

**A GENERALIZED RIEMANN-STIELTJES INTEGRAL  
ON TIME SCALES AND DISCONTINUOUS  
DYNAMICAL EQUATIONS**

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**Abstract:** This paper is concerned with a generalization of the Riemann-Stieltjes integral on time scales for deal with some aspects of discontinuous dynamic equations in which Riemann-Stieltjes integral does not works.

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## 1. Introduction

When modeling dynamic phenomena, researchers have traditionally used model in which either continuous or discrete time is considered like for instance in the differential and difference equations.

In recent years it became clear that for a more accurate description of phenomena it is necessary to go beyond this dichotomy in scales time.

Nowadays, several examples in which more general time scales appear, can be found in the literature [1-4].

With the purpose of deal with this situation S. Hilger introduced in 1990 the *calculus on time scales* (or on measure chains), see [11]. This kind of calculus showed the possibility to manage dynamic equations considering a very wide range of time scales transforming in this way the differential and difference calculus into special cases of a more general one. Examples of time scales are the real numbers, the integers, the sets having cluster points or even such as a Cantor set.

Regarding integral calculus on time scales the literature includes, among others, the Riemann delta and nabla-integral [8], alfa-integral [9], the Lebesgue and nabla-integrals [7], and the Henstock-Perron-Kurzweil [16] ones.

Considerations on Fredholm and Volterra integral systems using general time scales were done in [17] and [14].

In 2009, Mozyrska-Pawluszewicz-Torres, [15] defined and presented some initial properties for the Riemann-Stieltjes integral in the numerical case, opening in this way the possibility to develop the theory of Fredholm and Volterra-Stieltjes integral equations on time scales, even in Banach spaces. However, we still have some difficulties inherent to this kind of integrals when dealing with dynamic equations in which non-continuous functions may appear.

This situation call, in this way, for a more general vision on the subject and it is reflected by the necessity to consider a more general notion of integral, extending the Riemann-Stieltjes one. Such extension, still in the numerical case, is proposed in Section 4 below.

This paper is organized as follows: in Section 2 we introduce the Riemann-Stieltjes integrals. In Section 3 we show important gap in the theory of the Riemann-Stieltjes integrals when considering discontinuous functions. In Section 4 we define the odd-meshed filter integral. A comparison between these two types of integrals is done in Section 5. Finally, in Section 6 we give an application in a dynamic equation, considering a boundary value problem and in the 7-th one we point future perspectives when using this kind of integral.

## 2. The Riemann-Stieltjes (RS) Integral on Time Scales

From now we will be taking the notations and results presented in [15]. For general definitions, results and notations, see also [5], [11].

Let  $\mathbf{T}$  be a time scale (that is: a closed non-void subset of  $\mathbb{R}$ ) with  $a, b \in \mathbf{T}$ ,  $a < b$  and  $\mathbf{I} = [a, b]_{\mathbf{T}}$ , where  $[a, b]_{\mathbf{T}}$  denotes  $[a, b] \cap \mathbf{T}$  in  $\mathbb{R}$ ).

**Definition 2.1.** A partition of  $\mathbf{I}$  is any finite ordered subset

$$P = \{a = t_0 < t_1 < \dots < t_n = b\} \subseteq [a, b]_{\mathbf{T}} .$$

The length of  $P$  is denoted by  $|P| = n$ . We will be denoting by  $\wp$  the class of all partitions of  $\mathbf{I}$ . Note that it is defined the relation  $P \geq Q$  for  $P, Q \in \wp$  meaning:  $P \supseteq Q$ .

The definition of the Riemann-Stieltjes integral over  $\mathbf{I}$  is done in [15] by using the notion of the upper and lower Darboux-Stieltjes integrals. The Riemann-Stieltjes integral of  $f$  respect to  $g$  over  $\mathbf{I}$  is denoted by  $\int_a^b f \square g$ .

We reproduce here the definition of such Riemann-Stieltjes integral, done in [15]:

Let  $g$  be a strictly increasing real-valued function on the interval  $\mathbf{I}$ . Then for the partition  $P \in \wp$  define, both:

$$g(P) = \{g(a) = g(t_0), g(t_1), \dots, g(t_{n-1}), g(t_n) = g(b)\} \subset g(\mathbf{I}) ,$$

and

$$\Delta g_j = g(t_j) - g(t_{j-1}) .$$

Note that  $g(P)$  is a partition of  $[g(a), g(b)]_{\mathbf{T}}$ .

