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Citation: Journal of Mathematical Physics **36**, 307 (1995); doi: 10.1063/1.531294
View online: http://dx.doi.org/10.1063/1.531294
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/36/1?ver=pdfcov
Published by the AIP Publishing

The Korteweg–de Vries hierarchy and long water-waves

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(Received 20 June 1994; accepted for publication 9 September 1994)

By using the multiple scale method with the simultaneous introduction of multiple times, we study the propagation of long surface-waves in a shallow inviscid fluid. As a consequence of the requirements of scale invariance and absence of secular terms in each order of the perturbative expansion, we show that the Korteweg–de Vries hierarchy equations do play a role in the description of such waves. Finally, we show that this procedure of eliminating secularities is closely related to the renormalization technique introduced by Kodama and Taniuti. © *1995 American Institute of Physics.*

I. INTRODUCTION

In 1967, Gardner, Greene, Kruskal, and Miura,[1] by making use of the ideas of direct and inverse scattering, showed that the Korteweg–de Vries (KdV) equation could be solved exactly as an initial value problem. Shortly after, Lax[2] generalized these ideas to a general evolution equation of the type

$$\frac{\partial u}{\partial t}=K[u], \tag{1}$$

with K a nonlinear operator that could be written in the form

$$K[u]=[B,\ L], \tag{2}$$

and with B and L self-adjoint linear operators depending on u. As a consequence of this formalism, Lax showed the existence of an infinite sequence of integrable partial differential equations of the form

$$\frac{\partial u}{\partial t}=K_n[u], \quad n=1,2,..., \tag{3}$$

where

$$K_n=[B_n,\ L] \tag{4}$$

with

$$B_n=D^{2n+1}+\sum_{j=1}^{n}\ (b_jD^{2j+1}+D^{2j+1}b_j) \tag{5}$$

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0022-2488/95/36(1)/307/14/$6.00

and

$$L = D^2 + \tfrac{1}{6}u. \tag{6}$$

In these equations, b_j are coefficients that can be determined,[2] and $D = \partial/\partial x$. For $n=1$, the KdV equation is obtained. For $n \geqslant 2$, we have the higher order KdV equations. This infinite sequence of integrable nonlinear partial differential equations form the so called KdV hierarchy.[2] Along with KdV, they are all integrable by the inverse scattering method, and they have the same integrals of motion as KdV. However, in contrast to KdV, which has been shown to govern the nonlinear long-wave dynamics of general dispersive systems, the physical relevance of the higher order equations of the KdV hierarchy has remained, up to now, obscure.

We will show in this paper that the equations of the KdV hierarchy do appear in the description of physical systems. More specifically, we will first argue that the same amplitude that satisfies the KdV equation, also satisfies the higher order equations of the KdV hierarchy, each one in a different time-scale. These time-scales are defined from the basic time variable t by $\tau_1 = \epsilon^{1/2}t$, $\tau_3 = \epsilon^{3/2}t$, $\tau_5 = \epsilon^{5/2}t,\ldots$, where ϵ is a parameter satisfying $\epsilon \ll 1$. The primary reason for introducing them is that they may be used as a tool to eliminate the secularities coming from the soliton parts that appear in the problem when a perturbative solution for the higher order terms of the amplitude is searched. However, when a function of a single time variable $F(t)$ is extended to a function of several time variables $F(\tau_1, \tau_3, \tau_5, ..)$, the problem of knowing the allowed evolution in each one of the different time-scales is posed.[3] In fact, as we are going to see, these evolutions are not arbitrary, but are determined by a scale invariance requirement which must hold to ensure the ordering of the expansion, and by the compatibility condition

$$\frac{\partial^2 F}{\partial \tau_3 \partial \tau_{2n+1}} = \frac{\partial^2 F}{\partial \tau_{2n+1} \partial \tau_3}.$$

If, for a physical system, the evolution of a certain amplitude is governed by the KdV equation in the time τ_3, we will show that the above compatibility condition constraints the evolution on higher order times of that amplitude to be governed by the higher order equations of the KdV hierarchy, leaving only one free parameter at each order, which is related to the possibility of redefining the time τ_{2n+1} by a multiplicative factor α_{2n+1}. By choosing specific values for the free parameters, the task of secularity elimination can then be accomplished. As a consequence, our construction of a *bona fide* perturbative expansion will provide a link between the equations of the KdV hierarchy and the evolution of a physical quantity. The physical system studied here, that of long surface water-waves, is a kind of classical system where our ideas are well illustrated. However, the results obtained are, to a certain extent, model independent, and in this sense the study made here can be considered as representative of a general physical situation.

The paper is organized as follows. In Sec. II, we obtain the basic equations describing surface water-waves in the multiple time formalism. In Sec. III, the first few evolution equations are obtained. In Sec. IV, by using the multiple time formalism, we examine how the symmetry of the time derivatives determines the evolution of the wave amplitude in any time scale. The use of the KdV hierarchy equations to eliminate the secularities in the evolution of the higher order terms of the wave amplitude is discussed in Sec.V. In Sec. VI, we show with an explicit example that the method of eliminating the soliton related secularities by using these equations can be connected to the results of Kodama and Taniuti,[4] where a renormalization technique was introduced to obtain a secular free perturbative expansion. And finally, in Sec. VII, we summarize and discuss the results obtained.

