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The Reformulation of Nonlinear Complementarity Problems using the Fischer-Burmeister Function

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Abstract—A bounded-level-set result for a reformulation of the box-constrained variational inequality problem proposed recently by Facchinei, Fischer and Kanzow is proved. An application of this result to the (unbounded) nonlinear complementarity problem is suggested. © 1999 Elsevier Science Ltd. All rights reserved.

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1. MAIN RESULTS

Given $F : \mathbb{R}^n \to \mathbb{R}^n$, $F \in C^1(\mathbb{R}^n)$, the Box-Constrained Variational Inequality Problem (BVIP) consists on finding $x \in \Omega$ such that

$$\langle F(x), z - x\rangle \geq 0, \qquad \text{for all } z \in \Omega, \tag{1}$$

where Ω is the compact box

$$\Omega = \{x \in \mathbb{R}^n \mid \ell \leq x \leq r\}. \tag{2}$$

The Nonlinear Complementarity Problem (NCP) is problem (1) when $\Omega = \{x \in \mathbb{R}^n \mid x \geq 0\}$.

Facchinei, Fischer and Kanzow [1] proposed a reformulation of the BVIP as the following optimization problem:

$$\text{Minimize } \|F(x) - u + v\|^2 + \sum_{i=1}^{n} \varphi(u_i, x_i - \ell_i)^2 + \sum_{i=1}^{n} \varphi(v_i, r_i - x_i)^2, \tag{3}$$

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where $\|\cdot\|$ is the Euclidian norm and φ is the Fischer-Burmeister function

$$\varphi(a,b) = \sqrt{a^2+b^2} - a - b, \qquad \text{for all } a,b \in \mathbb{R}.$$

The importance of the Fischer-Burmeister reformulation of complementarity problems lies on the fact that the original problem is reduced to a simple problem for which many effective techniques exist. (Of course, this characteristic is shared by other reformulations that have been proposed in recent literature.) If the number of variables is large and the Jacobian of F is not very sparse, it is interesting to use matrix-free algorithms to solve (3). It is worth mentioning that interior-point techniques that do not rely on reformulations (see [2,3] and references therein) seem to be very efficient for solving complementarity problems, at least when handling (sparse) factorizations of matrices is possible.

From now on we call $f(x,u,v)$ the objective function of (3). It is easy to see that $f(x^*,u^*,v^*)=0$ if and only if, x^* is a solution of the BVIP. In [1] it was proved that if (x^*,u^*,v^*) is a stationary point of f and $F'(x^*)$ is a P_0-matrix it necessarily holds that $f(x^*,u^*,v^*)= 0$. The first result of this note will be to prove that, below a critical value, the level sets of f are bounded.

THEOREM 1. *Assume that* $-\infty < \ell_i < r_i < \infty$, *for all* $i = 1,\dots,n$ *and*

$$\alpha < \frac{(r_i - \ell_i)}{\sqrt{2}}, \qquad \text{for all } i = 1,\dots,n.$$

Then the set $S \equiv \{(x,u,v) \in \mathbb{R}^{3n} \mid f(x,u,v) \le \alpha^2\}$ *is bounded.*

PROOF. Let (x,u,v) be such that $f(x,u,v) \le \alpha^2$. Suppose that $x_i - \ell_i < -\alpha$. Then

$$\alpha \le \sqrt{u_i^2+\alpha^2} + \alpha - u_i < \sqrt{u_i^2 + (x_i-\ell_i)^2} - (x_i - \ell_i) - u_i = \varphi(u_i, x_i - \ell_i).$$

This implies that $f(x,u,v) > \alpha^2$. Therefore, if $(x,u,v) \in S$ we necessarily have that

$$x_i \ge \ell_i - \alpha, \qquad \text{for all } i = 1,\dots,n. \tag{4}$$

The same reasoning allows us to prove that

$$u_i \ge -\alpha, \quad x_i \le r_i - \alpha, \quad \text{and} \quad v_i \ge -\alpha, \tag{5}$$

for all $i = 1,\dots,n$.

Suppose now that S is unbounded. Therefore S contains an unbounded sequence (x^k,u^k,v^k). By (4),(5) this implies that there exists $i \in \{1,\dots,n\}$ such that $u_i^k \to \infty$ or there exists $i \in \{1,\dots,n\}$ such that $v_i^k \to \infty$. Consider the first case. We have that

$$\alpha^2 \ge f(x,u,v) \ge \left(\left[F\left(x^k\right)\right]_i - u_i^k + v_i^k\right)^2.$$

So,

$$\alpha \ge \left|\left[F\left(x^k\right)\right]_i - u_i^k + v_i^k\right| \ge \left|u_i^k\right| - \left|v_i^k\right| - \left|\left[F\left(x^k\right)\right]_i\right|.$$

Since $|[F(x^k)]_i|$ is bounded, this implies that $v_i^k \to \infty$. Analogously, if we assume $v_i^k \to \infty$ we necessarily obtain that $u_i^k \to \infty$, too. So, we can assume that there exists $i \in \{1,\dots,n\}$ such that both $u_i^k \to \infty$ and $v_i^k \to \infty$. Taking an appropriate subsequence, assume, without loss of generality, that $\{x_i^k\}$ is convergent, say, $x_i^k \to x_i^*$. Therefore,

$$\lim_{k\to\infty} \sqrt{\left(u_i^k\right)^2 + \left(x_i^k - \ell_i\right)^2} - u_i^k$$

$$= \lim_{k\to\infty} \frac{\left[\sqrt{\left(u_i^k\right)^2 + \left(x_i^k - \ell_i\right)^2} - u_i^k\right]\left[\sqrt{\left(u_i^k\right)^2 + \left(x_i^k - \ell_i\right)^2} + u_i^k\right]}{\left[\sqrt{\left(u_i^k\right)^2 + \left(x_i^k - \ell_i\right)^2} + u_i^k\right]}$$

$$= \lim_{k\to\infty} \frac{\left(u_i^k\right)^2 + \left(x_i^k - \ell_i\right)^2 - \left(u_i^k\right)^2}{\left[\sqrt{\left(u_i^k\right)^2 + \left(x_i^k - \ell_i\right)^2} + u_i^k\right]} = 0.$$

So,

$$\lim_{k\to\infty} \sqrt{\left(u_i^k\right)^2 + \left(x_i^k - \ell_i\right)^2} - \left(x_i^k - \ell_i\right) - u_i^k = \ell_i - x_i^*. \tag{6}$$

Analogously,

$$\lim_{k\to\infty} \sqrt{\left(v_i^k\right)^2 + \left(r_i - x_i^k\right)^2} - \left(r_i - x_i^k\right) - v_i^k = x_i^* - r_i. \tag{7}$$

Now,

$$\alpha^2 \geq f\left(x^k, u^k, v^k\right) \geq \varphi\left(u_i^k, x_i^k - \ell_i\right)^2 + \varphi\left(v_i^k, r_i - x_i^k\right)^2$$

$$= \left[\sqrt{\left(u_i^k\right)^2 + \left(x_i^k - \ell_i\right)^2} - \left(x_i^k - \ell_i\right) - u_i^k\right]^2 + \left[\sqrt{\left(v_i^k\right)^2 + \left(r_i - x_i^k\right)^2} - \left(r_i - x_i^k\right) - v_i^k\right]^2.$$

Therefore, by (6),(7),

$$\alpha^2 \geq \left(x_i^* - \ell_i\right)^2 + \left(x_i^* - r_i\right)^2. \tag{8}$$

But the minimum value of the right-hand side of (8) is $(r_i - \ell_i)^2/2$. So, by the definition of α, we arrived to a contradiction. ∎

