. This new class of potentials were called quasiexact solvable potentials (QES) or partially algebrized [11–15]. This last name was due to the observed fact that it was only possible to generate algebraically the exact part of its spectrum [13–15].

Now a natural question is that of wondering if there is a dynamical algebra behind the exact solvability of the CES class of potentials. In fact, we are going to show that the algebraic method used before was not quite as general as one could have thought and we will improve this method in order to enlarge as wide as possible the class of dynamically generated so(2,1) potentials. The result is that the method that we are going to obtain can generate both ES, CES, and QES potentials. This is possible through a redefinition of the resolvent operator with an arbitrary scaling and the consequent relaxing of a condition related to the fact that the Hamiltonian of a nonrelativistic system usually does not possess first-order space derivatives.

Furthermore, we hope it is possible to do a systematization of the quest for new CES potentials through the use of dynamical algebras. For this we will improve the method developed in Ref. 4, and also get the Green's functions for these CES potentials.

This work is organized as follows: in the second section we perform the necessary improvements in the approach used in Ref. [4], in order to deal with CES potentials. In the third section we obtain the CES from the so(2,1) Lie algebra in a systematized way. Section four is devoted to obtaining the Green's function of the CES potentials and in the last section we do our final considerations.

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II. THE ALGEBRAIC METHOD AND THE so(2,1) GENERATORS

The imposition of a dynamical algebra on a quantum system, corresponds to saying that its Hamiltonian must be written as a combination of the corresponding generators. In the present case we use the so(2,1) Lie algebra, whose generators satisfy the relations,

$$[T_1, T_2] = -iT_1, \quad [T_2, T_3] = -iT_3, \quad [T_1, T_3] = -iT_2.$$
$$(2.1)$$

Besides we will use a second-order differential operator realization, in such a way that the Hamiltonian shall be constructed as a linear combination of the generators. In general, second-order generators obeying (2.1) can be written [4] as

$$T_1 = \alpha_2\, x^{2-j}\, \frac{d^2}{dx^2} + \alpha_1\, x^{1-j}\, \frac{d}{dx} + \alpha_0\, x^{-j}, \quad (2.2a)$$

$$T_2 = -\frac{i}{j} x\, \frac{d}{dx} - i\,\beta, \quad (2.2b)$$

$$T_3 = \lambda\, x^j, \quad (2.2c)$$

where $\beta = \left(\frac{1}{2j}\right)\left(\frac{\alpha_1}{\alpha_2} + j - 1\right)$ and $\lambda = -(2\alpha_2\, j^2)^{-1}$. Naturally the parameters α_i and j ($\neq 0$), will be chosen conveniently to construct a desired Hamiltonian. In fact a usual Hamiltonian does not possess a first-order space derivative so in practice we will set $\alpha_1 = 0$. However, in the following for generality, we left α_1 free but at the end of the calculations the above restriction will be applied.

Performing a point canonical transformation $x = F(u)$, the generators (2.2) become [4]

$$T_1 = \alpha_2 \frac{F(u)^{2-j}}{\dot F(u)^2}\, \frac{d^2}{du^2}$$
$$+ \frac{F(u)^{1-j}}{\dot F(u)}\left(\alpha_1 - \alpha_2 \frac{\ddot F(u)F(u)}{\dot F(u)^2}\right)\frac{d}{du}$$
$$+ \alpha_0\, F(u)^{-j}, \quad (2.3a)$$

$$T_2 = -\frac{i}{j}\frac{F(u)}{\dot F(u)}\, \frac{d}{du} - i\,\beta, \quad (2.3b)$$

$$T_3 = \lambda\, F(u)^j, \quad (2.3c)$$

where the dot stands for derivatives in variable u. At this point, one can see the appearance of a linear derivative term in the generator T_1 (even when $\alpha_1 = 0$). As in general the Hamiltonian for an one-dimensional Schrödinger equation, does not have such a term, it is necessary to eliminate it. So we started with a system free of first-order space derivatives, implement a canonical transformation which brings up such a term and then we need a second transformation to end up with a Hamiltonian again without first-order space derivatives. For this we begin by remembering that the Green's function obeys

the nonhomogeneous equation:

$$(\mathcal{H} - \mathcal{E})\, G_{\mathcal{E}}(x, x') = \delta(x - x'); \quad (2.4)$$

and consequently, due to the form of the generators T_i, it is necessary to introduce the resolvent operator [4],

