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Multipartite quantum correlations in open quantum systems

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Abstract. In this paper, we present a measure of quantum correlation for a multipartite system, defined as the sum of the correlations for all possible partitions. Our measure can be defined for quantum discord (QD), geometric quantum discord or even for entanglement of formation (EOF). For tripartite pure states, we show that the multipartite measures for the QD and the EOF are equivalent, which allows direct comparison of the distribution and the robustness of these correlations in open quantum systems. We study dissipative dynamics for two distinct families of entanglement: a W state and a GHZ state. We show that, for the W state, the QD is more robust than the entanglement, while for the GHZ state, this is not true. It turns out that the initial genuine multipartite entanglement present in the GHZ state makes the EOF more robust than the QD.

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Contents

1. Introduction

Entanglement in composite quantum systems leads to many puzzling paradoxes in quantum theory [1–4]. The importance of entanglement is universally recognized, but it is well known that even separable quantum states possess correlations that cannot be simulated by classical systems—for instance, the nonclassical correlations captured by quantum discord (QD) [5]. For bipartite states, many attempts have been made to detect and measure QD [6, 7] and to find the connection between it and entanglement [8–10]. Furthermore, attention is also being paid to the measurement of quantum correlation in multipartite systems [11, 12]. Such measures help us to understand the distribution of quantum correlations and provide a way of studying dissipative dynamics in many-part quantum systems.

Although it is claimed that the QD is more robust than the entanglement in open quantum systems, little concrete evidence has been published to corroborate this claim as far as multipartite systems are concerned. Actually, for two qubits, there is a good deal of evidence that this is true [13], but can we extend this rule to multipartite quantum systems? For three qubits, for example, there exist two distinct families of entanglement [14]; how robust, then, is the QD for members of each family? Could entanglement be more robust than discord, against noise arising from the environment, in multipartite systems? Little has been written on these questions in the literature [15] and that is the focus of this paper.

To investigate the robustness of the quantum correlations in open quantum systems for tripartite states, we define a multipartite measure of quantum correlation slightly different from that employed by Rulli and Sarandy [11]. Those authors define a measure of global multipartite QD as the maximum of the quantum correlations that exist among all possible bipartitions. Here, to attain an average measure, we define global QD as the sum of correlations for all possible bipartitions. Despite being only subtly different, our measure now accounts for how the quantum correlation is distributed in the tripartite system and certainly gives a better insight into the robustness of these correlations in open quantum systems.

This paper is organized as follows. In section 2, we present the formal definition of the entanglement of formation (EOF), the usual QD and the geometric measure of quantum discord

(GQD). In section 3, we define our multipartite measure of quantum correlations (MMQC) based on the sum of correlations for all possible bipartitions. In section 4, we present an analytical solution to the MMQC for a three-qubit pure state and we show that for a general tripartite pure state the MMQC based on the usual QD and on the EOF are equivalent. In section 5, we extend our analysis to a tripartite mixed state, taking the dissipative dynamics of three qubits into account, and in section 6, we summarize our results.

2. Quantum correlations

In this paper, we consider three well-known measures of quantum correlations: EOF, usual QD and GQD. Here, we present the formal definition of each of these measurements.

2.1. Entanglement of formation

EOF is a measure of entanglement defined more than 15 years ago by Bennett *et al* [16]. Although very different from QD, EOF is connected with the latter by a monogamic relation [8, 17] and has a nice operational interpretation. It is defined as follows. Given a bipartite system A and B, consider all possible pure-state decompositions of the density matrix ρ_{AB}, that is, all ensembles of states $|\Psi_i\rangle$ with probability p_i such that $\rho_{AB} = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$. For each pure state, the entanglement is defined as the von Neumann entropy of either of the two subsystems A and B, such that $E(\Psi) = S_A = S_B$, where $S_A := S(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A)$, ρ_A being the partial trace over B, and there is an analogous expression for S_B. The EOF for a mixed state is the average entanglement of the pure states minimized over all possible decompositions, i.e.

$$E(\rho_{AB}) = \min \sum_i p_i E(\Psi_i). \tag{1}$$

Although this is very hard to calculate for a general bipartite system, for two qubits there is an analytical solution given in the seminal Wootters [18] paper.

