Some integrals occurring in a topology change problem

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(Received 11 February 2000; published 24 July 2000)

In a paper presented a few years ago, De Lorenci *et al.* showed, in the context of canonical quantum cosmology, a model which allowed space topology changes. The purpose of this present work is to go a step further in that model, by performing some calculations only estimated there for several compact manifolds of constant negative curvature, such as the Weeks and Thurston spaces and the icosahedral hyperbolic space (Best space).

PACS number(s): 98.80.Hw, 02.40.−k

I. INTRODUCTION

A few years ago De Lorenci *et al.* [1] presented a model of quantum cosmology which allowed space topology changes, having as the main idea the use of the ''conditional probability interpretation'' to establish selection rules for the possible changes of topology; the wave functions involved in the process were of the type

$$\Psi=\Psi(\alpha,\beta,\xi,\phi)=A_k(a,\phi)e^{\xi F_k}, \tag{1}$$

where α and β are appropriated canonical variables built upon the more common set of spherical coordinates (χ,θ,φ), the scale factor a and the curvature k; ξ and ϕ are, respectively, a dust field describing a ''distribution of irrotational dust particles'' and a scalar field, both representing the matter content of the model, and F_k is basically a numerical coefficient obtained by integration of certain functions constructed upon the ''value'' $\chi_0(\theta,\varphi;V^3)$ of the radial coordinate of the fundamental polyhedron's boundary of the three-dimensional manifold V^3, of curvature k, considered, written explicitly as

$$F_k=\frac{a}{2\pi\hbar m}\int_{V^3}\frac{\sin 2\sqrt{k}\chi_0(\theta,\varphi;V^3)}{2\sqrt{k}}d\theta\sin\theta d\varphi. \tag{2}$$

The topology changes would occur at some value ξ of the the dust field, when $a=\overline{a}$ and $\phi=\overline{\phi}$, such that the conditional probability of having $k=-1$, 0 or $+1$ would be

$$P_c(k|\overline{a},\overline{\phi})=\frac{|\Psi(k,\overline{a},\overline{\phi})|^2}{\sum_{k'=0,\pm1}|\Psi(k',\overline{a},\overline{\phi})|^2}=\frac{A_k^2(\overline{a},\overline{\phi})e^{2\xi F_k}}{\sum_{k'=0,\pm1}A_{k'}^2(\overline{a},\overline{\phi})e^{2\xi F_{k'}}}. \tag{3}$$

So, when $\xi\to\pm\infty$ one has one of the $P_c(k|\overline{a},\overline{\phi})$ equal to one and the other two null, depending upon the value of F_k.

In [1] the values of the functions F_k were only estimated for two different compact manifolds, the Poincaré dodecahedral space D^3, of positive curvature, and the hyperbolic icosaedral space I^3 (also known as Best space), of negative curvature; since there the authors claimed that ''it is not possible to calculate the F_i's exactly'' for these manifolds, the importance of the present work is in the exact calculation of the functions F_k for several compact manifolds of constant negative curvature, including the cited I^3.

II. SOME CALCULUS IN COMPACT MANIFOLDS

The functions F_k, such as presented in Eq. (2), are probably uncomputable since the specific form of the functions $\chi_0(\theta,\varphi;V^3)$ are difficult, if not impossible, to determine; however, one can simply establish the following limits for the F_k's:

$$4\pi\frac{\sin 2\sqrt{k}\chi_{\min}}{2\sqrt{k}}\leqslant\frac{F_k}{(a/2\pi\hbar m)}\leqslant 4\pi\frac{\sin 2\sqrt{k}\chi_{\max}}{2\sqrt{k}}, \tag{4}$$

where $\chi_{\min}$ and $\chi_{\max}$ are, respectively, the radii of the inscribed and circumscribed circumference of the fundamental cell of the manifold in consideration. In [1] the functions χ_0 appear after performing an ''integration with respect to the variable χ,'' using as the interval of integration $[0,\chi_0(\theta,\varphi;V^3)]$; so, in order to obtain a numerical value for the functions F_k, it is easy to see that one can start with the integral

$$F_k=-\frac{a}{2\pi\hbar m}\int_{V^3}[\sin^2\sqrt{k}\chi-\cos^2\sqrt{k}\chi]d\chi d\theta\sin\theta d\varphi. \tag{5}$$