Counterexample

In this counterexample, we show that the previous result is sharp. That is to say, the level set defined by $f(x,u,v) \leq \beta^2$ can be unbounded, where

$$\beta = \min\left\{\frac{(r_i - \ell_i)}{\sqrt{2}},\ i = 1,\dots,n\right\}.$$

Define $F(x) = \varepsilon x$ ($\varepsilon \geq 0$) and $\Omega = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$. So,

$$f(x,u,v) = (\varepsilon x - u + v)^2 + \varphi(u,x)^2 + \varphi(v, 1-x)^2.$$

The sequence $\{y^k \equiv (1/2, k, k),\ k = 0,1,2,\dots\}$ is unbounded. However,

$$f\left(y^k\right) = \left(\frac{\varepsilon}{2} - k + k\right)^2 + 2\varphi\left(k, \frac{1}{2}\right)^2$$

$$= \frac{\varepsilon^2}{4} + 2\left(\sqrt{\frac{1}{4} + k^2} - \frac{1}{2} - k\right)^2 \leq \frac{\varepsilon^2}{4} + \frac{1}{2} = \frac{\varepsilon^2}{4} + \beta^2.$$

2. APPLICATION TO THE NCP

The Fischer-Burmeister function has been applied by many authors to the nonlinear complementarity problem NCP by means of the reformulation

$$\text{Minimize} \sum_{i=1}^{m} \varphi(x_i, [F(x)]_i)^2. \tag{9}$$

See the references of [1]. If x^* is a stationary point of (9) and $F'(x^*)$ is a P_0-matrix, it can be ensured that x^* is a solution of the NCP. If F is a uniform P-function it can also be proved that the objective function of (9) has bounded level sets, but this property could not hold under weaker assumptions. In fact, consider the NCP defined by $F(x) = 1 - e^{-x}$. This function is strictly monotone and 0 is a solution. However, the level sets of the function (9) are not bounded. In fact, for all $x > 0$,

$$\varphi(x, F(x))^2 = \left(\sqrt{x^2 + (1 - e^{-x})^2} - (x + 1 - e^{-x})\right)^2 \leq 1.$$

So, it is natural to ask whether the bounded-level-set result proved in the previous section can help to establish bounded-level-set reformulations of the NCP using the Fischer-Burmeister function.

Let us define $\underline{L} = (L, \ldots, L) \in \mathbb{R}^n$ and consider the box

$$\Omega_L = \{x \in \mathbb{R}^n \mid 0 \leq x \leq \underline{L}\}.$$

The NCP is naturally connected with the BVIP defined by F and Ω_L. In the Facchinei-Fischer-Kanzow reformulation, the corresponding objective function is

$$f(x, u, v) = \|F(x) - u + v\|^2 + \sum_{i=1}^{n} \varphi(u_i, x_i)^2 + \sum_{i=1}^{n} \varphi(v_i, L - x_i)^2. \tag{10}$$

Clearly, $f(0,0,0) = \|F(0)\|^2$. Therefore, if we take $L > \|F(0)\|^2/2$, Theorem 1 guarantees that

$$\{x \in \mathbb{R}^n \mid f(x,u,v) \leq f(0,0,0)\} \text{ is bounded.}$$

This implies that standard unconstrained minimization algorithms, which usually generate sequences satisfying $f(x^{k+1}, u^{k+1}, v^{k+1}) \leq f(x^k, u^k, v^k)$ for all k will generate bounded sequences, if $(x^0, u^0, v^0) = (0,0,0)$. As a consequence, algorithms of that class will find stationary points, which, under the assumptions of [1], will be solutions of the BVIP defined by F and Ω_L. So, in order to solve the NCP we only need to guarantee that solutions of this BVIP are solutions of the NCP. An answer to this question is given in the following theorem.

THEOREM 2. *Assume that there exists a solution of the NCP that belongs to Ω_L and that, for all $x, y \in \mathbb{R}^n$,*

$$([F(x)]_i - [F(y)]_i)(x_i - y_i) \leq 0, \qquad \text{for all } i = 1, \ldots, n$$

implies that

$$([F(x)]_i - [F(y)]_i)(x_i - y_i) = 0, \qquad \text{for all } i = 1, \ldots, n.$$

Then, every stationary point of f (defined by (10)) is a solution of the NCP.

PROOF. Let x^* be a stationary point of f. The condition imposed to F in the hypothesis implies that

$$\max_{i: x_i \neq y_i} \{(F_i(x) - F_i(y))(x_i - y_i)\} \geq 0.$$

Therefore, by Theorem 5.8 of [3], F is a is P_0-function and $F'(x)$ is a P_0-matrix for all $x \in \mathbb{R}^n$. In particular, $F'(x^*)$ is a P_0-matrix. So, by [1], x^* is a solution of the BVIP defined by Ω_L. That is

$$[F(x^*)]_i \geq 0, \qquad \text{if } x_i^* = 0, \tag{11}$$

$$[F(x^*)]_i = 0, \qquad \text{if } 0 < x_i^* < L, \tag{12}$$

and

$$[F(x^*)]_i \leq 0, \qquad \text{if } x_i^* = L. \tag{13}$$

Assume that $\bar{x}$ is a solution of the NCP. This implies that

$$[F(\bar{x})]_i \geq 0, \quad \text{if } \bar{x}_i = 0 \qquad \text{and} \qquad [F(\bar{x})]_i = 0, \quad \text{if } \bar{x}_i > 0. \tag{14}$$

By (11)–(14) we have that

$$([F(x^*)]_i - [F(\bar{x})]_i)(x_i^* - \bar{x}_i) \leq 0, \tag{15}$$

for all $i = 1, \ldots, n$.

Let us define

$$\mathcal{I} = \{i \in \{1, \ldots, n\} \, [F(x^*)]_i < 0, \; x_i^* = L\}. \tag{16}$$

Assume, by contradiction, that $\mathcal{I} \neq \emptyset$. Then there exists $j \in \{1, \ldots, n\}$ such that $[F(x^*)]_j < 0$ and $x_j^* = L$. Therefore,

$$\begin{aligned} 0 > [F(x^*)]_j x_j^* &\geq [F(x^*)]_j x_j^* - [F(x^*)]_j \bar{x}_j - [F(\bar{x})]_j x_j^* + [F(\bar{x})]_j \bar{x}_j \\ &= \left([F(x^*)]_j - [F(\bar{x})]_j\right)(x_j^* - \bar{x}_j). \end{aligned} \tag{17}$$

But (15) and (17) contradict the hypothesis of the theorem.

Therefore, $\mathcal{I} = \emptyset$. Since x^* is a solution of the BVIP, this implies that x^* is a solution of the NCP, as we wanted to prove. ∎

REMARKS. In the linear case ($F(x) = Mx + q$) the hypothesis of Theorem 2 means that the matrix M is column-sufficient. If F is monotone ($\langle F(x) - F(y), \; x - y\rangle \geq 0$ for all $x, y \in \mathbb{R}^n$) or, even, if F is a P-function ($\max_{1 \leq i \leq n}\{(F_i(x) - F_i(y))(x_i - y_i)\} > 0$) this hypothesis holds, but the reciprocal is not true. For example, the matrix:

$$M = \begin{bmatrix} 0 & -3 \\ 0 & 1 \end{bmatrix}$$

is column-sufficient, but not positive semidefinite, therefore, F is not monotone. Moreover, M is not a P-matrix either, so F is not a P-function.

Finally, let us show that the hypothesis of this theorem is sharp and cannot be relaxed to, say, P_0-function. In fact, consider the following matrix

$$M = \begin{bmatrix} 0 & 0 \\ -3 & 1 \end{bmatrix}, \qquad q = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This is a P_0-matrix, but $F(x) \equiv Mx + q$ does not verify the hypothesis of Theorem 2, since M is not column-sufficient. Obviously, $(0,0)$ is a solution of the NCP. However, taking $L = 2$, we have that all the points of the form $(t, 2)$ for $t \in [2/3, 2]$, are solutions of the BVIP defined by $F(x) = Mx + q$, $\ell = 0$, $r = \underline{L}$. So, these points are stationary points of the associated optimization problem but, clearly, they are not solutions of the NCP.

CONCLUSIONS

When, for some nonlinear programming reformulation of a complementarity or variational inequality problem, it is known that every stationary point is a solution, it can be conjectured that standard minimization algorithms will be effective for finding a solution, since these algorithms generally find, in the limit, stationary points. However, at least from the theoretical point of view, the effectiveness of the minimization approach is not proved unless, eventually, a bounded level set can be reached. Otherwise, there could be no convergent subsequence at all. In this paper we proved that a reformulation of nonlinear complementarity problems satisfies the desired requirements under weaker conditions than the ones established in previous works.

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