$$\lambda = \frac{F(u)^{2-j}}{\dot F(u)^2}\, (\mathcal{H} - \mathcal{E})$$
$$= g_0 + g_1\, T_1 + g_2\, T_2 + g_3\, T_3, \quad (2.5)$$

where g_i ($i = 0, 1, 2, 3$) are arbitrary constants, that will be fixed in each case. From this resolvent operator we can apply the algebraic method and find the corresponding Green's function [4]

$$G_{\mathcal{E}}(u, u') = -j \sum_{\nu'} e^{2\pi i \nu}(4k\lambda)^{\nu+1}$$
$$\times [\dot F(u)\dot F(u')]^{-1/2}[F(u)F(u')]^{\frac{j\nu+1}{2}}$$
$$\times \sum_{n=0}^{\infty} \frac{n!\, L_n^\nu[4k\lambda F(u)^j]\, L_n^\nu[4k\lambda F(u')^j]}{\Gamma(n + \nu + 1)\, [g_0 + k(\nu + 1 + 2n)]}$$
$$\times \exp\{-2k\lambda[F(u)^j + F(u')^j]\}, \quad (2.6)$$

where $L_n^\nu(z)$ are the Laguerre's polynomials [16] and

$$k = \sqrt{\frac{g_1\, g_3}{2}}, \quad \nu = \pm\frac{1}{j}\sqrt{\left(1 - \frac{\alpha_1}{\alpha_2}\right)^2 - \frac{4\alpha_0}{\alpha_2}}. \quad (2.7)$$

Note that the poles of the Green's function determine the energy spectrum through

$$g_0 + k(\nu + 1 + 2n) = 0 \quad (n = 0, 1, 2, \ldots), \quad (2.8)$$

once the energy $\mathcal{E}$ is related to these parameters.

Here, the idea is to redefine the Green's function, and also the resolvent operator, in such a way as to eliminate the first-order derivative term in the generator T_1 which appears in Eq. (2.5). From this point we change the approach presented in Ref. [4]. The easiest way of discarding the linear derivative term in (2.3a) is to choose

$$\alpha_1 = \alpha_2 \frac{\ddot F(u)F(u)}{\dot F(u)^2},$$

but this will restrict undesirably the class of potentials that could be generated by the algebraic method. A simple way to circumvent this restriction is to introduce an arbitrary scaling in the Green's function,

$$G_{\mathcal{E}}(x, x') = e^{h(x) + h(x')}S_{\mathcal{E}}(x, x'), \quad (2.9)$$

so that we can choose $h(x)$ in order that $S_{\mathcal{E}}(x, x')$ obeys an equation where only second-order derivatives appear. Consequently we have that

$$\bar\Lambda = (g_0 + g_1\, T_1 + g_2\, T_2 + g_3\, T_3)\, e^{h(x) + h(x')}$$
$$= e^{h(x) + h(x')}(g_0 + g_1\, \tilde T_1 + g_2\, \tilde T_2 + g_3\, \tilde T_3). \quad (2.10)$$

Under this change the generators become

$$\tilde{T}_1 = a_2 \frac{d^2}{du^2} + [2\, a_2\, \dot{h}(u) + a_1]\frac{d}{du} + A_0, \quad (2.11a)$$

$$\tilde{T}_2 = -\frac{i}{j}\frac{F(u)}{\dot{F}(u)}\frac{d}{du} - i\left(\beta + \frac{1}{j}\frac{F(u)}{\dot{F}(u)}\dot{h}(u)\right), \quad (2.11b)$$

$$\tilde{T}_3 = \lambda\, F(u)^j, \quad (2.11c)$$

where we defined

$$a_2 \equiv \alpha_2\,\frac{F(u)^{2-j}}{\dot{F}(u)^2}, \quad (2.12a)$$

$$a_1 \equiv \frac{F(u)^{1-j}}{\dot{F}(u)}\left(\alpha_1 - \alpha_2\frac{\ddot{F}(u)F(u)}{\dot{F}(u)^2}\right), \quad (2.12a)$$

and

$$A_0 \equiv a_2\,[\ddot{h}(u) + \dot{h}(u)^2] + a_1\,\dot{h}(u) + \alpha_0\, F(u)^j. \quad (2.12b)$$

