Search for semiclassical gravity effects in relativistic stars

Daniel A. T. Vanzella and George E. A. Matsas
Instituto de Física Teórica, Universidade Estadual Paulista, Rua Pamplona 145, 01405-900, São Paulo, SP, Brazil
(Received 10 June 1999; published 22 May 2000)

We discuss the possible influence of gravity in the neutronization process \( p^+ e^- \rightarrow n \nu_e \), which is particularly important as a cooling mechanism of neutron stars. Our approach is semiclassical in the sense that leptonic fields are quantized on a classical background spacetime, while neutrons and protons are treated as excited and unexcited nucleon states, respectively. We expect gravity to have some influence wherever the energy content carried by the in state is barely above the neutron mass. In this case the emitted neutrinos would be soft enough to have a wavelength of the same order as the space curvature radius.

PACS number(s): 97.60.Jd, 04.40.Dg, 04.62.+v

\[ j^\mu = q u^\mu \delta (u - a^{-1}), \]  

where \( q \) is a small coupling constant and \( u^\mu = (a, 0) \) is the nucleon four-velocity. Next, in order to allow the current above to describe the proton-neutron transition, we shall consider the nucleon as a two-level system \([3]\). In this vein, neutrons \(|n\rangle \) and protons \(|p\rangle \) will be excited and unexcited eigenstates of the nucleon Hamiltonian \( \hat{H} \):

\[ \hat{H} = m_n |n\rangle + m_p |p\rangle, \]

where \( m_n \) and \( m_p \) are the neutron and proton masses, respectively. Hence current (2) will be replaced by

\[ j^\mu = \hat{q}(\tau) u^\mu \delta (u - a^{-1}), \]

where \( \hat{q}(\tau) = \exp(i\hat{H}\tau) \hat{q}_0 \exp(-i\hat{H}\tau) \) is a Hermitian monopole. The two-dimensional Fermi constant \( G_F = \langle \langle p | \hat{q}_0 | n\rangle \rangle = 9.198 \times 10^{-13} \) is determined \([4]\) by imposing that the mean proper lifetime of inertial neutrons is 887 s \([5]\).

In order to calculate the neutronization rate we shall quantize the lepton field in the Rindler wedge. The lepton field is expressed as \([6]\)

\[ \hat{\Psi}(\tau, u) = \sum_{\sigma = \pm} \int_0^{\infty} d\omega \left[ \hat{b}_{\omega \sigma} \hat{\psi}_{\omega \sigma}(\tau, u) + \hat{d}_{\omega \sigma}^\dagger \hat{\psi}_{-\omega -\sigma}(\tau, u) \right], \]

where \( \hat{\psi}_{\omega \sigma}(\tau, u) = f_{\omega \sigma}(u) e^{-i\omega \tau} \) are positive (\( \omega > 0 \)) and negative (\( \omega < 0 \)) frequency solutions of the Dirac equation, with respect to the boost Killing field \( \partial / \partial \tau \), with polarizations \( \sigma = \pm \). We recall that the absolute value of Rindler frequencies may assume arbitrary positive real values. In particular there are massive Rindler particles with arbitrarily small frequencies. (See Ref. \([7]\) for a discussion on zero-frequency Rindler particles.) Here

\[ f_{\omega \sigma}(u) = A_{\omega \sigma}, \]

\[ A_{\omega \sigma} = \begin{pmatrix} K_{i, a}^{} + 1/2 (mu) & i K_{i, a}^{} - 1/2 (mu) \\ 0 & 0 \end{pmatrix}, \]

where \( K_{i, a}^{} (mu) \) are constants.

\* Email address: vanzella@ift.unesp.br

\† Email address: matsas@ift.unesp.br
As a final step, we take the limit

\[ m \rightarrow 0 \]

where \( m \) is the lepton mass and the normalization constants

\[ A_+ = A_- = \left[ \frac{m \cosh(\pi \omega/a)}{2 \pi^2 a} \right]^{1/2} \]

were chosen such that the annihilation and creation operators satisfy the following simple anticommutation relations

\[ \{ b_{\omega\sigma}, b_{\omega'\sigma'}^\dagger \} = \delta(\omega - \omega') \delta_{\sigma\sigma'} , \]

\[ \{ b_{\omega\sigma}, b_{\omega'\sigma'} \} = \{ \tilde{b}_{\omega\sigma}, \tilde{b}_{\omega'\sigma'}^\dagger \} = \{ \tilde{b}_{\omega\sigma}, \tilde{b}_{\omega'\sigma'} \} = 0. \]