Take  $f$  a real-valued and bounded function on the interval  $\mathbf{I}$ . Let  $P = \{t_0, t_1, \dots, t_n\} \in \wp$ .

**Definition 2.2.** The upper and lower Darboux-Stieltjes sums with respect to  $g$ , are:

$$\text{(the upper)} \quad U(P; f; g) = \sum_{j=1}^n \sup_{[t_{j-1}, t_j]_{\mathbf{I}}} f(t) \Delta g_j , \tag{2.1}$$

$$\text{(the lower)} \quad L(P; f; g) = \sum_{j=1}^n \inf_{[t_{j-1}, t_j]_{\mathbf{I}}} f(t) \Delta g_j . \tag{2.2}$$

**Definition 2.3.** The upper Darboux-Stieltjes -integral of  $f$  with respect to function  $g$  in  $\mathbf{I}$  is the number  $\inf_{P \in \wp} U(P; f; g)$  and the lower one is  $\sup_{P \in \wp} L(P; f; g)$ .

If both the values are the same, then the Riemann-Stieltjes integral is defined and its value is this common value.

### 3. Remarks on the Riemann-Stieltjes (RS) Integral

At a first glance we observe an important fact in the definition of the Riemann-Stieltjes integral. It concerns the using of this kind of integrals on systems in which  $f$  and  $g$  are not continuous on  $\mathbf{I}$ .

In fact:

**Proposition 3.1.** *Let*

$$\mathbf{I} = [-1, 1]_{\mathbf{T}} = \mathcal{H} \cup -\mathcal{H} \cup \{0\},$$

where

$$\mathcal{H} = \left\{ \frac{1}{2^k}; k \in \mathbb{N}^* \right\},$$

and define

$$g(t) = \begin{cases} 1 + t, & t \in \mathcal{H} \cup \{0\}, \\ t, & \text{otherwise,} \end{cases} \tag{3.1}$$

and

$$f(t) = \begin{cases} 1, & t \in \mathcal{H} \cup \{0\}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Then we have:  $\int_{-1}^1 f(t) \square g(t)$  does not exist.

*Proof.* The two first equalities are immediate from the definition of the Riemann-Stieltjes integral. For the third one observe that for any interval  $[-\frac{1}{2^r}, \frac{1}{2^s}]$  with  $r, s \in \mathbb{N}$  we get in the upper and lower Darboux-Stieltjes sums of  $f$  to  $g$ , that:

$$\begin{aligned} \sup_{[-\frac{1}{2^r}, \frac{1}{2^s}]_{\mathbf{T}}} f(t) \cdot \left( 1 + \frac{1}{2^s} + \frac{1}{2^r} \right) &= \left( 1 + \frac{1}{2^s} + \frac{1}{2^r} \right), \\ \inf_{[-\frac{1}{2^r}, \frac{1}{2^s}]_{\mathbf{T}}} f(t) \cdot \left( 1 + \frac{1}{2^s} + \frac{1}{2^r} \right) &= 0, \end{aligned}$$

respectively. □

A more incisive example in this direction will be done in Section 6 below.

With the intention of – among other arguments – avoid such inconvenience just described in Proposition 3.1, we introduce another integral of the Riemann-Stieltjes type, extending properly the Riemann-Stieltjes integral.

### 4. The Odd-Meshed (OM) Integral on Time Scales

We are going to define an integral inspired by the Dushnik integral, defined by B. Dushnik in 1931, see [10], p. 96.

Let as above,  $\wp$  be the class of all partitions of  $\mathbf{I} = [a, b]_{\mathbf{T}}$ .

Let us take a subset of  $\wp$ ,  $\wp^*$ , where  $\hat{P} \in \wp^*$  if and only if  $|\hat{P}|$  is an odd number, and define the relation  $\hat{Q} \succeq \hat{R}$  in  $\wp^*$  fulfilling both conditions:

1.  $\hat{Q} \supseteq \hat{R}$ ,
2. the odd-indexed points  $t_{2k+1}$  ( $k = 0, 1, 2, \dots, n - 1$ ) in  $\hat{R} = \{a = t_0 < t_1 < \dots < t_{2n} = b\} \subseteq [a, b]_{\mathbf{T}}$  are also odd-indexed points  $\tau_{2r+1}$  ( $r = 0, 1, 2, \dots, m - 1$ ) in  $\hat{Q} = \{a = \tau_0 < \tau_1 < \dots < \tau_{2m} = b\} \subseteq [a, b]_{\mathbf{T}}$  ( $m \geq n$ ).