Finally, we can choose $h(u)$ in order to get a vanishing first-order derivative term in the generator $\tilde{T}_1$, so we get

$$\dot{h}(u) = -\frac{\dot{F}(u)}{2\,\alpha_2\, F(u)}\left(\alpha_1 - \alpha_2\frac{\ddot{F}(u)\,F(u)}{\dot{F}(u)^2}\right), \quad (2.13)$$

whose solution is given by

$$h(u) = \frac{1}{2}\ln\left(\frac{\dot{F}(u)}{F(u)^{\frac{\alpha_1}{\alpha_2}}}\right). \quad (2.14)$$

Substituting this $h(u)$ in Eq. (2.12b) we get

$$A_0 = \frac{a_2}{2}\left[\frac{\dddot{F}(u)}{\dot{F}(u)} - \frac{1}{2}\left(\frac{\ddot{F}(u)}{\dot{F}(u)}\right)^2 - \frac{3\,\alpha_1}{\alpha_2}\frac{\ddot{F}(u)}{F(u)}\right.$$
$$\left. + \frac{\alpha_1}{\alpha_2}\left(1 + \frac{\alpha_1}{2\,\alpha_2}\right)\left(\frac{\dot{F}(u)}{F(u)}\right)^2\right]$$
$$+ \frac{a_1}{2}\left(\frac{\ddot{F}(u)}{\dot{F}(u)} - \frac{\alpha_1}{\alpha_2}\frac{\dot{F}(u)}{F(u)}\right) + \alpha_0\, F(u)^j. \quad (2.15)$$

From now on we will make some particularizations on the generator (2.11a) and the resolvent operator $\bar{\Lambda}$, Eq. (2.10). Without any loss of generality we take $\alpha_1 = g_2 = 0$ and $g_1 = 1$. The first choice reflects the fact that an usual nonrelativistic Hamiltonian of a general form does not possess first-order space derivatives and the second is a normalization constant. In this case we obtain

$$\bar{\Lambda} = e^{h(u)\,+\,h(u')}\left(\frac{F(u)^{2\,-\,j}}{\dot{F}(u)^2}\right)[\mathcal{H} - \mathcal{E}], \quad (2.16)$$

where

$$\mathcal{H} - \mathcal{E} = -\frac{\hbar^2}{2\,\mu}\frac{d^2}{du^2} + \Delta V(u)$$
$$+ \left(\frac{\dot{F}(u)}{F(u)}\right)^2[\alpha_0 + g_0 F(u)^j - g_3\lambda F(u)^{2j}], \quad (2.17)$$

with

$$\Delta V(u) = \frac{\hbar^2}{\mu}\left[-\frac{1}{4}\frac{\dddot{F}(u)}{\dot{F}(u)} + \frac{3}{8}\left(\frac{\ddot{F}(u)}{\dot{F}(u)}\right)^2\right]. \quad (2.18)$$

Now we are prepared to generate the one-dimensional family of so(2,1) potentials including the CES class. This is going to be done in the next section.

III. CES POTENTIALS FROM so(2,1) LIE ALGEBRA

In this section we will obtain a family of one-dimensional so(2,1) potentials, with the method developed above. This family includes the classes of ES, QES, and CES potentials. The first two were extensively studied in the literature [1–4,11–15] and we will concentrate on the CES potentials [6,7] which is our main interest here. So we are going to see how the so(2,1) Lie algebra can generate this new class of potentials. With this in mind, we redefine the arbitrary function $F(u)$ through a translation

$$F(u) = f(u) + a, \quad (3.1)$$

where a is an arbitrary constant. It is easy to see that apart from the kinetic term, the Hamiltonian in Eq. (2.17) has a potential of the form

$$V(u) = \mathcal{E} + \Delta V(u)$$
$$+ \left(\frac{\dot{f}(u)}{f(u) + a}\right)^2\{\alpha_0 + g_0\,[f(u) + a]^j$$
$$+ g_3\,\lambda\,[f(u) + a]^{2j}\}, \quad (3.2)$$

with $\Delta V(u)$ remaining unchanged in its form, because $\dot{F}(u) = \dot{f}(u)$. The above expression for the potential $V(u)$ can accommodate a large family of potentials including ES, QES, and CES. In fact, its form is quite general and for practical purposes we will make some restrictions on it, in order to elucidate its analysis. Remembering that the potentials discussed in Ref. [7] were mapped into a harmonic oscillator, and also that for the harmonic oscillator, the constant j is taken to be equal to 2 [4], we make this choice in Eq. (3.2), obtaining

$$V(u) = \mathcal{E} + \Delta V(u) + \dot{f}(u)^2\left[\alpha + 2\sigma\, f(u)\right.$$
$$\left. + g_3\lambda f(u)^2 + \frac{\alpha_0}{[f(u) + a]^2}\right] \quad (3.3)$$

where we defined $\alpha \equiv g_0 + \lambda\, g_3\, a^2$ and $\sigma \equiv \lambda\, a\, g_3$. Note that with the choice $j = 2$ we can only describe potentials which could be mapped into a harmonic oscillator. This can easily be seen taking $f(u) = u - a$ which turns $V(u)$ into a simple harmonic oscillator plus a centrifugal barrier. Other choices for j would lead to different potentials. For example, taking $j = 1$ and $f(u) = u - a$ in Eq. (3.2) we get the Coulomb potential plus a centrifugal barrier [4].

As we are looking for exact potentials, the potential $V(u)$ in (3.3) cannot be energy dependent, contrary to what happens with the quasiexactly solvable potentials [11–15]. So, at least one of the terms in (3.3) must be a constant. This is necessary to cancel the explicit dependence of $V(u)$ in the energy.

This condition is expressed through the following non-linear differential equations

$$\sigma\, \dot{f}(u)^2\, f(u) = \gamma_1{}^2, \tag{3.4a}$$

$$-\lambda\, g_3\, \dot{f}(u)^2\, f(u)^2 = \gamma_2{}^2, \tag{3.4b}$$

$$\frac{\alpha_0\, \dot{f}(u)^2}{[f(u) + a]^2} = \gamma_3{}^2, \tag{3.4c}$$

$$\frac{3\, \hbar^2}{8\, \mu} \left(\frac{\ddot{f}(u)}{\dot{f}(u)} \right)^2 = \gamma_4{}^2, \tag{3.4d}$$

$$\frac{\hbar^2}{4\, \mu} \frac{\dddot{f}(u)}{\dot{f}(u)} = \gamma_5{}^2, \tag{3.4e}$$

$$\frac{\hbar^2}{\mu} \left[-\frac{1}{4} \frac{\dddot{f}(u)}{\dot{f}(u)} + \frac{3}{8} \left(\frac{\ddot{f}(u)}{\dot{f}(u)} \right)^2 \right] = \gamma_6{}^2. \tag{3.4f}$$

The solution of each one of the above equations will give us a transformation linking the harmonic oscillator with a particular potential. In some cases it will be an exactly solvable, in other ones it will be a CES and may also be a QES one.

The first three equations have the solutions:

$$f(u) = u^{2/3}, \tag{3.5a}$$

$$f(u) = u^{1/2}, \tag{3.5b}$$

$$f(u) = e^{c\, u} - a, \tag{3.5c}$$

where we choose the constants conveniently, without any loss of generality. The third transformation (3.5c), leads us to the Morse potential, as described in Ref. [4]. The other two lead us, respectively, to the two CES studied formerly [6,7]. The first transformation gives the following potential,

$$V^{(2/3)}(u) = -\frac{5\hbar^2}{72\mu} \frac{1}{u^2} + \mathcal{E} + \frac{4}{9} \left[2\sigma + \lambda g_3 u^{\frac{2}{3}} + \frac{\alpha}{u^{\frac{2}{3}}} \right.$$
$$\left. + \frac{\alpha_0}{u^{\frac{2}{3}} \left(u^{\frac{2}{3}} + a \right)^2} \right]. \tag{3.6}$$

So, we must take $\mathcal{E} = -8\sigma/9$, in order to keep $V^{(2/3)}(u)$ energy independent. This general potential is, in fact, not exactly solvable since the presence of the term with coefficient α_0 depends on the energy $\mathcal{E}$. In order to get a complete solution we must take $\alpha_0 = 0$ rendering this QES potential a CES one, since the strength of the centrifugal barrier has a very special value of $-5\hbar^2/72\mu$ [7]. Substituting these choices in Eq. (2.8) we find the energy spectrum ($\nu = \pm 1/2$)

$$\mathcal{E} = \pm\frac{8}{9}\sqrt{\alpha + \sqrt{\frac{g_3}{2}}\left(n + \frac{1}{2}\right)} \quad (n = 0, 1, 2, \ldots), \tag{3.7}$$