II. THE MULTIPLE TIME FORMALISM FOR WATER WAVES

We consider a two-dimensional inviscid incompressible fluid in a constant gravitational field. The space coordinates are denoted by (x,z) and the corresponding components of the velocity $\mathbf{v}$ by (u,w). The gravitational acceleration $\mathbf{g}$ is in the negative z direction. The equations describing such a fluid are:[5] the incompressibility equation

$$\nabla\cdot\mathbf{v}=0, \tag{7}$$

and the Euler equation

$$\frac{D\mathbf{v}}{Dt}\equiv\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\frac{1}{\rho}\nabla p+\mathbf{g}, \tag{8}$$

where ρ is the fluid density, p is the presure and $\mathbf{g}=-g\hat{k}$, with $\hat{k}$ an unit vector in the z direction. If we assume the fluid to be irrotational,

$$\nabla\times\mathbf{v}=0, \tag{9}$$

and consequently a velocity potential ϕ can be introduced through

$$\mathbf{v}=\nabla\phi. \tag{10}$$

Using this definition, Eq. (7) becomes

$$\nabla^2\phi=0, \tag{11}$$

while Eq. (8), after an integration, reads

$$\frac{p-p_0}{\rho}=-\frac{\partial}{\partial t}\,\phi-\frac{1}{2}(\nabla\phi)^2-gz, \tag{12}$$

with p_0 an integration constant.

We consider the case of a fluid of height h, limited above by a passive gas exerting a constant pressure p_0 on it, and let the upper surface to be described by

$$z=\zeta(x,t). \tag{13}$$

The kinematic boundary condition at this surface is then written in the form

$$\frac{D\zeta}{Dt}\equiv\zeta_t+\phi_x\zeta_x=\phi_z, \tag{14}$$

with the subscripts denoting partial derivatives. However, there is also a dynamic boundary condition obtained from Eq. (12), which reads

$$\phi_t+\tfrac{1}{2}[(\phi_x)^2+(\phi_z)^2]+g\rho=0, \tag{15}$$

on $z=\zeta(x,t)$. Finally, the lower boundary is supposed to be a rigid horizontal flat botton, localized at $z=-h$. In this case, the corresponding boundary condition implies that the normal velocity of the fluid must vanish:

$$\phi_z=0. \tag{16}$$

We now wish to consider the long-wave in shallow-water approximation to the above equations. This may be done, for instance, by using the the reductive perturbation method of Taniuti,[6] which introduces slow space and time variables. The slow space variable is given by

$$\xi' = \epsilon^{1/2} x, \tag{17}$$

with ϵ a small parameter. Next, we introduce an infinite sequence of slow time variables:[7]

$$\tau_1 = \epsilon^{1/2} t, \quad \tau_3 = \epsilon^{3/2} t, \quad \tau_5 = \epsilon^{5/2} t, \dots . \tag{18}$$

In addition, we expand ζ and ϕ in a suitable power series in the parameter ϵ:

$$\zeta = \epsilon \hat{\zeta} \equiv \epsilon(\zeta_0 + \epsilon \zeta_1 + \epsilon^2 \zeta_2 + \cdots), \tag{19}$$

$$\phi = \epsilon^{1/2} \hat{\phi} \equiv \epsilon^{1/2}(\phi_0 + \epsilon \phi_1 + \epsilon^2 \phi_2 + \cdots). \tag{20}$$

Introducing these expansions and the slow variables into the water wave equations, we obtain

$$\epsilon \hat{\phi}_{\xi'\xi'} + \hat{\phi}_{zz} = 0, \quad -h < z < \epsilon \hat{\zeta}, \tag{21}$$

$$\hat{\phi}_z = 0, \quad z = -h, \tag{22}$$

$$\hat{\phi}_z = \epsilon \hat{\zeta}_{\tau_1} + \epsilon^2 \hat{\zeta}_{\tau_3} + \epsilon^3 \hat{\zeta}_{\tau_5} + \cdots + \epsilon^2 \hat{\phi}_{\xi'} \hat{\zeta}_{\xi'}, \quad z = \epsilon \hat{\zeta}, \tag{23}$$

$$2g\hat{\zeta} + 2\hat{\phi}_{\tau_1} + 2\epsilon \hat{\phi}_{\tau_3} + 2\epsilon^2 \hat{\phi}_{\tau_5} + \cdots + \epsilon \hat{\phi}_{\xi'} \hat{\phi}_{\xi'} + \hat{\phi}_z \hat{\phi}_z = 0, \quad z = \epsilon \hat{\zeta}. \tag{24}$$

These are the basic equations we are going to use to obtain the evolution equations describing the system.

III. THE FIRST EVOLUTION EQUATIONS

Equation (21) for the velocity potential can be solved for the boundary condition (22), independently of Eqs. (23) and (24). At order ϵ^0, the solution is

$$\phi_0 = \mathscr{F}, \tag{25}$$

with $\mathscr{F} = \mathscr{F}(\xi'; \tau_1, \tau_3, \tau_5, \dots)$ an arbitrary function. At order ϵ^1, it is given by

$$\phi_1 = -\left(\frac{z^2}{2} + hz\right) \mathscr{F}_{\xi'\xi'} + \mathscr{G}, \tag{26}$$

with $\mathscr{G} = \mathscr{G}(\xi'; \tau_1, \tau_3, \tau_5, \dots)$ another arbitrary function. At order ϵ^2, we get

$$\phi_2 = -\tfrac{1}{24}(z^4 + 4hz^3 - 8h^3 z) \mathscr{F}_{(4\xi')} - \tfrac{1}{2}(z^2 + 2hz) \mathscr{G}_{\xi'\xi'} + \mathscr{H}, \tag{27}$$

with $\mathscr{H} = \mathscr{H}(\xi'; \tau_1, \tau_3, \tau_5, \dots)$ another arbitrary function. At order ϵ^3, the solution is

$$\begin{aligned} \phi_3 = &-\tfrac{1}{720}(z^6 + 6hz^5 - 40h^3 z^3 + 16h^6) \mathscr{F}_{(6\xi')} \\ &+ \tfrac{1}{24}(z^4 + 4hz^3 - 8h^3 z) \mathscr{G}_{(4\xi')} - \tfrac{1}{2}(z^2 + 2hz) \mathscr{H}_{\xi'\xi'} + \mathscr{I}, \end{aligned} \tag{28}$$

with $\mathscr{I} = \mathscr{I}(\xi'; \tau_1, \tau_3, \tau_5, \dots)$ again an arbitrary function. We could proceed further and calculate $\phi_4, \phi_5, \dots$, but we stop here since this is all we are going to need.

We pass now to Eqs. (23) and (24), and solve the first few orders in our perturbative scheme. At order ϵ^0, Eq. (23) gives nothing, while Eq. (24) reads

$$g\zeta_0+\phi_{0\tau_1}=0. \tag{29}$$

Substituting ϕ_0, we get

$$\zeta_0=-\frac{1}{g}\mathscr{F}_{\tau_1}. \tag{30}$$

At order ϵ^1, Eq. (23) is

$$\mathscr{F}_{\tau_1\tau_1}-gh\mathscr{F}_{\xi'\xi'}=0. \tag{31}$$

Defining

$$gh\equiv c^2, \tag{32}$$

with c a velocity, its solution can be written in the form

$$\mathscr{F}=F(\xi'-c\tau_1)+G(\xi'+c\tau_1),$$

where the sign $-(+)$ refers to a wave moving to the right (left) with a velocity c. For definiteness, let us choose the wave moving to the right, and let us define a new coordinate system by

$$\xi=\xi'-c\tau_1. \tag{33}$$

Therefore,

$$\mathscr{F}_{\tau_1}=-c\mathscr{F}_{\xi}, \tag{34}$$

and from Eq. (30) we get

$$\zeta_0=\frac{h}{c}\mathscr{F}_{\xi}. \tag{35}$$

Consequently,

$$\zeta_{0\tau_1}+c\zeta_{0\xi}=0, \tag{36}$$

which is the usual linear wave equation.

Now, using Eqs. (17) and (18), we can rewrite Eq. (33) in the form:

$$\xi=\epsilon^{1/2}(x-ct), \tag{37}$$

from which we see that ξ is a coordinate system moving with the velocity c of the linear waves, in relation to the laboratory coordinate x. Therefore, from now on, all dependent variables will be assumed to depend on (ξ',τ_1) through ξ only, which automatically implies that they satisfiy Eq. (34).

At order ϵ^1, Eq. (24) gives

$$2g\zeta_1-2c\mathscr{G}_{\xi}+2\mathscr{F}_{\tau_3}+\mathscr{F}_{\xi}\mathscr{F}_{\xi}=0, \tag{38}$$

where we have already used Eq. (34) for $\mathscr{G}$. Derivating in relation to ξ, and using Eq. (35), we obtain

$$c\zeta_{1\xi}-h\mathscr{G}_{\xi\xi}=-\zeta_{0\tau_3}-\frac{c}{h}\zeta_0\zeta_{0\xi}. \tag{39}$$

Now we go to the order ϵ^2. From Eq. (23), we have

$$c\zeta_{1\xi}-h\mathscr{G}_{\xi\xi}=\zeta_{0\tau_3}+2\frac{c}{h}\zeta_0\zeta_{0\xi}+\frac{ch^2}{3}\zeta_{0\xi\xi\xi}. \tag{40}$$

Substituting into Eq. (39), we get

$$\zeta_{0\tau_3}+\frac{3c}{2h}\zeta_0\zeta_{0\xi}+\frac{ch^2}{6}\zeta_{0\xi\xi\xi}=0, \tag{41}$$

which is the KdV equation. It is the first nontrivial equation in the KdV hierarchy. At this point we can see the importance of introducing multiple times: while the linear wave Eq. (36) describes the evolution of ζ_0 in the time τ_1, the KdV equation describes the evolution of ζ_0 in the time τ_3. In other words, the phenomena described by each one of these equations occur in different time scales. And this will be the case for all the higher order equations of the KdV hierarchy. Now, from Eq. (24) at order ϵ^2, we obtain

$$2g\zeta_2+2c^2\zeta_0\zeta_{0\xi\xi}-2c\mathscr{H}_\xi+2\mathscr{G}_{\tau_3}+2\mathscr{F}_{\tau_5}+2\mathscr{F}_\xi\mathscr{G}_\xi+c^2\zeta_{0\xi}\zeta_{0\xi}=0. \tag{42}$$

Derivating in relation to ξ, and using Eq. (35), it becomes

$$c\zeta_{2\xi}-h\mathscr{H}_{\xi\xi}+\frac{h}{c}\mathscr{G}_{\xi\tau_3}+(\zeta_{0\xi}\mathscr{G}_\xi+\zeta_0\mathscr{G}_{\xi\xi})=-\zeta_{0\tau_5}-2ch\zeta_{0\xi}\zeta_{0\xi\xi}-ch\zeta_0\zeta_{0\xi\xi\xi}. \tag{43}$$