2.2. Usual quantum discord

QD is a well-known measure of quantum correlation defined by Ollivier and Zurek [5] about 10 years ago. It is defined as

$$\delta_{AB}^{\leftarrow} = I_{AB} - J_{AB}^{\leftarrow}, \tag{2}$$

where $I_{AB} = S_A + S_B - S_{AB}$ is the mutual information and $J_{AB}^{\leftarrow}$ is the classical correlation [19]. Explicitly,

$$J_{AB}^{\leftarrow} = \max_{\{\Pi_k\}} \left[S_A - \sum_k p_k S(\rho_{A|k}) \right], \tag{3}$$

where $\rho_{A|k} = \mathrm{Tr}_B(\Pi_k \rho_{AB} \Pi_k)/\mathrm{Tr}_{AB}(\Pi_k \rho_{AB} \Pi_k)$ is the local post-measurement state after obtaining the outcome k in B with probability p_k. QD measures the amount of mutual information that is not accessible locally [20, 21] and generally is not symmetric, i.e. $\delta_{AB}^{\leftarrow} \neq \delta_{BA}^{\leftarrow}$.

2.3. Geometric measure of quantum discord

Intuitively, QD can be viewed as a measure of the minimum loss of correlation due to measurements in the sense of quantum mutual information. A state with zero discord, i.e. $\delta_{AB}^{\leftarrow} = 0$, is a state whose information is not disturbed by local measurements; it is known as a classical-quantum (CQ) state. A CQ state is of the form [5]

$$\chi = \sum_{k=1}^{m} p_k |k\rangle\langle k| \otimes \rho_k^B, \tag{4}$$

where $\{p_k\}$ is a probability distribution, $|k\rangle$ an arbitrary orthonormal basis in subsystem A and ρ_k^B a set of arbitrary density matrices in subsystem B.

Denoting the set of all CQ states on $H_A \otimes H_B$ as Ω_0, it is natural to think that the farther a state ρ is from Ω_0, the higher is its QD. Indeed, we can use the distance from ρ to the nearest state in Ω_0 as a measure of discord for state ρ, and this is the idea behind GQD [22]. Thus, GQD was introduced as

$$D(\rho) = \min_{\chi \in \Omega_0} ||\rho - \chi||^2, \tag{5}$$

where Ω_0 denotes the set of zero-discord states and $||X - Y||^2 = \mathrm{tr}(X - Y)^2$ stands for the squared Hilbert–Schmidt norm. Note that the maximum value reached by the GQD is $1/2$ for two-qubit states, so it is appropriate to consider $2D$ as a measure of GQD hereafter, in order to compare it with other measures of correlation [23].

Interestingly, an explicit expression for GQD in a general two-qubit state can be written [22]. In the Bloch representation, any two-qubit state ρ can be represented as follows:

$$\rho = \frac{1}{4}\left(\mathbb{I}\otimes\mathbb{I} + \sum_{i=1}^{3} x_i\sigma^i\otimes\mathbb{I} + \sum_{i=1}^{3} y_i\mathbb{I}\otimes\sigma^i + \sum_{i,j=1}^{3} t_{ij}\sigma^i\otimes\sigma^j\right), \tag{6}$$

where $\mathbb{I}$ is the identity matrix, σ^i $(i = 1, 2, 3)$ are the three Pauli matrices, $x_i = \mathrm{tr}(\sigma^i\otimes\mathbb{I})\rho$ and $y_i = \mathrm{tr}(\mathbb{I}\otimes\sigma^i)\rho$ are the components of the local Bloch vectors $\vec{x}$ and $\vec{y}$, respectively, and $t_{ij} = \mathrm{tr}(\sigma^i\otimes\sigma^j)\rho$ are components of the correlation matrix T. Then the GQD of ρ is given by

$$D(\rho) = \frac{1}{4}\left(||\vec{x}||^2 + ||T||^2 - \lambda_{\max}\right), \tag{7}$$

$\lambda_{\max}$ being the largest eigenvalue of the matrix $K = \vec{x}\vec{x}^{\mathrm{t}} + TT^{\mathrm{t}}$ and $||T||^2 = \mathrm{tr}(TT^{\mathrm{t}})$. The superscript t denotes the transpose of vectors or matrices. Furthermore, it is important to mention that an analytical solution for a bipartite system of dimension $2 \times N$ has been given in [24].

3. Multipartite measure of quantum correlations (MMQC)

Definition 1. *For an arbitrary N-partite state $\hat{\rho}_{1,\dots,N}$, the multipartite measure of quantum correlation $\mathcal{Q}(\hat{\rho}_{1,\dots,N})$ is defined as follows.*

Let ρ be an N-partite state, and μ and ν be any subsets among all possible partitions. The MMQC is defined as the sum of the quantum correlations for all possible bipartitions,

$$\mathcal{Q}\left(\hat{\rho}_{1,\dots,N}\right) = \sum_{\mu\neq\nu=1}^{N} M_{\mu(\nu)}, \tag{8}$$

where $M_{\mu(\nu)}$ is a measure of quantum correlation that can be given by the GQD, the usual QD or the EOF. Here, the subset between $(\cdot)$ is the measured one in the case of GQD or QD and can be ignored for the EOF. It can be seen that the measure defined in equation (8) is symmetrical and, more importantly, it is zero if the state has just classical–classical correlations.

To elucidate the MMQC defined above, let us consider the case of a tripartite state. Explicitly,

$$
\begin{aligned}
\mathcal{Q}\left(\hat{\rho}_{ABC}\right) = {} & M_{A(B)} + M_{B(A)} + M_{A(C)} + M_{C(A)} \\
& + M_{B(C)} + M_{C(B)} + M_{A(BC)} + M_{BC(A)} \\
& + M_{B(AC)} + M_{AC(B)} + M_{C(AB)} + M_{AB(C)}.
\end{aligned}
\tag{9}
$$

4. MMQC analytical solution for three-qubit pure states

Our starting point is an analytical solution to the MMQC for three-qubit pure states. For the case of GQD, an analytical solution can be found with the help of the result presented in [24]. Actually, from that result, an analytical solution can be obtained for GQD even for three-qubit mixed states. The analytical solution for three qubits can also be obtained for the usual EOF. With the help of concurrence, each entanglement measurement involving two sets of one subsystem (e.g. $E_{B(C)}$) can be obtained trivially by calculating the Wootters formula [18]. Furthermore, for tripartite pure states, we note that the entanglement measure involving a set of one subsystem and a set of two (e.g. $E_{A(BC)}$) is given by the von Neumann entropy of one of the partitions. For example, $E_{A(BC)} = S_A = S_{BC}$.

Finally, to calculate the MMQC for the usual QD, we note the result given in [21], where the authors show that the sum of the QD for all possible bipartitions involving sets of one subsystem is equal to the sum of the EOF for all possible bipartitions,

$$
\begin{aligned}
& E_{A(B)} + E_{B(A)} + E_{A(C)} + E_{C(A)} + E_{B(C)} + E_{C(B)} \\
& = \delta^{\leftarrow}_{A(B)} + \delta^{\leftarrow}_{B(A)} + \delta^{\leftarrow}_{A(C)} + \delta^{\leftarrow}_{C(A)} + \delta^{\leftarrow}_{B(C)} + \delta^{\leftarrow}_{C(B)}.
\end{aligned}
\tag{10}
$$

Moreover, since we are considering pure states, the QD measure involving a set of one and a set of two subsystems is given by the entropy of one of the partitions, exactly as the EOF. For example, $E_{A(BC)} = \delta^{\leftarrow}_{A(BC)} = \delta^{\leftarrow}_{BC(A)} = S_A = S_{BC}$.

Thus, for general tripartite pure states, the MMQC defined in equation (9) is identical for the QD and the EOF, resulting in an analytical solution for the MMQC for three-qubit pure states, for the QD as well. To confirm this, we note the result given in equation (10), which is valid irrespective of the system dimension of the subsystems. On the other hand, for general tripartite states (not three-qubit), an analytical solution does not exist, either for the EOF or for the QD.

5. MMQC for three-qubit mixed states

To calculate the MMQC for three-qubit mixed states, we limit ourselves to studying a rank-2 density matrix. In this case, as we show below, a simple strategy can be used to calculate all terms of equation (9). Since we are considering three qubits, to calculate the MMQC for sets of one subsystem ($E_{A(B)}$, $E_{A(C)}$, $\delta^{\leftarrow}_{A(B)}$, $\delta^{\leftarrow}_{A(C)}$, etc) is trivial. Since in this case the terms are composed of two qubits, the EOF can be calculated analytically by means of concurrence [18],

and the QD can be calculated numerically by using positive-operator valued measurements (POVMs)[6]. So, the question is: how can we calculate, for a three-qubit mixed state, the MMQC for the terms that involve a set of one subsystem and a set of two, i.e. $E_{A(BC)}$, $E_{B(AC)}$, $\delta_{A(BC)}^{\leftarrow}$, $\delta_{B(AC)}^{\leftarrow}$, etc? In this case, in contrast to that of pure states, the von Neumann entropy cannot be used to calculate these terms. As we will show below, the answer to the above question is given by the monogamic relation between EOF and QD [8, 17].

In the dissipative dynamics that we will study below, at all times ρ_{ABC} is a rank-2 density matrix. As a consequence, the extra subsystem that purifies ρ_{ABC} is a two-level subsystem that we define here as E. Then, to calculate the MMQC, for the terms that involve a set of one subsystem and a set of two, the strategy is to calculate the quadripartite pure state ρ_{ABCE} that involves four qubits and, in sequence, use the monogamic relation. The monogamic relation implies that the EOF between two partitions is connected with the QD between one of the partitions and the third one that purifies the pair,

$$E_{A(BC)} = \delta_{A(E)}^{\leftarrow} + S_{A|E}. \tag{11}$$

To purify ABC, the first step is to write ρ_{ABC} in its diagonal form,

$$\rho_{ABC} = \lambda_1 |\Phi_1\rangle\langle\Phi_1| + \lambda_2 |\Phi_2\rangle\langle\Phi_2|, \tag{12}$$

where λ_i and $|\Phi_i\rangle$ are, respectively, the density matrix eigenvalues and eigenvectors, for $i = 1, 2$. The pure state is then written as

$$|\Psi_{ABCE}\rangle = \sqrt{\lambda_1}|\Phi_1\rangle|0\rangle + \sqrt{\lambda_2}|\Phi_2\rangle|1\rangle, \tag{13}$$

where the states $|0\rangle$ and $|1\rangle$ pertain to the two-level system Hilbert space of E. It is easy to verify that $\mathrm{Tr}_E\{|\Psi_{ABCE}\rangle\langle\Psi_{ABCE}|\} = \rho_{ABC}$.

Given the purification procedure, below we present in detail the strategy used to calculate the QD and the EOF for the rank-2 tripartite mixed state ρ_{ABC}. Here, we show how to calculate just the terms $E_{A(BC)}$, $\delta_{A(BC)}^{\leftarrow}$ and $\delta_{BC(A)}^{\leftarrow}$, but the strategy is analogous for all the other terms in equation (9) involving a set of one subsystem and a set of two. To calculate the EOF, we use the monogamic relation given by equation (11) where, as pointed out above, $A(E)$ involves a two-qubit system since ABC is of rank 2. In this case, to calculate $E_{A(BC)}$, we compute $\delta_{A(E)}^{\leftarrow}$ numerically and $S_{A|E}$ analytically. To calculate the QD, on the other hand, the result can be reached analytically by the expression

$$\delta_{A(BC)}^{\leftarrow} = E_{A(E)} + S_{A|E}, \tag{14}$$

where $E_{A(E)}$ can be computed by means of Wootter's concurrence. So, the remaining question is: how can we calculate $\delta_{BC(A)}^{\leftarrow}$? Once more, we use the monogamic relation. First we note that

$$\delta_{BC(A)}^{\leftarrow} = E_{BC(E)} + S_{BC|E}, \tag{15}$$

which relates the QD with the entanglement between BC and E. To calculate $E_{BC(E)}$, we recall that the EOF is symmetric, i.e. $E_{BC(E)} = E_{E(BC)}$, and use the monogamic relation

$$E_{E(BC)} = \delta_{E(A)}^{\leftarrow} + S_{E|A}. \tag{16}$$

---

[6] It is important to observe that to calculate QD for the two-qubit state, in the maximization of equation (3), it is necessary to use general POVMs rather than projection measurements to evaluate the maximum of the classical correlation. However, as well noted in [25], in this case, the difference is tiny.
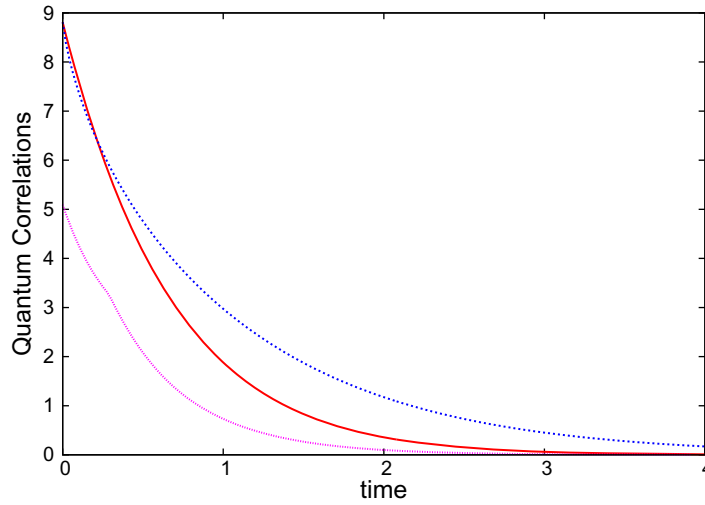
Figure 1. The red curve (solid) shows the EOF dynamics, while the blue curve (dotted) shows QD and the cyan curve (traced) the GQD: initial W state subjected to independent amplitude-damping channels.

Thus, substituting the equation above in equation (15) and noting that $S_{BCE} = S_A$, we obtain

$$\delta^{\leftarrow}_{BC(A)} = \delta^{\leftarrow}_{E(A)} + S_{A|E}, \tag{17}$$

which connects $\delta^{\leftarrow}_{BC(A)}$ with $\delta^{\leftarrow}_{E(A)}$. Note that the latter term involves two qubits and can be calculated numerically. Thus, following the recipe above, it is straightforward to calculate the quantum correlations for all terms of equation (9).

6. MMQC in open quantum systems

To study the MMQC in open quantum systems, we analyze two special situations: firstly, a three-qubit W state subjected to independent amplitude-damping channels and secondly, the GHZ state with independent phase damping. The reason for this specific choice is that, throughout the whole dissipative process, we have a rank-2 density matrix [26]. In this case, we can use the strategy explained in section 5 to calculate the MMQC for the usual QD and the EOF.

For these specific channels, the dissipative dynamics can be calculated straightforwardly by means of Kraus operators [27]. Since we assume independent environments for each qubit, given an initial state for three qubits $\rho(0)$, its evolution can be written as

$$\rho(t) = \sum_{\alpha,\beta,\gamma} E_{\alpha,\beta,\gamma} \rho(0) E^{\dagger}_{\alpha,\beta,\gamma}, \tag{18}$$

where the so-called Kraus operators $E_{\alpha,\beta,\gamma} \equiv E_\alpha \otimes E_\beta \otimes E_\gamma$ satisfy $\sum_{\alpha,\beta,\gamma} E^{\dagger}_{\alpha,\beta,\gamma} E_{\alpha,\beta,\gamma} = \mathbb{I}$ for all t. The operators $E_{\{\alpha\}}$ describe the one-qubit quantum channel effects. We first consider a W state subjected to independent amplitude damping. This damping describes the exchange of energy between the system and the environment and is described by the Kraus operators $E_0 = \sqrt{p}(\sigma_x + i\sigma_y)/2$ and $E_1 = \text{diag}(1, \sqrt{1-p})$, where $p = 1 - e^{-\Gamma t}$, Γ denoting the decay rate, and σ_x and σ_y are Pauli matrices. In figure 1, we show the dissipative dynamics of the MMQC for an initial state given by $|W\rangle = (|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$. We see that in this
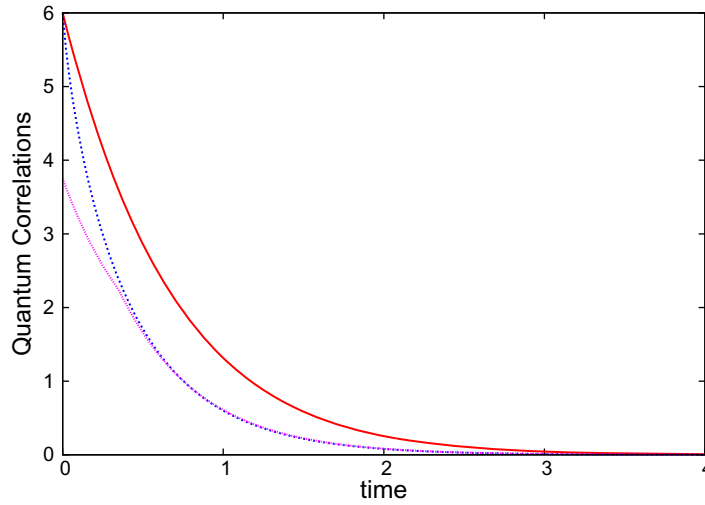
Figure 2. The red curve (solid) shows the EOF dynamics, whereas the blue curve (dotted) shows QD and the cyan curve (traced) represents the GQD: initial GHZ state subjected to independent phase-damping channels.

situation the QD is actually more robust than EOF and GQD. For a very short time, the EOF resists, but it decays fast while QD maintains greater robustness. This result corroborates the idea that the QD is more robust than the EOF in open quantum systems.

This peculiar situation occurs because of the conservative relation between the EOF and the QD [8],

$$E_{A(BC)} + E_{A(E)} = \delta_{A(BC)}^{\leftarrow} + \delta_{A(E)}^{\leftarrow}, \tag{19}$$

where $ABCE$ is a pure state with the three-qubit system represented by ABC and the environment by E. For the W state subjected to the amplitude-damping channel, figure 1 shows that QD is sustained in the system since the QD MMQC becomes greater than the entanglement MMQC. It means that, to maintain the conservative relation, the subsystems become entangled with the environment but create less discord. In equation (19), for example, if we have $E_{A(BC)} < \delta_{A(BC)}^{\leftarrow}$, i.e. the QD is greater than the EOF in the system ABC, necessarily $E_{A(E)} > \delta_{A(E)}^{\leftarrow}$, i.e. the entanglement between parts of the system and the environment is greater. Indeed, this property is valid for any bipartition present in the definition of equation (9), since it is impossible to create or destroy some amount of EOF (QD) in the system without destroying or creating the same amount of QD (EOF) with the environment.

In the second, and more important case, we consider an initial GHZ state subjected to phase damping. The dephasing channel induces a loss of quantum coherence without any energy exchange. In this case the Kraus operators are given by $E_0 = \text{diag}(1/\sqrt{2}, \sqrt{1-p})$ and $E_1 = \text{diag}(1/\sqrt{2}, \sqrt{p})$. In figure 2 we show the dissipative dynamics of the MMQC for an initial state given by $|GHZ\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$. Here, we see a very interesting result. Contrary to what is claimed in the literature, the EOF is in fact more robust than the QD for this kind of initial condition and quantum channel. EOF is sustained for a longer time than QD. This occurs because the phase-damping channel does not create any entanglement or discord between two parts of the subsystem. In other words, for any given time, $E_{A(B)} = E_{A(C)} = E_{B(C)} = 0$, $\delta_{A(B)}^{\leftarrow} = \delta_{A(C)}^{\leftarrow} = \delta_{B(C)}^{\leftarrow} = 0$ and $\delta_{B(A)}^{\leftarrow} = \delta_{C(A)}^{\leftarrow} = \delta_{C(B)}^{\leftarrow} = 0$. With this peculiar property, in this case

the MMQC for the EOF or the QD is given by

$$\mathcal{Q}\left(\hat{\rho}_{ABC}\right) = M_{A(BC)} + M_{BC(A)} + M_{B(AC)} + M_{AC(B)} + M_{C(AB)} + M_{AB(C)}. \tag{20}$$

To calculate equation (20) for the QD, we use, as explained above, the monogamic relation between the EOF and the QD. In this case, each of the six terms is given by

$$
\begin{aligned}
\delta_{A(BC)}^{\leftarrow} &= \delta_{BC(A)}^{\leftarrow} = E_{A(E)} + S_{A|E}, \\
\delta_{B(AC)}^{\leftarrow} &= \delta_{AC(B)}^{\leftarrow} = E_{B(E)} + S_{B|E}, \\
\delta_{C(AB)}^{\leftarrow} &= \delta_{AB(C)}^{\leftarrow} = E_{C(E)} + S_{C|E},
\end{aligned}
\tag{21}
$$

where the QD is symmetric because in this situation $S_A = S_{BC}$, $S_B = S_{AC}$ and $S_C = S_{AB}$.

Given the results in the equations above, what can we say about the entanglement of each part of the system (A, B or C) with the environment E, when the quantum state is initially in a GHZ state and is subjected to independent phase damping? In other words, are $E_{A(E)}$, $E_{B(E)}$ and $E_{C(E)}$ different from zero during the dissipative dynamics process? To answer this question, we calculate explicitly the density matrices ρ_{AE}, ρ_{BE} and ρ_{CE}. When an initial three-qubit GHZ state is subjected to independent dephasing environments the global state (system plus environment) is pure and is given by equation (13). In this situation, the eigenvalues are $\lambda_1(t) = 1 - e^{-3\Gamma t}$ and $\lambda_2(t) = 1 + e^{-3\Gamma t}$ and the eigenvectors are $|\Phi_1\rangle = (|111\rangle - |000\rangle)/\sqrt{2}$ and $|\Phi_2\rangle = (|111\rangle + |000\rangle)/\sqrt{2}$. Thus,

$$|\Psi_{ABCE}(t)\rangle = \sqrt{\lambda_1(t)}\left(\frac{|111\rangle - |000\rangle}{\sqrt{2}}\right)|0\rangle + \sqrt{\lambda_2(t)}\left(\frac{|111\rangle + |000\rangle}{\sqrt{2}}\right)|1\rangle. \tag{22}$$

Defining the density matrix $\rho_{ABCE}(t) = |\Psi_{ABCE}(t)\rangle\langle\Psi_{ABCE}(t)|$ and tracing out the subsystems B and C, the density matrix $\rho_{AE}(t) = \text{Tr}_{BC}\{\rho_{ABCE}(t)\}$ is given by

$$\rho_{AE}(t) = \frac{1}{2}|\phi_1(t)\rangle\langle\phi_1(t)| + \frac{1}{2}|\phi_2(t)\rangle\langle\phi_2(t)| \tag{23}$$

with $|\phi_1(t)\rangle = |1\rangle(\sqrt{\lambda_1(t)}|0\rangle + \sqrt{\lambda_2(t)}|1\rangle)$ and $|\phi_2(t)\rangle = -|0\rangle(\sqrt{\lambda_1(t)}|0\rangle - \sqrt{\lambda_2(t)}|1\rangle)$. Thus, since $|\phi_1(t)\rangle$ and $|\phi_2(t)\rangle$ are separable states for any time t, $\rho_{AE}(t)$ is also separable. Furthermore, tracing out the subsystems A and C or A and B from $\rho_{ABCE}(t)$ we note that $\rho_{AE}(t) = \rho_{BE}(t) = \rho_{CE}(t)$ which proves that $E_{A(E)} = E_{B(E)} = E_{C(E)} = 0$. So, the proof above shows that *none* of the subsystems of the GHZ state becomes entangled with the environment, if each one is subjected to an independent phase-damping channel. In this case, the MMQC for the QD can be calculated analytically by means of the conditional entropy. Indeed,

$$\mathcal{Q}_{QD}\left(\hat{\rho}_{ABC}\right) = 2(S_{A|E} + S_{B|E} + S_{C|E}). \tag{24}$$

For the EOF, on the other hand, the situation is a little different. In this case, the six terms are given by

$$
\begin{aligned}
E_{A(BC)} &= E_{BC(A)} = \delta_{A(E)}^{\leftarrow} + S_{A|E}, \\
E_{B(AC)} &= E_{AC(B)} = \delta_{B(E)}^{\leftarrow} + S_{B|E}, \\
E_{C(AB)} &= E_{AB(C)} = \delta_{C(E)}^{\leftarrow} + S_{C|E}.
\end{aligned}
\tag{25}
$$

The crucial difference between equations (21) and (25) is that, while each part of the system does not become entangled with the environment, it does create QD with it. In other words,

the phase-damping channel, acting independently over each qubit, creates QD between the subsystems and the environment. Furthermore, given the initial symmetry of the GHZ state, we find that $\delta_{A(E)}^{\leftarrow} = \delta_{B(E)}^{\leftarrow} = \delta_{C(E)}^{\leftarrow} \neq 0$ and, consequently,

$$\mathcal{Q}_{EOF}\left(\hat{\rho}_{ABC}\right) = \mathcal{Q}_{QD}\left(\hat{\rho}_{ABC}\right) + 2\left(\delta_{A(E)}^{\leftarrow} + \delta_{B(E)}^{\leftarrow} + \delta_{C(E)}^{\leftarrow}\right).$$

This is an important result and a direct consequence of the conservative relation between the EOF and the QD where we are concerned with the quadripartite system $ABCE$ (three qubits plus environment). For the GHZ state subjected to phase damping, there is no entanglement between each subsystem (A, B or C) and the environment (E) but there is QD. Thus, the QD that is created with the environment needs to be compensated by the entanglement retained in the system, making the EOF more robust than the QD in this particular situation. It must be emphasized that this is a direct consequence of the GHZ being a genuine multipartite entangled state, which means that the EOF between any set of two parts ($E_{A(B)}$, $E_{A(E)}$, $E_{B(C)}$, etc) is always zero during dissipative dynamics.

7. Conclusion

In this paper, we have presented an alternative measure of multipartite quantum correlations. Our measure gives a novel and intuitive means of comparing the robustness of entanglement and discord in multipartite systems, against the detrimental interaction with the environment. We analyze two distinct initial conditions, which involve different kinds of multipartite entanglement. We show that the robustness of the EOF depends on the family of entanglement present in the initial state, raising the question of whether it is greater than the robustness of QD in open quantum systems. Actually, for a three-qubit W state, QD proves to be more robust, but the same cannot be said about the GHZ state. We show that this behavior is related to the way that the multipartite quantum state is quantum correlated with the environment. For the GHZ state subjected to independent phase-damping channels, the individual qubits do not become entangled with the environment, but to create QD with it. Thus entanglement is preserved for a longer time than the QD. We believe that the discussion presented here may contribute further to the understanding of the distribution of entanglement and discord in open quantum systems.

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Note added. While finishing this paper we became aware of related work [28] where the author reached some complementary conclusions to those presented here.

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