Noticing now that

$$dV=\frac{\sin^2\sqrt{k}\chi}{k}d\chi d\theta\sin\theta d\varphi \tag{6}$$

is simply the element of the volume for the spatial part of a Friedmann-Robertson-Walker metric, written in spherical coordinates, there are two possible ways to proceed, one plainer and the other a little more sophisticated; in both, however, one needs to redefine the coordinates and limits of the integration used. So, the next step consists in the use of

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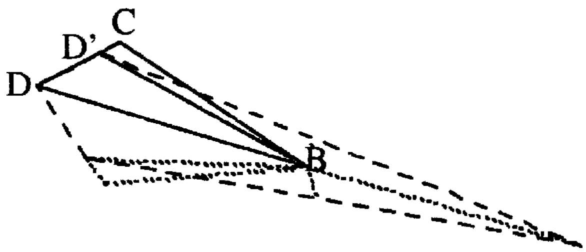

FIG. 1. Division of one face of the Weeks manifold (quadrilateral area bounded by the segmented lines) in several triangles; the triangles labeled BCD and BCD' form, when connected to the center A of the fundamental polyhedron, the basis of one positive and one negative tetrahedron, respectively. Notice that the point A, not shown in the figure, belongs to a different plane.

cylindrical coordinates (ρ,φ,z), related to the spherical coordinates (χ,θ,φ) by means of the relations

$$\cos\sqrt{k}\chi=\cos\sqrt{k}\rho\,\cos\sqrt{k}z,\qquad \sin\sqrt{k}\chi\,\sin\theta=\sin\sqrt{k}\rho, \tag{7}$$

or

$$d\chi^2+\frac{\sin^2\sqrt{k}\chi}{k}[d\theta^2+\sin^2\theta d\varphi^2] =d\rho^2+\cos^2\sqrt{k}\rho dz^2+\frac{\sin^2\sqrt{k}\rho}{k}d\varphi^2. \tag{8}$$

Now, one has the interval of integration $[0,\rho_0(z,\varphi;V^3)]$ for the coordinate ρ; the expression for $\rho_0(z,\varphi;V^3)$ is easily obtainable, since it is only a matter of using trigonometrical identities in the plane, i.e., in the triangles that compose the faces of each tetrahedron in which the fundamental polyhedron can be divided[1] using the following procedure:

For each face draw a geodesic line perpendicular to it, connecting it to the center A of the polyhedron, and crossing it or its plane in a point B (this line AB gives the height z of the tetraedron); for each edge draw a geodesic line perpendicular to it and connecting it to the point B of the face to which the edge belongs, crossing the edge or its extension in a point C; complete the tetrahedron with one of the two vertices of the edge, naming it D.

These steps will create some ''negative'' tetrahedra, covering also regions outside the polyhedron, and some ''positive,'' covering only regions of the polyhedron, each one of them having four right-angled triangles, one of which (named here BCD) is the base of the tetrahedron; integration on the compact manifold represented by the polyhedron is the difference between the sums of the integrations on all of the positive tetrahedra and the integrations on all of the negative tetrahedra (see Fig. 1).

[1]For more information on trigonometric identities in non-Euclidean spaces one can see Refs. [2–6]; [3] is a classical book of cosmology with one section on spherical trigonometry.

The easiest path of integration consists of simply making

$$F_k=-\frac{a}{2\pi\hbar m}\int_{V^3}[2\sin^2\sqrt{k}\chi-1]d\chi d\theta\sin\theta d\varphi =-\frac{a}{2\pi\hbar m}\left[2kv-\int_{V^3}\frac{kdV}{\sin^2\sqrt{k}\chi}\right], \tag{9}$$

where v is the volume of the compact manifold where the integration is being performed. The remaining integral in the right-hand side of the last equality must be done in the new set of cylindrical coordinates,[2] where the limits of integration for the particular case of negative curvature $(k=-1)$ are, in each tetrahedron,[3]

$$0\leq z\leq z_0\equiv d_{AB},\quad 0\leq\varphi\leq C\hat{B}D, \tag{10}$$

and

$$0\leq\rho\leq\rho_0(z,\varphi)=\text{arctanh}\left[\tan B\hat{A}C\,\frac{\sinh z}{\cos\varphi}\right]. \tag{11}$$

The integration in the coordinate ρ is easily done and gives finally

$$\frac{F_{-1}}{a/2\pi\hbar m}=2v+\int_0^{C\hat{B}D}d\varphi\int_0^{d_{AB}}\frac{dz}{\cosh^2 z}\times\ln\sqrt{\frac{\cos^2\varphi+\tan^2 B\hat{A}C}{\cos^2\varphi-\tan^2 B\hat{A}C\sinh^2 z}} \tag{12}$$

from where numerical results can be obtained by plain numerical integration.