We pass now to the order ϵ^3. From Eq.(23), we have

$$c\zeta_{2\xi}-h\mathscr{H}_\xi-(\zeta_0\mathscr{G}_{\xi\xi}+\zeta_{0\xi}\mathscr{G}_\xi)-\frac{c}{h}(\zeta_{0\xi}\zeta_1+\zeta_0\zeta_{1\xi})-\frac{h^3}{3}\mathscr{G}_{(4\xi)}=\zeta_{0\tau_5}+\frac{2}{15}ch^4\zeta_{0(5\xi)}. \tag{44}$$

Equations (43) and (44) can be combined to yield an evolution equation involving $\zeta_{1\tau_3}$, $\zeta_{0\tau_5}$ and $\mathscr{G}_{\xi\tau_3}$:

$$\begin{aligned}&2\zeta_{0\tau_5}+\tfrac{2}{15}ch^4\zeta_{0(5\xi)}+2ch\zeta_{0\xi}\zeta_{0\xi\xi}+ch\zeta_0\zeta_{0\xi\xi\xi}+\zeta_{1\tau_3}\\&\quad+\frac{c}{h}(\zeta_{0\xi}\zeta_1+\zeta_0\zeta_{1\xi})+\frac{h}{c}\mathscr{G}_{\xi\tau_3}+2(\zeta_0\mathscr{G}_{\xi\xi}+\zeta_{0\xi}\mathscr{G}_\xi)+\frac{h^3}{3}\mathscr{G}_{(4\xi)}=0.\end{aligned} \tag{45}$$

Now, making use of Eq. (41) to describe $\mathscr{F}_{\tau_3}$, Eq. (38) can be put in the form

$$\zeta_1-\frac{h}{c}\mathscr{G}_\xi=\frac{1}{4h}\zeta_0^2+\frac{h^2}{6}\zeta_{0\xi\xi}. \tag{46}$$

We can then use this equation to eliminate $\mathscr{G}_\xi$ from Eq. (45). The result is

$$\zeta_{1\tau_3}+\frac{3c}{2h}(\zeta_0\zeta_1)_\xi+\frac{ch^2}{6}\zeta_{1\xi\xi\xi}=S_1(\zeta_0), \tag{47}$$

where

$$S_1(\zeta_0)=-\zeta_{0\tau_5}-\frac{19}{360}ch^4\zeta_{0(5\xi)}-\frac{5}{12}ch\zeta_0\zeta_{0\xi\xi\xi}-\frac{23}{24}ch\zeta_{0\xi}\zeta_{0\xi\xi}+\frac{3c}{8h^2}\zeta_0^2\zeta_{0\xi}. \tag{48}$$

Equation (47), as it stands, can not be viewed as an evolution equation for ζ_1 in the time τ_3 because the nonhomogeneous term involves $\zeta_{0\tau_5}$, and the evolution of ζ_0 in τ_5 is not known up to this point. Moreover, the term proportional to $\zeta_{0(5\xi)}$ in $S_1(\zeta_0)$ is a resonant term when ζ_0 is a solitary–wave solution of the KdV equation. In other words, it is a secular producing term to the solution ζ_1.[4] We will show in the next sections how the evolution of ζ_0 in τ_5 may be determined, and how it may be used to cancel the secular term.

IV. THE SYMMETRY OF TIME DERIVATIVES AND THE KORTEWEG–DE VRIES HIERARCHY

As is widely known,[8] the KdV equation

$$\zeta_t=6\zeta\zeta_x-\zeta_{xxx} \tag{49}$$

is a Hamiltonian system, and can be written in the form

$$\zeta_t=\frac{\partial}{\partial x}\left(\frac{\delta H}{\delta\zeta}\right), \tag{50}$$

with the Hamiltonian H given by

$$H=\int_{-\infty}^{+\infty}\left[\frac{(\zeta_x)^2}{2}+\zeta^3\right]dx. \tag{51}$$

On the other hand, as is also well known, the KdV equation has an infinite sequence of conservation laws,[9,10] with the integrals of motion given by

$$I_n[\zeta]=\int_{-\infty}^{+\infty}\Delta_n\,dx,\quad n=0,1,2,3,\ldots\ , \tag{52}$$

where Δ_n, the conserved density, is a polynomial in ζ,ζ_x,ζ_{xx}, etc. Their explicit form can be obtained through a perturbative scheme developed by Miura, Gardner, and Kruskal,[10] which is based on expansions on a small parameter ϵ. In this method, each integral of motion I_n appears in a different order of the perturbation parameter ϵ. Now, the infinite sequence of equations of the KdV hierarchy can be expressed in terms of these integrals of motion according to[11]

$$\zeta_t=\frac{\partial}{\partial x}\left(\frac{\delta I_n}{\delta\zeta}\right). \tag{53}$$

For $n=1$, the integral I_1 coincides with the Hamiltonian given by Eq. (51), and the KdV equation is obtained, which is the first nontrivial member of the KdV hierarchy. From the explicit form of I_n,[10] it is easy to show that each one of Eqs.(53) is invariant, up to a Galilean term, under the transformations

$$\xi=\epsilon^{1/2}(x-ct), \tag{54}$$

$$\tau_{2n+1}=\epsilon^{n+1/2}t;\quad n=1,2,3,\ldots \tag{55}$$

$$\zeta=\epsilon\zeta_0. \tag{56}$$

In these new coordinates, called slow variables, Eq. (53) is written in the form

$$\zeta_{0\tau_{2n+1}}=\frac{\partial}{\partial\xi}\left(\frac{\delta I_n}{\delta\zeta_0}\right). \tag{57}$$