in accordance with previous results [6,7]. The second transformation, Eq. (3.5b), gives the potential

$$V^{(1/2)}(u) = -\frac{3\hbar^2}{32\mu}\frac{1}{u^2} + \mathcal{E} + \frac{1}{4}\left[g_3\lambda + \frac{2\sigma}{u^{1/2}} + \frac{\alpha}{u} \right.$$
$$\left. + \frac{\alpha_0}{u(u^{1/2} + a)} \right], \tag{3.8}$$

with the same choices as before. Again the term with α_0 brings this potential a QES one. Taking $\alpha_0 = 0$ it becomes a CES one with the strength of the centrifugal barrier fixed to $-3\hbar^2/32\mu$ [7]. The energy spectrum is given by Eq. (2.8):

$$\alpha + 4\mathcal{E}a + \left(-\frac{2\mathcal{E}}{\lambda}\right)^{1/2}\left(n + \frac{1}{2}\right) = 0 \quad (n = 0, 1, 2, \ldots), \tag{3.9}$$

which is easily shown to be

$$\frac{2\mathcal{E}^3}{\lambda}\left(n + \frac{1}{2}\right)^{1/2} + \alpha\mathcal{E}^2 + \frac{\alpha\sigma^2\mathcal{E}}{2} + \frac{\sigma^4}{16} = 0 \,(n = 0, 1, 2, \ldots), \tag{3.10}$$

as was expected [6,7].

The remaining three conditions (3.4d)–(3.4f) lead to the following transformations:

$$f(u) = ke^{\gamma u} \tag{3.11a}$$

for both (3.4d) and (3.4e), and

$$f(u) = tgh(\gamma u) \tag{3.11b}$$

corresponding to (3.4f). The transformation (3.11a) leads to the following potential ($\alpha_0 = 0$):

$$V_{\mathcal{E}}(u) = \mathcal{E} + \frac{\hbar^2\gamma^2}{8\mu}$$
$$+ k^2\gamma^2 g_3\lambda e^{2\gamma u}[k^2 e^{2\gamma u} + 2kae^{\gamma u} + a^2], \tag{3.12}$$

which is a QES potential since $\mathcal{E} = -\hbar^2\gamma^2/8\mu$ and the constant γ appears in all the potential terms leaving $V_{\mathcal{E}}(u)$ energy dependent. Notice that no choice for the other parameters in (3.12) can remedy this situation.

The last transformation (3.11b) leads also to a QES potential because the same relation between energy and parameters is present here, as can easily be verified with its substitution in the general potential, Eq. (3.2). Many other QES potentials can also be generated by taking for example $j \neq 2$ in (3.2), but we will concentrate on the CES ones since for this class we can find the analytical

solution for the entire spectrum. Let us now discuss the Green's function for the CES found above.

IV. GREEN'S FUNCTION FOR THE CES POTENTIALS

Now, once we have established the one-dimensional so(2,1) CES potentials we are going to present their Green's function. Applying the rescaling (2.9) and the shift (3.1) into Eq. (2.6) we have the Green's function for an arbitrary so(2,1) potential ($\alpha_1 = 0$):

$$G_{\mathcal{E}}(u, u') = -j \sum_{\nu} e^{2\pi i\nu}(4k\lambda)^{\nu+1}\{[f(u) + a][f(u') + a]\}^{\frac{i\nu+1}{2}}$$

$$\times \sum_{n=0}^{\infty} \frac{n! \, L_n^{\nu}\{4k\lambda[f(u) + a]^j\} \, L_n^{\nu}\{4k\lambda[f(u') + a]^j\}}{\Gamma(n + \nu + 1) \, [g_0 + k(\nu + 1 + 2n)]} \exp\left(-2k\lambda\{[f(u) + a]^j + [f(u') + a]^j\}\right). \tag{4.1}$$

This general form can be restricted to the cases discussed in the preceding section. For $V^{(2/3)}(u)$, Eq. (3.6), with $f(u)$ given by (3.5a) we find ($\alpha_0 = 0$; $j = 2$; $k = \sqrt{g_3/2}$)

$$G_{\mathcal{E}}^{(2/3)}(u, u') = -2 \sum_{\nu=-1/2}^{+1/2} e^{2\pi i\nu}(4k\lambda)^{\nu+1}[(u^{2/3} + a)(u'^{2/3} + a)]^{\frac{2\nu+1}{2}}$$

$$\times \sum_{n=0}^{\infty} \frac{n! \, L_n^{\nu}[4k\lambda(u^{2/3} + a)^2] \, L_n^{\nu}[4k\lambda(u'^{2/3} + a)^2]}{\Gamma(n + \nu + 1) \, [\alpha - a\sigma + k(\nu + 1 + 2n)]} \exp\{-2k\lambda[(u^{2/3} + a)^2 + (u'^{2/3} + a)^2]\}. \tag{4.2}$$

From this Green's function it is easy to obtain the wave functions (up to a phase) which satisfy the Schrödinger equation for that potential:

$$\Psi_{n,\nu}^{(2/3)}(u) = \sqrt{\frac{2n!}{\Gamma(n + \nu + 1)}}(4k\lambda)^{\frac{\nu+1}{2}}(u^{2/3} + a)^{\frac{2\nu+1}{2}} L_n^{\nu}[4k\lambda(u^{2/3} + a)^2] \exp\{-2k\lambda(u^{2/3} + a)^2\}. \tag{4.3}$$

Note that n ($= 0, 1, 2, ...$) is the usual principal quantum number and ν ($= \pm 1/2$) defines the parity of the solution. Owing to the relation between Laguerre and Hermite polynomials [16], one can see that this solution coincides with the previous results for this potential appearing in the literature [6,7].

An analogous situation occurs for the potential $V^{(1/2)}(u)$, Eq. (3.8). With the transformation (3.5b) we find its Green's function ($\alpha_0 = 0$, $j = 2, k = \sqrt{g_3/2}$)

$$G_{\mathcal{E}}^{(1/2)}(u, u') = -2 \sum_{\nu=-1/2}^{+1/2} e^{2\pi i\nu}(4k\lambda)^{\nu+1}[(u^{1/2} + a)(u'^{1/2} + a)]^{\frac{2\nu+1}{2}}$$

$$\times \sum_{n=0}^{\infty} \frac{n! \, L_n^{\nu}[4k\lambda(u^{1/2} + a)^2] \, L_n^{\nu}[4k\lambda(u'^{1/2} + a)^2]}{\Gamma(n + \nu + 1) \, [\alpha - a\sigma + k(\nu + 1 + 2n)]} \exp\{-2k\lambda[(u^{1/2} + a)^2 + (u'^{1/2} + a)^2]\},$$

$$\tag{4.4}$$

and the corresponding wave functions

$$\Psi_{n,\nu}^{(1/2)}(u) = \sqrt{\frac{2n!}{\Gamma(n + \nu + 1)}}(4k\lambda)^{\frac{\nu+1}{2}}(u^{1/2} + a)^{\frac{2\nu+1}{2}}$$

$$\times L_n^{\nu}[4k\lambda(u^{1/2} + a)^2]$$

$$\times \exp\{-2k\lambda(u^{1/2} + a)^2\}, \tag{4.5}$$

in accordance, as before, with the literature [6,7]. We limit ourselves in discussing the Green's function for the CES potentials since it is well known that it is meaningless to construct such a function for a QES potential once

only part of its spectrum is exactly solvable. The other exactly solvable potentials were discussed before [4].

V. CONCLUSIONS

In this paper, we have shown that the class of potentials which can be described by the so(2,1) Lie algebra can be enlarged, adding to it the CES potentials to the previously known ES and QES ones. The approach used here led to obtaining the Green's function for these potentials, from which we have found their spectra and wave

functions in accordance with the literature [6,7].

So far, we have only considered one-dimensional systems but the technique employed here can be extended to systems in two, three, or arbitrary dimensions with symmetries described by direct sums of the so(2,1) Lie algebra, as is well known for the ES and QES potentials [4,5,15]. This is presently under investigation, and we hope to report on it in the near future.

As a final comment let us mention that the discussion presented here about the so(2,1) potentials is restricted to the ones for which the Hamiltonian can be written in terms of the second-order generators of so(2,1) Lie algebra. This essentialy takes care of the potentials which can be mapped into a harmonic oscillator. However, another approach based on first-order generators is also known

and in this case the Hamiltonian is related to the Casimir operator of the so(2,1) Lie algebra. This second approach leads to another family of so(2,1) potentials which are deformable into Posch-Teller potentials [17]. At present it is not clear how the discussion of this paper can be applied to this second family of so(2,1) potentials.

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