Now we are ready to calculate the neutronization amplitude

\[ A = \langle n | \langle \nu_{\omega\sigma} | \hat{S}_I \hat{\Psi}_e \hat{\Psi}_\nu | \bar{p} \rangle \rangle, \]

where we minimally couple the nucleon current (4) to the leptonic fields \( \hat{\Psi}_e \) and \( \hat{\Psi}_\nu \) through the Fermi interaction action

\[ \hat{S}_I = \int d^2 x \sqrt{-g} \hat{j}_\mu \gamma^\mu \hat{\Psi}_e \gamma^\rho \hat{\Psi}_\nu + \hat{\Psi}_e \gamma^\rho \hat{\Psi}_\nu. \]

In the Rindler wedge \( \gamma^\rho_0 = (e_0)^\rho_0 \) with tetrads \( (e_0)^\rho = u^{-1} \delta_0^\rho \) and \( (e)^\rho = \delta_\rho^\mu \), where \( \gamma^\rho \) are the usual Dirac matrices. By using Eq. (11), we obtain the following amplitude:

\[ A_{ac} = G_F \int_{-\infty}^{+\infty} d\tau e^{i\Delta m \tau} \langle \nu_{\omega,\sigma} | \hat{\Psi}_e^\dagger (\tau, a^{-1}) \hat{\Psi}_e (\tau, a^{-1}) | \nu_{\omega,\sigma} \rangle , \]

where \( \Delta m = m_n - m_p \). Next, by using Eq. (5) we obtain

\[ A_{ac} = G_F \delta_{\sigma \sigma'} \int_{-\infty}^{+\infty} d\tau e^{i\Delta m \tau} \langle \nu_{\omega,\sigma} | \hat{\Psi}_e^\dagger (\tau, a^{-1}) \hat{\Psi}_e (\tau, a^{-1}) \rangle. \]

Using now explicitly \( \psi_{\omega,\sigma}(\tau, u) \) to perform the integral, we obtain

\[ A_{ac} = \frac{4 G_F}{\pi a} \sqrt{m_e m_\nu} \cosh(\pi \omega_e / a) \cosh(\pi \omega_\nu / a) \]

\[ \times \Re \left[ K_{i\omega \mu - \Delta m / a} K_{i\omega \nu + \Delta m / a} \right] \delta_{\sigma \sigma'} \]

\[ \times \delta(\omega_e - \omega_\nu - \Delta m). \]

This result will be used to calculate the total reaction rate

\[ \Gamma_{ac}(\omega, T_e) = \frac{1}{\tau} \sum_{\sigma_e \sigma_\nu} \sum_{\sigma'_{\sigma_e} \sigma'_{\sigma_\nu}} \int_{0}^{+\infty} d\omega_e \int_{0}^{+\infty} d\omega_\nu \left| A_{ac} \right|^2 n_F(\omega, T_e) \]

\[ \times \left[ 1 - n_F(\omega, T_e) \right], \]

where \( \tau = 2 \pi \delta(0) \) is the total nucleon proper time [8], and \( n_F(\omega, T) = 1/[1 + \exp(\omega/T)] \) is the usual fermionic thermal factor. We shall consider further two cases. In the first one, we assume \( T_e = 10^9 \) K and \( T_\nu = 0 \) K, i.e., the neutron star would be cold enough to be transparent to the neutrinos. In the second one, we assume \( T_e = T_\nu = 10^{10} \) K, i.e., electrons and neutrinos would be in thermal equilibrium. By using Eq. (15) in Eq. (16), we obtain

\[ \Gamma_{ac}(\omega, T_e) = \frac{4 G_F^2 m_e m_\nu}{\pi^3 a^2} \int_{-\infty}^{+\infty} d\omega_e \cosh(\pi \omega_e / a) \cosh(\pi \omega_e - \Delta m / a) \cosh(\pi (\omega_e - \Delta m) / 2 T_e) \cosh(\pi (\omega_e - \Delta m) / 2 T_\nu) \]

\[ \times \left\{ \Re \left[ K_{i\omega_e - \Delta m / a} K_{i\omega_\nu + \Delta m / a} \right] \right\}^2. \]

As a final step, we take the limit \( m_\nu \rightarrow 0 \) in Eq. (17) (see Ref. [9]):

\[ \Gamma_{ac}(\omega) = \frac{G_F^2 m_e}{\pi^2 a} \int_{-\infty}^{+\infty} d\omega_e \cosh(\pi \omega_e / a) \cosh(\pi \omega_e - \Delta m / a) \cosh(\pi (\omega_e - \Delta m) / 2 T_e) \cosh(\pi (\omega_e - \Delta m) / 2 T_\nu) \]

\[ \times \left\{ K_{i\omega_e - \Delta m / a} K_{i\omega_\nu + \Delta m / a} \right\}^2. \]
In order to compare the reaction rate above with the usual one obtained in inertial frames, we calculate next the reaction rate for \( a = 0 \) using plain quantum field theory in Minkowski spacetime. This will be used also as a consistency check since we will compare it with the \( a \rightarrow 0 \) limit obtained from Eq. (18).