Notice that an infinite sequence of slow time variables τ_3, τ_5, τ_7, ... was introduced in the above transformations. As a consequence, any dependent variable $F(x;t)$ will be a function of these multiple times:

$$F(x;t)=F(x;\tau_1,\tau_3,\tau_5,...). \tag{58}$$

Therefore, the time derivative of $F(x;t)$ turns out to be

$$\frac{\partial F}{\partial t}=\left(\epsilon^{3/2}\frac{\partial}{\partial\tau_3}+\epsilon^{5/2}\frac{\partial}{\partial\tau_5}+\epsilon^{7/2}\frac{\partial}{\partial\tau_7}+...\right)F. \tag{59}$$

Let us now return to our physical system. As we have already seen, the evolution of ζ_0 in the time τ_3 was found to be the KdV equation

$$\zeta_{0\tau_3}=\alpha_1\zeta_{0\xi\xi\xi}+\beta_1\zeta_0\zeta_{0\xi}, \tag{60}$$

with

$$\alpha_1=-\frac{ch^2}{6},\quad \beta_1=-\frac{3c}{2h}. \tag{61}$$

Let us then examine how the evolution of ζ_0 in times $\tau_5,\tau_7,...$ can be obtained. The first crucial point of this analysis is to note that, for the evolution of ζ_0 in the time τ_3, the two terms in the r.h.s. of Eq.(60) exhaust all possible terms such that, when transforming back from the slow to the laboratory variables, the resulting equation does not depend on ϵ.

In the very same way, the evolution equation of ζ_0 in the time τ_5 must only involve terms presenting the same property as the terms in the KdV equation. In other words, this evolution equation must be formally the same in the slow (ξ,τ_5,ζ_0) as well as in the laboratory coordinates (x,t,ζ), establishing thus a scale invariance requirement. If this were not so, the ordering of the perturbative series would not be ensured. A simple analysis shows that the only possible terms are

$$\zeta_{0\tau_5}=\alpha_2\zeta_{0(5\xi)}+\beta_2\zeta_0\zeta_{0\xi\xi\xi}+(\beta_2+\gamma_2)\zeta_{0\xi}\zeta_{0\xi\xi}+\delta_2\zeta_0^2\zeta_{0\xi}, \tag{62}$$

where $\alpha_2,\beta_2,\gamma_2,\delta_2$ are constants. Now comes the second crucial point: the coefficients $\alpha_2,\beta_2,\gamma_2,\delta_2$ are not completely arbitrary if ζ_0 is to satisfy also the KdV equation in the time τ_3. Instead, they are constrainted by the relations arising when the natural (in the multiple time formalism) compatibility condition

$$(\zeta_{0\tau_3})_{\tau_5}=(\zeta_{0\tau_5})_{\tau_3} \tag{63}$$

is imposed. This condition only makes sense in the multiple time formalism since otherwise it would be redundant and would lead to a trivial identity. A straightforward calculation shows that the relations arising from this condition are

$$\frac{\beta_2}{\alpha_2}=\frac{5}{3}\frac{\beta_1}{\alpha_1},\quad \frac{\gamma_2}{\alpha_2}=\frac{5}{3}\frac{\beta_1}{\alpha_1},\quad \frac{\delta_2}{\alpha_2}=\frac{5}{6}\left(\frac{\beta_1}{\alpha_1}\right)^2. \tag{64}$$

Substituting into Eq.(62) leads to

$$\zeta_{0\tau_5}=\alpha_2\left[\zeta_{0(5\xi)}+\frac{5}{3}\left(\frac{\beta_1}{\alpha_1}\right)\zeta_0\zeta_{0\xi\xi\xi}+\frac{10}{3}\left(\frac{\beta_1}{\alpha_1}\right)\zeta_{0\xi}\zeta_{0\xi\xi}+\frac{5}{6}\left(\frac{\beta_1}{\alpha_1}\right)^2\zeta_0^2\zeta_{0\xi}\right], \tag{65}$$

which is the 5th order equation of the KdV hierarchy. Different choices of the free parameter α_2 correspond to different normalizations of the time τ_5. In particular, for $\alpha_2=6$ it acquires the canonical form[12]

$$\zeta_{0\tau_5}=6\zeta_{0(5\xi)}+10\alpha\zeta_0\zeta_{0\xi\xi\xi}+20\alpha\zeta_{0\xi}\zeta_{0\xi\xi}+5\alpha^2\zeta_0^2\zeta_{0\xi}, \tag{66}$$

with $\alpha=\beta_1/\alpha_1$.

This procedure can be extended to any higher order. In other words, if ζ_0 satisfies the KdV equation in the time τ_3, by making use of the scale invariance described above, we can first select all possible terms appearing in the evolution of ζ_0 in the time τ_{2n+1}. Then, by imposing the general time symmetry condition

$$\frac{\partial^2\zeta_0}{\partial\tau_3\partial\tau_{2n+1}}=\frac{\partial^2\zeta_0}{\partial\tau_{2n+1}\partial\tau_3}, \tag{67}$$

the parameters $\beta_n,\gamma_n,...$ appearing in that equation can be uniquely determined in terms of α_n, α_1, and β_1, and the resulting equation is found to be the $(2n+1)$th order equation of the KdV hierarchy. As before, α_n is left as a free parameter responsible for the arbitrariness in the scale of the time τ_{2n+1}. We do not intend here to give a general proof valid for any order, but one can get easily convinced of the validity of this result by performing an explicity calculation for the first few equations.

Before proceeding further, let us remark that one could *a priori* envisage, in place of Eq. (62) a new one including also a dependence on ζ_1, and still keep the agreement with the scale invariance requirement. However, when introducing this back into the physical Eq. (60), the compatibility condition (63) makes all the coefficients of these supplementary terms vanish.

V. HIGHER ORDER EVOLUTION EQUATIONS: THE ELIMINATION OF THE SECULAR PRODUCING TERMS

Let us now return to the evolution equation for ζ_1 in the time τ_3, given by Eqs.(47)–(48). As we have seen, there were two problems related to it: the evolution of ζ_0 in τ_5, and the secular producing term. The last section gave the clue to the first problem. As we have seen, the evolution of ζ_0 in τ_5 can not be chosen arbitrarily, but it is restricted to the KdV hierarchy equations. We show now how the soliton related secular producing term can be eliminated. A comparison between Eq. (48) for $S_1(\zeta_0)$ and the 5th order KdV Eq. (65), immediately shows that the choice

$$\alpha_2=-\tfrac{19}{360}ch^4 \tag{68}$$

has the property of eliminating the terms

$$\zeta_{0\tau_5}-\tfrac{19}{360}ch^4\zeta_{0(5\xi)} \tag{69}$$

from the nonhomogeneous term $S_1(\zeta_0)$. Consequently, $S_1(\zeta_0)$ acquires the form

$$S_1(\zeta_0)=\frac{9}{24}ch\zeta_0\zeta_{0\xi\xi\xi}+\frac{5}{8}ch\zeta_{0\xi}\zeta_{0\xi\xi}+\frac{63}{16}\frac{c}{h^2}\zeta_0^2\zeta_{0\xi}, \tag{70}$$

and we see that it does not present a secular producing term anymore. The secular–free solution ζ_1 can then be found by solving the linear equation (47) with $S_1(\zeta_0)$ given by Eq. (70).

In the next orders,[13] the elimination of the secular terms becomes more and more involved, but the general procedure is basically the same. For example, in the next order we obtain for ζ_2 in the time τ_3 a linear evolution equation of the form:

$$\zeta_{2\tau_3}+\frac{3c}{2h}(\zeta_0\zeta_2)_\xi+\frac{ch^2}{6}\zeta_{2\xi\xi\xi}=S_2(\zeta_0,\zeta_1), \tag{71}$$

where

$$S_2(\zeta_0,\zeta_1)=-\zeta_{0\tau_7}-\zeta_{1\tau_5}-s_2^{(1)}\zeta_{0(7\xi)}-s_2^{(2)}\zeta_{1(5\xi)}-\cdots, \tag{72}$$

with $s_2^{(1)}$ and $s_2^{(2)}$ known constants. All the remaining terms of $S_2(\zeta_0,\zeta_1)$ are nonlinear terms involving ζ_0 and ζ_1. At this order, there appears two terms that will give rise to secularities, namely, $\zeta_{0(7\xi)}$ and $\zeta_{1(5\xi)}$. These secularities come from the solitonic nature assumed for ζ_0, and from the solution of the nonhomogeneous linear equation for ζ_1 in the time τ_3, which can be found in Ref. 4. For the elimination of these secularities, the first step is to determine the evolution of ζ_1 in the time τ_5. By making use of the already discussed scale invariance between slow and laboratory variables, we are able to write all the terms to possibly appear in this equation. A simple analysis shows that they are

$$\zeta_{1\tau_5}-a_2[\zeta_{1(5\xi)}+b_2\zeta_1\zeta_{1\xi}+c_2(\zeta_0\zeta_{1\xi})_\xi+d_2(\zeta_0\zeta_{1\xi\xi\xi}+\zeta_1\zeta_{0\xi\xi\xi})+e_2(\zeta_0^2\zeta_1)_\xi]=T_2(\zeta_0), \tag{73}$$

where

$$\begin{aligned}T_2(\zeta_0)=&f_2\zeta_{0(7\xi)}+g_2\zeta_{0\xi\xi}\zeta_{0\xi\xi\xi}+h_2\zeta_{0\xi}\zeta_{0(4\xi)}+i_2\zeta_0\zeta_{0(5\xi)}+j_2\zeta_0^2\zeta_{0\xi\xi\xi}\\&+k_2\zeta_0\zeta_{0\xi\xi}+l_2(\zeta_{0\xi})^3+m_2\zeta_0^3\zeta_{0\xi}\end{aligned} \tag{74}$$

with $a_2,b_2,\ldots,m_2$ constants to be determined. Then, since we already know the equation for $\zeta_{1\tau_3}$, we can use the symmetry condition

$$(\zeta_{1\tau_3})_{\tau_5}=(\zeta_{1\tau_5})_{\tau_3} \tag{75}$$

to obtain all the constants appearing in the evolution equation for ζ_1 in the time τ_5. This is a laborious step, and the result is

$$a_2=-\frac{19ch^4}{360},\quad b_2=0,\quad c_2=\frac{10}{3}\alpha,\quad d_2=\frac{5}{3}\alpha,\quad e_2=\frac{5}{6}\alpha^2, \tag{76}$$

where $\alpha=\beta_1/\alpha_1$, which shows that the l.h.s. of Eq. (73) is the linearized 5th order KdV hierarchy equation. The remaining constants are given by

$$g_2=\frac{247}{288}ch^3+\frac{105}{h^3}f_2,\quad h_2=\frac{19}{36}ch^3+\frac{63}{h^3}f_2,\quad i_2=\frac{19}{96}ch^3+\frac{21}{h^3}f_2,\quad j_2=\frac{57}{16}c+\frac{315}{2h^6}f_2$$

$$k_2=\frac{437}{32}c+\frac{630}{h^6}f_2,\quad l_2=\frac{209}{64}c+\frac{315}{2h^6}f_2,\quad m_2=\frac{285\,c}{16\,h^3}+\frac{945}{h^9}f_2$$

with f_2 a free parameter. Notice that the nature of f_2 is completely different from the free parameter α_2 which arose from the symmetry condition (63). While the latter is related to different normalizations of τ_5, the former expresses the relevance of the term $\zeta_{0(7\xi)}$ in the equation for

$\zeta_{1\tau_5}$. The importance of this fact can be seen in the following way. Considering ζ_0 as a pure one-soliton solution, we see that $\zeta_{0(7\xi)}$ will be a secular term for the evolution of ζ_1 in the time τ_5. Therefore, if we want to make sense of Eqs. (73)–(74) as a physical equation, we can choose $f_2=0$ so to eliminate that secularity. Then, substituting this equation into Eq. (72), $S_2(\zeta_0,\zeta_1)$ acquires the form

$$S_2(\zeta_0,\zeta_1)=-\zeta_{0\tau_7}-s_2^{(1)}\zeta_{0(7\xi)}-\tilde{s}_2^{(2)}\zeta_{1(5\xi)}-\cdots \tag{77}$$

Note that the coefficient of $\zeta_{1(5\xi)}$ has been modified due to the contribution coming from the substitution of $\zeta_{1\tau_5}$. Now, for $\zeta_0\sim\text{sech}^2\,\eta$, both terms $\zeta_{0(7\xi)}$ and $\zeta_{1(5\xi)}$ will contribute with secular terms of the same kind to $S_2(\zeta_0,\zeta_1)$. The sum of these secular contributions can be written in the form[4]

$$s_2^{(3)}\ \text{sech}^2\ \eta\ \text{tgh}\ \eta,$$

with $s_2^{(3)}$ a known constant. The next step is to remember that, according to the discussion presented in Sec. IV, the evolution of ζ_0 in the time τ_7 is given by the 7th order equation of the KdV hierarchy, which is of the form

$$\zeta_{0\tau_7}=\alpha_3[\zeta_{0(7\xi)}+\cdots], \tag{78}$$

with α_3 a free parameter. Since $\zeta_{0(7\xi)}$ contains a term proportional to $\text{sech}^2\,\eta\ \text{tgh}\,\eta$, a properly chosen α_3 turns possible the use of this equation to eliminate all the secular terms from $S_2(\zeta_0,\zeta_1)$. In other words, the 7th order equation of the KdV hierarchy is able to eliminate all the secular terms at this order. As the resonant terms at any order of the perturbative scheme are always the linear terms of S_n,[4] the elimination of the secular producing terms by choosing appropriate values for the free parameters in the higher order KdV equations remains possible at any higher order.

VI. RENORMALIZATION OF THE SOLITON VELOCITY

We have already assumed, in the context of the multiple time formalism, that

$$\zeta_0=\zeta_0(\xi;\tau_3,\tau_5,\tau_7,...), \tag{79}$$

where

$$\xi=\xi'-c\tau_1, \tag{80}$$

so that ζ_0 automatically satisfies the linear wave equation

$$\zeta_{0\tau_1}+c\zeta_{0\xi}=0. \tag{81}$$

Our concern now will be the solutions to the higher order equations of the KdV hierarchy. Hereafter, for simplicity, we will assume that c and h can be set equal to unity, so that the KdV equation reads

$$\zeta_{0\tau_3}=\alpha_1\zeta_{0\xi\xi\xi}+\beta_1\zeta_0\zeta_{0\xi}, \tag{82}$$

with

$$\alpha_1=-\tfrac{1}{6},\quad \beta_1=-\tfrac{3}{2}. \tag{83}$$

We now look for a traveling wave solution ζ_0 that satisfies this equation in the time τ_3, but that satifies also the higher order equations of the KdV hierarchy in the times $\tau_5, \tau_7, \ldots$. This *multi-solution* can be written in the form

$$\zeta_0 = \frac{k^2}{3} \operatorname{sech}^2(k\Lambda) \tag{84}$$

with the argument Λ given by

$$\Lambda = \xi - \frac{k^2}{A_1}\tau_3 - \frac{k^4}{A_2}\tau_5 - \frac{k^6}{A_3}\tau_7 - \cdots, \tag{85}$$

and with $A_1, A_2, A_3, \ldots$ parameters depending respectively on the constants α_1, α_2, $\alpha_3, \ldots$. This argument is to be interpreted in the following way: when the evolution equation under consideration is concerned to the time τ_{2n+1}, the relevant argument is

$$\Lambda = \xi - \sum_{i=1}^{n} \frac{k^{2i}}{A_i}\tau_{2i+1} - \theta_n,$$

with θ_n a phase involving all the remaining terms of the sum. Then, for the KdV Eq. (82), $A_1 = (1/\alpha_1) = 6$, and the solution is

$$\zeta_0 = \frac{k^2}{3} \operatorname{sech}^2\left[k\left(\xi - \frac{k^2}{6}\tau_3 - \theta_3(k;\tau_5,\ldots) \right) \right], \tag{86}$$

where θ_3 is a phase. On the other hand, the traveling wave solution satisfying simultaneously the KdV equation (82) in the time τ_3, and the 5th order KdV equation (65) in the time τ_5, is given by

$$\zeta_0 = \frac{k^2}{3} \operatorname{sech}^2\left[k\left(\xi - \frac{k^2}{6}\tau_3 - \frac{k^4}{A_2}\tau_5 - \theta_5(k;\tau_7,\ldots) \right) \right], \tag{87}$$

and so on, to any higher order.

Once we have found the solutions to any member of the KdV hierarchy, let us return to the equation for ζ_1 in the time τ_3, which is now $(c=h=1)$ written in the form

$$\zeta_{1\tau_3} + \tfrac{3}{2}(\zeta_0\zeta_1)_\xi + \tfrac{1}{6}\zeta_{1\xi\xi\xi} = S_1(\zeta_0), \tag{88}$$

with

$$S_1(\zeta_0) = \tfrac{9}{24}\zeta_0\zeta_{0\xi\xi\xi} + \tfrac{5}{8}\zeta_{0\xi}\zeta_{0\xi\xi} + \tfrac{63}{16}\zeta_0^2\zeta_{0\xi}. \tag{89}$$

It is important to remember that the linear term of $S_1(\zeta_0)$, which is the secular producing term in the equation for ζ_1, was eliminated through the use of the 5th order KdV Eq. (65), with α_2 given by Eq. (68). Therefore, the ζ_0 appearing in Eqs. (88)–(89) is that given by Eq. (87), since it must be a solution of KdV as well as of the KdV hierarchy 5th order Eq. (65).

Finally, let us take the solutions ζ_0, and let us make the transformation back from the slow (ξ, τ, ζ_0) to the laboratory coordinates (x, t, ζ). For the case of the solution (86) to the KdV equation, we get

$$\zeta = \frac{k^2}{3}\epsilon \operatorname{sech}^2[k\epsilon^{1/2}(x - V_3 t)], \tag{90}$$

where

$$V_3 = c + \epsilon \frac{k^2}{6} \tag{91}$$

is the solitary-wave velocity in the laboratory coordinates. For the case of the solution (87) to the 5th order KdV equation, we get

$$\zeta = \frac{k^2}{3} \epsilon \operatorname{sech}^2[k\epsilon^{1/2}(x - V_5 t)], \tag{92}$$

where now

$$V_5 = V_3 + \epsilon^2 \frac{k^4}{A_2}. \tag{93}$$

Since A_2 depends on the parameter α_2, to choose α_2 means to define the velocity renormalization itself. In this sense we can say that there is a unique velocity renormalization leading to a secular–free perturbation scheme for ζ_1. From these properties, we can see now that this method is equivalent to the renormalization technique developed by Kodama and Taniuti,[4] in which the secular-free higher order effects were also given by the renormalization of the KdV soliton velocity. Moreover, in the same way as in the method of Kodama and Taniuti, if higher order scales are introduced, it is possible to continue the secular-free perturbation to higher orders by using the higher order equations of the KdV hierarchy. And in general, depending on the order considered in the perturbative scheme, the renormalized solitary-wave velocity in the laboratory coordinates is given by

$$V_{2n+1} = c + \sum_{i=1}^{n} \epsilon^i \frac{k^{2i}}{A_i}. \tag{94}$$

The analysis for multi–soliton solutions proceeds in a similar fashion as well.

VII. FINAL REMARKS

By using the reductive perturbation method of Taniuti,[6] with the introduction of an infinite sequence of slow time variables $\tau_1, \tau_3, \tau_5, \ldots$, we studied the propagation of long surface waves in a shallow inviscid fluid. The three main ingredients of our analysis were: (i) the scale-invariance argument, which restricts the possible forms of the evolution equations, and which is necessary for the coherence of the perturbative expansion; (ii) the compatibility condition (67), which appears when a function of a single time–variable is extended to a multiple time–variable; (iii) the secularity elimination procedure, without which the perturbative expansion would be meaningless. By using (i) and (ii) we have shown that, if the amplitude ζ_0 satisfies the KdV equation, it also satisfies all the higher–order equations of the KdV hierarchy. Then, by using the 5th order equation of the KdV hierarchy with a properly chosen α_2, we have shown that the soliton related secular producing terms of the equation for ζ_1 in the time τ_3 could be eliminated. As higher orders are reached, the number of resonant terms increases. Despite of this, we have shown that a similar procedure can be used to remove all these secularities up to any higher order. As the secularity elimination is mandatory for a physical theory, being essencially a finiteness requirement, the results obtained in this paper allowed us to give a physical meaning to the KdV hierarchy equations. Thereafter, by considering a solitary wave solution, we have shown that the elimination of secularities through the use of higher order KdV equations corresponds, in the laboratory coordinates, to a renormalization of the soliton velocities, as obtained previously by Kodama and Taniuti.[4]

The study of long-waves in shallow water is representative of a wide class, that of the weak nonlinear dispersive systems, where the KdV equation has a kind of universal character. In this sense, we can say that the results of the present paper do not depend on the specific physical system under consideration, or, which is the same, on the specific form of the basic equations, except for the values of the coefficients in the perturbative expansion. It is, therefore, legitimate to conjecture that they might be extended to the above mentioned larger class of systems.

ACKNOWLEDGMENTS

The authors are grateful to J. Léon and J. C. Montero for useful discussions. Two of the authors (R.A.K. and J.G.P.) would like to thank CNPq, Brazil, for partial support. R.A.K. also thanks CAPES, Brazil, and M.A.M. thanks FAPESP, Brazil for financial support.

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[13] We thank A. Degasperis and P. M. Santini for useful comments on this point.