**Definition 4.1.** For the functions  $f$  and  $g$ , the (Cauchy) sum associated to  $\hat{P} \in \wp^*$  of  $f$  relatively to  $g$  is the number (if finite)

$$\sigma_{\hat{P}}(f; g) = \sum_{i=0}^{\frac{|\hat{P}|-1}{2}} [g(t_{2i+2}) - g(t_{2i})] \cdot f(t_{2i+1}) . \tag{4.1}$$

We define the odd meshed integral of  $f$  relatively to  $g$ , and write  $OM - \int_{[a,b]_{\mathbf{T}}} \cdot \Delta_s g(s) \cdot f(s)$ , or then simply  $\int_{[a,b]_{\mathbf{T}}} \cdot \Delta_s g(s) \cdot f(s)$ , the following limit (if existing):

$$\int_{[a,b]_{\mathbf{T}}} \cdot \Delta_s g(s) \cdot f(s) = \lim_{\hat{P} \in \wp^*} \sigma_{\hat{P}}(f; g) , \tag{4.2}$$

where  $\lim_{\hat{P} \in \wp^*} \sigma_{\hat{P}}(f; g) = z$  means: for every neighborhood  $V$  of  $z$ , there exists  $\hat{P}_V \in \wp^*$  such that for every  $\hat{P} \in \wp^*$  with  $\hat{P} \succeq \hat{P}_V$ , we must have  $\sigma_{\hat{P}}(f; g) \in V$ .

**Remark 4.1.** Normally we will be treating discontinuous functions, and then the scale  $\mathbf{T}$  must to be enumerable, at least. Despite of this, we define  $\int_{[a,b]_{\mathbf{T}}} \cdot \Delta_s g(s) \cdot f(s)$  for  $\mathbf{T}$  having an (finite) even number of points, as being the value  $RS - \int_a^b f \square g$ .

**Remark 4.2.** From now on,

1. we will be considering only the  $\Delta$  part (concerning the delta derivative) implicit in  $\square$ ,
2. we will be using the more suggestive notations  $\int_{[a,b]_{\mathbf{T}}} \Delta_s g(s) \cdot f(s)$  instead of the considered above  $\int_a^b f \square g$ , and

3. we will be referring to  $\int_{[a,b]_{\mathbf{T}}} \Delta_s g(s) \cdot f(s)$  or  $\int_{[a,b]_{\mathbf{T}}} \Delta_s g(s) \cdot f(s)$ , as the RS or OM-integral, respectively.

In the coming theorem we establish classes of functions  $f$  and  $g$  for which there exists the OM-integral. Let us before define:

**Definition 4.2.** The function  $f$  is said to be regulated if it has only discontinuities of the first kind in its domain.

**Definition 4.3.** Let  $\mathbf{T}$  be a time scale with  $a, b \in \mathbf{T}$  and  $\mathbf{I} = [a, b]_{\mathbf{T}}$ ,  $\wp$  the class of all partitions of  $\mathbf{I}$  and consider  $P = \{t_0, t_1, \dots, t_n\} \in \wp$ . Define the number  $V_P(g) = \sum_{t=1}^n \|g(t_1) - g(t_{i-1})\|$ .

We say that  $g : \mathbf{I} \rightarrow X$  is of bounded variation in  $\mathbf{I}$  if  $V_{\mathbf{I}}(g)_{P \in \wp} = \sup V_P(g) < \infty$ .

**Theorem 4.1.** Let  $f, g : \mathbf{I} \rightarrow \mathbb{R}$ . If  $f$  is regulated and  $g$  is of bounded variation on  $\mathbf{I}$  then there exists  $\int_{[a,b]_{\mathbf{T}}} \Delta_s g(s) \cdot f(s)$ .

*Proof.* If  $f$  is a regulated function on  $\mathbf{I}$ , then it is well known that  $f$  must be bounded on  $\mathbf{I}$ .

In the Cauchy sum associated to  $\hat{P} \in \wp^*$  of  $f$  relatively to  $g$ , (4.1), we see in an immediate way that  $\sigma_{\hat{P}}(f; g) \leq V_{\mathbf{I}}(g) \max_{\mathbf{I}} f = \text{constant} < \infty$ , and then according (4.2) we get the result in the theorem.  $\square$

### 5. Connections between $\int_{[a,b]_{\mathbf{T}}} \Delta_s g(s) \cdot f(s)$ and $\int_{[a,b]_{\mathbf{T}}} \Delta_s g(s) \cdot f(s)$

In this section we will be showing that if the RS-integral there exists then the OM-integral exists too. Further, the OM-integral extends properly the RS-integral.

**Theorem 5.1.** The existence of the RS-integral implies the existence of the OM-integral.

*Proof.* In an immediate way we see that if

$$\int_{[a,b]_{\mathbf{T}}} \Delta_s g(s) \cdot f(s)$$

is defined, then

$$\int_{[a,b]_{\mathbf{T}}} \Delta_s g(s) \cdot f(s)$$

is defined too, and we get:

$$\int_{[a,b]_{\mathbf{T}}} \Delta_s g(s) \cdot f(s) = \int_{[a,b]_{\mathbf{T}}} \Delta_s g(s) \cdot f(s) .$$

In fact: observe both

1. for a sequence of points  $t_{2i}, t_{2i+1}, t_{2i+2}$  in the partition  $\hat{P} \in \wp^* \subseteq \wp$ , we have

$$\begin{aligned}
 & [g(t_{2i+2}) - g(t_{2i})] \cdot f(t_{2i+1}) \\
 &= [g(t_{2i+2}) - g(t_{2i+1})] \cdot f(t_{2i+1}) + [g(t_{2i+1}) - g(t_{2i})] \cdot f(t_{2i+1})
 \end{aligned}$$

and

2. for every  $q \in [r, s]_{\mathbf{T}} : [g(t_s) - g(t_r)] \cdot \inf_{\sigma \in [r, s]_{\mathbf{T}}} f(\sigma) \leq [g(t_s) - g(t_r)] \cdot f(q)$   
and  $[g(t_s) - g(t_r)] \cdot \sup_{\sigma \in [r, s]_{\mathbf{T}}} f(\sigma) \leq [g(t_s) - g(t_r)] \cdot f(q)$ .

Gathering the stated in 1 and 2, we get that the upper and the lower Darboux-Stieltjes sums for  $\hat{P}$  decreases or increases, respectively, when considering the Cauchy sum (4.1) in the OM-integral. In this way we get that in fact, if  $\int_{[a,b]_{\mathbf{T}}} \Delta_s g(s) \cdot f(s)$  exists then  $\int_{[a,b]_{\mathbf{T}}} \cdot \Delta_s g(s) \cdot f(s)$  exists too.

Further, we have

**Theorem 5.2.** *The OM-integral extends properly the RS-integral.*

*Proof.* As shown in Section 3 above when considering  $f$  and  $g$  as in (3.2) and (3.1) respectively, we have that

$$\int_{[-1,1]_{\mathbf{T}}} \Delta_s g(s) \cdot f(s)$$

does not exist. But on other hand

$$\int_{[-1,1]_{\mathbf{T}}} \cdot \Delta_s g(s) \cdot f(s) = 1 .$$

To see this, just define an initial partition of  $\mathbf{I}$  say:  $\hat{Q} = \{-1, 0, 1\}$  and observe that all refinements  $\hat{P}$  of  $\hat{Q}$  that preserve the point 0 as an odd-indexed element will maintain the value of  $\sigma_{\hat{P}}(f; g)$  constantly equal to 1. □

### 6. The OM-Integral and Discontinuous Dynamic Equations

We shall investigate the following boundary value problem (BVP) for  $t \in \mathbf{I} = [-1, 1]_{\mathbf{T}} = \mathcal{H} \cup -\mathcal{H} \cup \{0\}$ ,

$$\begin{cases} x^{\Delta\Delta} + A(t)x^{\Delta} = h(t), \\ x(-1) = 0, \\ x(1) = 1, \end{cases} \tag{6.1}$$

where

$$A(t) = \begin{cases} 1, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0, \end{cases} \tag{6.2}$$

and  $h(t)$  satisfies

$$\int_{[-1,t]_{\mathbf{T}}} (t-s)h(s)ds + \frac{t+1}{2(1-\int_{\mathbf{I}}[(1-s)h(s)ds])} = \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{6.3}$$

It is known that

**Theorem 6.1.** (see [13], [3]) *The BVP (6.1) can be represented by a Fredholm-Stieltjes integral equation*

$$x(t) = \int_{\mathbf{I}} (\cdot)\Delta_s \alpha(t,s).x(s) = m(t), \quad t \in \mathbf{I}, \tag{6.4}$$

with  $(\cdot)\Delta_s$  denoting either  $\Delta_s$  if the RS-integral is considered or  $\cdot\Delta_s$  if we consider the OM one, where

$$\alpha(t,s) = \begin{cases} \frac{t-1}{2}[A(s) + (s+1)A^\Delta(s)], & \text{if } t > s, \\ \frac{t+1}{2}[A(s) + (s-1)A^\Delta(s)], & \text{if } t < s, \end{cases} \tag{6.5}$$

and

$$m(t) = \int_{[-1,t]_{\mathbf{T}}} (t-s)h(s)ds + \frac{t+1}{2[1-\int_{\mathbf{I}}[(1-s)h(s)ds]]}.$$

**Theorem 6.2.** *If in (6.4) we have the OM-integral then we can show that*

$$x(t) = \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{otherwise,} \end{cases}$$

is a solution of the problem (6.1). On the other hand observe that it is not possible to represent this solution with the RS-integral.

*Proof.* The delta derivative of the function  $A$  in (6.2) (on time scales),  $A^\Delta$ , satisfies  $A^\Delta(t) = 0$  for every  $t \in \mathbf{I}$ .

Then  $\alpha$  in (6.5) is the function:

$$\alpha(t,s) = \begin{cases} \text{for } t > s : \begin{cases} \frac{t-1}{2}, & \text{if } s \geq 0, \\ 0, & \text{if } s < 0, \end{cases} \\ \text{for } t < s : \begin{cases} \frac{t+1}{2}, & \text{if } s \geq 0, \\ 0, & \text{if } s < 0, \end{cases} \end{cases}$$



showing in this way that by using

$$x(t) = \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{otherwise,} \end{cases}$$

the integral term in (6.4) is equal to zero, since we are using  $\cdot\Delta_s$  in the integral.

But  $m(t) = \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{otherwise,} \end{cases}$  and then  $x(t) = m(t)$ , and in this way such function  $x$  is a solution of the initial BVP(6.1).

Ending the proof of the theorem: if, on the contrary the RS-integral is considered in (6.4) then, in the same way that in the example in Section 3 above, we see that

$$\int_{[-1,0]_{\mathbf{T}}} \Delta_s \alpha(t, s).x(s) = 0, \quad t \in \mathbf{I},$$

and

$$\int_{[0,1]_{\mathbf{T}}} \Delta_s \alpha(t, s).x(s) = 0, \quad t \in \mathbf{I},$$

but

$$\int_{\mathbf{I}} \Delta_s \alpha(t, s).x(s) \quad t \in \mathbf{I},$$

does not exist. □

### 7. Conclusion and Perspectives by Considering the OM-Integral in Dynamic Equations

When considering the time scale  $\mathbf{T} = \mathbb{R}$ , the OM-integral coincides with the Dushnik integral one, as shown in [3].

This version arises when we are looking for linear equations of the form

$$y + ky = f ,$$

where  $y, f$  are elements of a general Banach space  $F([a, b], X)$ . It is well known (see [12]) that if the function  $k$  is causal then taking  $F([a, b], X)$  being the set of all regulated functions the operator  $k$  has a simple characterization

$$(ky)(t) = \int_{[a,t]_{\mathbf{T}}} \cdot\Delta_s K(t, s).y(s).$$

The operator  $K$  in the numerical case  $X = \mathbb{R}$  is of bounded variation in the second variable and the bilinearity linking  $y$  and  $K$  is just the Dushnik integral.

Then we are able in this way to handle a wide class of Fredholm-Stieltjes integral equations on spaces of discontinuous functions (see [3]).

Due to all that, we see that a very good perspective for a future use of the OM-integral in discontinuous dynamic equations on general time scales, is open.

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