Let us briefly outline the Minkowski calculation. The leptonic fields will be expressed in terms of the usual Minkowski coordinates \((t, z)\) as

\[
\hat{\Psi}(t, z) = \sum_{\sigma = \pm} \int_{-\infty}^{+\infty} dk \left[ \hat{b}_{k\sigma} \psi_{k\sigma}^{(+\omega)}(t, z) + \hat{d}_{k\sigma} \psi_{k\sigma}^{(-\omega)}(t, z) \right],
\]

(19)

where \( \hat{b}_{k\sigma} \) and \( \hat{d}_{k\sigma} \) are annihilation and creation operators of fermions and antifermions, respectively, with momentum \( k \) and polarization \( \sigma \). In the inertial frame, energy, momentum and mass \( m \) are related as usual: \( \omega = \sqrt{k^2 + m^2} > 0 \), \( \psi_{k\sigma}^{(+\omega)}(t, z) \) and \( \psi_{k\sigma}^{(-\omega)}(t, z) \) are positive and negative frequency solutions of the Dirac equation with respect to \( \partial \partial t \), respectively. In the Dirac representation (see, e.g., Ref. [8]), we find

\[
\psi_{k+}^{(+\omega)}(t, z) = \frac{e^{i(\omega t + kz)}}{\sqrt{2\pi}} \begin{pmatrix}
\pm \sqrt{(\omega + m)/2} \\
0 \\
k \sqrt{2\omega(\omega - m)} \\
0
\end{pmatrix}
\]

(20)

and

\[
\psi_{k-}^{(-\omega)}(t, z) = \frac{e^{i(\omega t - kz)}}{\sqrt{2\pi}} \begin{pmatrix}
0 \\
\pm \sqrt{(\omega + m)/2} \\
-k \sqrt{2\omega(\omega + m)} \\
0
\end{pmatrix},
\]

(21)

where the normalization constants were chosen such that the creation and annihilation operators satisfy

\[
\{ \hat{b}_{k\sigma}, \hat{d}^\dagger_{k'\sigma'} \} = \{ \hat{d}_{k\sigma}, \hat{b}^\dagger_{k'\sigma'} \} = \delta(k - k') \delta_{\sigma \sigma'}
\]

(22)

and

\[
\{ \hat{b}_{k\sigma}, \hat{b}^\dagger_{k'\sigma'} \} = \{ \hat{d}_{k\sigma}, \hat{d}^\dagger_{k'\sigma'} \} = \{ \hat{b}_{k\sigma}, \hat{d}_{k'\sigma'} \} = \{ \hat{d}_{k\sigma}, \hat{b}_{k'\sigma'} \} = 0.
\]

(23)

The neutronization amplitude for inertial nucleons in the Minkowski spacetime,

\[
A_{in} = G_F \int_{-\infty}^{+\infty} dt e^{i\Delta mt} \psi_{k\sigma}^{(+\omega)}(t, 0) \psi_{k\sigma}^{(+\omega)}(t, 0),
\]

(24)

is calculated by using the interaction action (12) in Eq. (11), where \( \gamma^\mu \) is replaced by the usual \( \gamma^\mu \) Dirac matrices, and the current is given by \( j^\mu = \hat{q}(t) \rho \delta(z) \) with \( \rho = 1 \). This leads us straightforwardly to the following neutronization rate for inertial nucleons:

\[
\Gamma_{in} = \frac{2G_F^2}{\pi} \int_{k_0}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega t - kz}}{1 + e^{\omega/\lambda_T}} \frac{1}{1 + e^{(\omega - \Delta m)/\lambda_T}}.
\]

(25)

where \( m = 0 \), \( L = \sqrt{m^2 - m^2} \), and we recall that \( \omega_c = \sqrt{k_c^2 + m^2} \).