Notice that one could go a step further with analytical integration, making in Eq. (12) the integration in the variable z, yielding a formula for the volume of each tetrahedron of the manifold,[4] and finishing with an expression for F_{-1} consisting of one single integral in the variable φ. However, the intermediate results of this procedure are somewhat lengthy, and so, instead, one can start doing

$$F_k=-\frac{ak}{2\pi\hbar m}\int_{V^3}[1-\cot^2\sqrt{k}\chi]dV =-\frac{a}{2\pi\hbar m}\left[kv+\int_{V^3}\nabla_\mu V^\mu dV\right], \tag{13}$$

[2]In the new set of coordinates the element of the volume to be used is

$$dV=\sinh\rho\cosh\rho d\rho dz d\varphi,$$

obtained from Eq. (8).

[3]The trigonometric identities that lead to such a result are shown in the appendix at the end of this work.

[4]The final result for the volume is presented in the last section of this work.

TABLE I. Data for each manifold studied.

Manifold	Volume	χ_{min}	χ_{max}
Weeks	0.942707	0.519162	0.752470
Thurston	0.981369	0.535437	0.748538
$m036(-3,2)$	2.029883	0.675646	1.014814
$m016(-4,3)$	2.343017	0.691286	0.895576
$m036(-2,3)$	2.568971	0.726205	0.895576
Best	4.686034	0.868298	1.382571
$v3469(+3,1)$	5.137941	0.808931	1.45241

TABLE II. Summary of the results.

Manifold	$F_{-1}/(a/2\pi\hbar m)$	$2\pi\sinh 2\chi_{min}$	$2\pi\sinh 2\chi_{max}$
Weeks	9.28474	7.76109	13.4518
Thurston	9.48385	8.09029	13.3355
$m036(-3,2)$	13.4897	11.3208	23.4987
$m016(-4,3)$	14.5526	11.7314	18.3142
$m036(-2,3)$	15.3167	12.6901	20.6181
Best	21.4948	17.2847	49.6976
$v3469(+3,1)$	22.5418	15.2178	57.1996

where V^μ is a vector satisfying the first-order differential equation

$$\nabla_\mu V^\mu = (\partial_\mu + \Gamma^\rho_{\mu\rho})V^\mu = -k\cot^2\sqrt{k}\chi. \tag{14}$$

This equation has, in principle, several solutions; if one assumes, quite arbitrarily, that in spherical coordinates V^μ is a vector with only a radial component and that all constants of integration appearing in the solution of the differential equation can be set equal to zero, one obtains as a solution, in spherical coordinates,[5]

$$\begin{aligned} V^\chi &= -\frac{1}{2}[\sqrt{k}\cot\sqrt{k}\chi + k\chi\csc^2\sqrt{k}\chi] \\ &= -\frac{1}{\chi} - \sum_{l=1}^{\infty}\frac{k^2\chi^3}{(k\chi^2-\pi^2\ell^2)^2}, \end{aligned} \tag{15}$$

where the last equality shows clearly the behavior of the solution when $k=0$. This result permits the use of Stokes's theorem [8] to make

$$-\int_{V^3} k\cot^2\sqrt{k}\chi\, dV = \int_{S=\partial V^3} g_{\mu\nu}V^\mu n^\nu dA, \tag{16}$$

where n^ν is a vector normal to the boundary S of the fundamental cell of the compact manifold V^3, obeying the constraint $n^\mu n_\mu = 1$.

In the procedure presented here the faces of the fundamental polyhedron that represents a compact manifold appear, by construction, as surfaces of constant z, allowing one to use as an element of area

$$dA = \frac{\sin\sqrt{k}\rho}{\sqrt{k}}\, d\rho\, d\varphi. \tag{17}$$

Finally, to carry out the integration the vector V^χ must be written in cylindrical coordinates; only the component $V^z = V^\chi \partial_\chi z$, normal to the base of the tetrahedron, is important. This procedure can be used also to give the volume of each tetrahedron, which allows one to write, in the case of negative curvature,

$$\frac{F_{-1}}{a/2\pi\hbar m} = \int_0^{C\hat{B}D}\frac{d\varphi}{\coth z_0}\ln\sqrt{\frac{\cos^2\varphi + \tan^2 B\hat{A}C}{\cos^2\varphi - \tan^2 B\hat{A}C\sinh^2 z_0}}, \tag{18}$$

where $z_0 = d_{AB}$; this result is the same one would obtain from plain integration of Eq. (12), validating therefore the assumptions made in the choice of the solution for V^μ.

III. NUMERICAL RESULTS

To obtain numerical results the data—volumes and coordinates of all vertices for several hyperbolic compact manifolds—contained in the literature were used (see, for instance, [9] and [10]) together with those of the software SNAPPEA[6] [11]; part of the data used are presented in Table I. The manifolds chosen present in some way a degree of symmetry which simplified the calculus, but, in principle, the approach followed can be used to any compact manifold. All results are presented in Table II where they are compared with estimates done as in [1]; the result obtained for the Weeks manifold was used in [12].

IV. CONCLUSION

There are several formulations of quantum cosmology and the intention of this work is to shed some new light on a particular one, showing that the wave functions built by the procedure of [1] present a dependence on the volume of the compact manifold in consideration; aside that, such wave functions have an additional dependence on the *shape* of the fundamental cell of the manifold, due to a surface term that does not appear in several other models.

To finish, it is also interesting to notice that the results presented here, though of specific relevance for a particular model of quantum cosmology, can be seen in a more generalized context, since this work presents a method that allows one to easily calculate the volume of the fundamental poly-

[5]Here the identity used is

$$\frac{\pi^2}{4m^2}\csc^2\frac{\pi}{m} + \frac{\pi}{4m}\cot\frac{\pi}{m} - \frac{1}{2} = \sum_{k=1}^{\infty}\frac{1}{(1-k^2m^2)^2}$$

found as Eq. (1.423) of Ref. [7].

[6]SNAPPEA is an electronic catalog of thousands of hyperbolic compact manifolds, each one of them identified by volume and a code such as $m036(-3,2)$.

hedron of a compact manifold. Explicitly, for the particular case of negative curvature, the volume of each tetrahedron in which the fundamental polyhedron can be divided is

$$v=\int_0^{C\hat{B}D}\frac{d\varphi}{2}\times\left\{\frac{\operatorname{arctanh}[\tanh z_0\sqrt{1+\sec^2\varphi\tan^2 B\hat{A}C}]}{\sqrt{1+\sec^2\varphi\tan^2 B\hat{A}C}}-z\right\},\qquad(19)$$

where again $z_0=d_{AB}$; alternatively,

$$v=\int_0^{d_{AB}}\frac{dz}{2}\times\left\{\frac{\operatorname{arctanh}[\tan C\hat{B}D(\cot^2 B\hat{A}C\operatorname{csch}^2 z-1)^{-1/2}]}{\sqrt{\cot^2 B\hat{A}C\operatorname{csch}^2 z-1}}\right\}.\qquad(20)$$

These results must be compared with the more traditional ones given in [4], [5], and [6].

ACKNOWLEDGMENTS

The author wants to thank the Brazilian state of São Paulo, which gave him financial support through Grant No. 96/0052-3 of the Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP); the author is also deeply grateful for the help of R.G. Teixeira and Professor Helio V. Fagundes.

APPENDIX: TRIGONOMETRIC IDENTITIES

In the non-Euclidean geometry the trigonometric identities valid for a triangle XYZ, with right angle Z, of sides x, y, and hypotenuse z are [4]

$$\sin Y=\frac{\sin\sqrt{k}y}{\sin\sqrt{k}z};\quad \cos Y=\frac{\tan\sqrt{k}x}{\tan\sqrt{k}z};\quad \tan Y=\frac{\tan\sqrt{k}y}{\sin\sqrt{k}x}.\qquad(A1)$$

Using the second identity in the right-angled triangle BCD, of right angle C, and the third one in the right-angled triangle ABC, of right angle B, one can write, for the tetrahedron $ABCD$ built as in Sec. II,

$$\cos C\hat{B}D=\cos\varphi=\frac{\tan\sqrt{k}d_{BC}}{\tan\sqrt{k}\rho}=\frac{\tan B\hat{A}C\sin\sqrt{k}d_{AB}}{\tan\sqrt{k}\rho}\qquad(A2)$$

from where one obtains Eq. (11), after identification of d_{AB} with z.

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