In order to clearly analyze the influence of the frame acceleration on the neutronization process, let us use Eqs. (18) and (25) to define the following relative reaction rate:

\[
\mathcal{R}(a) = \frac{\Gamma_{in}(a) - \Gamma_{in}}{\Gamma_{in}}.
\]

(26)

FIG. 1. The relative reaction rate \( \mathcal{R}(a) \) is plotted as a function of the frame acceleration \( a \) for temperatures \( T_e = 10^9 \) K and \( T_v = 0 \) K. Note that \( \mathcal{R}(a \rightarrow 0) \rightarrow 0 \), as expected. After an oscillatory regime the relative reaction rate tends to the asymptotic value \( \mathcal{R}(a \rightarrow \Delta m, T_e) \approx -7.2\% \). The maximum value reached by \( |\mathcal{R}(a)| \) is about 30%.

\[
\Gamma_{in} = \frac{2G_F^2}{\pi} \int_{k_0}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega t - kz}}{1 + e^{\omega/\lambda_T}} \frac{1}{1 + e^{(\omega - \Delta m)/\lambda_T}}.
\]

(25)

\[
\text{FIG. 2. The relative reaction rate } \mathcal{R}(a) \text{ is plotted as a function of the frame acceleration } a \text{ for temperatures } T_e = T_v = 10^{10} \text{ K. After an oscillatory regime the relative reaction rate tends to the asymptotic value } \mathcal{R}(a \rightarrow \Delta m, T_e) \approx -3.5\%. \text{ The maximum value reached by } |\mathcal{R}(a)| \text{ is about } 10\%.}
\]
In Figs. 1 and 2 we plot $\mathcal{R}(a)$ for the two aforementioned cases: (i) $T_e = 10^9$ K and $T_r = 0$ K, and (ii) $T_e = T_r = 10^{10}$ K. First we note from the figures that $\Gamma_{ac}(a \rightarrow 0)$ is in agreement with the expression obtained for $\Gamma_{in}$ since $\mathcal{R}(a \rightarrow 0)$ $\to 0$. Figures 1 and 2 exhibit a complicated oscillatory pattern up to $a \approx 1$ MeV. Indeed, the frame acceleration plays its most important role in this region: $|\mathcal{R}(a)|$ reaches about 30% and 10% for cases (i) and (ii), respectively. For large enough accelerations, $a \gg \Delta m, T_e$, we obtain from Eq. (18) an asymptotic expression for $\Gamma_{ac}$, namely,

$$\Gamma_{ac}(a \gg \Delta m, T_e) \approx \frac{2 G_F^2}{\pi} \int_{\Delta m}^{+\infty} d\omega_e \left(1 + e^{\omega_e / T_e} \right) \left(1 + e^{(\omega_e - \Delta m) / T_e} \right).$$

(27)

By using Eq. (27) in Eq. (26), we can compute the asymptotic relative reaction rate, namely, $\mathcal{R}(a \gg \Delta m, T_e)$. We find that $\mathcal{R}(a \gg \Delta m, T_e) \approx -7.2\%$ and $\mathcal{R}(a \gg \Delta m, T_e) \approx -3.5\%$ for cases (i) and (ii), respectively, i.e., according to our toy model, ultrahigh accelerations damp the neutronization rate by a few percents.

In summary, we have looked for gravity effects in the neutronization process which frequently occurs in the interior of neutron stars. The reaction rate obtained by means of a simplified model exhibits a complicated oscillatory pattern up to $a \approx 1$ MeV. Afterwards it tends to an asymptotic value which indicates that the reaction is somewhat damped. We note that proper accelerations of the order $a \approx 1$ MeV are much beyond what would be expected in the interior of relativistic stars. Indeed a proton at the surface of a typical neutron star with radius $R \approx 10^4$ m and mass $M \approx 2M_\odot$ would have a proper acceleration of about $a \approx M(1 - 2M/R)^{-1/2}/R^2 \approx 10^{-17}$ MeV. Thus, as far as our toy model is concerned, the gravitational field of a neutron star would play a negligible role in the neutronization process. We emphasize, however, that only a four-dimensional Schwarzschild calculation would be realistic enough to precisely determine the whole influence of gravity in the neutronization reaction and other similar processes. In a more realistic calculation, for instance, effects due to the space curvature itself, which is absent here, should show up wherever the emitted neutrinos are soft enough to ‘feel’ the global background geometry. In this case, even reactions taking place at the star core, where $a \approx 0$, would be influenced by gravity. More detailed investigations on the role played by gravity in particle processes taking place in relativistic stars would be welcome.

D.V. was fully supported by Fundação de Amparo à Pesquisa do Estado de São Paulo while G.M. was partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico.