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On the Bäcklund and Darboux transformations for the Tzitzéica model

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Para minhas tias, Lúcia & Luciene

. . . once or twice she had peeped into the book . . . but it had no pictures or conversations in it, "and what is the use of a book," thought Alice "without pictures or conversation?"

Alice's Adventures in Wonderland (Lewis Carroll)

"Reason must approach nature with the view, indeed of receiving information from it, not however in the character of a pupil, who listens to all that his master chooses to tell him, but in that of a judge, who compels the witnesses to reply to those questions which he himself thinks fit to propose."

Critique of Pure Reason (Kant)

"I think nature's imagination is so much greater than man's, she's never going to let us relax."
"But you gotta stop and think about it, about the complexity to really get the pleasure and it's all really there, the inconceivable nature of nature."

Fun to Imagine (Richard Feynman)

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Resumo

Consideramos dois métodos, chamados transformações de Darboux e Bäcklund, para geração de soluções solitonicas no modelo integrável de Tzitzéica. No contexto de modelos com defeitos, tratamos essas transformações e percebemos que elas estão escondidas no sistema sob a forma de condições sobre o defeito. Por fim, usando as profundas relações entre teorias clássicas de superfícies e solitons, mostramos que os métodos de Bäcklund e Darboux estão intimimante relacionadas com a clássica transformação de Tzitzéica.

Palavras Chaves: Solitons; Tzitzéica; Bäcklund; Darboux; Invariância de Calibre.

Áreas do conhecimento: Física Matemática; Teoria de Campos; Modelos Integráveis.

Abstract

We consider two methods, called Darboux and Bäcklund transformations, for generating of solitonic solutions in the Tzitzéica integrable model. In the context of models with defects, we treat these transformations and we realized that they are hidden under the form of conditions over the defect. At the end, using the deep relations between classical theories of surfaces and solitons, we show that the Bäcklund and Darboux methods are intimately related with the classical Tzitzéica transformation.

Key Words: Solitons; Tzitzéica; Bäcklund; Darboux; Gauge Invariance.

Areas: Mathematical Physics; Field Theory; Integrable Models.

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Chapter 1

Introduction and General Overview

NON-LINEAR THEORIES play an important role in modern science, the phenomena described for some sorts of such theories, are endowed with two kinds of behaviour which are not mutually exclusive, "chaotic" and "deterministic". We are interested in the second type and *Solitons* are within this class of phenomena. Physicists and mathematicians share a deep interest in systems with regular behaviour, besides, it is not unusual for biologists to apply solitons in their works.

Therefore, besides several applications in physical phenomena, such as *solitary wave in water* and *signal in optical fibre*, in biological systems, such as *chemical energy transport in proteins*, solitons provide an effective mathematical lab, for studying of non-linear partial differential equations, differential geometry of surfaces, Lie theory and so on [1, 2].

A well known story, is that *John Scott Russell*, in 1834, observing a boat being drawn by two horses along a channel in Edinburgh, noticed that, after the boat suddenly stopped, a bow wave, with a shape of a solitary wave continued its motion forward - without changing its form or velocity. Russell, perhaps because he was a naval engineer, realized that such a wave was not an ordinary phenomenon and, he followed that solitary wave, until lost it, two miles later.

The scientist built a tank where he kept studying what he named *wave of translation*, and he could notice some of its properties, through several experiments; however, the length of the tank and consequently, the short duration of the wave, did not allow the scientist to be able to see the interaction of two solitons [1, 3, 4].

In 1895, Korteweg and de Vries derived the equation, known today as *KdV equation*, which describes water waves in shallow channels, however, a less known fact is that, a pair of equivalent equations had already been found by Boussinesq at 1870, which describe waves propagation in rectangular channels, and independently, by Rayleigh that studied and explained mathematically the Russell's observation [1, 4].

Zabusky and Kruskal, via a computational approach, could rediscover the KdV equation in the continuum limit of an anharmonic lattice model and they could consider its solutions. By the way, they named these solutions as *solitons*, to denote the particle-like behaviour of these solitary waves under interaction, in this sense, this is just a "parody" of protons, electrons, neutrons and so on [1, 4].

The modern study of theory of solitons began in this context, when the computational tools could be used, and solitons - at this time, still a curiosity - have become a useful mathematical tool.

But, what really is a soliton? This question does not have a unique answer and today there are several uses of this term in some different contexts. The solitons that we are interested in, can be described as localized and stable solutions of a completely integrable system. By *localized and stable*, we mean that they look like solitary waves or lumps, which can be scattered without loss of their characteristics including, their shape, even when interact with other solitons. The stability of these solutions are guaranteed by infinitely many dynamical conserved quantities in the system [5, 6].

Another type of solitons (that we will not deal in this work), are those which have their stability guaranteed by discrete homotopy invariants, in this sense, they are topological solitons. *Kinks in one dimension, vortices in two dimensions, monopoles, skyrmions and instantons* are some examples of topological solitons which have a huge value in modern science [5, 7, 8].

$*\quad*\quad*$

On the other hand, the relation between theory of surfaces and theory of solitons, is a well understood subject today, and maybe, due the large "overlap" that exists between these two areas, sometimes they appear to be the same study under different perspectives [4, 9, 10, 11, 12].

Differential geometry of curves and surfaces have its origin in the early XIX century, mainly in the works of *Gaspard Monge*$(1746 - 1818)$ and *Carl Friedrich*

Gauss or simply *Gauß* (1777 – 1855). Monge, whose main work in differential geometry of surfaces is compiled at *Application de l'analyse à la géométrie*, published in 1807 [13], was responsible for a descriptive study of curves and surfaces. He was an engineer, and it was an important detail in his work, because, Monge has seen the surfaces strongly related with the surrounding space and tried to make a relation between these surfaces and partial differential equations. Most of his work were generated by this attempt.

One of the greatest scientists of all times, certainly was Gauss. His approach to surfaces theory started with the attempt to measure quantities, such as distance, on the surface of the Earth. Thereby, he realized that to study a generic two dimensional surface, he does not necessarily need to attach the surface to a three dimensional space, but one may assign to that surface, its own intrinsic geometric properties [14]. As we know today, this was a paradigm that grew up and from which emerged the development of non-Euclidean geometries – that have found several applications in physics, including the so charming *General Relativity.* The fundamental contribution of Gauss in surfaces theory, *Disquisitiones generales circa superficies curvas* (*General investigations of curved surfaces*), published in 1827 [15], is a mathematical masterpiece still today.

Gauss set down a system of equations, known evidently as *Gauss equations*, that are fundamental in the description and analysis of surfaces. Furthermore, from the symmetries and compatibility of this system of equations – for a special class of surfaces – arise the remarkable connection between classical differential geometry of surfaces and solitons theory.

The special surfaces referred in last paragraph are named *hyperbolic surfaces*: those with negative Gaussian curvature[1]. For instance, the celebrated *sine-Gordon* equation

$$\omega_{\rho\varrho} = \frac{1}{a^2}\sin\omega,$$

that was born in the Edmond Bour description of hyperbolic surfaces with constant curvature $\kappa := -1/a^2$ in 1862, and in this sense, a *pseudo-spherical* surface [1, 3, 4, 16]. Independently, Bonnet in 1867 and Enneper in 1868, through the same consideration of pseudo-spherical surfaces, rediscovered the sine-Gordon equation.

[1]It is worth remembering that the total curvature of a surface S is given by $\int\int_S \kappa dS$, where κ is the Gaussian curvature. As this is an easy formula to remember, we will avoid to use the term *total curvature*.

Underlying the hyperbolic surfaces description in terms of differential equations systems, such as Gauss equations, and its relation with solitons theory, the success of these two areas – and of the differential geometry as a whole –, are due the procedures of generating iteratively surfaces of a type, from a previous one of the same type, called *seed* [17]. Albert Victor Bäcklund $(1845-1922)$, Gaston Darboux $(1842-1917)$ among others important mathematicians, investigated these mechanism of generating surfaces and the classes of surfaces which admit those kind of transformations. Moreover, the transformations that bear theirs names are today a well established subject in differential geometry of hyperbolic surfaces, also, the invariance under these transformations seems to be a shared property of the solitonic equations [4, 9, 16].

In the sine-Gordon case, for example, the iterative procedure comes from a geometric construction of pseudo-spherical surfaces. Furthermore, the new solution of the sine-Gordon equation is result of the compatibility conditions, that the system of equations which describe the new surface recently constructed, must satisfy. Rogers & Schief, at [16], summarize wonderfully this effort as

> *"... if a point P is taken on an initial pseudo-spherical surface Σ and a line segment PP' of constant length and tangential to Σ at P is constructed in a manner dictated by a Bäcklund transformation ... then the locus of the points P' as P traces out Σ is another pseudo-spherical surface Σ' with the same total curvature as Σ. The procedure may be iterated to generate a sequence of pseudo-spherical surfaces all with the same total curvature as the original seed surface Σ."*

This procedure yields the well known Bäcklund transformation for the sine-Gordon equation

$$\left(\frac{\omega'-\omega}{2}\right)_\rho = \frac{\beta}{a}\sin\left(\frac{\omega'+\omega}{2}\right),$$
$$\left(\frac{\omega'+\omega}{2}\right)_\varrho = \frac{1}{\beta\, a}\sin\left(\frac{\omega'-\omega}{2}\right),$$

where β is what is known as *Bäcklund parameter*, $a=(-\kappa)^{-1/2}$ is the *pseudo-radius* and $\omega,\ \omega'$ are two solutions of the sine-Gordon equation. It is worthwhile to comment that the Bäcklund parameter is far to be a negligible constant. Actually, it has a deep connection with a non-linear superposition principle, that is embodied in something known as *permutability theorem*, and has as a consequence, the commutativity of two successive Bäcklund transformations.

* * *

Sustained by these relations between solitons and surfaces theories, the statement mission of this work is to consider the transformations of the *Tzitzéica equation* under these two perspectives, the *solitonic* and the *geometric*. Along this work we will speak a little more about the historical contribution of the Romanian mathematician Tzitzéica under these two approaches, pointing some important aspects, the real motivation and the modern interest in this model.

It is worth commenting that, Tzitzéica started studying a class of hyperbolic surfaces whose have Gaussian curvature satisfying the condition that bears his name,

$$\kappa \sim -d^4,$$

where d is the distance from the origin to a generic point in the surface. Due its importance in affine geometry, the Tzitzéica surface is called *affinsphären* or *affine-spheres*. Remarkably, Tzitzéica set down not only a linear representation of this surface – equivalent in modern language to the Lax pair – but also, he found a Bäcklund-like transformation that bears his name.

The solitonic Tzitzéica equation

$$(\ln v)_{\rho\varrho} = v - v^{-2}$$

arises from the compatibility conditions of the Gauss system for the surface, as we already know, and as example of a physical system which this equation is naturally adopted, consider the *anisentropic gasdynamics system*. Also, this last equation keeps important connections with *TODA lattice model*, as shown in [4, 16].

The first part of this work, is devoted to the solitonic approach. Then, in the next chapter, we give the basic framework for the study of integrable models. We present the notion of integrability, Poisson structure, Lax pair and zero curvature equation for continuous systems. We chose a non standard way to present these concepts in view of completeness, however, several good texts discuss the same issues in a more algebraic fashion [5, 6, 18].

Chapter 3 is devoted to the Tzitzéica model, the criteria which made us to divide this text in part I and II will be clear in that chapter. We discuss the Lax pair and the zero curvature condition, after, using the gauge invariance, we propose the Darboux and Bäcklund transformation for this beautiful model.

The application of the Bäcklund transformation may be realized in the chapter 4. The method of Corrigan & Zambon is dealt here, where we consider an integrable system built by two domains and its interface is thought as a defect. The conditions in the defect will determine the behaviour of the whole system and of its conservation laws, including energy and momenta [19, 20]. The key point in this chapter is that, at the defect position, the Bäcklund transformation naturally appears as conditions over the defect. Parallel to this work, we have found at [21], infinitely many conserved quantities, which guarantees that the introduction of the defect for the Tzitzéica model does not spoil its integrability.

The solitonic context is abandoned and a less natural way to deal with these kind of systems is started in the second part, under the perspective of surfaces. Then, a new introduction will be necessary now, and we describe in the chapter 5, the basic theory of surfaces, following the methodology of Monge and Gauss.

In the next chapter, we show the notion of hyperbolic surfaces and affinsphären. Also, the prototype of a generalization of the treatment of soliton theory is tasted, in this sense, we can see that the soliton theory can admit an approach under the concepts of surface [4, 16].

Finally, in the chapter 7, we show that the Bäcklund transformation, previously found in the third chapter, hides the classical Tzitzéica transformation. And we consider at the end, a naive example of this transformation.

In order to be coherent and self-consistent, we would like to use some mathematical concepts and terms that are not too familiar among most of the physicists. Then, we offer some mathematical appendices, that are not mandatory but can be very useful for some readers and evidently to the author itself. Thus, from the main text together with the appendices, we hope that the whole work can be read, just based in the work itself.

Part I

Solitonic Context

Chapter 2

Integrable Models

One can, in a heuristic way, define the subject that we are interested in, saying: *An integrable model consists of non-linear differential equations which can be solved analytically, at least "in principle".*
Since mathematicians as well as physicists are quite interested in these models, today there are several works, looking through their theoretical and experimental aspects. The formal characteristics of the integrable models, concepts such as *infinite dimensional Lie algebras and their representations* and new subjects that were born in the core of differential geometry as well as in the Sturm-Liouville problem – like *Bäcklund, Moutard, Darboux transformations* and so on –, are the main interests shared by mathematicians. Physicists, naturally, are interested in the possibility of applying these models in physical phenomena, besides, the solitonic solutions of these models emerge as a good opportunity to test new ideas in the areas of *non-linear optics, hydrodynamics, condensed matter, continuous mechanics, plasma physics and high energy physics* [4, 5, 6]. In fact, solitons are the strongest tools for non-perturbative approach in various theories, from the hydrodynamics to string theory.

In this chapter, the goal is to present in a very succinct fashion, the classical ideas of integrable systems. We start with the general aspects of a Hamiltonian system and present the *Liouville theorem*, which states what an integrable system is. After, we talk a little bit about the geometry of *phase space*, we point the general aspects of a *symplectic manifold* and finally, the *Lax pair* is presented. It is worth remembering that we will try in this chapter, give just the fundamental concepts that we will need to attack the Tzitzéica model, however, for those who are not familiar with the subject of integrable models, we strongly recommend a reading of the books [5, 6] for a complete treatment of this topic.

2.1 Hamiltonian System

The evolution of some mechanical systems with n degrees of freedom can be described by the Hamilton's equations, in such a case, one says that those systems are *Hamiltonian systems*. The motion of such a system can be analyzed in a $2n$-dimensional space, which is spanned by the coordinate functions $(q_i, p_i),\ i = 1, \ldots, n$, i.e. the canonical coordinates and the momenta respectively. This space, that one denotes by $\mathcal{M}$, is called *phase space*, and it is locally euclidean, it means that in a neighborhood of a point, this space looks like the $\mathbb{R}^{2n}$, but globally, it can be a non-trivial manifold[1].

One can consider differentiable functions (that are called *dynamical variables*) $f, g : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ such that $f = f(q,p;t)$ and $g = g(q,p;t)$ [2], where p, q are the coordinates of the phase space $\mathcal{M}$ and t is the evolution parameter, which is usually called *time*. Also, one can define now, the famous *Poisson bracket* as

$$\{f, g\} := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right), \tag{2.1}$$

that satisfies two important properties,

$$\begin{aligned} &\{f, g\} = -\{g, f\} && \text{skew-symmetry} \\ &\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0. && \text{Jacobi identity} \end{aligned}$$

When $\{f, g\} = 0$, one says that f and g are in *involution*.

[1]In a mathematical language, the configuration space where the system lives in, is the manifold $\mathcal{N}$ and the points of such a manifold are denoted by q. In addition, one knows that the momenta, in the Lagrangian mechanics, are defined by $p := \frac{\partial L}{\partial \dot{q}}$, so the momenta lives in the cotangent space $\mathcal{TN}_q^*$. Thus, the phase space is the cotangent bundle $\mathcal{M} = \mathcal{TN}^* := \{(q,p) \mid q \in \mathcal{N},\ p \in \mathcal{TN}_q^*\}$ [22].

[2]The dependence of a dynamical variable f on the parameter t is usually given through the coordinate functions q and p, so it is usual to split this dependence, writing $f = f(q,p;t) \equiv f(q(t), p(t))$ instead of $f = f(q,p,t)$. This last notation, one often uses when there exists an explicit dependence on t.

The well known results about Hamiltonian system are summed by: Given a system with n degrees of freedom, coordinate functions (q_i, p_i), $i = 1, \ldots, n$ and a Hamiltonian $H = H(q, p; t)$, the Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{2.2}$$

since the Poisson bracket of the coordinate functions (q_i, p_i) satisfies

$$\{q_i, q_j\} = 0 = \{p_i, p_j\} \quad \text{and} \quad \{q_i, p_j\} = \delta_{ij}.$$

Considering again, the function $f = f(q, p; t)$, one can write the total derivative

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial f}{\partial p_i} \frac{dp_i}{dt} \right) \\ &= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \\ &= \{f, H\}, \end{aligned} \tag{2.3}$$

where the Hamilton's equations of motion have been used and $\partial f / \partial t = 0$, since the function f has no explicit dependence on the time.

A function $f = f(q, p; t)$ such that $\dot{f} = 0$ when (2.3) is valid, is called a *first integral* or as physicists usually call, a *constant of motion*. Evidently, the Hamiltonian itself is a constant of motion, since $\{H, H\} = 0$. In addition, in the case when the dynamical variable f is equal the coordinate function p_i or q_i, one has the Poisson bracket

$$\dot{q}_i = \{q_i, H\} \quad \text{and} \quad \dot{p}_i = \{p_i, H\}.$$

Let us now, enunciate two important theorems,[3]

Theorem 1 (Liouville) *A Hamiltonian system is integrable by the method of quadratures, if and only if, it is a $2n$ dimensional phase space $\mathcal{M}$ with a set of n functionally independent functions that are in involution, i.e. $\kappa_1, \ldots, \kappa_n$, such that*

$$\{\kappa_i, \kappa_j\} = 0, \quad i, j = 1, \ldots, n.$$

so, the functions κ_i are constants of motion.

[3]The reader can find the proofs in [5], further references are in [23, 24].

Theorem 2 (Arnold-Liouville) *Consider now, the integrable system*

$$(\mathcal{M}, \kappa_1, \ldots, \kappa_n)$$

and the Hamiltonian $H := \kappa_1$. Also, let

$$\mathcal{M}_{\kappa_i} := \{(p,q) \in \mathcal{M} \mid \kappa_i(p,q) = constant\} \qquad i = 1, \ldots, n,$$

be an n-dimensional level surface of constants of motion.

When $\mathcal{M}_{\kappa_i}$ is compact and connected, this surface is diffeomorphic to the torus

$$T^n := S^1 \times \cdots \times S^1,$$

and we can define the coordinates,

$$I_1, \ldots, I_n; \phi_1, \ldots, \phi_n, \qquad 0 \leq \phi_i \leq 2\pi,$$

that are known as action-angle *coordinates, respectively. The angle ϕ_i are coordinates on the level surface $\mathcal{M}_{\kappa_i}$ and the actions I_i are the constants of motion.*

With this construction, the Hamilton's equations are

$$\dot{I}_i = 0 \quad and \quad \dot{\phi}_i = \omega_i(I_1, \ldots, I_n).$$

Integrating these equations, we have

$$\phi_i(t) = \omega_i(I)t + \phi_i(0) \qquad and \qquad I_i(t) = I_i(0),$$

that are n circular motions with constant angular velocities. We can realize that, it is always possible to solve an integrable model through a sequence of algebraic operations and integrations.

2.2 Poisson Structures

The coordinate functions (q_i, p_i) in a $2n$-dimensional manifold $\mathcal{M}$ can be written in a most natural form when one combines the positions q_i with the momenta p_i by:

$$y^a := (q_i, p_i), \quad \begin{cases} i = 1, \ldots, n \\ a = 1, \ldots, 2n \end{cases}, \tag{2.4}$$

in a way that

$$y^i = q_i \quad \text{and} \quad y^{i+n} = p_i.$$

With this definition, the Poisson bracket of the coordinate functions are

$$\{y^a, y^b\} := \epsilon^{ab}, \tag{2.5}$$

where the antisymmetric $2n \times 2n$ matrix

$$\epsilon^{ab} := \begin{pmatrix} 0 & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & 0 \end{pmatrix}$$

has been defined.

In the same way, one can evaluate the Poisson bracket of two dynamical functions by

$$\begin{aligned}
\{f, g\} &= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} \right) \\
&= \sum_{a,b=1}^{2n} \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y^a}\frac{\partial y^a}{\partial q_i}\frac{\partial y^b}{\partial p_i}\frac{\partial g}{\partial y^b} - \frac{\partial g}{\partial y^b}\frac{\partial y^b}{\partial q_i}\frac{\partial y^a}{\partial p_i}\frac{\partial f}{\partial y^a} \right) \\
&= \sum_{a,b=1}^{2n} \left\{ \frac{\partial f}{\partial y^a} \left[\sum_{i=1}^{n} \left(\frac{\partial y^a}{\partial q_i}\frac{\partial y^b}{\partial p_i} - \frac{\partial y^b}{\partial q_i}\frac{\partial y^a}{\partial p_i} \right) \right] \frac{\partial g}{\partial y^b} \right\} \\
&:= \frac{\partial f}{\partial y^a}\{y^a, y^b\}\frac{\partial g}{\partial y^b} := \epsilon^{ab}\partial_a f\, \partial_b g,
\end{aligned} \tag{2.6}$$

where the implicit summation over repeated indices and

$$\partial_a \equiv \frac{\partial}{\partial y^a},$$

have been defined.

Naturally, given the Hamiltonian $H : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$, the generalized Hamilton's equations are

$$\dot{y}^a = \{y^a, H\} = \epsilon^{ab}\partial_b H. \tag{2.7}$$

We have seen that, with a little bit of sophistication, we could treat the Poisson bracket in a very natural way. When we thought in the Hamiltonian structure in a geometric fashion, the Poisson bracket seems to gain a "*cause*". After this little effort, we are going to consider the classical "road" for the generalization of the Hamiltonian systems.

Evidently, we will not to study each "minutia" and definitely, we will not arrive too far, but in the end of this chapter, we will to know exactly what is enough to understand the mathematical structure of an integrable model and its geometric approach, that is what we are really interested. From now on, the aim of this chapter is to present some geometric ideas of the Hamiltonian systems and their impact over the integrable models, that is, we are looking to the Lax Pair.

2.2.1 Symplectic Structure

The ideas behind the Poisson bracket provide a generalization of a geometric approach to the Hamiltonian systems, let us clarify this affirmation with "work", because, this is the way that physicists like. Consider, then, a p-dimensional manifold $\mathcal{M}$ with coordinates functions $(x^1, \ldots, x^p)$.
Definition: A skew-symmetric matrix $\omega^{ab} = \omega^{ab}(x)$ is called a *Poisson structure* if the Poisson bracket, defined by

$$\{f, g\} := \sum_{a,b=1}^{p} \omega^{ab}(x) \frac{\partial f}{\partial x^a} \frac{\partial g}{\partial x^b} = \omega^{ab}(x) \partial_a f \; \partial_b g,$$

satisfies the well known skew-symmetry property and the Jacobi identity. Evidently, one gets

$$\omega^{ab}(x) = \{x^a, x^b\}$$

when the coordinates functions are used in the Poisson bracket.

In this point of view, the Hamiltonian function is $H : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ and the generalized Hamilton's equations are

$$\begin{aligned} \dot{x}^a = \{x^a, H\} &= \omega^{cb} \partial_c x^a(x) \partial_b H = \omega^{cb} \delta_c{}^a \partial_b H \\ &= \omega^{ab}(x) \partial_b H. \end{aligned} \tag{2.8}$$

Furthermore, everything known about Hamiltonian system can be recovered, when one sets in the manifold $\mathcal{M}$, the dimension $p = 2n$ and one identifies the coordinate functions $y^a \in \mathcal{M}$ with the generalized phase space coordinates, i.e. $x^a \equiv y^a$. Whereas the functions y^a form a basis of the phase space $\mathcal{M}$, then, ω^{ab} must be non-singular, so, the inverse matrix $\omega_{ab} = (\omega^{ab})^{-1}$ can be defined by

$$\omega^{ab} \omega_{bc} := \delta^a_c := \omega_{cb} \omega^{ba}. \tag{2.9}$$

In addition, it is known that the Jacobi identity is satisfied, i.e.

$$\{y^a, \{y^b, y^c\}\} + \{y^b, \{y^c, y^a\}\} + \{y^c, \{y^a, y^b\}\} = 0,$$

and from this identity, follows the *Bianchi identity*.

$$\partial_a \omega_{bc} + \partial_b \omega_{ca} + \partial_c \omega_{ab} = 0,$$

In this case, ω_{ab} is called a symplectic structure and the space $\mathcal{M}$ endowed with such a structure is called a symplectic manifold. This symplectic structure ω_{ab} can be used in an analogous form to the Riemannian metric[4] $g_{\mu\nu}$, what we mean is that, this structure can be used to lower indices and in this sense, is the metric of a symplectic manifold. In the same way, the Poisson structure ω^{ab} can be used to raise indices in the same manifold $\mathcal{M}$.

It is worthwhile to comment that, formally, a symplectic manifold $\mathcal{M}$ is more general than we have defined here [25], so, the phase space is just, a special case of a symplectic manifold [6]. One of the most important features of the phase space of an integrable model can be summarized, when one realizes that, since we are dealing with a Hamiltonian system with n degrees of freedom and as it is integrable, we can think of each conserved quantity $\kappa_i,\ i = 1, \ldots, n$ as a Hamiltonian [18]. We conclude that, since for an integrable model there are at least two distinct Hamiltonians, in the symplectic manifold of this model there exist at least two distinct symplectic structures. Next section, we will start quantifying these last words and the general features of the phase space of an integrable model will be more evident.

2.3 Phase Space of an Integrable Model

Considering from now that there are two distinct symplectic structures in some dynamical system, this is equivalent to consider the dynamics generated by two distinct Hamiltonians, or more basically, two distinct Lagrangians, then

$$L = \pi_a(y)\dot{y}^a - H(y) \tag{2.10}$$

and

$$\bar{L} = \Pi_a(y)\dot{y}^a - \bar{H}(y), \tag{2.11}$$

[4]Appendix A page 76 for details.

where $a = 1, \ldots, 2n$, the generalized momenta are π and Π, and the dot over the coordinates functions, denotes the evolution parameter derivative, that is naturally, the *time*. The Euler-Lagrange equations for this system are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}^a} - \frac{\partial L}{\partial y^a} = 0 \;\Rightarrow\; (\partial_a \pi_b - \partial_b \pi_a)\dot{y}^b = \partial_a H,$$

and

$$\frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{y}^a} - \frac{\partial \bar{L}}{\partial y^a} = 0 \;\Rightarrow\; (\partial_a \Pi_b - \partial_b \Pi_a)\dot{y}^b = \partial_a \bar{H},$$

and from equations (2.8) and (2.9) one can write

$$\begin{aligned} \omega_{ab}\; \dot{y}^b &= \partial_a H, \\ \Omega_{ab}\; \dot{y}^b &= \partial_a \bar{H}, \end{aligned} \tag{2.12}$$

where the symplectic structures

$$\omega_{ab} := \partial_a \pi_b - \partial_b \pi_a$$

and

$$\Omega_{ab} := \partial_a \Pi_b - \partial_b \Pi_a,$$

have been defined, each generating its own Poisson structure [6]. Then, from the definition of Poisson bracket, one gets

$$\begin{aligned} \{f, g\}_L &= \omega^{ab}\; \partial_a f\; \partial_b g \\ \{f, g\}_{\bar{L}} &= \Omega^{ab}\; \partial_a f\; \partial_b g. \end{aligned} \tag{2.13}$$

Finally, the Hamilton's equations are

$$\begin{aligned} \dot{y}^a &= \omega^{ab}\partial_b H, \\ \dot{y}^a &= \Omega^{ab}\partial_b \bar{H}, \end{aligned}$$

that describe, naturally, the same dynamic

$$\omega^{ab}\partial_b H = \Omega^{ab}\partial_b \bar{H}.$$

2.3.1 The Lax Pair

Firstly, a new quantity in the manifold[5] must be defined with the symplectic structure above, then, let

$$\mathcal{S}_a^b := \omega_{ac}\Omega^{cb} \tag{2.14}$$

be such a quantity.

The next step is to consider the cross derivative of the equations (2.12), so, by consistency

$$\partial_a\, \partial_b H \equiv \partial_b\, \partial_a H,$$

which implies

$$\begin{aligned} 0 &= \partial_a(\omega_{bc}\, \dot{y}^c) - \partial_b(\omega_{ac}\, \dot{y}^c) \\ &= (\partial_a\omega_{bc})\dot{y}^c + \omega_{bc}\partial_a\dot{y}^c - (\partial_b\omega_{ac})\dot{y}^c - \omega_{ac}\partial_b\dot{y}^c \\ &= (\partial_a\omega_{bc} + \partial_b\omega_{ca})\dot{y}^c + \omega_{bc}\partial_a\dot{y}^c - \omega_{ac}\partial_b\dot{y}^c, \end{aligned} \tag{2.15}$$

using now, the Bianchi identity

$$\partial_a\omega_{bc} + \partial_b\omega_{ca} + \partial_c\omega_{ab} = 0,$$

and the following definition

$$\partial_a\dot{y}^b = \partial_a(\omega^{bc}\partial_c H) = \partial_a(\Omega^{ab}\partial_b\bar{H}) := \mathcal{U}_a{}^b,$$

one can write (2.15) as

$$-\partial_c\omega_{ab}\,\dot{y}^c - \mathcal{U}_a{}^c\omega_{cb} + \mathcal{U}_b{}^c\omega_{ca} = 0,$$

or

$$\frac{d\omega_{ab}}{dt} = -\mathcal{U}_a{}^c\omega_{cb} + \mathcal{U}_b{}^c\omega_{ca}. \tag{2.16}$$

In the same way, one finds

$$\frac{d\Omega_{ab}}{dt} = -\mathcal{U}_a{}^c\Omega_{cb} + \mathcal{U}_b{}^c\Omega_{ca}. \tag{2.17}$$

To consider the expression for the Poisson structure, just take the derivative

$$\begin{aligned} \frac{d(\omega^{ab}\omega_{bc})}{dt} &= 0 \\ \frac{d\omega^{ab}}{dt}\omega_{bc} + \omega^{ab}\frac{d\omega_{bc}}{dt} &= 0 \end{aligned}$$

[5] One could define $\mathcal{S}_a^b := \Omega_{ac}\omega^{cb}$, that the calculation would be the same.

then

$$\frac{d\omega^{ab}}{dt}\omega_{bc} = \omega^{ab}(\mathcal{U}_b{}^e\omega_{ec} - \mathcal{U}_c{}^e\omega_{eb})$$

multiplying by the inverse ω^{cd}

$$\frac{d\omega^{ab}}{dt}\delta_b^d = \omega^{ab}(\mathcal{U}_b{}^e\omega_{ec} - \mathcal{U}_c{}^e\omega_{eb})\omega^{cd},$$

that can be written finally, as

$$\frac{d\omega^{ab}}{dt} = \omega^{ac}\mathcal{U}_c{}^b - \omega^{bc}\mathcal{U}_c{}^b, \tag{2.18}$$

in the same way, one finds

$$\frac{d\Omega^{ab}}{dt} = \Omega^{ac}\mathcal{U}_c{}^b - \Omega^{bc}\mathcal{U}_c{}^b. \tag{2.19}$$

When one takes the time derivative of (2.14) and carefully uses (2.16) and (2.19), will conclude that

$$\frac{d\mathcal{S}_a^b}{dt} = \mathcal{S}_a^c\mathcal{U}_c^b - \mathcal{U}_a^c\mathcal{S}_c^b,$$

which, in the matrix form, is

$$\frac{d\mathcal{S}}{dt} = [\mathcal{S}, \mathcal{U}]. \tag{2.20}$$

This last equation is known as *Lax equation* and the matrices $\mathcal{S}$ and $\mathcal{U}$ as *Lax Pair*.

The Lax pair is a powerful tool in the study of integrable models. The basic idea behind is that, if a non-linear equation is associated to the Hamiltonian system and one can find such matrices, then, to this dynamical system, a Schrödinger like-equation can be associated. With this effort, the *inverse scattering method*[6] can be used, which lead to the integrability of the Hamiltonian system.

Therefore, given the non-linear evolution equation of a system, one would like to find a linear operator $L(t)$ whose eigenvalues are constant under the non-linear dynamical evolution. In order to be coherent, the linear operator evolve in the Heisenberg picture[7] like

$$L(t) := U(t)L(0)U^{\dagger}(t),$$

[6]This method is used to solve a initial value problem of an integrable system and can be thought as an analogue of the Fourier transform, but in the non-linear case. We suggest, in crescent order of difficulty [18, 6, 26] for further reading.

[7]We are treating with a classical system, however, we allow ourselves to use this abuse of terminology.

then

$$U^{\dagger}(t)L(t)U(t) = L(0), \tag{2.21}$$

where $U(t)$ is an unitary operator. Then

$$U^{\dagger}(t)U(t) = 1 \;\Rightarrow\; \frac{\partial U^{\dagger}(t)}{\partial t}U(t) + U^{\dagger}(t)\frac{\partial U(t)}{\partial t} = 0.$$

In addition, one assumes that, there exists another operator M, not necessarily linear, such that

$$\frac{\partial U(t)}{\partial t} := -M(t)U(t),$$

which implies

$$\frac{\partial U^{\dagger}(t)}{\partial t} = U^{\dagger}(t)M.$$

Taking the derivative of (2.21),

$$\frac{\partial U^{\dagger}(t)}{\partial t}L(t)U(t) + U^{\dagger}(t)\frac{\partial L(t)}{\partial t}U(t) + U^{\dagger}(t)L(t)\frac{\partial U(t)}{\partial t} = 0,$$

and using the derivative of the evolution operator $U(t)$, one concludes that:

$$\frac{\partial L(t)}{\partial t} = [L(t), M(t)],$$

that is naturally, the Lax equation (2.20).

Finally, one would like that the eigenvalue of the linear operator was constant, in fact, easily this statement can be proven true. Assume firstly, that

$$L(t)\psi(t) = \lambda(t)\psi(t),$$

in a way that the eigenfunction evolves as

$$\psi(t) = U(t)\psi(0),$$

from where follows that

$$\frac{\partial\psi(t)}{dt} = \frac{\partial U(t)}{dt}\psi(0) = -M(t)U(t)\psi(0) = -M(t)\psi(t).$$

Taking the derivative of the eigenvalue equation

$$\begin{aligned}
\frac{\partial}{\partial t}(L(t)\psi(t)) &= \frac{\partial}{\partial t}(\lambda(t)\psi(t)) \\
\frac{\partial L(t)}{\partial t}\psi(t) + L(t)\frac{\partial\psi(t)}{\partial t} &= \frac{\partial\lambda(t)}{\partial t}\psi(t) + \lambda\frac{\partial\psi(t)}{\partial t} \\
\left(\frac{\partial L(t)}{\partial t} - L(t)M(t)\right)\psi(t) &= \left(\frac{\partial\lambda(t)}{\partial t} - M(t)L(t)\right)\psi(t),
\end{aligned}$$

which implies

$$\frac{\partial L(t)}{\partial t} = [L(t), M(t)] + \frac{\partial \lambda(t)}{\partial t},$$

and from the Lax equation one finally concludes that

$$\frac{\partial \lambda(t)}{\partial t} = 0.$$

The constant eigenvalue λ is usually called *spectral parameter*.

2.3.2 Continuous Systems

Up to now, we have considered systems with n degrees of freedom, however, the Poisson structure that we have just seen is useful when we want (or need) to work in a continuous system. There are a kind of pattern that help us to pass from a discrete system to a continuous one [5], we can sum up this scheme as:

◇ Replace the coordinates $y^i(t)$ from the discrete system to a dynamical variables $u(x,t)$, in a way that, the discrete index i becomes the continuous variable $x \in \mathbb{R}$;
◇ The phase space $\mathcal{M}$ is replaced by a space of smooth functions on a line;
◇ The summation over the indices a by integrals in x;
◇ The functions of the coordinates $f(y)$ by functional $F[u]$;
◇ The partial derivative by functional derivative.

One can write a functional as

$$F[u] = \int_{\mathbb{R}} f(u, u_x, u_{xx}, \dots)dx,$$

where the subscript denotes partial derivative, and the functional derivative is given by

$$\frac{\delta F}{\delta u(x)} = \frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial u_x} + \frac{\partial^2}{\partial x^2}\frac{\partial f}{\partial u_{xx}} + \dots$$

with

$$\frac{\delta u(z)}{\delta u(x)} = \delta(z - x),$$

where δ is the Dirac delta. Therefore, the ordinary differential equations, that one used to have, are replaced by partial differential equations.

By mimicking what was made in the discrete system, a Poisson bracket may be defined considering two functional through

$$\{F, G\} := \int_{\mathbb{R}^2} \omega(x, z, u) \frac{\delta F}{\delta u(x)} \frac{\delta G}{\delta u(z)} dxdz, \tag{2.22}$$

where, we require that $\omega(x, z, u)$ is a Poisson structure. A possible choice (but not the only one) is

$$\omega(x, z) = \frac{1}{2}\frac{\partial}{\partial x}\delta(x-z) - \frac{1}{2}\frac{\partial}{\partial z}\delta(x-z).$$

Now, with this definition, one could work a little bit with (2.22), so

$$\begin{aligned}
\{F, G\} &= \frac{1}{2}\int_{\mathbb{R}^2} \frac{\partial}{\partial x}\delta(x-z)\frac{\delta F}{\delta u(x)}\frac{\delta G}{\delta u(z)} dxdz - \frac{1}{2}\int_{\mathbb{R}^2} \frac{\partial}{\partial z}\delta(x-z)\frac{\delta F}{\delta u(x)}\frac{\delta G}{\delta u(z)} dxdz \\
&= -\frac{1}{2}\int_{\mathbb{R}^2} \delta(x-z)\left(\frac{\partial}{\partial x}\frac{\delta F}{\delta u(x)}\right)\frac{\delta G}{\delta u(z)} dxdz + \\
&\qquad\qquad + \frac{1}{2}\int_{\mathbb{R}^2} \delta(x-z)\frac{\delta F}{\delta u(x)}\left(\frac{\partial}{\partial z}\frac{\delta G}{\delta u(z)}\right) dxdz \\
&= -\frac{1}{2}\int_{\mathbb{R}} \left(\frac{\partial}{\partial z}\frac{\delta F}{\delta u(z)}\right)\frac{\delta G}{\delta u(z)} dz + \frac{1}{2}\int_{\mathbb{R}} \frac{\delta F}{\delta u(z)}\left(\frac{\partial}{\partial z}\frac{\delta G}{\delta u(z)}\right) dz \\
&= \frac{1}{2}\int_{\mathbb{R}} \frac{\delta F}{\delta u(z)}\left(\frac{\partial}{\partial z}\frac{\delta G}{\delta u(z)}\right) dz + \frac{1}{2}\int_{\mathbb{R}} \frac{\delta F}{\delta u(z)}\left(\frac{\partial}{\partial z}\frac{\delta G}{\delta u(z)}\right) dz,
\end{aligned}$$

thus, one concludes that

$$\{F, G\} = \int_{\mathbb{R}} \frac{\delta F}{\delta u(z)}\left(\frac{\partial}{\partial z}\frac{\delta G}{\delta u(z)}\right) dz, \tag{2.23}$$

from which follows that the Hamilton's equations are

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \{u, H[u]\} = \int_{\mathbb{R}} \frac{\delta u(x)}{\delta u(z)}\left(\frac{\partial}{\partial z}\frac{\delta H[u]}{\delta u(z)}\right) dz \\
&= \frac{\partial}{\partial x}\frac{\delta H[u]}{\delta u(x)}.
\end{aligned} \tag{2.24}$$

2.4 The Zero-Curvature Formulation

Among various integrable models, we are sure that the most interesting and mathematically richer are those continuous. Naturally, we would like to treat a large number of these models with only a few mathematical tools, we mean that, we would like to have a way to unify a description of a large number of continuous integrable models. When Zakharov and Shabat were working with one of these models, namely, non-linear Schrödinger, they have used an approach, that later was generalized by Ablowitz, Kaup, Newell and Segur, which allows to work with several others integrable models.

This approach uses a Lax operator that is of first order in $\frac{\partial}{\partial x}$. Remember that the Lax equation is

$$\frac{\partial L(t,x)}{\partial t} = [L(t,x), M(t,x)],$$

that one can write as

$$[\partial_t + M(t,x), L(t,x)] = 0, \tag{2.25}$$

where we have denoted the dependence on the continuous variable x. Also, we already know that

$$\partial_t \psi(t,x) = -M(t)\psi(t,x) := -A_t\ \psi(t,x)$$

and when one supposes that there exists an operator $N(t,x)$ such that

$$L(t,x)\psi(t,x) = (\partial_x + N(t,x))\psi(t,x) = \lambda\psi(t,x),$$

one gets the equation

$$\partial_x \psi(t,x) = -(N-\lambda)\psi(t,x) := -A_x\ \psi(t,x),$$

finally, from the Lax equation presented in the form (2.25), one has

$$[\partial_t + A_t(t,x), \partial_x + A_x(t,x)] = 0.$$

This last equation can be written as

$$\partial_t A_x - \partial_x A_t + [A_t, A_x] = 0, \tag{2.26}$$

that is often called *zero-curvature equation*[8]. The origin of this usual name comes from the differential geometry and even the study of this name would lead us to a long road in the mathematics realm. Generically this equation means that the curvature of the *connection* $A := A_x\,dx + A_t\,dt$ vanishes. Connections, on the other hand, are necessary structures to lead to a "well defined" notion of derivative in the manifold[9].

[8]So do *zero-curvature condition* and *Zakharov-Shabat equation*.

[9]We suggest [27, 28] for a succinct reading about this topic.

Summing up In this chapter we have dealt with the notion of integrable models; also, with some mathematical rigour, we have presented the Lax pair and the zero-curvature condition that are the main classical results needed for the following work.

Chapter 3

The Model and its Transformations

ONE OF THE few relativistic integrable models in $1+1$ dimensions with one scalar field, is the *Tzitzéica model*; and it is what we will consider in this chapter. This model, as well as *sine-Gordon* model, first appeared in the study of hyperbolic surfaces and has been rediscovered in the solitonic context some years later [4, 16, 29, 30, 31].

This model is a good alternative to extend the developments done in the research of the sine-Gordon model, actually, the Tzitzéica model can be a laboratory to develop and test new ideas in non-perturbative methods in physics. Besides, this model awakes interest in its confinement mechanism and the conformal invariance extension [32]. Therefore, if we take account these cited applications, the phenomenological use in gas-dynamics and the enormous value in the affine differential geometry [4, 16], we justify any effort done by physicists and mathematicians to a better understanding of this model.

In this chapter, we try to present the basic aspects of the Tzitéica model, from its Lagrangian density to the Lax pair, also, we will consider a particular type of gauge transformation, and we derive a system (of differential equations) that makes a relation between two Tzitzéica solitons solutions [30], this differential relation in known as Bäcklund transformation.

On one hand, there are in the literature, several Bäcklund transformations available for the Tzitzéica model – for instance [29, 33, 34] – which have a strong non-linear structure and do not allow effective calculations. On the other hand, there was a kind of misunderstanding that do not exist Bäcklund transformations for the

Tzitzéica model. What is true in this story is that: *It does not exist a transformation similar to the Bäcklund of the sine-Gordon model*[1]. A necessary condition for the existence of similar transformation [35, 36, 37, 38], states that:

> *The non-linear Klein-Gordon equation*
>
> $$\phi_{+-} = F(\phi)$$
>
> *has a Bäcklund similar to the Bäcklund of the sine-Gordon model, if and only if $F(\phi)$ satisfies the linear condition*
>
> $$\frac{d^2F(\phi)}{d\phi^2} + kF(\phi) = 0.$$

The Tzitzéica model does not satisfy this requirement, then, it does not have a Bäcklund similar to the Bäcklund of the sine-Gordon model. The misunderstanding comes from the belief that this requirement is a little bit stronger: *The only integrable non-linear Klein-Gordon equations, are those which satisfy the linear condition above.* Again, what is true is that: *This is a sufficient requirement for the integrability, but not a necessary condition.*

Then, the Bäcklund transformation that Borisov et. al. has found [30] is not so simple, indeed, it will depend of an auxiliary field. Besides the relationship with integrable defects that we are going to explain in the next chapter, the advantage of this transformation over the previously obtained at [29, 33, 34], is in the geometric context, in fact, Borisov et. al. showed how one can reduce his Bäcklund transformation to another classical transformation, the *Tzitzéica-Moutard*, that as the name suggests, is a Moutard-type transformation that was obtained by Gheorghe Tzitzéica [39, 40, 41, 42] and we will explain latter.

3.1 The Tzitzéica model

The starting point for this study is the Lagrangian density that was defined in [32] by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\left(2e^{\phi} + e^{-2\phi}\right), \tag{3.1}$$

[1] In the introduction we have presented such a Bäcklund transformation; also, we will often omit, by simplicity, the word "transformation".

which gives the following field equation

$$\partial_+\partial_-\phi \equiv \partial_\mu\partial^\mu\phi \equiv \partial^2\phi = -e^{\phi} + e^{-2\phi}, \tag{3.2}$$

where, the light-cone coordinates:

$$x_\pm := \frac{1}{2}\left(t \pm x\right), \tag{3.3}$$

have been defined. From this definition, one gets the following derivatives

$$\partial_\pm = \partial_t \pm \partial_x \Rightarrow \partial_+\partial_- = \partial_t^2 - \partial_x^2,$$

that will be used exhaustively.

Tsarev noticed that in a series of papers between 1907 and 1910 [39, 40, 41, 42], the geometer Tzitzéica analyzed surfaces associated with an equation in the form

$$\partial_\rho\partial_\varrho\left(\ln v\right) = v - \frac{1}{v^2}.$$

When one defines $\ln v := \phi$, last equation gets a new form

$$\partial_\rho\partial_\varrho\left(\phi\right) = e^{\phi} - e^{-2\phi}.$$

that is the field equation (3.2) above considered, with an appropriate global sign. This sign one can be recovered when one chooses[2] $\rho \to -x_+$ and $\varrho \to x_-$, then

$$\partial_+\partial_-\left(\ln v\right) = -v + \frac{1}{v^2}, \tag{3.4}$$

that one can write as

$$\partial^2 v - \frac{\partial_+ v\, \partial_- v}{v} = -v^2 + \frac{1}{v}.$$

Equations (3.2) and (3.4) are usually called *Tzitzéica equation*[3].

In the second part of this work we will turn to talk about the Tzitzéica model, but under a different perspective, we will study the Tzitzéica surfaces, that will be useful to construct transformations; but for while, this is everything what we have to talk about them.

As a last comment in this section, the argument of the fields has been hidden, but always keep in mind the dependence

$$\phi \equiv \phi\left(x, t\right) \equiv \phi\left(x_+, x_-\right).$$

[2]Evidently, the choice $\rho \to x_+$ and $\varrho \to -x_-$ would provide the same sign, however, the equations considered in the chapter 6, show that this choice is wrong.

[3]There are in the literature, some people that call (3.2) as *Bullough-Dodd equation* and (3.4) as Tzitzéica equation.

3.2 Lax Pair

In the previous chapter, we have considered the whole theory of integrable models in a very short version, however, the most remarkable aspect of this chapter is that we can understand that, if we have the Lax pair of one given model, then such a model is integrable. Thus, we will avoid any consideration about the integrability of the Tzitzéica model and we will justify it, giving its Lax pair.

In this short section, the Lax pair considered by [32] is presented and as one would expect, we will finish this section with the zero curvature equation. Therefore, let the Lax pair[4] be:

$$A_+ = -\left(\frac{\sqrt{2}}{2}\, e^{\phi}\,\lambda\,\mathbf{T}_+ + e^{-2\phi}\,\lambda\,\mathbf{L}_{-2}\right) = \begin{pmatrix} 0 & -i\,\lambda e^{\phi} & 0 \\ 0 & 0 & -i\,\lambda e^{\phi} \\ \lambda e^{-2\phi} & 0 & 0 \end{pmatrix}$$

and

$$A_- = -\partial_-\phi\,\mathbf{T}_3 + \frac{\sqrt{2}}{2}\,\frac{1}{\lambda}\,\mathbf{T}_- + \frac{1}{\lambda}\,\mathbf{L}_2 = \begin{pmatrix} -\partial_-\phi & 0 & -\frac{1}{\lambda} \\ -\frac{i}{\lambda} & 0 & 0 \\ 0 & -\frac{i}{\lambda} & \partial_-\phi \end{pmatrix},$$

where λ is the spectral parameter and the pair is living in the Lie Algebra $\mathfrak{su}(3)$ with $\mathbf{T}_3,\ \mathbf{T}_\pm,\ \mathbf{L}_{\pm 2}$ among its generators[5]. When one decides to use the field $v = e^{\phi}$, the Lax pair will be presented as

$$A_+ = \begin{pmatrix} 0 & -i\,\lambda\,v & 0 \\ 0 & 0 & -i\,\lambda\,v \\ \lambda\,v^{-2} & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_- = \begin{pmatrix} -\frac{1}{v}\partial_- v & 0 & -\frac{1}{\lambda} \\ -\frac{i}{\lambda} & 0 & 0 \\ 0 & -\frac{i}{\lambda} & \frac{1}{v}\partial_- v \end{pmatrix}. \tag{3.5}$$

Now, taking the system

$$\partial_\pm \Psi = -A_\pm \Psi\,, \tag{3.6}$$

or yet

$$(\partial_\pm + A_\pm)\Psi := P_\pm \Psi = 0,$$

and considering the relation

$$\begin{aligned} [P_+, P_-]\,\Psi &= [\partial_+ + A_+, \partial_- + A_-]\,\Psi \\ &= \left(\partial_+ A_- - \partial_- A_+ + [A_+, A_-]\right)\Psi = 0, \end{aligned}$$

[4] Formally, the Lax pair is $P_\pm := \partial_\pm + A_\pm$, but, without loss of generality, just the connections $A_\pm$ can be called as Lax pair.

[5] Appendix B, page (100).

the Zero-curvature condition naturally appears

$$\partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0. \tag{3.7}$$

Direct substitution of the potentials $A_\pm$ (3.5) in the zero-curvature equation gives the Tzitzéica equation.

3.3 Gauge Transformations

A new solution of the system (3.6) is considered[6], when one defines the *gauge transformation*

$$\bar{\Psi} = \mathbf{K}\,\Psi,$$

where $\mathbf{K}$ is an element of the Lie group $SU(3)$. Follows from this last system, that

$$\begin{aligned}\partial_\pm \bar{\Psi} &= \partial_\pm(\mathbf{K}\,\Psi) = (\partial_\pm \mathbf{K})\,\Psi + \mathbf{K}\,(\partial_\pm \Psi) = (\partial_\pm \mathbf{K})\,\mathbf{K}^{-1}\bar{\Psi} - \mathbf{K}\,A_\pm \Psi \\ &= [(\partial_\pm \mathbf{K})\,\mathbf{K}^{-1} - \mathbf{K}\,A_\pm \mathbf{K}^{-1}]\bar{\Psi} = -\bar{A}_\pm \bar{\Psi},\end{aligned}$$

then, the system (3.6) is invariant under the gauge transformation

$$\begin{aligned}\Psi &\Rightarrow \quad \bar{\Psi} = \mathbf{K}\Psi \\ A_\pm &\Rightarrow \bar{A}_\pm = \mathbf{K}\,A_\pm \mathbf{K}^{-1} - (\partial_\pm \mathbf{K})\,\mathbf{K}^{-1},\end{aligned}$$

that yields

$$\partial_\pm \mathbf{K} = \mathbf{K}\,A_\pm - \bar{A}_\pm\,\mathbf{K}. \tag{3.8}$$

Where we have considered that there exists another solution $\bar{\phi} = \ln \bar{v}$ of the Tzitzéica equation such that, the new lax pair is given by

$$\bar{A}_+ = \begin{pmatrix} 0 & -i\,\lambda\,\bar{v} & 0 \\ 0 & 0 & -i\,\lambda\,\bar{v} \\ \lambda\,\bar{v}^{-2} & 0 & 0 \end{pmatrix} \quad \text{and} \quad \bar{A}_- = \begin{pmatrix} -\frac{1}{\bar{v}}\partial_- \bar{v} & 0 & -\frac{1}{\lambda} \\ -\frac{i}{\lambda} & 0 & 0 \\ 0 & -\frac{i}{\lambda} & \frac{1}{\bar{v}}\partial_- \bar{v} \end{pmatrix}.$$

Choosing the operator $\mathbf{K}$ depending on the spectral parameter λ, one can construct the *Bäcklund transformation* [1, 4] that can be defined as:

[6] Among the variety of works explaining what we are going to start now, we recommend [3].

Definition:

Consider two non-linear operators $\mathfrak{P}$ and $\mathfrak{Q}$ such that $\mathfrak{P}[\varphi(x,t)] = 0$ and $\mathfrak{Q}[\vartheta(x,t)] = 0$. A Bäcklund Transformation is a pair of relations

$$\mathfrak{R}_i(\varphi, \vartheta, \varphi_x, \vartheta_x, \varphi_t, \vartheta_t) = 0, \quad i = 1,2$$

which is integrable for φ when $\mathfrak{Q}(\vartheta) = 0$ and the resulting φ satisfies $\mathfrak{P}(\varphi) = 0$, and vice-versa. In the particular case $\mathfrak{P} = \mathfrak{Q}$ one calls $\mathfrak{R}_i, \quad i = 1,2$ a *self-Bäcklund transformation*[7].

This type of gauge transformation is called *Darboux Transformation* and the operator $\mathbf{K}$, which is called Darboux matrix, must satisfy some requirements, including the one that fixes its form.

Indeed, the Darboux matrix associated with a Lax pair which is polynomial in the parameter λ and in the inverse λ^{-1}, has its general form given by[8] [4, 10]

$$\mathbf{K} = \frac{1}{\lambda^n} K_n + \sum_{i=0}^{n-1} \frac{1}{\lambda^i} K_i, \tag{3.9}$$

where K_n is a constant diagonal matrix, that without loss of generality we will choose to be the identity matrix $\mathbb{I}$. Since this type of transformation have been considered along the years [3, 19, 43, 44], we are going to start now, an extension of the same problem, but with the degree of the Darboux matrix equal to three, $n = 3$. This number has been found by the unglamorous method of trial and error.

3.3.1 Gauge transformation for the Tzitzéica model

Now we use the technology that was used by [30] to construct the Darboux transformation related to the Lax pair (3.5). When we consider a full matrix $\mathbf{K}$, relating the matrices $A_\pm$ and $\bar{A}_\pm$ with components in the form

$$k_{ij} = \alpha_{ij} + \beta_{ij}\frac{1}{\lambda} + \delta_{ij}\frac{1}{\lambda^2} + \gamma_{ij}\frac{1}{\lambda^3},$$

[7] As we are interested just in self-Bäcklund transformation, we will speak just Bäcklund transformation, but keep in mind the terminology

[8] The positive expansion in λ is admissible and gives the same result - see in the appendix C the way that we can use this expansion and convince yourself that, what is really important in this calculation are the equations; and that their order in λ, assumes a place in the organization of the system, what can be done by positive or negative powers in λ.

we find three uncoupled systems of PDE involving the variables

$$\begin{aligned}
1^{\rm st}-\{&\alpha_{11}:=\alpha_1;\alpha_{22}:=\alpha_2;\alpha_{33}:=\alpha_3;\beta_{13}:=\beta_1;\beta_{21}:=\beta_2;\beta_{32}:=\beta_3;\\
&\delta_{12}:=\delta_1;\delta_{23}:=\delta_2;\delta_{31}:=\delta_3;\gamma_{11}:=\gamma_1;\gamma_{22}:=\gamma_2;\gamma_{33}:=\gamma_3\},\\
2^{\rm nd}-\{&\alpha_{12};\alpha_{23};\alpha_{31};\beta_{11};\beta_{22};\beta_{33};\delta_{13};\delta_{21};\delta_{32};\gamma_{12};\gamma_{23};\gamma_{31}\},\\
3^{\rm rd}-\{&\alpha_{13};\alpha_{21};\alpha_{32};\beta_{12};\beta_{23};\beta_{31};\delta_{11};\delta_{22};\delta_{33};\gamma_{13};\gamma_{21};\gamma_{32}\}.
\end{aligned}$$

As we just said, we work with the first system because the lower order matrix with $n=3$ must be diagonal, then the others terms naturally vanish, hence

$$\mathbf{K}=\begin{pmatrix}\alpha_1&0&0\\0&\alpha_2&0\\0&0&\alpha_3\end{pmatrix}+\begin{pmatrix}0&0&\beta_1\\\beta_2&0&0\\0&\beta_3&0\end{pmatrix}\frac{1}{\lambda}+\begin{pmatrix}0&\delta_1&0\\0&0&\delta_2\\\delta_3&0&0\end{pmatrix}\frac{1}{\lambda^2}+\begin{pmatrix}\gamma_1&0&0\\0&\gamma_2&0\\0&0&\gamma_3\end{pmatrix}\frac{1}{\lambda^3}.$$

When we put this *Ansatz* in (3.8), we will find several coupled PDEs, which when solved give[9]:

$$\mathbf{K}=\begin{pmatrix}\alpha+\nu\,\lambda^{-3} & \frac{2\,\xi\,\bar v\,\nu}{\alpha\gamma}\left(\alpha+\xi\right)\,\lambda^{-2} & \frac{2\,\xi^2\bar v^2\,\nu}{\alpha^2\gamma^2}\left(\alpha+\xi\,\right)^2\lambda^{-1}\\ \frac{\alpha\gamma}{\xi\bar v}\lambda^{-1} & \xi+\nu\,\lambda^{-3} & \frac{2\,\xi\,\bar v\,\nu}{\alpha\gamma}\left(\alpha+\xi\right)\,\lambda^{-2}\\ \frac{\alpha\,\gamma^2}{2\bar v^2\xi^2}\,\lambda^{-2} & \frac{\gamma}{\bar v}\lambda^{-1} & \xi^2\,\frac{1}{\alpha}+\nu\,\lambda^{-3}\end{pmatrix},$$

where ξ and ν are constants and the fields α and γ must satisfy the equations

$$\partial_+\alpha-i\,\frac{\alpha\,\gamma}{\xi}-\frac{2\,\nu}{\gamma^2}\left(\alpha+\xi\,\right)^2=0 \tag{3.10}$$

$$\frac{1}{\bar v}\partial_+\gamma-\frac{\gamma}{\bar v^2}\partial_+\bar v+\frac{2\,\nu\,\xi}{\alpha\,\bar v\,\gamma}\left(\alpha+\xi\,\right)+i\,\frac{\gamma^2}{2\,\bar v\,\xi}=0 \tag{3.11}$$

$$\partial_-\alpha-i\,\frac{2\,\bar v\,\xi}{\gamma}\left(\alpha+\xi\,\right)+\frac{\alpha^2\gamma^2}{2\,\nu\,\bar v^2\,\xi^2}=0 \tag{3.12}$$

$$\frac{\partial_-\gamma}{\bar v}+i\,\xi\left(\frac{\xi}{\alpha}-1\right)=0 \tag{3.13}$$

and

$$\alpha=\frac{\bar v}{v}\xi. \tag{3.14}$$

It is worthwhile to say that equations (3.10) to (3.13) also could be called Darboux Transformation, that together with (3.14) may be used to find new solutions

[9]In the appendix, a kind of sketch for this calculation will be presented, that, although it is easy, it is annoying too.

for the Tzitzéica equation through a previous one. Nevertheless, this is not a simple system to solve, as easily one can notice throughout its cumbersome structure.

We can define two useful functions

$$p := \frac{\phi + \bar{\phi}}{2} \qquad q := \frac{\phi - \bar{\phi}}{2} \tag{3.15}$$

whose definition allows us to write

$$\alpha = exp\,(-2\,q)\,\xi, \tag{3.16}$$

from which follow the equations (3.10) to (3.13) given now by

$$\begin{aligned}
\partial_+ q &= -\frac{1}{2}\left[i\,\frac{\gamma}{\xi} + \frac{2\nu\,\xi}{\gamma^2}\left(e^q + e^{-q}\right)^2\right],\\
\partial_+\gamma - \gamma\,\partial_+ p &= -\frac{\nu\xi\,(e^{2q} - e^{-2q})}{\gamma},\\
\partial_- q &= -\frac{1}{2}\left[i\,\frac{2\,\xi\,e^p}{\gamma}\left(e^q + e^{-q}\right) - \frac{\gamma^2}{2\xi\nu}e^{-2p}\right],\\
\partial_-\gamma &= i\,\xi\,e^p\,\left(e^{-q} - e^q\right).
\end{aligned}$$

As a guess, we set

$$\gamma := e^{\Lambda},$$

thus, the above equations give

$$\partial_+ q = -\frac{1}{2}\left[\frac{i}{\xi}e^{\Lambda} + 2\,\nu\,\xi\,e^{-2\Lambda}\left(e^q + e^{-q}\right)^2\right] \tag{3.17}$$

$$\partial_+\left(\Lambda - p\right) = -\nu\,\xi\,e^{-2\,\Lambda}\,\left(e^{2\,q} - e^{-2\,q}\right) \tag{3.18}$$

$$\partial_- q = -\frac{1}{2}\left[2i\,\nu\,\xi\,e^{p-\Lambda}\left(e^q + e^{-q}\right) - \frac{e^{2\Lambda - 2p}}{2\,\xi}\right] \tag{3.19}$$

$$\partial_-\Lambda = i\,\xi\,e^{-\Lambda + p}\,\left(e^{-q} - e^q\right). \tag{3.20}$$

Finally, the compatibility condition

$$\left(\partial_+\partial_-\right)q = \left(\partial_-\partial_+\right)q\,,$$

is satisfied and in order to satisfy (3.9), we choose $\nu = 1$.

Again, we remember that a natural question that arises is: What would happen if we had taken $\mathbf{K}$ with positive expansion in λ? In other words, if we choose $\mathbf{K}$ as

$$\mathbf{K} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}\lambda^3 + \begin{pmatrix} 0 & 0 & \beta_1 \\ \beta_2 & 0 & 0 \\ 0 & \beta_3 & 0 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & \delta_1 & 0 \\ 0 & 0 & \delta_2 \\ \delta_3 & 0 & 0 \end{pmatrix}\lambda + \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix},$$

how would be the Darboux Transformation (DT) now? From what we have said about the form of the matrix (3.9), we can realize that the DT will not change, and naturally, the matrix **K** is

$$\mathbf{K} = \begin{pmatrix} \alpha\,\lambda^3 + \nu & \frac{2\,\xi\,\bar{v}\,\nu}{\alpha\gamma}\,(\alpha+\xi)\,\lambda & \frac{2\,\xi^2\bar{v}^2\,\nu}{\alpha^2\gamma^2}\,(\alpha+\xi\,)^2\,\lambda^2 \\ \frac{\alpha\gamma}{\xi\bar{v}}\lambda^2 & \xi\,\lambda^3+\nu & \frac{2\,\xi\,\bar{v}\,\nu}{\alpha\gamma}\,(\alpha+\xi)\,\lambda \\ \frac{\alpha\,\gamma^2}{2\bar{v}^2\xi^2}\,\lambda & \frac{\gamma}{\bar{v}}\lambda^2 & \xi^2\,\frac{1}{\alpha}\,\lambda^3+\nu \end{pmatrix}.$$

The Bäcklund transformation, could be considered just writing (3.17) and (3.19) in terms of the fields, so

$$\partial_+(\phi-\bar{\phi}) = -\left[\frac{i}{\xi}e^{\Lambda} + 2\xi e^{-2\Lambda}(e^{\frac{1}{2}(\phi-\bar{\phi})} + e^{-\frac{1}{2}(\phi-\bar{\phi})})^2\right], \tag{3.21}$$

$$\partial_-(\phi-\bar{\phi}) = -\left[2i\xi e^{-\Lambda}(e^{\phi}+e^{\bar{\phi}}) - \frac{e^{-2\Lambda}}{2\xi}e^{-\phi-\bar{\phi}}\right]. \tag{3.22}$$

Summing up In this chapter, we have presented the Tzitzéica model through its Lagrangian density, also, we have presented a particular gauge transformation called Darboux transformation, which allowed us to define a pair of differential equations, which make a relation between two known solutions of the Tzitzéica equation.

Chapter 4

The Tzitzéica Model With Defects

Some years ago, it was introduced the notion of models that remain integrable even when there exists an internal boundary condition, or using the standard nomenclature, a *defect* or *jump defect* [19]. In this sense, one could think of the whole system as the junction of two domains, each one, integrable by itself.

In this approach, the sine-Gordon and Liouville models have been considered and the conserved charges required for the integrability of the whole system were found [19, 45]. Also, from the defect conditions (internal boundary conditions) between these domains emerge the Bäcklund transformation at the defect.

Despite the success in some integrable non-linear Klein-Gordon equations, this approach does not work with the Tzitzéica model and needed to be generalized. With this aim, Corrigan & Zambon [20] allowed to the defect a kind of "well defined" degree of freedom. Therefore, in the first section of this chapter, we will show what they did in an effective fashion and later, we will apply it in the Tzitzéica model.

After, we easily conclude that exists a deep connection between the formulation considered here and the gauge formulation considered in the previous chapter, i.e. there exists a relation between the Darboux transformation and the defect conditions that emerge from the integrable defects.

4.1 Introducing Defects

Consider, first of all, the action

$$\mathcal{S}[\phi,\bar{\phi},\Lambda] := \int d^2x\ \theta(-x)\mathcal{L}_\phi(\phi,\partial_\mu\phi) + \int d^2x\ \theta(x)\mathcal{L}_{\bar{\phi}}(\bar{\phi},\partial_\mu\bar{\phi}) + \\ + \int d^2x\ \delta(x)\mathcal{L}_{\mathcal{D}}(\phi,\partial_t\phi,\bar{\phi},\partial_t\bar{\phi},\Lambda,\partial_t\Lambda),$$

where $\theta(x)$ is the Heaviside function and $\delta(x)$ de Dirac function[1]. Making a variation $\delta\mathcal{S}=0$, considering the Gauss theorem and that the variation of the field vanishes at the boundary of the system, one can write

$$\begin{aligned} 0 = & \int d^2x\ \theta(-x)\left[\frac{\partial\mathcal{L}_\phi}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}_\phi}{\partial\partial_\mu\phi}\right)\right]\delta\phi + \int d^2x\ \theta(-x)\partial_\mu\left(\frac{\partial\mathcal{L}_\phi}{\partial\partial_\mu\phi}\delta\phi\right) + \\ & + \int d^2x\ \theta(x)\left[\frac{\partial\mathcal{L}_{\bar{\phi}}}{\partial\bar{\phi}} - \partial_\mu\left(\frac{\partial\mathcal{L}_{\bar{\phi}}}{\partial\partial_\mu\bar{\phi}}\right)\right]\delta\bar{\phi} + \int d^2x\ \theta(x)\partial_\mu\left(\frac{\partial\mathcal{L}_{\bar{\phi}}}{\partial\partial_\mu\bar{\phi}}\delta\bar{\phi}\right) + \\ & + \int_{\mathbb{R}} dt\left\{\left[\frac{\partial\mathcal{L}_{\mathcal{D}}}{\partial\phi} - \partial_t\left(\frac{\partial\mathcal{L}_{\mathcal{D}}}{\partial\partial_t\phi}\right)\right]\delta\phi + \left[\frac{\partial\mathcal{L}_{\mathcal{D}}}{\partial\bar{\phi}} - \partial_t\left(\frac{\partial\mathcal{L}_{\mathcal{D}}}{\partial\partial_t\bar{\phi}}\right)\right]\delta\bar{\phi}\right\}\Bigg|_{x=0} + \\ & + \int_{\mathbb{R}} dt\left[\frac{\partial\mathcal{L}_{\mathcal{D}}}{\partial\Lambda} - \partial_t\left(\frac{\partial\mathcal{L}_{\mathcal{D}}}{\partial\partial_t\Lambda}\right)\right]\delta\Lambda\Bigg|_{x=0} + \underbrace{\int_{\mathbb{R}} dt\partial_t\left(\frac{\partial\mathcal{L}_{\mathcal{D}}}{\partial\partial_t\phi}\delta\phi\right)\Bigg|_{x=0}}_{(=0)} + \\ & + \underbrace{\int_{\mathbb{R}} dt\partial_t\left(\frac{\partial\mathcal{L}_{\mathcal{D}}}{\partial\partial_t\bar{\phi}}\delta\bar{\phi}\right)\Bigg|_{x=0}}_{(=0)} + \underbrace{\int_{\mathbb{R}} dt\partial_t\left(\frac{\partial\mathcal{L}_{\mathcal{D}}}{\partial\partial_t\Lambda}\delta\Lambda\right)\Bigg|_{x=0}}_{(=0)}, \end{aligned}$$

also, one can define the currents

$$j^\mu = (j_t, j_x) := \frac{\partial\mathcal{L}_\phi}{\partial\partial_\mu\phi}\delta\phi,$$

and

$$J^\mu = (J_t, J_x) := \frac{\partial\mathcal{L}_{\bar{\phi}}}{\partial\partial_\mu\bar{\phi}}\delta\bar{\phi},$$

[1] Pay attention here, the Dirac delta $\delta(x)$ has an argument, while the variation δ does not have. Also, as d^2x means $dxdt$, the Heaviside function must be taken, just in the space component.

so, one finds the expression

$$
\begin{aligned}
0= & \int d^2x\ \theta(-x)\left[\frac{\partial \mathcal{L}_\phi}{\partial \phi}-\partial_\mu\left(\frac{\partial \mathcal{L}_\phi}{\partial \partial_\mu \phi}\right)\right]\delta\phi+\int_{-\infty}^{0} dx\int_{\mathbb{R}} dt\left[\partial_t j_t+\partial_x j_x\right]+\\
& +\int d^2x\ \theta(x)\left[\frac{\partial \mathcal{L}_{\bar\phi}}{\partial \bar\phi}-\partial_\mu\left(\frac{\partial \mathcal{L}_{\bar\phi}}{\partial \partial_\mu \bar\phi}\right)\right]\delta\bar\phi+\int_{0}^{\infty} dx\int_{\mathbb{R}} dt\left[\partial_t J_t+\partial_x J_x\right]+\\
& +\int_{\mathbb{R}} dt\left\{\left[\frac{\partial \mathcal{L}_{\mathcal{D}}}{\partial \phi}-\partial_t\left(\frac{\partial \mathcal{L}_{\mathcal{D}}}{\partial \partial_t \phi}\right)\right]\delta\phi+\left[\frac{\partial \mathcal{L}_{\mathcal{D}}}{\partial \bar\phi}-\partial_t\left(\frac{\partial \mathcal{L}_{\mathcal{D}}}{\partial \partial_t \bar\phi}\right)\right]\delta\bar\phi\right\}\bigg|_{x=0}+\\
& +\int_{\mathbb{R}} dt\left[\frac{\partial \mathcal{L}_{\mathcal{D}}}{\partial \Lambda}-\partial_t\left(\frac{\partial \mathcal{L}_{\mathcal{D}}}{\partial \partial_t \Lambda}\right)\right]\delta\Lambda\bigg|_{x=0}
\end{aligned}
$$

finally

$$
\begin{aligned}
0= & \int d^2x\ \theta(-x)\left[\frac{\partial \mathcal{L}_\phi}{\partial \phi}-\partial_\mu\left(\frac{\partial \mathcal{L}_\phi}{\partial \partial_\mu \phi}\right)\right]\delta\phi+\int d^2x\ \theta(x)\left[\frac{\partial \mathcal{L}_{\bar\phi}}{\partial \bar\phi}-\partial_\mu\left(\frac{\partial \mathcal{L}_{\bar\phi}}{\partial \partial_\mu \bar\phi}\right)\right]\delta\bar\phi+\\
& +\int_{\mathbb{R}} dt\left\{\left[\frac{\partial \mathcal{L}_{\mathcal{D}}}{\partial \phi}-\partial_t\left(\frac{\partial \mathcal{L}_{\mathcal{D}}}{\partial \partial_t \phi}\right)\right]\delta\phi+j_x+\left[\frac{\partial \mathcal{L}_{\mathcal{D}}}{\partial \bar\phi}-\partial_t\left(\frac{\partial \mathcal{L}_{\mathcal{D}}}{\partial \partial_t \bar\phi}\right)\right]\delta\bar\phi-J_x\right\}\bigg|_{x=0}+\\
& +\int_{\mathbb{R}} dt\left[\frac{\partial \mathcal{L}_{\mathcal{D}}}{\partial \Lambda}-\partial_t\left(\frac{\partial \mathcal{L}_{\mathcal{D}}}{\partial \partial_t \Lambda}\right)\right]\delta\Lambda\bigg|_{x=0}.
\end{aligned}
$$

Then, the *Euler-Lagrange equations* are

$$
\begin{array}{l}
\frac{\partial \mathcal{L}_\phi}{\partial \phi}-\partial_\mu\left(\frac{\partial \mathcal{L}_\phi}{\partial \partial_\mu \phi}\right)=0 \quad x<0,\\
\frac{\partial \mathcal{L}_{\bar\phi}}{\partial \bar\phi}-\partial_\mu\left(\frac{\partial \mathcal{L}_{\bar\phi}}{\partial \partial_\mu \bar\phi}\right)=0 \quad x>0
\end{array}
\tag{4.1}
$$

together with the boundary $(x=0)$ conditions

$$
\begin{aligned}
\frac{\partial \mathcal{L}_{\mathcal{D}}}{\partial \phi}-\partial_t\left(\frac{\partial \mathcal{L}_{\mathcal{D}}}{\partial\left(\phi_t\right)}\right) &= -\frac{\partial \mathcal{L}_\phi}{\partial\left(\phi_x\right)},\\
\frac{\partial \mathcal{L}_{\mathcal{D}}}{\partial \bar\phi}-\partial_t\left(\frac{\partial \mathcal{L}_{\mathcal{D}}}{\partial\left(\bar\phi_t\right)}\right) &= \frac{\partial \mathcal{L}_{\bar\phi}}{\partial\left(\bar\phi_x\right)},\\
\frac{\partial \mathcal{L}_{\mathcal{D}}}{\partial \Lambda}-\partial_t\left(\frac{\partial \mathcal{L}_{\mathcal{D}}}{\partial\left(\Lambda_t\right)}\right) &= 0.
\end{aligned}
\tag{4.2}
$$

The Lagrangian of the integrable defect is defined as[2]

$$
\mathcal{L}_{\mathcal{D}}=-\frac{\phi\bar\phi_t-\phi_t\bar\phi}{2}-\Lambda\left(\phi-\bar\phi\right)_t+\Lambda_t\left(\phi-\bar\phi\right)+\mathcal{D}\left(\phi,\bar\phi,\Lambda\right),
$$

[2]It is worthwhile to comment that, this definition of the Lagrangian differs by a minus from the Lagrangian defined by the original work [20]. The reason of this difference is the consistence with the notation of the article [32]; if we had decided to use the Lagrangian defined by the first paper, we would have to change the Lax pair of the second work by $A_+ \rightleftharpoons A_-$, to make both articles work consistently at the same time.

then, the boundary (or defect) conditions are

$$\begin{aligned}
\phi_x &= -\bar{\phi}_t + 2\,\Lambda_t + \frac{\partial\mathcal{D}}{\partial\phi}, \\
\bar{\phi}_x &= -\phi_t + 2\,\Lambda_t - \frac{\partial\mathcal{D}}{\partial\bar{\phi}}, \\
\phi_t &= \bar{\phi}_t + \frac{1}{2}\frac{\partial\mathcal{D}}{\partial\Lambda}.
\end{aligned} \tag{4.3}$$

Since time translation invariance is not broken, the energy $\mathcal{E} - \mathcal{D}$ (where $\mathcal{E}$ is the bulk energy of the fields ϕ and $\bar{\phi}$) is conserved. On the other hand, the usual total momentum is not conserved, then, some contribution over the defect must save this conservation law. Corrigan & Zambon [20] have considered the following momentum

$$\mathcal{P} := \int_{-\infty}^{0} dx \phi_x \phi_t + \int_{0}^{\infty} dx \bar{\phi}_x \bar{\phi}_t.$$

Taking the time derivative and using the Euler-Lagrange equations (4.1) with the usual Lagrangian densities, one has

$$\begin{aligned}
\dot{\mathcal{P}} &= \int_{-\infty}^{0} dx (\phi_{xt}\phi_t + \phi_x\phi_{tt}) + \int_{0}^{\infty} dx (\bar{\phi}_{xt}\bar{\phi}_t + \bar{\phi}_x\bar{\phi}_{tt}) \\
&= \int_{-\infty}^{0} dx \left(\phi_{xt}\phi_t + \phi_x\phi_{xx} - \phi_x \frac{\partial V(\phi)}{\partial\phi} \right) + \\
&\qquad\qquad + \int_{0}^{\infty} dx \left(\bar{\phi}_{xt}\bar{\phi}_t + \bar{\phi}_x\bar{\phi}_{tt} - \bar{\phi}_x \frac{\partial \bar{V}(\bar{\phi})}{\partial\bar{\phi}} \right) \\
&= \frac{1}{2}\left(\phi_t^2 + \phi_x^2 - 2V(\phi)\right)\Big|_{x=0} - \frac{1}{2}\left(\bar{\phi}_t^2 + \bar{\phi}_x^2 - 2\bar{V}(\bar{\phi})\right)\Big|_{x=0},
\end{aligned} \tag{4.4}$$

keeping in mind that this equation is considered at the point $x = 0$, one can go on, and uses the defect conditions (4.3), so

$$\begin{aligned}
\dot{\mathcal{P}} = -\,\bar{\phi}_t \frac{\partial\mathcal{D}}{\partial\phi} - \phi_t \frac{\partial\mathcal{D}}{\partial\bar{\phi}} + 2\Lambda_t \left(\frac{\partial\mathcal{D}}{\partial\phi} + \frac{\partial\mathcal{D}}{\partial\bar{\phi}} + \frac{1}{2}\frac{\partial\mathcal{D}}{\partial\Lambda} \right) + \\
+ \frac{1}{2}\left[\left(\frac{\partial\mathcal{D}}{\partial\phi}\right)^2 - \left(\frac{\partial\mathcal{D}}{\partial\bar{\phi}}\right)^2 \right] - V(\phi) + \bar{V}(\bar{\phi}).
\end{aligned} \tag{4.5}$$

Considering that $\dot{\mathcal{P}}$ is the total derivative of some function $-\Upsilon$ such that

$$\begin{aligned}\frac{\partial\Upsilon}{\partial\phi} =& \frac{\partial\mathcal{D}}{\partial\bar{\phi}} - F,\\ \frac{\partial\Upsilon}{\partial\bar{\phi}} =& \frac{\partial\mathcal{D}}{\partial\phi} + F,\\ \frac{\partial\Upsilon}{\partial\Lambda} =& -2\left(\frac{\partial\mathcal{D}}{\partial\phi} + \frac{\partial\mathcal{D}}{\partial\bar{\phi}} + \frac{1}{2}\frac{\partial\mathcal{D}}{\partial\Lambda}\right),\end{aligned} \tag{4.6}$$

for some $F = F(\phi, \bar{\phi}, \Lambda)$, one has

$$\begin{aligned}\dot{\mathcal{P}} =& -\bar{\phi}_t\frac{\partial\Upsilon}{\partial\bar{\phi}} - \phi_t\frac{\partial\Upsilon}{\partial\phi} - \Lambda_t\frac{\partial\Upsilon}{\partial\Lambda} +\\ &+ \frac{1}{2}\left[\left(\frac{\partial\mathcal{D}}{\partial\phi}\right)^2 - \left(\frac{\partial\mathcal{D}}{\partial\bar{\phi}}\right)^2\right] - V(\phi) + \bar{V}(\bar{\phi}) + (\bar{\phi}_t - \phi_t)F\\ =& -\frac{d\Upsilon}{dt} + \underbrace{\frac{1}{2}\left[\left(\frac{\partial\mathcal{D}}{\partial\phi}\right)^2 - \left(\frac{\partial\mathcal{D}}{\partial\bar{\phi}}\right)^2\right] - V(\phi) + \bar{V}(\bar{\phi}) - \frac{1}{2}\frac{\partial\mathcal{D}}{\partial\Lambda}F}_{=0}\end{aligned}$$

and finally

$$\left(\frac{\partial\mathcal{D}}{\partial\phi}\right)^2 - \left(\frac{\partial\mathcal{D}}{\partial\bar{\phi}}\right)^2 = 2(V - \bar{V}) + \mathcal{D}_\Lambda F. \tag{4.7}$$

Also, using the functions previously defined

$$p := \frac{\phi + \bar{\phi}}{2} \qquad \text{and} \qquad q := \frac{\phi - \bar{\phi}}{2},$$

and with some algebra, one writes (4.6) and (4.7) as

$$\begin{aligned}\frac{\partial\Upsilon}{\partial p} &= \frac{\partial\mathcal{D}}{\partial p}\\ \frac{\partial\Upsilon}{\partial q} &= -\frac{\partial\mathcal{D}}{\partial q} - 2F\\ \frac{\partial\Upsilon}{\partial\Lambda} &= -\frac{\partial\mathcal{D}}{\partial\Lambda} - 2\frac{\partial\mathcal{D}}{\partial p}.\end{aligned}$$

Eliminating Υ from the previous system, one gets

$$\begin{aligned}\frac{\partial\mathcal{D}}{\partial q\partial p} &= -\frac{\partial F}{\partial p}\\ \frac{\partial\mathcal{D}}{\partial\Lambda\partial p} &= -\frac{\partial^2\mathcal{D}}{\partial p^2}\\ \frac{\partial F}{\partial\Lambda} &= \frac{\partial\mathcal{D}}{\partial q\partial p} = -\frac{\partial F}{\partial p}.\end{aligned}$$

From these equations, Corrigan & Zambon at [20] found, by a heuristic argument, that

$$\mathcal{D} = f(p - \Lambda, q) + g(q, \Lambda), \tag{4.8}$$

together with

$$F = -\frac{\partial f}{\partial q} \qquad \text{and} \qquad \Upsilon = f - g$$

then, equation (4.7) is

$$\mathcal{D}_p \mathcal{D}_q = 2(V - \bar{V}) + (f_\Lambda + g_\Lambda)F,$$

which can be written as

$$f_q g_\Lambda - f_\Lambda g_q = 2(V - \bar{V}). \tag{4.9}$$

Remark: Every dependence on Λ is in the left-hand side. This last equation will be fundamental in our future considerations about transformations of the Tzitzéica model.

4.2 Lagrangian approach

We already know that the Lagrangian to be considered is

$$\mathcal{L} = \theta(-x)\,\mathcal{L}_\phi + \theta(x)\,\mathcal{L}_{\bar{\phi}} + \delta(x)\,\mathcal{L}_\mathcal{D}, \tag{4.10}$$

where the integrable defect will be given through

$$\mathcal{L}_\mathcal{D} = -\frac{\phi\bar{\phi}_t - \phi_t\bar{\phi}}{2} - \Lambda\left(\phi - \bar{\phi}\right)_t + \Lambda_t\left(\phi - \bar{\phi}\right) + \mathcal{D}\left(\phi, \bar{\phi}, \Lambda\right),$$

with fields implemented by the following Lagrangian densities

$$\mathcal{L}_\phi = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi),$$

and

$$\mathcal{L}_{\bar{\phi}} = \frac{1}{2}\partial_\mu\bar{\phi}\partial^\mu\bar{\phi} - \bar{V}(\bar{\phi}),$$

where the field ϕ is taken for $x < 0$ and, obviously, $\bar{\phi}$ will be defined for $x > 0$. The Tzitzéica potentials are given by

$$V(\phi) = \frac{1}{2}\left(2e^{\phi} + e^{-2\phi}\right) = \frac{1}{2}\left(2e^{p+q} + e^{-2p-2q}\right) \tag{4.11}$$

$$V\left(\bar{\phi}\right) = \frac{1}{2}\left(2e^{\bar{\phi}} + e^{-2\bar{\phi}}\right) = \frac{1}{2}\left(2e^{p-q} + e^{-2p+2q}\right). \tag{4.12}$$

With some algebraic manipulations, from the boundary conditions (4.3) and the definition (3.15) we can set the following relations

$$\partial_+ q = \frac{1}{2}\left(\frac{\partial \mathcal{D}}{\partial p} + \frac{\partial \mathcal{D}}{\partial \Lambda}\right), \tag{4.13}$$

$$\partial_- q = -\frac{1}{2}\left(\frac{\partial \mathcal{D}}{\partial p}\right). \tag{4.14}$$

Last section, we have shown as Corrigan & Zambon [20], by a heuristic argument, have proved that $\mathcal{D}$ can be set down as

$$\mathcal{D} = f + g,$$

where the function g depends[3] on q and Λ, and f depends on q and $p - \Lambda$. Also, such functions must satisfy the bracket (4.9), i.e.

$$f_q\, g_\Lambda - f_\Lambda\, g_q = 2\left(V - \bar{V}\right),$$

which, any dependence on Λ contained in the left hand side of the equation must cancel out [20]. With the potentials given by (4.11) and (4.12), the most general *Ansatz* is

$$f = \Delta\, e^{2\,\Lambda - 2\,p} + \Omega\, e^{p-\Lambda} \quad \text{and} \quad g = \Xi\, e^{-2\,\Lambda} + \Pi\, e^{\Lambda}, \tag{4.15}$$

where Δ, Ω, Ξ and Π are functions of q. When we put (4.11), (4.12) and (4.15) into (4.9), we find the constraints

$$\Delta_q\, \Pi = 2\,\Delta\, \Pi_q; \qquad \Xi_q\, \Omega = 2\,\Xi\, \Omega_q;$$

$$2\left(\Delta\, \Xi\right)_q = \left(e^{2\,q} - e^{-2\,q}\right); \qquad \left(\Omega\, \Pi\right)_q = 2\left(e^{q} - e^{-q}\right),$$

where the solution can be constructed by

$$\begin{aligned}
(\Omega\Pi)_q = 2(e^q - e^{-q}) \quad &\Rightarrow \quad \Omega\Pi = 2(e^q + e^{-q}),\\
2(\Delta\Xi)_q = (e^{2q} - e^{-2q}) \quad &\Rightarrow \quad 2\Delta\Xi = \frac{1}{2}(e^{2q} + e^{-2q}),\\
\Delta_q\, \Pi = 2\,\Delta\, \Pi_q \quad &\Rightarrow \quad \Delta = \kappa\, \Pi^2,
\end{aligned}$$

[3]Pay attention that at this point, the function Λ here is, *a priori*, a different function from the previously defined function Λ at chapter 3. This Greek letter has been chosen just for convenience, because, in few lines, it will be proved that they are the same function.

thus, the general solution is

$$\Omega = \frac{2}{\Pi}\left(e^{q}+e^{-q}\right), \quad \Delta = \kappa\,\Pi^2, \quad \Xi = \frac{1}{4\,\kappa\,\Pi^2}\,\left(e^{q}+e^{-q}\right)^2.$$

Hence,

$$\begin{aligned}\mathcal{D} &= f+g \\ &= \kappa\,\Pi^2\,e^{2\,\Lambda-2\,p} + \frac{2}{\Pi}\,\left(e^{q}+e^{-q}\right)\,e^{p-\Lambda} + \frac{1}{4\,\kappa\,\Pi^2}\,\left(e^{q}+e^{-q}\right)^2\,e^{-2\,\Lambda} + \Pi\,e^{\Lambda}.\end{aligned}$$

Taking this last equation, placing it in (4.13) and (4.14) we arrive at the following equations

$$\partial_+ q = -\frac{1}{2}\left[\frac{1}{2\,\kappa\,\Pi^2}\,e^{-2\Lambda}\,\left(e^{q}+e^{-q}\right)^2 - \Pi\,e^{\Lambda}\right], \tag{4.16}$$

$$\partial_- q = -\frac{1}{2}\left[-2\,\kappa\,\Pi^2\,e^{2\,\Lambda-2\,p} + \frac{2}{\Pi}\,\left(e^{q}+e^{-q}\right)\,e^{p-\Lambda}\right], \tag{4.17}$$

that we must compare with the Darboux Transformation, in particular, with equations (3.17) and (3.19). Now, we can realize that

$$\Pi = -\frac{i}{\xi} \quad \text{and} \quad \Pi^2 = \frac{1}{4\,\xi\,\kappa},$$

remembering that we have put $\nu = 1$, therefore

$$\kappa = -\frac{\xi}{4}.$$

Finally, the whole Lagrangian is given by

$$\begin{aligned}\mathcal{L} =& \theta\,(-x)\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\left(2e^{\phi}+e^{-2\phi}\right)\right) + \theta\,(x)\left(\frac{1}{2}\partial_\mu\bar{\phi}\partial^\mu\bar{\phi} - \frac{1}{2}\left(2e^{\bar{\phi}}+e^{-2\bar{\phi}}\right)\right) + \\ &+ \delta\,(x)\left[-\frac{\phi\bar{\phi}_t - \phi_t\bar{\phi}}{2} - \Lambda\left(\phi-\bar{\phi}\right)_t + \Lambda_t\left(\phi-\bar{\phi}\right) + \frac{1}{4\,\xi}\,e^{2\,\Lambda-2\left(\phi-\bar{\phi}\right)} + \right. \\ &\left. + 2\,i\,\xi\left(e^{2\,\phi}+e^{-\left(2\,\bar{\phi}\right)}\right)e^{-\Lambda} + \xi\left(e^{\phi-\bar{\phi}}+e^{\bar{\phi}-\phi}\right)^2 e^{-2\,\Lambda} - \frac{i}{\xi}e^{\Lambda}\right].\end{aligned} \tag{4.18}$$

It has been proved so, that from the Lagrangian above constructed, the Darboux transformation naturally appears. Then, the Lagrangian with this kind of defect (that was analyzed by [20]), remains integrable and the relation between the solitons in different domains is given by the Darboux Transformation associated to the linear problem, i.e. to the system that arises from the Lax Pair. The Darboux, or even the Bäcklund transformation is a weak criterion to prove the integrability of the whole system; we need to find the conserved charges. In fact, in our recent work [21], we have found the infinite number of conserved charges, which prove that this whole system is integrable, even with the existence of this kind of defect.

Summing up In this chapter, we have considered a method to introduce integrable defects, broad enough to encompass the model important in this work - the Tzitzéica model. Also, we have found a Lagrangian density which describes the integrable defect in the system. And finally, we have proved that the Darboux transformation (and hence the Bäcklund Transformation) obtained by Borisov agreed with the defect conditions obtained by Corrigan & Zambon.

Part II

Geometric Context

Chapter 5

Theory of Surfaces

AFTER THE GREAT EFFORT that we have done up to now, we arrive at the easiest chapter of this work. Certainly there are non-trivial results here, however, theory of curves and surfaces is a topic with a strong visual appeal and in this sense, the subject is more pleasant. The first section is devoted to curves and the principal result is the Frenet equations. The second section, naturally, the ideas of curves are extended and we deal with surfaces. In the last two sections the Gauss-Weingarten equations as well as the Mainardi-Codazzi equations are presented.

This chapter represents a sort of training for the next chapter. There, we will use the technology that we are going to develop here and we will look the Tzitzéica model through its original point of view, the geometric one. Therefore, the idea of surface and the equations that describe a generic surfaces must be known, so, in this chapter, the aim is to present those ideas and, that is the reason why we need this little introduction to the theory of surfaces. We have created this short chapter, following the classical books of *Do Carmo* [47] and *Struik* [14].

5.1 Curves

Before start studying surfaces itself, let us talk a little about curves in $\mathbb{R}^3$, this is the conventional approach and we will use that. As was pointed by Struik at [14],

> *One can think curve in space as paths of a point in motion.*

The coordinates (x_1, x_2, x_3) of a point can be expressed as functions of a parameter $u \in I \subseteq \mathbb{R}$

$$x_i := x_i(u) \qquad i = 1, 2, 3,$$

where the evolution parameter u is usually called *time*. So, the curve $\mathcal{C}$ is represented by the map

$$\mathbf{r}(u) := (x_1(u), x_2(u), x_3(u)) : I \to \mathbb{R}^3.$$

One can define the length of this curve as

$$s = s(u) := \int_{u_0}^{u} \left| \frac{d\mathbf{r}}{du} \right| du,$$

and from the fundamental theorem of calculus

$$\frac{ds}{du} = \left| \frac{d\mathbf{r}}{du} \right|.$$

This equation defines a representation of the curve $\mathcal{C}$ in terms of the *arc length* or *natural representation*, so in this representation one writes $\mathbf{r} = \mathbf{r}(s)$ and

$$\left| \frac{d\mathbf{r}}{ds} \right| = 1,$$

thus, $\frac{d\mathbf{r}}{ds} \equiv \dot{\mathbf{r}}$ is a unit vector, since

$$\left| \frac{d\mathbf{r}}{ds} \right|^2 = \frac{d\mathbf{r}}{ds} \cdot \frac{d\mathbf{r}}{ds} = \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = 1.$$

The vector $\dot{\mathbf{r}}$ is called *unit tangent vector* and it is denoted by

$$\mathbf{t} \equiv \dot{\mathbf{r}}.$$

Now, one can take the derivative of the tangent vector and one defines the *curvature vector*,

$$\mathbf{k} = \mathbf{k}(s) = \dot{\mathbf{t}}.$$

The magnitude of this vector is called *curvature* of the curve $\mathcal{C}$ at the point $P \in \mathbf{x}(s)$ and it is presented as

$$k = |\mathbf{k}(s)|.$$

Also, one can think of a unit vector $\mathbf{n}$ in the direction of the curvature vector and naturally, perpendicular to the unit tangent vector $\mathbf{t}$, so

$$\mathbf{k} = k\mathbf{n}.$$

The vector $\mathbf{n}$ is called *principal normal vector.* Finally, since one has the orthogonal vectors $\mathbf{t}$ and $\mathbf{n}$, a new vector, perpendicular to both, called *unit binormal vector* can be defined as

$$\mathbf{b} := \mathbf{t} \times \mathbf{n}.$$

The triad $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ is known as *moving trihedron* of the curve $\mathcal{C}$.

As $\mathbf{b} \cdot \mathbf{t} = 0$, taking its derivative

$$\frac{d}{ds}(\mathbf{b} \cdot \mathbf{t}) = \dot{\mathbf{b}} \cdot \mathbf{t} + \mathbf{b} \cdot \dot{\mathbf{t}} = 0,$$

so

$$\dot{\mathbf{b}} \cdot \mathbf{t} = -\mathbf{b} \cdot \dot{\mathbf{t}} = -k\mathbf{b} \cdot \mathbf{n} = 0,$$

then, $\dot{\mathbf{b}}$ is perpendicular to $\mathbf{t}$, and from $\mathbf{b} \cdot \mathbf{b} = 1$ one concludes that $\dot{\mathbf{b}}$ is orthogonal to $\mathbf{b}$ too, since

$$\dot{\mathbf{b}} \cdot \mathbf{b} = 0,$$

one concludes so, that $\mathbf{b}$ is proportional to $\mathbf{n}$, what allows one to write

$$\frac{d\mathbf{b}}{ds} = \dot{\mathbf{b}} := -\tau\mathbf{n},$$

and one calls τ the *torsion* of the curve $\mathcal{C}$ at the point $P \in \mathbf{x}(s)$.

Evidently, the derivative of the unit vector $\mathbf{n}$ is needed. For this deal, it is enough to notice that one can write this vector as

$$\mathbf{n} = \mathbf{b} \times \mathbf{t},$$

so, its derivative is just

$$\dot{\mathbf{n}} = \dot{\mathbf{b}} \times \mathbf{t} + \mathbf{b} \times \dot{\mathbf{t}} = -\tau\mathbf{n} \times \mathbf{t} + k\mathbf{b} \times \mathbf{n} = -k\mathbf{t} + \tau\mathbf{b}.$$

One can sum up these last results as

$$\begin{array}{rcccc} \frac{d\mathbf{t}}{ds} & = & & k\mathbf{n} & \\ \frac{d\mathbf{n}}{ds} & = & -k\mathbf{t} & & \tau\mathbf{b} \\ \frac{d\mathbf{b}}{ds} & = & & -\tau\mathbf{n} & \end{array} \tag{5.1}$$

and

$$\frac{d\mathbf{r}}{ds} = \mathbf{t}. \tag{5.2}$$

These equations are called *Frenet Formulas* or *Serre-Frenet Formulas* and describe the motion of the moving trihedron along the curve.

5.2 Surfaces

Heuristically, one can understand a *surface* Σ as the set of points in the euclidean space $\mathbb{R}^3$ that in certain sense resembles deformed sheets placed together, in a way that there are no sharp points, cutting, self intersections and so on [1].

Intelligently, one may mimic what was done in curves and try to extend it for surfaces. So, a map from a space parameter, that without loss of generality can be a subset of the plane $S \subseteq \mathbb{R}^2$, onto a set of points in $U \subset \mathbb{R}^3$, is a representation of a surface in the euclidean space $\mathbb{R}^3$. Expressing the rectangular functions of $\mathbb{R}^3$ in terms of two parameters $\rho,\ \varrho \in S$, one has

$$x_i := x_i(\rho, \varrho), \qquad i = 1, 2, 3;$$

and in vector form

$$\mathbf{r} := \mathbf{r}(\rho, \varrho) = x_1\mathbf{e_1} + x_2\mathbf{e_2} + x_3\mathbf{e_3} : S \subseteq \mathbb{R}^2 \to U \subset \mathbb{R}^3.$$

Over this surface, one can define a curve $\mathcal{C}$; in other words, on the surface one can glue a curve, to be determined by functions with a parametric forms $\rho = \rho(t)$ and $\varrho = \varrho(t)$. In this sense, the functions ρ and ϱ draw a curve in $S \subseteq \mathbb{R}^2$ and then, x_i draws the curve on the surface,

$$x_i \circ \rho,\ x_i \circ \varrho : \mathbb{R} \to \mathbb{R}^3 \qquad i = 1, 2, 3.$$

The tangent vector[2] to this curve on the surface is defined as

$$d\mathbf{r} := \frac{d\mathbf{r}}{d\rho}d\rho + \frac{d\mathbf{r}}{d\varrho}d\varrho \equiv \mathbf{r}_\rho d\rho + \mathbf{r}_\varrho d\varrho.$$

And the distance between two points on the curve is already known

$$s = \int \left| \frac{d\mathbf{r}}{dt} \right| dt.$$

that one would have obtained integrating

$$\sqrt{ds^2} = \sqrt{d\mathbf{r} \cdot d\mathbf{r}},$$

[1] Regular surface, but the adjective is hidden, because the curves treated here, are so smooth and well behaved that the notions of calculus can be extended.

[2] In the appendix A, page 83, we explain the differential as a map between tangent spaces.

along the curve. The quantity to be considered now, is

$$I = ds^2 = d\mathbf{r} \cdot d\mathbf{r} = Ed\rho^2 + 2Fd\rho d\varrho + Gd\varrho^2, \tag{5.3}$$

where: $E = \mathbf{r}_\rho \cdot \mathbf{r}_\rho$, $F = \mathbf{r}_\rho \cdot \mathbf{r}_\varrho$ and $G = \mathbf{r}_\varrho \cdot \mathbf{r}_\varrho$, that one could sum up through

$\cdot$	$\mathbf{r}_\rho$	$\mathbf{r}_\varrho$	$\mathbf{N}$
$\mathbf{r}_\rho$	E	F	0
$\mathbf{r}_\varrho$	F	G	0
$\mathbf{N}$	0	0	1

The function $I = ds^2$ is known as *First Fundamental Form* and indicates how the surface inherits the inner product of the euclidean space $\mathbb{R}^3$. Also, notice that when the parametric curves are orthogonal $\mathbf{r}_\rho \perp \mathbf{r}_\varrho$ then $F = 0$. Another important result is the positivity of

$$\begin{aligned} EG - F^2 &= (\mathbf{r}_\rho \cdot \mathbf{r}_\rho)(\mathbf{r}_\varrho \cdot \mathbf{r}_\varrho) - (\mathbf{r}_\rho \cdot \mathbf{r}_\varrho)(\mathbf{r}_\rho \cdot \mathbf{r}_\varrho), \\ &= \left(\delta^j_m r^j_\rho r^m_\varrho\right)\left(\delta^k_n r^k_\rho r^n_\varrho\right) - \left(\delta^j_n r^j_\rho r^n_\varrho\right)\left(\delta^k_m r^k_\rho r^m_\varrho\right) \\ &= \left(\delta^j_m \delta^k_n - \delta^j_n \delta^k_m\right) r^j_\rho r^n_\varrho r^k_\rho r^m_\varrho \end{aligned}$$

and using

$$\epsilon^{ijk}\epsilon_{imn} = \delta^j_m \delta^k_n - \delta^j_n \delta^k_m,$$

easily one concludes that

$$EG - F^2 = \left|\mathbf{r}_\rho \times \mathbf{r}_\varrho\right|^2 := H^2 > 0. \tag{5.4}$$

Those familiar with special relativity can recognize that when one deals here with ds, it is doing the same considerations that are usually done there. The only difference is the coefficients E, F, G that depend of the space structure, which in the relativity case, is Minkowski space. In relativity, the quantity ds is written as

$$ds^2 = -c^2 dt^2 + d\mathbf{r} \cdot d\mathbf{r},$$

that is the space-time interval, an invariant quantity as well as the length is an invariant in the euclidean space. Also, we know that the function ds is enough to our considerations in the relativity, mainly because the Minkowski space is not embedded in a space of higher dimension. When we treat spaces embedded in some space of higher dimension, we need some information about the normal vector to the surface in each of its points. This information will be counted when one defines another invariant quantity, but at this time, besides the tangent vectors to some curve on this space, the normal vector must be considered. This invariant quantity is called *Second Fundamental Form.*

Evidently, a generic two-dimensional surface, that is the goal of this section, must be thought as a space embedded in the euclidean three-dimensional space. The second invariant quantity will be considered by setting a normal unit vector to the surface

$$\mathbf{N} = \frac{\mathbf{r}_\rho \times \mathbf{r}_\varrho}{|\mathbf{r}_\rho \times \mathbf{r}_\varrho|}. \tag{5.5}$$

Considering now the curvature vector of a curve $\mathcal{C}$ at the point P, we can decompose

$$\frac{d\mathbf{t}}{ds} = \ddot{\mathbf{r}} = \mathbf{k} = \mathbf{k}_n + \mathbf{k}_t,$$

where $\mathbf{k}_n$ is normal and $\mathbf{k}_t$ is tangent to the surface. Naturally, one can define a proportionality constant k_n such that

$$\mathbf{k}_n := k_n \mathbf{N}, \tag{5.6}$$

this vector is called *normal curvature vector* and its component k_n is the *normal curvature*, by definition. The tangent vector $\mathbf{t} := \dot{\mathbf{r}}$ is orthogonal to $\mathbf{N}$, then

$$\frac{d}{ds}(\mathbf{t}\cdot\mathbf{N}) = \frac{d\mathbf{t}}{ds}\cdot\mathbf{N} + \mathbf{t}\cdot\frac{d\mathbf{N}}{ds} = 0.$$

As, the curvature vector is $\mathbf{k} = \frac{d\mathbf{t}}{ds}$, one has

$$\mathbf{k}\cdot\mathbf{N} = -\frac{d\mathbf{r}}{ds}\cdot\frac{d\mathbf{N}}{ds},$$

using the natural representation, defined by $\frac{d\mathbf{r}}{ds} = 1$, finally

$$k_n = -\frac{d\mathbf{r}\cdot d\mathbf{N}}{d\mathbf{r}\cdot d\mathbf{r}}. \tag{5.7}$$

Only de numerator is unknown, the denominator is the first fundamental form, then

$$d\mathbf{N} := \mathbf{N}_\rho d\rho + \mathbf{N}_\varrho d\varrho \quad\text{and}\quad d\mathbf{r} := \mathbf{r}_\rho d\rho + \mathbf{r}_\varrho d\varrho,$$

what allows to set down

$$II = -d\mathbf{r}\cdot d\mathbf{N} = e d\rho^2 + 2f d\rho d\varrho + g d\varrho^2, \tag{5.8}$$

where $e = -\mathbf{r}_\rho \cdot \mathbf{N}_\rho$, $f = -\frac{1}{2}(\mathbf{r}_\rho \cdot \mathbf{N}_\varrho + \mathbf{r}_\varrho \cdot \mathbf{N}_\rho)$ and $g = -\mathbf{r}_\varrho \cdot \mathbf{N}_\varrho$.

Now, using

$$\mathbf{r}_\rho \cdot \mathbf{N} = 0 = \mathbf{r}_\varrho \cdot \mathbf{N},$$

one can write $e = \mathbf{r}_{\rho\rho} \cdot \mathbf{N}$, $f = \mathbf{r}_{\rho\varrho} \cdot \mathbf{N}$ and $g = \mathbf{r}_{\varrho\varrho} \cdot \mathbf{N}$. This last function II, is what one calls *Second Fundamental Form.* The general aspects of the surface in the three-dimensional space can be determined just with these quantities, the first and second fundamental forms.

Another important result is the *Gaussian Curvature*,

$$\kappa := \frac{eg - f^2}{EG - F^2}, \tag{5.9}$$

that says the behaviour of the curvature in a given point. Follows from this definition that each point in the surface is a

Hyperbolic point if:	$eg - f^2 < 0 \;\Rightarrow\; \kappa < 0,$
Parabolic or planar point if:	$eg - f^2 = 0 \;\Rightarrow\; \kappa = 0,$
Elliptic point if:	$eg - f^2 > 0 \;\Rightarrow\; \kappa > 0,$

since (5.4) sets that $EG - F^2 > 0$. The derivation of the expression for the Gaussian curvature is not complicated, however, it would demand a short deviation of the real aim of this section, this derivation is made in a wonderful fashion in [14, 48].

Asymptotic Curves

Now, let us talk about asymptotic curves very quickly. Asymptotic directions occurs when:

$$k_n = 0 \;\Rightarrow\; II = 0 \;\Rightarrow\; ed\rho^2 + 2f d\rho d\varrho + g d\varrho^2 = 0,$$

and curves having these directions are called *asymptotic curves.* They occurs, for instance, if there exists a straight line on the surface.

When $e = 0 = g$, and $f \neq 0$, i.e. in a hyperbolic point, the asymptotic curves are given by

$$d\rho = 0 \Rightarrow \rho = c_0 \in \mathbb{R},$$

$$d\varrho = 0 \Rightarrow \varrho = c_1 \in \mathbb{R},$$

i.e, ρ and ϱ constants. Therefore, *In a neighborhood of a hyperbolic point on a surface, there exist two distinct families of asymptotic lines.*

The asymptotic curves in the case of a hyperbolic point are useful parametrizations. For elliptic points, there are no real curves that satisfy $k_n = 0$.

5.3 The Gauss-Weingarten equations

One can think about the Gauss-Weingarten equations for surfaces analogously to the Frenet equations for curves. It means that, as in the case of curves that one is able to express the vectors $\dot{\mathbf{t}}$, $\dot{\mathbf{n}}$ and $\dot{\mathbf{b}}$ in terms of the orthonormal moving trihedron for curves $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ (5.1) one would like to express $\mathbf{r}_{\rho\rho}$, $\mathbf{r}_{\rho\varrho}$, $\mathbf{r}_{\varrho\varrho}$ and $\mathbf{N}_\rho$, $\mathbf{N}_\varrho$ in terms of a new linearly independent (not necessarily orthonormal) moving trihedron for surfaces, $\mathbf{r}_\rho$, $\mathbf{r}_\varrho$, and $\mathbf{N}$. These equations can be defined as

$$\left.\begin{aligned}
\mathbf{r}_{\rho\rho} &:= \Gamma^1_{11}\mathbf{r}_\rho + \Gamma^2_{11}\mathbf{r}_\varrho + \alpha_{11}\mathbf{N}\\
\mathbf{r}_{\rho\varrho} &:= \Gamma^1_{12}\mathbf{r}_\rho + \Gamma^2_{12}\mathbf{r}_\varrho + \alpha_{12}\mathbf{N}\\
\mathbf{r}_{\varrho\varrho} &:= \Gamma^1_{22}\mathbf{r}_\rho + \Gamma^2_{22}\mathbf{r}_\varrho + \alpha_{22}\mathbf{N}
\end{aligned}\right\} \textit{Gauss equations}$$

$$\left.\begin{aligned}
\mathbf{N}_\rho &:= \beta^1_1\mathbf{r}_\rho + \beta^2_1\mathbf{r}_\varrho + \gamma_1\mathbf{N}\\
\mathbf{N}_\varrho &:= \beta^2_1\mathbf{r}_\rho + \beta^2_2\mathbf{r}_\varrho + \gamma_2\mathbf{N}
\end{aligned}\right\} \textit{Weingarten equations}$$

where Γ^i_{jk}, α_{ij}, β^j_i, γ_i for $i, j, k = 1, 2$ must be determined.

Explicit calculations for determination of Γ^i_{jk}, α_{ij}, β^j_i, γ_i were carefully made by [48]. This achievement is not complicated, however, demands a little effort. The key point of this achievement is to consider the orthogonality relations

$$\mathbf{N}\cdot\mathbf{N}_\rho = \mathbf{N}\cdot\mathbf{N}_\varrho = 0 = \mathbf{N}\cdot\mathbf{r}_\varrho = \mathbf{N}\cdot\mathbf{r}_\rho$$

and that $\mathbf{N}$ is a unit vector.

The Gauss equations associated with a surface Σ in $\mathbb{R}^3$ are

$$\begin{aligned}
\mathbf{r}_{\rho\rho} &= \Gamma^1_{11}\mathbf{r}_\rho + \Gamma^2_{11}\mathbf{r}_\varrho + e\mathbf{N}\\
\mathbf{r}_{\rho\varrho} &= \Gamma^1_{12}\mathbf{r}_\rho + \Gamma^2_{12}\mathbf{r}_\varrho + f\mathbf{N}\\
\mathbf{r}_{\varrho\varrho} &= \Gamma^1_{22}\mathbf{r}_\rho + \Gamma^2_{22}\mathbf{r}_\varrho + g\mathbf{N}.
\end{aligned} \tag{5.10}$$

While the Weingarten equations are

$$\begin{aligned}\mathbf{N}_\rho &= \frac{fF-eG}{EG-F^2}\mathbf{r}_\rho + \frac{eF-fE}{EG-F^2}\mathbf{r}_\varrho,\\ \mathbf{N}_\varrho &= \frac{gF-fG}{EG-F^2}\mathbf{r}_\rho + \frac{fF-gE}{EG-F^2}\mathbf{r}_\varrho.\end{aligned} \tag{5.11}$$

The coefficients Γ^i_{jk}, called *Christoffel symbols*, are given[3] by

$$\begin{aligned}\Gamma^1_{11} &= \frac{GE_\rho - 2FF_\rho + FE_\varrho}{2H^2}, & \Gamma^2_{11} &= \frac{2EF_\rho - EE_\varrho - FE_\rho}{2H^2},\\ \Gamma^1_{12} &= \frac{GE_\varrho - FG_\rho}{2H^2}, & \Gamma^2_{12} &= \frac{EG_\rho - FE_\varrho}{2H^2},\\ \Gamma^1_{22} &= \frac{2GF_\varrho - GG_\rho - FG_\varrho}{2H^2}, & \Gamma^2_{22} &= \frac{EG_\varrho - 2FF_\varrho + FG_\rho}{2H^2}.\end{aligned} \tag{5.12}$$

The compatibility conditions $(\mathbf{r}_{\rho\rho})_\varrho = (\mathbf{r}_{\rho\varrho})_\rho$ and $(\mathbf{r}_{\varrho\varrho})_\rho = (\mathbf{r}_{\rho\varrho})_\varrho$ make arise the Mainardi-Codazzi equations:

$$\begin{aligned}e_\varrho - f_\rho &= \Gamma^1_{12}e + \left(\Gamma^2_{12} - \Gamma^1_{11}\right) f - \Gamma^2_{11} g,\\ f_\varrho - g_\rho &= \Gamma^1_{22}e + \left(\Gamma^2_{22} - \Gamma^1_{12}\right) f + \Gamma^2_{21} g.\end{aligned} \tag{5.13}$$

There are several analytical expressions for the Mainardi-Codazzi equation, in this work, one particularly useful can be obtained considering the derivative

$$\begin{aligned}\left(\frac{e}{H}\right)_{\rho(\varrho)} &= \frac{e_{\rho(\varrho)}}{H} - \frac{e}{H^2}H_{\rho(\varrho)} \Rightarrow e_{\rho(\varrho)} = \left(\frac{e}{H}\right)_{\rho(\varrho)} H + \frac{e}{H}H_{\rho(\varrho)}\\ \left(\frac{f}{H}\right)_{\rho(\varrho)} &= \frac{f_{\rho(\varrho)}}{H} - \frac{f}{H^2}H_{\rho(\varrho)} \Rightarrow f_{\rho(\varrho)} = \left(\frac{f}{H}\right)_{\rho(\varrho)} H + \frac{f}{H}H_{\rho(\varrho)}\\ \left(\frac{g}{H}\right)_{\rho(\varrho)} &= \frac{g_{\rho(\varrho)}}{H} - \frac{g}{H^2}H_{\rho(\varrho)} \Rightarrow g_{\rho(\varrho)} = \left(\frac{g}{H}\right)_{\rho(\varrho)} H + \frac{g}{H}H_{\rho(\varrho)}\end{aligned}$$

and plugging these equations in (5.13). Hence, one gets another formula for the Mainardi-Codazzi equations [14]

$$\begin{aligned}\left(\frac{e}{H}\right)_\varrho - \left(\frac{f}{H}\right)_\rho + \frac{e}{H}\Gamma^2_{22} - 2\frac{f}{H}\Gamma^2_{12} + \frac{g}{H}\Gamma^2_{11} &= 0,\\ \left(\frac{g}{H}\right)_\rho - \left(\frac{f}{H}\right)_\varrho + \frac{e}{H}\Gamma^1_{22} - 2\frac{f}{H}\Gamma^1_{12} + \frac{g}{H}\Gamma^1_{11} &= 0.\end{aligned} \tag{5.14}$$

[3]We physicists usually write the Christoffel symbols in a particular fashion, for this, consider the definition $g_{11} = E,\ g_{22} = G,\ g_{12} = F$, see [14, 48] for details.

Summing up The general aspects of surfaces embedded in the euclidean space $\mathbb{R}^3$ have been considered here. We learn about the Gauss approach for surfaces including the Gaussian curvature and the Gauss-Weingarten equations, also from the compatibility conditions for these equations, emerge the Mainardi-Codazzi equations.

Chapter 6

From geometry to Tzitzéica equation

In this brief chapter, the focus are surfaces with negative Gaussian curvature, or hyperbolic surfaces, in particular, surfaces whose the Gaussian curvature at a generic point, is proportional to the fourth power of the distance from origin to this point. There are a particular interest in these surfaces called *affine spheres* or *affinsphären*, because, from the compatibility condition of the equations which describe this surface, emerge the equation that we know as Tzitzéica equation. This was the approach used by Tzitzéica in his work, which culminated in the origin of a new branch of differential geometry of surfaces, the *affine geometry*, therefore, affinsphären are as important for affine geometry as the ordinary spheres are for Riemannian geometry.

In this chapter, we will derive the set of equations which describe the Tzitzéica surface and naturally, we will find its equation from the compatibility conditions of the Gauss equations. The present chapter was composed in the light of the books of Rogers & Shief [4] and Coley et.al. [16].

6.1 Affinesphären and Hyperbolic Surfaces

Consider a hyperbolic surface Σ parametrized in terms of orthogonal ($F = 0$) asymptotic ($e = 0 = g$) curves; in this system, the Gauss equations are

$$\begin{aligned}\mathbf{r}_{\rho\rho} &= \Gamma^1_{11}\mathbf{r}_\rho + \Gamma^2_{11}\mathbf{r}_\varrho,\\ \mathbf{r}_{\rho\varrho} &= \Gamma^1_{12}\mathbf{r}_\rho + \Gamma^2_{12}\mathbf{r}_\varrho + f\mathbf{N},\\ \mathbf{r}_{\varrho\varrho} &= \Gamma^1_{22}\mathbf{r}_\rho + \Gamma^2_{22}\mathbf{r}_\varrho,\end{aligned} \tag{6.1}$$

the Mainardi-Codazzi equations are

$$\begin{aligned}\left(\frac{f}{H}\right)_\rho + 2\left(\frac{f}{H}\right)\Gamma^2_{12} = 0,\\ \left(\frac{f}{H}\right)_\varrho + 2\left(\frac{f}{H}\right)\Gamma^1_{12} = 0,\end{aligned} \tag{6.2}$$

and finally, the Gaussian curvature is

$$\kappa = -\frac{f^2}{H^2}. \tag{6.3}$$

From the Mainardi-Codazzi equations, we have

$$-\left(\frac{f}{H}\right)\Gamma^2_{12} = \frac{1}{2}\left(\frac{f}{H}\right)_\rho \Rightarrow -\Gamma^2_{12} = \frac{1}{2}\left(\frac{f}{H}\right)^{-1}\left(\frac{f}{H}\right)_\rho \Rightarrow -\Gamma^2_{12} = \frac{2}{4}\left[\ln\left(\frac{f}{H}\right)\right]_\rho$$

$$\Rightarrow -\Gamma^2_{12} = \frac{1}{4}\left[\ln\left(\frac{f^2}{H^2}\right)\right]_\rho = \frac{1}{4}\left[\ln(-\kappa)\right]_\rho$$

and in the same way for Γ^1_{12}, therefore

$$\Gamma^2_{12} = -\frac{1}{4}\left[\ln(-\kappa)\right]_\rho, \tag{6.4}$$

$$\Gamma^1_{12} = -\frac{1}{4}\left[\ln(-\kappa)\right]_\varrho. \tag{6.5}$$

The cross derivative $\mathbf{r}_{\rho\varrho}$ can be write now

$$\mathbf{r}_{\rho\varrho} = -\frac{1}{4}\left[\ln(-\kappa)\right]_\varrho \mathbf{r}_\rho - \frac{1}{4}\left[\ln(-\kappa)\right]_\rho \mathbf{r}_\varrho + f\mathbf{N}. \tag{6.6}$$

So, the Gauss equation involving the cross derivative was written in terms of the Gaussian curvature κ. The next step is to consider the distance from origin to a tangent plane to Σ at a generic point P; that distance, obviously, is

$$d = \mathbf{N}\cdot\mathbf{r},$$

together with the derivatives

$$d_\rho = \mathbf{N}_\rho \cdot \mathbf{r} \quad \text{and} \quad d_\varrho = \mathbf{N}_\varrho \cdot \mathbf{r}.$$

One could think about these equations as projections of the vector $\mathbf{r}$ onto $\mathbf{N}$, $\mathbf{N}_\rho$ and $\mathbf{N}_\varrho$. Considering the Weingarten equations

$$\mathbf{N}_\rho = -\frac{fE}{H^2}\mathbf{r}_\varrho, \tag{6.7}$$

$$\mathbf{N}_\varrho = -\frac{fG}{H^2}\mathbf{r}_\rho, \tag{6.8}$$

one has

$$d_\rho = \mathbf{N}_\rho \cdot \mathbf{r} = -\frac{fE}{H^2}\mathbf{r}_\varrho \cdot \mathbf{r} \;\Rightarrow\; \mathbf{r}_\varrho \cdot \mathbf{r} = -\frac{H^2}{fE}d_\rho,$$

$$d_\varrho = \mathbf{N}_\varrho \cdot \mathbf{r} = -\frac{fG}{H^2}\mathbf{r}_\rho \cdot \mathbf{r} \;\Rightarrow\; \mathbf{r}_\rho \cdot \mathbf{r} = -\frac{H^2}{fG}d_\varrho,$$

which imply

$$\mathbf{r} = -\frac{H^2 d_\varrho}{fE}\mathbf{r}_\rho - \frac{H^2 d_\rho}{fE}\mathbf{r}_\varrho + \mathbf{N}d.$$

From the relations, which define the coefficient $f = \mathbf{r}_{\rho\varrho} \cdot \mathbf{N}$ of the second fundamental form and the orthogonality relations $\mathbf{r}_\rho \cdot \mathbf{N} = 0 = \mathbf{r}_\varrho \cdot \mathbf{N}$, one concludes that

$$\mathbf{N}_\rho \cdot \mathbf{r}_\varrho = -f = \mathbf{N}_\varrho \cdot \mathbf{r}_\rho$$

$$-\frac{fE}{H^2} = -f = -\frac{fG}{H^2},$$

hence $E = G = 1$. Finally, the vector $\mathbf{r}$ is

$$\mathbf{r} = -\frac{d_\varrho}{f}\mathbf{r}_\rho - \frac{d_\rho}{f}\mathbf{r}_\varrho + \mathbf{N}d. \tag{6.9}$$

Using this last equation, one can subtract $\frac{f}{d}\mathbf{r}$ from (6.6); then

$$\mathbf{r}_{\rho\varrho} - \frac{f}{d}\mathbf{r} = \left(\frac{d_\varrho}{d} - \frac{1}{4}\left[\ln(-\kappa)\right]_\varrho\right)\mathbf{r}_\rho + \left(\frac{d_\rho}{d} - \frac{1}{4}\left[\ln(-\kappa)\right]_\rho\right)\mathbf{r}_\varrho$$

$$= -\frac{1}{4}\left[\ln\left(-\frac{\kappa}{d^4}\right)\right]_\varrho \mathbf{r}_\rho - \frac{1}{4}\left[\ln\left(-\frac{\kappa}{d^4}\right)\right]_\rho \mathbf{r}_\varrho.$$

Using now, the *Tzitzéica condition*, that is, when we suppose that the Gaussian curvature κ, is proportional to the fourth power of the distance d, i.e.

$$-\frac{\kappa}{d^4} = c^2 \in \mathbb{R}_+^*$$

one arrives to the equation

$$\mathbf{r}_{\rho\varrho} = v\mathbf{r} \tag{6.10}$$

where one defines $v := f/d$.

The compatibility conditions $(\mathbf{r}_{\rho\varrho})_\rho = (\mathbf{r}_{\rho\rho})_\varrho$ and $(\mathbf{r}_{\rho\varrho})_\varrho = (\mathbf{r}_{\varrho\varrho})_\rho$ imply

$$\Gamma^1_{11} = \frac{v_\rho}{v}, \quad \Gamma^2_{11} = \frac{a(\rho)}{v}, \quad \Gamma^1_{22} = \frac{b(\varrho)}{v} \quad \text{and} \quad \Gamma^2_{22} = \frac{v_\varrho}{v}$$

and the field v must satisfy the following equation

$$(\ln v)_{\rho\varrho} = v - \frac{ab}{v^2}.$$

This is exactly the Tzitzéica equation, when $ab = 1$. With this parametrization, the Gauss equations can be presented as

$$\begin{aligned}
\mathbf{r}_{\rho\rho} &= \frac{v_\rho}{v}\mathbf{r}_\rho + \frac{a}{v}\mathbf{r}_\varrho, \\
\mathbf{r}_{\rho\varrho} &= v\mathbf{r}, \\
\mathbf{r}_{\varrho\varrho} &= \frac{v_\varrho}{v}\mathbf{r}_\varrho + \frac{a^{-1}}{v}\mathbf{r}_\rho.
\end{aligned} \tag{6.11}$$

Summing up We have found here, the Tzitzéica equation from the compatibility conditions of the Gauss equations. This approach gives some additional information, because, besides the Tzitzéica equation itself, now we have a system of equations which describe a hyperbolic surface and it is fundamental for the construction of the Bäcklund transformation that we are looking for.

Chapter 7

Transformation of Surfaces

ONE OF THE MOST PROMINENT PROBLEMS around the Tzitzéica equation is about its transformations. As we said before, Tzitzéica studied a particular class of surfaces associated with this equation, that has a negative Gaussian curvature - hyperbolic surfaces - in $\mathbb{R}^3$.

In his work, Tzitzéica found a linear representation of the equations that describe the surface (that we have already seen) and he has established a transformation that allowed him to find new solutions of the equation, having at least, one trivial solution as a seed (that we will see now). In a modern point of view, he found the Lax Pair and the Moutard transformation for his equation.

All along this work, we have cited *Bäcklund and Darboux like-transformations*, now, we speak about *Tzitzéica and Moutard like-transformations*, that we will define in this chapter. There are a kind of overlap among these transformations, as was pointed by [29].

In a heuristic mode, Darboux Transformation generates a new solution, that is expressed via an older solution and the eigenfunction of the Lax operator associated to the first solution; while the Bäcklund Transformation does not require any solution of the Lax Pair problem associated, however, to the contrary to the Darboux Transformation, the Bäcklund does not represent an explicit solution of the equation, but a differential relationship between the new and older solutions.

Loosely speaking, we can say that these kind of transformations, allow the iterative construction of hyperbolic surfaces. Then, we now use this idea to our problem. Therefore, in this chapter, the last one, we will show how to reduce the transformations that we have found in the first part of our work, to the transformation that Tzitzéica found in his original work.

7.1 On the Tzitzéica Transformation

Analogously to what was done at [30], the Darboux transformation, previously found at the chapter 3 through the system (3.10) - (3.13), can be written in a different way, via the following approach:

Explicit calculation shows that the determinant of the Darboux matrix $\mathbf{K}$ is given by

$$\det \mathbf{K} = \left(-\frac{1}{\lambda^3} + \xi\right)^2 \left(\frac{1}{\lambda^3} + \xi\right).$$

According to [26, 30, 46], when $\xi = \frac{1}{\lambda^3}$, the $\det \mathbf{K}$ is equal to zero and the rank of the Darboux matrix is the unity, that is

$$\det \mathbf{K}\Big|_{\xi=\frac{1}{\lambda^3}} = 0.$$

An arbitrary matrix function $J(\mu)$ is said to have a zero at μ_0 if $\det J(\mu_0) = 0$, then, the inverse matrix $J^{-1}(\mu)$ has a singularity at this point, i.e. a pole of finite order. If μ_0 is a simple pole, then one can expand

$$J^{-1}(\mu) = \frac{A}{\mu - \mu_0} + C + \ldots,$$

and in the neighborhood of μ_0 the function $J(\mu)$ can be expanded as

$$J(\mu) = B + D\,(\mu - \mu_0) + \ldots$$

Consider now the problem with $J_1(\mu)$ and $J_2(\mu)$ such that $J_1(\mu)J_2(\mu) = 1$, $J_1(\infty) = 1$ and $J_2(\infty) = 1$. Now, let μ_0 and ν_0 be zeros of $J_1(\mu)$ and $J_2(\mu)$ respectively. As $J_1(\mu)J_2(\mu) = 1$, the zero of $J_1(\mu)$ is the pole of $J_2(\mu)$ and vice versa, thus one can write

$$J_1(\mu) = 1 + \frac{A_1}{\mu - \nu_0},$$
$$J_2(\mu) = 1 + \frac{A_2}{\mu - \mu_0}.$$

Then

$$J_1(\mu)J_2(\mu) = \left(1 + \frac{A_1}{\mu - \nu_0}\right)\left(1 + \frac{A_2}{\mu - \mu_0}\right) = 1.$$

from which one easily concludes that

$$A_1 = -A_2$$

and

$$A_1 - \frac{A_1 A_2}{\mu_0 - \nu_0} = 0, \qquad A_2 + \frac{A_1 A_2}{\mu_0 - \nu_0} = 0.$$

This system has the following solution

$$A_1 = -(\mu_0 - \nu_0)P$$

and

$$A_2 = \quad (\mu_0 - \nu_0)P,$$

where $P^2 = P$, it means that P is a projection operator.

Finally, one writes

$$J_1(\mu) = 1 - \frac{\mu_0 - \nu_0}{\mu - \nu_0}P,$$

$$J_2(\mu) = 1 + \frac{\mu_0 - \nu_0}{\mu - \mu_0}P,$$

where

$$J_1(\mu_0) = 1 - P = J_1(\nu_0).$$

Obviously, one can think of these operators in a N-dimensional linear complex space V^N. In such a space, let there be a linear operator U, then

$$KerU = \{\vec{u}_0 \ \in \ V^N \mid U\vec{u}_0 = \vec{0}\},$$

and

$$ImU = \{\vec{u} \ \in V^N \mid U\vec{u} = \vec{w} \ \in \ V^N\};$$

i.e. the *null space* (kernel) and the *range* (image) of the linear operator U, respectively. Evidently, the null vector $\vec{0}$ is present in both spaces and if $\vec{u}_0^{\,i} \ \in \ KerU$ and $\vec{u}^{\,j} \ \in \ ImU$, for any $i, j \ \in \ \mathbb{N}$, then

$$\sum_i a_i \vec{u}_0^{\,i} \ \in KerU$$

and

$$\sum_j b_j \vec{u}^{\,j} \ \in ImU,$$

$\forall\ a_i,\ b_j\ \in \mathbb{C}$. It means that $KerU$ and ImU are linear subspaces of V^N and

$$V^N = KerU \oplus ImU.$$

In the *non*degenerate case, i.e. $\det U \neq 0 \Rightarrow kerU = 0$ and $ImU = V^N$ and in a degenerate case, $\det U = 0$, $kerU$ and ImU are non-trivial.

Now, if U is a projector, i.e. $U^2 = U$, then there are two important properties to be considered:

1. $\forall\ \vec{u}_I \in ImU \Rightarrow U\vec{u}_I = \vec{u}_I$;

2. As $V^N = KerU \oplus ImU$ one can decompose a vector $\vec{u} \in V^N$ as $\vec{u} = \vec{u}_0 + \vec{u}_I$ where $\vec{u}_0 \in KerU$ and $\vec{u}_I \in ImU$. Then

$$\vec{u}_I = U\vec{u}$$

and

$$\vec{u}_0 = \vec{u} - U\vec{u} = (1 - U)\,\vec{u}.$$

So, if U is a projector, then, $1 - U$ also will be.

Now, notice that $ImU = \{\vec{u}_I\}$ and $(1 - U)\,\vec{u}_I = \vec{0}$, then $Ker\,(1 - U) = \{\vec{u}_I\}$, then

$$ImU = \{\vec{u}_I\} \equiv Ker\,(1 - U)\,,$$

in the same way, one concludes that

$$KerU = \{\vec{u}_0\} \equiv Im\,(1 - U)\,.$$

After this long conversation, easily one concludes that the Darboux matrix $\mathbf{K}$ at $\xi = \frac{1}{\lambda^3}$, is a projection matrix. In this situation, the matrix can be written as [1]

$$\mathbf{K}_{i\,j}|_{\xi=\frac{1}{\lambda^3}} = m_i\, n_j, \tag{7.1}$$

which implies the following theorem.

[1] Although, mathematicians may be familiar with this formula, it can cause a strange reaction for physicists, but if one writes it as $\mathbf{K}_{i\,j} = |m_i\rangle\langle n_j|$, everything will be fine. However, the first way to write this expression was chosen, because, the second form is often used for quantum systems and in present context, is everything classic.

Theorem 1 The solution of equation (3.8) for $\xi = \frac{1}{\lambda^3}$, can be represented in the form (7.1), where $\vec{m}$ and $\vec{n}$ are solutions of

$$\left| \begin{array}{l} \partial_- \vec{m} = -\bar{A}_- \, \vec{m} \\ \partial_+ \vec{m} = -\bar{A}_+ \, \vec{m} \end{array} \right. \qquad \left| \begin{array}{l} \partial_- \vec{n}^\dagger = A_-^\dagger \, \vec{n}^\dagger \\ \partial_+ \vec{n}^\dagger = A_+^\dagger \, \vec{n}^\dagger \end{array} \right. , \tag{7.2}$$

where

$$\vec{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \quad \text{and} \quad \vec{n} = \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix}.$$

Proof. As $\mathbf{K} = \vec{m}\,\vec{n}$, its derivative will be

$$\partial_\pm \mathbf{K} = (\partial_\pm \vec{m})\,\vec{n} + \vec{m}\,(\partial_\pm \vec{n}) = -\bar{A}_\pm \,\vec{m}\,\vec{n} + \vec{m}\,\vec{n}\,A_\pm = \mathbf{K}\,A_\pm - \bar{A}_\pm\,\mathbf{K}.$$

■

Using the Lax pair (3.5), the following system of equations is found

$$\left| \begin{array}{l} \partial_- n_1 = -\frac{\partial_- v}{v} n_1 - \frac{i}{\lambda} n_2 \\ \partial_- n_2 = -\frac{i}{\lambda} n_3 \\ \partial_- n_3 = \frac{\partial_- v}{v} n_3 - \frac{1}{\lambda} n_1 \end{array} \right. \qquad \left| \begin{array}{l} \partial_+ n_1 = \frac{\lambda}{v^2} n_3 \\ \partial_+ n_2 = -i\,\lambda\,v\,n_1 \\ \partial_+ n_3 = -i\lambda\,v\,n_2 \end{array} \right. . \tag{7.3}$$

Considering the cross derivative in this system and eliminating n_1 and n_3, the following equation appears

$$\begin{aligned} \partial^2 n_2 :&= \partial_- \partial_+ n_2 \\ &= i\lambda(\partial_- v) n_1 + i\lambda v \partial_- n_1 \\ &= i\lambda(\partial_- v) n_1 + i\lambda v \left(-\frac{\partial_- v}{v} n_1 + \frac{i}{\lambda} n_2 \right) \\ &= -v\,n_2 \equiv \partial_+ \partial_- n_2. \end{aligned} \tag{7.4}$$

Also, the following relations are available

$$\partial_-^2 n_2 = \frac{\partial_- v \partial_- n_2}{v} - \frac{1}{\lambda^3\, v} \partial_+ n_2 \tag{7.5}$$

$$\partial_+^2 n_2 = \frac{\partial_+ v \partial_+ n_2}{v} + \frac{\lambda^3}{v} \partial_- n_2. \tag{7.6}$$

From the equation (7.1) and $\beta := \gamma/\bar{v}$, the following relations are invoked

$$\begin{aligned} \frac{\mathbf{K}_{13}}{\mathbf{K}_{12}} &\equiv \frac{n_3}{n_2} = \frac{\xi\,(\alpha + \xi)}{\alpha\beta} \frac{1}{\sqrt[3]{\xi}} \\ \frac{\mathbf{K}_{12}}{\mathbf{K}_{11}} &\equiv \frac{n_2}{n_1} = \frac{2\xi}{\alpha\beta} \sqrt[3]{\xi^2}, \end{aligned} \tag{7.7}$$

from which follows

$$\frac{n_3}{n_2} = \frac{\xi(\alpha+\xi)}{\alpha\beta}\frac{1}{\sqrt[3]{\xi}} = \frac{(\alpha+\xi)}{\sqrt[3]{\xi}}\frac{1}{2\sqrt[3]{\xi^2}}\frac{n_2}{n_1},$$

then, α can be presented as a function of n_j, $j = 1, 2, 3$, i.e.

$$\alpha = \xi\left(\frac{2n_1n_3}{n_2^2} - 1\right); \tag{7.8}$$

using now, the previously definition of α (3.14), the derivatives $\partial_\pm n_2$ from (7.3) and the following relation

$$\partial_+\partial_- \ln h = -\frac{\partial_+h\partial_-h}{h^2} + \frac{\partial_+\partial_-h}{h}, \qquad \forall h \equiv h(x_+, x_-), \tag{7.9}$$

finally the transformation equation is found

$$\bar{v} = v + 2\partial_+\partial_-(\ln n_2). \tag{7.10}$$

This is the transformation that Tzitzéica has found in his work and it is a Moutard-type transformation; in modern literature, it is known as Tzitzéica transformation. This equation allows to construct iteratively solutions of the equation (3.4), that is, this equation gives a new Tzitzéica surface from a previous one, or in solitonic context, a new Tzitzéica soliton from a previous one. For that realization, it is enough to solve the equation (7.4).

7.1.1 On the Moutard and the Tzitzéica transformations

Now, one is able to understand the general way of iterative generating of solutions. Finally, the Moutard and the Tzitzéica transformations will be explained with some sort of precision.

Theorem 2 (Moutard Transformation) *The hyperbolic equation*

$$\mathbf{r}_{\rho\varrho} = v\mathbf{r} \tag{7.11}$$

is form invariant under the transformation

$$\begin{aligned} \mathbf{r} \to \mathbf{r}' &= \mathbf{r} - 2\frac{\mathbf{m}}{\eta} \\ v \to v' &= v - 2(\ln\eta)_{\rho\varrho}, \end{aligned} \tag{7.12}$$

where $\mathbf{m}$ *is defined by*

$$\mathbf{m}_\rho := \eta_\rho \, \mathbf{r}, \qquad \mathbf{m}_\varrho := \eta \, \mathbf{r}_\varrho \tag{7.13}$$

and η *is a particular solution of a scalar version of (7.11) , that means that* η *satisfies*

$$\eta_{\rho\varrho} = v\eta.$$

Proof. With the aim to proof this theorem, just consider the derivative

$$\begin{aligned}
\mathbf{r}'_{\rho\varrho} &= \left(\mathbf{r} - 2\frac{\mathbf{m}}{\eta}\right)_{\rho\varrho} \\
&= \mathbf{r}_{\rho\varrho} - 2\left(\frac{\mathbf{m}}{\eta}\right)_{\rho\varrho} = h\mathbf{r} - 2\left(\frac{\mathbf{m}}{\eta}\right)_{\varrho\rho} = v\mathbf{r} - 2\left(\frac{\mathbf{m}_\varrho}{\eta} - \frac{\mathbf{m}}{\eta^2}\eta_\varrho\right)_\rho \\
&= v\mathbf{r} - 2\mathbf{r}_{\varrho\rho} + 2\left(\frac{\mathbf{m}}{\eta^2}\eta_\varrho\right)_\rho = -v\mathbf{r} + 2\left(\frac{\mathbf{m}_\rho}{\eta^2}\eta_\varrho + \frac{\mathbf{m}}{\eta^2}\eta_{\varrho\rho}\right) - 4\frac{\mathbf{m}\eta_\varrho\eta_\rho}{\eta^3} \\
&= -v\mathbf{r} + 2\left(\frac{\eta_\varrho\eta_\rho}{\eta^2}\mathbf{r} + \frac{\mathbf{m}}{\eta^2}v\eta - 2\frac{\mathbf{m}\,\eta_\varrho\eta_\rho}{\eta^3}\right) \\
&= \mathbf{r}\left(-v + 2\frac{\eta_\rho\eta_\varrho}{\eta^2}\right) + 2\frac{\mathbf{m}}{\eta}\left(v - 2\frac{\eta_\rho\eta_\varrho}{\eta^2}\right) = \left(-v + 2\frac{\eta_\rho\eta_\varrho}{\eta^2}\right)\left(\mathbf{r} - 2\frac{\mathbf{m}}{\eta}\right),
\end{aligned}$$

now, using (7.9)

$$\mathbf{r}'_{\rho\varrho} = \left(v - 2\left(\ln\eta\right)_{\rho\varrho}\right)\left(\mathbf{r} - 2\frac{\mathbf{m}}{\eta}\right) = v'\mathbf{r}'.$$

Q.E.D.

■

Suppose now that $\mathbf{r}$ satisfies (7.11) and the equations

$$\begin{aligned}
\mathbf{r}_{\rho\rho} &= \frac{v_\rho}{v}\mathbf{r}_\rho + \frac{a}{v}\mathbf{r}_\rho \\
\mathbf{r}_{\varrho\varrho} &= \frac{v_\varrho}{v}\mathbf{r}_\varrho + \frac{a^{-1}}{v}\mathbf{r}_\rho,
\end{aligned}$$

with the real parameter a. Also, consider that the field η satisfies a scalar version of these equations but with the real parameter b.

Choosing the potential $\mathbf{m}$ in an appropriate way

$$\mathbf{m} := \frac{1}{a+b}\left[b\eta\mathbf{r} + a\frac{\eta_\rho}{v}\mathbf{r}_\varrho - b\frac{\eta_\varrho}{v}\mathbf{r}_\rho\right],$$

then, easily one finds

$$\mathbf{r}' = \frac{a-b}{a+b}\left[\mathbf{r} - \frac{2}{(a-b)v}\left(a\frac{\eta_\rho}{\eta}\mathbf{r}_\varrho - b\frac{\eta_\varrho}{\eta}\mathbf{r}_\rho\right)\right],$$

with $\frac{a-b}{a+b}$ as an irrelevant constant factor. Finally, the Tzitzéica Transformation can be presented.

Theorem 3 (Tzitzéica Transformation) *The Gauss equations*

$$\begin{aligned} \mathbf{r}_{\rho\rho} &= \frac{v_\rho}{v}\mathbf{r}_\rho + \frac{a}{v}\mathbf{r}_\varrho \\ \mathbf{r}_{\rho\varrho} &= v\mathbf{r} \\ \mathbf{r}_{\varrho\varrho} &= \frac{v_\varrho}{v}\mathbf{r}_\varrho + \frac{a^{-1}}{v}\mathbf{r}_\rho, , \end{aligned} \tag{7.14}$$

together with the Tzitzéica equation

$$(\ln v)_{\rho\varrho} = v - \frac{1}{v^2}, \tag{7.15}$$

are invariant under the transformation

$$\begin{aligned} \mathbf{r} \to \mathbf{r}' &= \mathbf{r} - \frac{2}{(a-b)v}\left(a\frac{\eta_\rho}{\eta}\mathbf{r}_\varrho - b\frac{\eta_\varrho}{\eta}\mathbf{r}_\rho\right) \\ v \to v' &= v - 2\left(\ln\eta\right)_{\rho\varrho}, \end{aligned} \tag{7.16}$$

where η is a particular solution of the scalar version of (7.14), but with b as a parameter.

Then, we already know that we can make $\rho \to -x_+$ and $\varrho \to x_-$, and when we identify the field η with n_2, we get the Tzitzéica-Moutard transformation (7.10).

7.1.2 Zero to one soliton - a naive example

As a simple but necessary example, the construction of one soliton solution from the trivial solution is now considered. When $v = 1$ ($\phi \equiv 0$), let

$$n_2 = exp\left[\frac{i}{2}\left(\varepsilon\, x_+ + \frac{x_-}{\varepsilon}\right)\right]\cosh\left[\frac{\sqrt{3}}{2}\left(\varepsilon\, x_+ - \frac{x_-}{\varepsilon}\right)\right]$$

be a particular solution of (7.4) with ε an arbitrary constant. Since one puts this function in (7.10), the following solution arises

$$\bar{v} = 1 - \frac{3}{2\cosh^2\left[\frac{\sqrt{3}}{2}\left(\varepsilon\, x_+ - \frac{x_-}{\varepsilon}\right)\right]}\ . \tag{7.17}$$

With a computational help, this last expression can be evaluated as a solution of equation (3.4), in fact, this is the one soliton solution of the Tzitzéica equation, as one might expect. The example considered here is the easiest, of course; a second application of the Tzitzéica transformation will demand to solve the equation (7.4) with (7.17) as a seed - and it is not so simple. The construction of explicit new solution through the Tzitzéica procedure has been considered at [31], where those solutions were expressed in terms of the Weierstrass functions.

Summing up We have shown in this chapter that the Tzitzéica-Moutard transformation is in the heart of the Bäcklund transformation which we have found in chapter 3. This transformation is so useful, that we have considered a simple example as an upshot of this chapter.

Chapter 8

Conclusions

At the end of each chapter, we have considered a sort of conclusion, what makes this formal conclusion very friendly. The main objective of this work was to consider the transformations that allow to find a new solution from a previous one. We have considered these transformations from different points of view and at the end, we conclude that we have a systematic way to construct them.

First of all, we found the Darboux and Bäcklund transformation via gauge transformation and after, we saw that from the Lagrangian approach for integrable systems with jump defects, we could obtain the same set of transformations. In this way, indirectly we have found that the defect does not spoil the integrability of the whole system. The integrability of the system would be guaranteed if we had the conserved quantities, and actually, recently we have found these infinitely many conserved charges at [21].

In the second part of this work, we started considering our model from the geometric aspect and we have shown how to reduce the transformations already obtained at the first part to the transformation that Tzitzéica himself, found. The advantage of the geometric approach consists in the freedom available to us, that is, besides the Tzitzéica equation and the Lax pair - available in the solitonic context -, at the geometric approach, we have equations that describe the surface, then, we have some additional informations.

As we said before at the beginning of third chapter, there exist some misunderstanding about the Bäcklund transformation for the model. While some researchers say that there are no Bäcklund transformation for Tzitzéica equation, others have found ineffective transformations. The transformations that we have dealt here in this work, were obtained by Borisov [30] and we have shown that his transformations can be identified as the defect conditions obtained by Corrigan & Zambon [20]. Also, these transformations were identified with the Moutard-Tzitzéica transformation, where we have an effective formula for the generating of solitons. This is the reason why we think that the transformations considered here in this work, are better than any other considered after the Tzitzéica original work.

Part III

Appendices

Appendix A

Manifolds

TO AVOID TO CITE TOO MANY references along this chapter, it is enough to say that the approach used here in all these definitions were completely based in the books of James Munkres [49] for topology, Frank W. Warner for differential geometry [50] - two wonderful books, with a clear text and several interesting examples, as every mathematics book should be. The book of Robert Wald [51], was also an important reference to us, mainly, because he is a physicist and has a special (and clever) way to define some mathematical concepts (compare, for example, the Warner's definition and Wald's definition of *tangent vector* on a manifold). The classnotes of Rui Loja Fernandes [27], Luiz Agostinho Ferreira [52] and the books of Nakahara [28], Seymour Lipschutz [53], Manfredo Perdigão do Carmo [54], Luiz San Martin [55], Fuchs [56] and Gilmore [57] were eventually used and are strongly recommended.

* * *

By *set*, we understand a collection of objects, with or without an additional structure. For example, a set of books, people, points and so on. A set can be finite or infinite. The pattern used in naive set theory is assumed here, for example, the lowercase usually denote the elements in the set, while uppercase mean the sets by themselves, i.e. a is an element and A is a set. Relations between *element-set* and *set-set* are denoted respectively by $\in$ (read *belongs*) and $\subset$ (read *subset*) together with their natural negations $\notin$ and $\not\subset$, for example, if a is an element of the set A, then $a \in A$, if not, $a \notin A$. If every element of the set A are in the set B, then $A \subset B$

and if $A \neq B$, A is a *proper subset* of B. Evidently, the *empty* set is denoted by the usual symbol $\varnothing$.

A typical set usually will be defined by the properties of their elements. For example, a set A of elements a that satisfy the generic property $\mathcal{P}$, in mathematical terms, this set can be write as $A := \{a \,|\, a \,\text{satisfies}\, \mathcal{P}\,\}$ where $:=$ means "*is defined as*" and the bar ($|$) stands "*such that*". Others three important symbols are $\exists$ (exists), $\equiv$ (equivalent) and $\forall$ (for all).

The expression $f : A \rightarrow B$ means the *mapping* (sometimes said map, function or application; soon, in a particular way, we will adapt a subtle distinction among them) of a set A into the set B, or in terms of the respective elements $a \mapsto f(a)$, where was used a special arrow. The set A is called *domain* of the map f and B is called *range* of the map. The *image* of the map, is the set $f(A) = \{b \in B \,|b = f(a) \,\forall\, a \in A\} \subset B$, while the *inverse image* of the element $b \in B$ is $f^{-1}(b) = \{a \in A|\ f(a) = b\}$. If $f : A \rightarrow B$ and $g : B \rightarrow C$ are mappings, the *composition* $g \circ f : A \rightarrow C$ is defined by $a \mapsto g \circ f(a) \equiv g(f(a))$.

The real importance of the next section is the definition of *Hausdorff paracompact topological space*. This space, observed as a *manifold*, admits a Riemannian metric, it allows to "globalize" many local properties and to define integration over the manifold [58] (in fact, a simpler explanation would be enough for a physicist, the spacetime where we live in, is assumed to be Hausdorff paracompact). The *manifold*, on the other hand, is important for the definition of *Lie algebra* and the concept of *integrability*, that is the heart of this work. Although the most of concepts used in next two sections will be used just there, this acquaintance is useful to avoid or solve some eventual pathological problem in our manifolds (see [49]).

Evidently, Lie algebra could be defined without any comment about topology or differential geometry, but notice that, the Bäcklund and Darboux transformations started in the differential geometry, and the aim of this work was to consider relations between them and gauge transformations. So, it is more natural to consider the present approach, which increases naturally, than another approach, which we would need to go to a branch math (like algebra) and then return to geometry. Similar reasons took us to consider integrability through the geometric approach.

Concepts on Topology

Definition 1: Consider A and B two sets, a map $f : A \to B$, is said to be *injective* if given two distinct points in A, their images in B under f are distinct, i.e. if $\forall\ a, e \in\ A\ , a \neq e$ then $f(a) \neq f(e)$, f is called *injective.* A *surjective* map is the one that, each point of B is an image of a point of A under f, i.e. $\forall\ b\ \in\ B\ \exists\ a \in\ A$ such that $f(a) = b$. A map that is both injective and surjective is called *bijective*. When f is bijective there exists a map from B to A called inverse of f and it is denoted by f^{-1}.

If these sets have some operation defined, as addition or product, and the map $f : A \to B$ preserves this operation, i.e. $a_i,\ a_j \in A$, so $a_i \cdot a_j \in A$, then, $f(a_i \cdot a_j) \equiv f(a_i) \cdot f(a_j) \in B$, then we say that f is a *homomorphism.* When the homomorphism f is bijective, then, we say that f is an *isomorphism.*

Actually, this is a naive definition, because the operation which the set B is endowed, is not necessarily the same of the set A, so, the concept of homomorphism is better understood, when we deal with a well defined algebraic structure (that we will define soon), for example, groups, rings, vector spaces and so on. With this point of view, the concept of homomorphism can be thought as the map that preserves the algebraic structure and not the operation of a set.

Definition 2: The topology τ of a set X is a collection of subsets of X with the following properties:
(i) X and $\varnothing\ \in\ \tau$;
(ii) Let be O_i a element of τ, the union $\bigcup_i O_i$ of an arbitrary number $i \in I \subset \mathbb{N}$ is in τ, i.e. $\bigcup_i^n O_{i=1} \subset \tau$, $\forall n \in \mathbb{N}$ (I is usually called index set);
(ii) Let be O_i an element of τ, the intersection $\bigcap_i O_i$ of a finite number $i \in I \subset \mathbb{N}$ is in τ, i.e. $\bigcup_i^n O_{i=1} \subset \tau$, $\forall\ n\ (< \infty) \in \mathbb{N}$.

A Topological Space, is a set X with a topology τ, usually denoted by (X, τ) or just X when there is no confusion source in the definition of the topology.

A subset U of X is called *open set* when U an element of τ. Now, one could think of a topological space as a collection of open sets such that $\varnothing$ and X are open, and an arbitrary union of open sets and a finite intersection of open sets are open.

Definition 3: A *basis* for a topology τ on X is a collection $\mathfrak{B}$ of subsets of X, called *basis elements*, with the properties:
(i) $\forall\ x\ \in\ X$, $\exists$ at least one basis element B containing x;

(ii) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \bigcap B_2$.

As was defined at [49], an *open* in X, is a subset U of X which, for each $x \in U$, there is a basis element $B \in \mathfrak{B}$, such that $x \in B$ and $B \subset U$. A subset V of X is called *closed* if the set $X \backslash V = X - V := \{x \in X | x \notin V\}$ (*complement*) is open. By this definition, we conclude that X and $\varnothing$ are open and closed simultaneously.

Definition 4: A neighborhood U of a point x, is an open set that contains x.

Definition 5: A Hausdorff space is a topological space X which, given any two distinct points a and b, always one can find two neighborhoods of a and b respectively, such that, the intersection between them is empty, in a mathematical way, a topological space is Hausdorff if: Given $a, b \in X$ with $a \neq b$, then $\exists\ U_a$ and U_b such that $a \in\ U_a$, $b \in\ U_b$ and $U_a \bigcap U_b = \varnothing$. When two sets satisfy $A \bigcap B = \varnothing$ one usually says that A and B are disjoints.

Definition 6: Let X and Y be two topological spaces, a map $f : X \to Y$ will be called continuous if given any open subset $V \subset Y$ its inverse image $f^{-1}(V)$ is an open subset $U \subset X$.

Definition 7: Given two topological spaces X and Y the bijective map $f : X \to Y$ will be called *homeomorphism* if f and f^{-1} are continuous.

Definition 8: If X is a topological space.

$\diamond$ **(D 8.1)** The *separation* is a pair U and V of disjoint nonempty subspaces of X, i.e. $U \bigcap V = \varnothing$, such that $U \bigcup V = X$.

$\diamond$ **(D 8.2)** The topological space X is said to be *connected* if there is no separation of X [1].

$\diamond$ **(D 8.3)** Given two distinct points x and y in the topological space X, a *path* in X from x to y is a continuous map $f : [a, b] \in \mathbb{R} \to X$, such that $f(a) = x$ and $f(b) = y$. The topological space x is said to be *path* (or *arcwise*) *connected*, if every pair of points in X can be joined by a path in X.

$\diamond$ **(D 8.4)** A *loop* in a topological space X is a path such that $f(a) = f(b)$. If any loop in X can me shrunk to a point of X, then X is called *simply connected.*

Definition 9: Let X be a topological space

$\diamond$ **(D 9.1)** A collection $\{O_i\}$ of subsets of X is a *cover* of the set $Y \subset X$ if $Y \subset \bigcup O_i$.

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[1] Actually, this definition is a property of the standard definition. A topological space X is *connected* if, and only if, the only subsets of X simultaneously open and closed are the empty set $\varnothing$ and X.

It is an *open cover* if each O_i is open. Let $\{\bar{O}_i\}$ be a sub collection of $\{O_i\}$ that still covers Y, $\{\bar{O}_i\}$ is called *subcover*.
$\diamond$ **(D 9.2)** The set X is *compact* if every open cover $\{O_i\}$ of X contains a finite subcover (a subcover with a finite number of elements).
$\diamond$ **(D 9.3)** A cover $\{U_j\}$ is a *refinement* of $\{O_i\}$, if whatever the U_j, always exists an O_i such that $U_j \subset O_i$. When the sets U_j are open, $\{U_j\}$ is called *open refinement* of $\{O_i\}$. Obviously, when they are closed, $\{U_j\}$ is called *closed refinement* of $\{O_i\}$.
$\diamond$ **(D 9.4)** A collection $\{C_i\}$ of subsets of X is said to be *locally finite* in X if every point $x \in X$ has a neighborhood W_x that intersects a finite number of elements of C_i, i.e. $W_x \bigcap C_i \neq \varnothing$ for a finite index $i \in I \subset \mathbb{N}$.
$\diamond$ **(D 9.4)** A topological space X is *paracompact* if every open cover $\{O_i\}$ has an open locally finite refinement $\{U_j\}$.

Basic Mathematical Structures

1. Group:

An abstract group G is a set of elements together with a composition law ($\circ$) such that if $f,\ g,\ h$ are arbitrary elements in G, then
(i) $g \circ h \ \in \ G$;
(ii) $f \circ (g \circ h) = (f \circ g) \circ h$;
(iii) $\exists\ e$ called *identity* such that $e \circ g = g = g \circ e$;
(iv) $\exists$ an *inverse* such that $\bar{g} \circ g = e = g \circ \bar{g}$, $\ \bar{g}$ is usually denoted as g^{-1}.

The group is called *abelian* or *commutative* when satisfies the extra condition:
(v) $g \circ h = h \circ g, \ \ \forall\ g, h \ \in G$.

2. Ring:

A set A of elements together with two operations

1. $+$ called *addition*,

2. $\cdot$ called *multiplication*,

that satisfies the conditions
(i) A is an abelian group under $+$;

$(ii.1)$ $\forall\, a,\, b\, \in\, A,\, a \cdot b\, \in\, A$ Closure;

$(ii.2)$ $\forall\, a,\, b,\, c\, \in\, A,\, a \cdot (b \cdot c) = (a \cdot b) \cdot c$ Associativity;

$(ii.3)$ $\exists\, \mathbb{I} \in A$, such that $\mathbb{I} \cdot a = a = a \cdot \mathbb{I},\ \forall\, a \in A$ Identity;

$(ii.4)$ $(a+b) \cdot (c+d) = a \cdot (c+d) + b \cdot (c+d)$

$= a \cdot c + a \cdot d + b \cdot c + b \cdot d$ Distributive Law.

A *commutative ring* is the one which satisfies the extra condition

$(ii.5)$ $a \cdot b = b \cdot a, \quad \forall a,\, b\, \in A.$

Now, if for all element a different from zero in the commutative ring, exists an inverse element $a^{-1} \in A$, such that, the multiplication $a \cdot a^{-1}$ is equal to the neutral element of the multiplication (i.e. the identity), then we shall call this ring by *Körper*[2]. In mathematical words, A körper K is an commutative ring A together with the property:

(iii) $\forall\, a \in\, A - \{0\},\ \exists b\, \in A - \{0\}$ such that $a \cdot b = \mathbb{I}$. We denote $b \equiv a^{-1}$.

Then, a körper A is an abelian group with respect to the addition and the neutral element of this operation (denoted by 0), is called *zero*, and $A - \{0\}$ is an abelian group with respect to the multiplication, with neutral element of multiplication (denoted by 1), and called *identity.* The real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ are examples of körper. The elements of a körper are usually called *scalars* and the symbol for multiplication $(\cdot)$, will be used just in this section, so along this work, the scalar multiplication $(a \cdot b)$ will be $(a\ b)$, with $a,\ b$ in the körper.

3. Linear Vector Spaces:

It is a collection $\mathbf{V}$, together the körper K (that we assume hereafter $\mathbb{R}$) and two operations

1. $+$ called *vector addition*,

2. $*$ called *scalar multiplication*,

such that, the following properties hold

(A) $\mathbf{V}$ is an abelian group under $+$:

(A.1) $\mathbf{v},\, \mathbf{u} \in \mathbf{V}$, then $\mathbf{v} + \mathbf{u}\ \in\ \mathbb{R}$;

[2]The English word for this set is *field*, however, to avoid confusion with so many definitions to a same word, we preferred, to use the German word *körper* that means *body*, and it is used to denote this commutative ring.

(A.2) $\mathbf{v} + (\mathbf{u} + \mathbf{w}) = (\mathbf{v} + \mathbf{u}) + \mathbf{w}$, $\mathbf{v}\ \mathbf{u}, \mathbf{w} \in \mathbf{V}$;
(A.3) $\mathbf{v}_0 + \mathbf{v} = \mathbf{v} = \mathbf{v} + \mathbf{v}_0$, we denote $\mathbf{v}_0 = \mathbf{0}$;
(A.4) $\mathbf{v} + (-\mathbf{v}) = \mathbf{0} = (-\mathbf{v}) + \mathbf{v}$;
(A.5) $\mathbf{v} + \mathbf{u} = \mathbf{u} + \mathbf{v}$.

Together with
(M.1) $\lambda \in \mathbb{R}$ and $\mathbf{v} \in \mathbf{V}$, then $\lambda * \mathbf{v} \in \mathbf{V}$;
(M.2) $\lambda,\ \mu \in \mathbb{R}$ and $\mathbf{v} \in \mathbf{V}$, then $\mu * (\lambda * \mathbf{v}) = (\mu \cdot \lambda) * \mathbf{v}$;
(M.3) $1 * \mathbf{v} = \mathbf{v} = \mathbf{v} * 1$;
(M.4) $(\lambda + \mu) * (\mathbf{v} + \mathbf{u}) = \lambda * (\mathbf{v} + \mathbf{u}) + \mu * (\mathbf{v} + \mathbf{u})$
$= \lambda * \mathbf{v} + \lambda * \mathbf{u} + \mu * \mathbf{v} + \mu * \mathbf{u}$.

A subset $\mathbf{W}$ of $\mathbf{V}$, that is itself a vector space with the same algebraic operations of $\mathbf{V}$, is called a *Vector Subspace* of $\mathbf{V}$ over $\mathbb{R}$.

3.1 Linear Dependence and Basis: Consider the set $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ of k vectors in $\mathbf{V}$, if the equation

$$a^1 * \mathbf{v}_1 + \cdots + a^k * \mathbf{v}_k = \mathbf{0},$$

implies that the scalars $a^i = 0$, then we say the set $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ is *linearly independent*; reciprocally, if these scalars are different from zero, $a^i \neq 0$ then we say they set$\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ is *linearly dependent*.

A set of linearly independent vectors $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is called a *basis* of $\mathbf{V}$, if any element $\mathbf{v} \in \mathbf{V}$ can be written in a unique way as a linear combination of $\{\mathbf{e}_i\}$:

$$\mathbf{v} = v^1 * \mathbf{e}_1 + \cdots + v^n * \mathbf{e}_n = \sum_{i=1}^{n} v^i * \mathbf{e}_i \equiv v^i * \mathbf{e}_i,$$

where we call *components* the real numbers v^i. And the number n is called the dimension of the Vector Space.

3.2 Linear Maps, Image and Kernel: Given two vector spaces $\mathbf{V}$ and $\mathbf{U}$, a map $f : \mathbf{V} \to \mathbf{U}$ is said to be *linear* if $f(\lambda * \mathbf{v_i} + \mu * \mathbf{v_j}) = \lambda * f(\mathbf{v_i}) + \mu * f(\mathbf{u_j})$, $\forall\ \mathbf{v_i},\ \mathbf{v_j} \in \mathbf{V}$ and $\lambda,\ \mu \in \mathbb{R}$. The *image* is obviously, $imf = \{f(\mathbf{V})\} \subseteq \mathbf{U}$ and the kernel is the set of elements in $\mathbf{V}$ mapped onto $\mathbf{0}$, i.e. $kerf = \{\mathbf{v} \in \mathbf{V} | f(\mathbf{v}) = \mathbf{0}\}$. Evidently $\mathbf{0}$ belongs to imf and $kerf$, then, they satisfy, independently, the properties of a linear vector space - they are called *linear subspaces* - and the dimension N of $\mathbf{V}$ is:

$$N = \dim \mathbf{V} = \dim(kerf) + \dim(imf).$$

Consider now the vector space $\mathbf{V}$ and T a linear map $T : \mathbf{V} \to \mathbf{V}$, we call this linear map as *linear operator*. An *eigenvalue* of T is a scalar $\alpha \in K$, such that, there is a non zero vector $\mathbf{v} \in \mathbf{V}$ that satisfies the eigenvalue equation defined by

$$T\mathbf{v} = \alpha \mathbf{v}.$$

If α is the eigenvalue of T, then, any $\mathbf{v}$ that satisfies the eigenvalue equation is called *eigenvector*.

3.3 Dual Vector Space: Let $\mathbf{w} : \mathbf{V} \to \mathbb{R}$ be a linear map from the vector space $\mathbf{V}$ to $\mathbb{R}$. The set of maps $\{\mathbf{w}_i\}$ have a natural structure of linear vector spaces, because, if $\mathbf{w}_j$, $\mathbf{w}_k$ are linear maps in this set, then

$$(\mathbf{w}_j + \mathbf{w}_k)(\lambda * \mathbf{v} + \mu * \mathbf{u}) = \lambda \cdot \mathbf{w}_j(\mathbf{v}) + \mu \cdot \mathbf{w}_j(\mathbf{u}) + \lambda \cdot \mathbf{w}_k(\mathbf{v}) + \mu \cdot \mathbf{w}_k(\mathbf{u}),$$

where λ, $\mu \in \mathbb{R}$ and $\mathbf{v}$, $\mathbf{u} \in \mathbf{V}$. Therefore, this set is in fact a linear vector space, denoted by $\mathbf{V}^* = \{\mathbf{w} \mid \mathbf{w} : \mathbf{V} \to \mathbb{R}\}$, and it is called *Dual Vector Space* to $\mathbf{V}$. Its elements $\mathbf{w}$ are called *dual vectors* .

If $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is a basis for $\mathbf{V}$, we can define, without loss of generality, a basis for $\mathbf{V}^*$ choosing $\{\mathbf{d}^1, \cdots, \mathbf{d}^n\} \in \mathbf{V}$, such that:

$$\mathbf{d}^i(\mathbf{e}_j) = \delta^i{}_j = \begin{cases} 1 & \text{if } \ i = j \\ 0 & \text{if } \ i \neq j \end{cases} .$$

Therefore, any dual vector $\mathbf{w} \in \mathbf{V}^*$ can be written as:

$$\mathbf{w} = w_1 * \mathbf{d}^1 + \cdots + w_n * \mathbf{d}^n = \sum_{i=1}^{n} w_i * \mathbf{d}^i = w_i * \mathbf{d}^i.$$

The action of $\mathbf{w} \in \mathbf{V}^*$ over $\mathbf{v} \in \mathbf{V}$ could be thought as a *scalar product* in $\mathbf{V}$, since

$$\mathbf{w}(\mathbf{v}) = w_i * \mathbf{d}^i(v^j * \mathbf{e}_j) = [(w_i \cdot v^j) * \mathbf{d}^i](\mathbf{e}_j) = [(w_i \cdot v^j)] \cdot [\mathbf{d}^i(\mathbf{e}_j)] = (w_i \cdot v^j) \cdot \delta^i{}_j$$

then

$$\mathbf{w}(\mathbf{v}) = w_i \cdot v^i \in \mathbb{R}.$$

The scalar product can be write as $\langle \ , \ \rangle : \mathbf{V}^* \times \mathbf{V} \to \mathbb{R}$, where the *Cartesian product* $\mathbf{V}^* \times \mathbf{V}$ is defined as all pairs $(\mathbf{w}, \mathbf{v}) \in \mathbf{V}^* \times \mathbf{V}$, where $\mathbf{w} \in \mathbf{V}^*$ and $\mathbf{v} \in \mathbf{V}$. Obviously, this idea of Cartesian product can be extended beyond the linear vector spaces theory. Using this notation, we can set

$$\mathbf{w}(\mathbf{v}) := \langle \mathbf{w}, \mathbf{v} \rangle = w_i \cdot v^i.$$

Let us call all the vectors belonging to $\mathbf{V}$ as *contravariant* vectors and *covariant* vectors those $\in \mathbf{V}^*$. We use the following notation, if $\mathbf{v} \in \mathbf{V}$, then $\mathbf{v} = v^i * \mathbf{e}_i \equiv v^i \mathbf{e}_i$ and if $\mathbf{u} \in \mathbf{V}^*$, then $\mathbf{u} = u_i * \mathbf{d}^i \equiv u_i \mathbf{d}^i$. The numbers v^i and u_i are called *contravariant* and *covariant* components respectively. So, the difference between these vectors can be expressed through their components, upper indices components to contravariant vectors and lower indices to covariant vectors.

If now we consider a mapping (an example of isomorphism) $g : \mathbf{V} \to \mathbf{V}^*$, then, the inner product of two vectors $\mathbf{v}, \mathbf{u} \in \mathbf{V}$

$$g(\ , \ ) : \mathbf{V} \times \mathbf{V} \to \mathbb{R},$$

can be expressed by

$$g(\mathbf{v}, \mathbf{u}) := \langle g(\mathbf{v}), \mathbf{u} \rangle = \langle \mathbf{v}^*, \mathbf{u} \rangle = v_i u^i,$$

we require the symmetry[3] of this mapping, then

$$g(\mathbf{v}, \mathbf{u}) := g(\mathbf{u}, \mathbf{v}) = \langle g(\mathbf{u}), \mathbf{v} \rangle = \langle \mathbf{u}^*, \mathbf{v} \rangle = u_i v^i.$$

Also, we require this mapping as *nondegenerate*, i.e. $g(\mathbf{v}, \mathbf{u}) = 0 \ \ \forall \ \mathbf{v}$, then $\mathbf{u} = \mathbf{0}$. This function is called *metric*, the reason of this name is that, if we choose its positivity (the positive defined metric is called *Riemannian* metric), we can recover the notion of distance between two points, however, we do not expect the positivity of our metric.

The simplest form of the function g is the following

$$g := \sum_{i,j=1}^{n} g_{ij} \mathbf{d}^i \otimes \mathbf{d}^j = \sum_{i,j=1}^{n} g_{ij} \mathbf{d}^i \mathbf{d}^j = g_{ij} \mathbf{d}^i \mathbf{d}^j,$$

where g_{ij} are the components of the metric, and the symbol $\otimes$, was to indicate an special multiplication between the basis of $\mathbf{V}^*$, in a most complete treatment, we could see this function g as a particular type of an object called *tensor* and this symbol as an *outer product*, however, this escapes of the scope of this work.

As we have seen, the action of g on a contravariant vector $\mathbf{v}$ leads it to a covariant vector $\mathbf{v}^*$. So $g(\mathbf{v}, \cdot) := \mathbf{v}^*$, then

$$g(\mathbf{v}, \cdot) = g_{ij} \mathbf{d}^i(\mathbf{v}) \mathbf{d}^j := \mathbf{v}^* = v_j \mathbf{d}^j$$

[3]This is not a mandatory requirement, actually, there are some important sorts of mappings like that [57], which, we do not impose any symmetry property and others that we impose the antisymmetry property.

$$g_{ij}\mathbf{d}^i(v^k\mathbf{e}_k)\mathbf{d}^j = v_j\mathbf{d}^j$$

$$g_{ij}v^k\mathbf{d}^i(\mathbf{e}_k)\mathbf{d}^j = g_{ij}v^k\delta^i_k\mathbf{d}^j = g_{ij}v^i\mathbf{d}^j = v_j\mathbf{d}^j,$$

then

$$g_{ij}v^i := v_j.$$

4. Linear Algebra:

Linear Algebra is a vector space $\mathbf{V}$ over the körper $\mathbb{R}$ (or generically K), together with three operations

1. $+$ called *vector addition*,

2. $*$ called *scalar multiplication*,

3. $\times$ called *vector multiplication*,

such that, besides the Linear Vector properties, the following properties hold
(N.1) $\mathbf{v},\ \mathbf{u}\ \in \mathbf{V}$ then $\mathbf{v}\times\mathbf{u}\ \in \mathbf{V}$;
(N.2) $(\mathbf{v}+\mathbf{u})\times(\mathbf{w}+\mathbf{x}) = \mathbf{v}\times(\mathbf{w}+\mathbf{x}) + \mathbf{u}\times(\mathbf{w}+\mathbf{x})$
$= \mathbf{v}\times\mathbf{w} + \mathbf{v}\times\mathbf{x} + \mathbf{u}\times\mathbf{w} + \mathbf{u}\times\mathbf{x}$.
Definition: A vector subspace $\mathbf{W}$ of an algebra $\mathbf{V}$, which is itself a algebra, i.e. $\mathbf{W}$ is closed under the vector multiplication, is called *Subalgebra*.

Differentiable Manifolds

In this section, by *function*, one must understand a mapping into the real numbers, by *map* the mapping into the Euclidean n-dimensional space (formally defined in the next paragraph) and by *application* the mapping into a generic manifold M. Notice that, this classification is very particular and it was adapted for a didactic purpose, then, in any other text, these terms can be used as synonyms, with a similar or exactly meaning used here (who knows?), so, the author reiterates that these criteria must be used carefully and preferably just in this section.

Consider the Euclidean n-dimensional space $\mathbb{R}^n = \{(x^1, x^2, \ldots, x^n)|\ x^i \in \mathbb{R}\ \forall i \in [1, n] \subset \mathbb{N}\}$ and the *canonical coordinate function* $x^i : \mathbb{R}^n \to \mathbb{R}$ defined by:

$$x^i(r) := r^i$$

where $r = (r^1, r^2, \ldots, r^n) \in \mathbb{R}^n$. The canonical coordinate function x^1 on $\mathbb{R}$ will be denoted by x, then $x^i(r) := x, \quad \forall\, r \in \mathbb{R}$. At the same time, if $f : A \subset \mathbb{R}^n \to \mathbb{R}^n$, one sets the i*th component function* of f by

$$f^i := x^i \circ f.$$

When $f : \mathbb{R} \to \mathbb{R}$ and $t \in \mathbb{R}$, the *derivative* of f at t is

$$\left.\frac{d}{dx}\right|_t (f) = \left.\frac{df}{dx}\right|_t.$$

If $f : \mathbb{R}^n \to \mathbb{R}$ and $t = (t^1, \ldots, t^n) \in \mathbb{R}^n$, the *partial derivative* of f with respect to x^i $(1 \leq i \leq n)$ at t is

$$\left.\frac{\partial}{\partial x^i}\right|_t (f) = \left.\frac{\partial f}{\partial x^i}\right|_t.$$

Finally, if $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a n-tuple of non-negative numbers, then one sets

$$[\alpha] = \sum \alpha_i,$$

and

$$\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial x^{[\alpha]}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

Definition 10: Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a function, f is called *differentiable of class C^k*, where k is a non-negative integer, when the partial derivatives $\partial^\alpha f / \partial x^\alpha$ exist and are continuous on A for $[\alpha] \leq k$. If $f : A \subset \mathbb{R}^n \to \mathbb{R}^n$, then f is *differentiable of class C^k* if each component function f^i is C^k. If f is C^k $\forall\, k \geq 0$, the f is C^∞.

A locally n-dimensional euclidean space $\mathcal{M}$ is a Hausdorff Topological Space, which in each of its points, one can find a neighborhood, homeomorphic to an open subset of $\mathbb{R}^n$, i.e. $\forall\, m \in \mathcal{M}$, $\exists\, U \subset \mathcal{M}$ such that $\phi : U \to \mathbb{R}^n$ is an homeomorphism. The homeomorphism ϕ is called *coordinate map*, the functions $\varphi^i := x^i \circ \phi : \mathcal{M} \to \mathbb{R}$ is called *coordinate functions* and one usually calls the pair $(U, \phi) \equiv (U, \varphi^1, \ldots, \varphi^n)$ by *coordinate system* (or *chart*, as mathematicians usually call). Always will be possible to treat x^i instead φ^i, so the coordinate system will be denoted by $(U, \phi) \equiv (U, x^1, x^2, \ldots, x^n)$. If $m \in \mathcal{M}$ and $\phi(p) = 0$ the coordinate system is said centered at p.

Definition 11: A differentiable structure $\mathfrak{F}$ of class C^k on a locally euclidean space $\mathcal{M}$ is a collection of coordinate systems $\mathfrak{C} = \{(U_i, \phi_i) | i \in I\}$, that satisfies the

following properties:
(i) $\bigcup_i U_i = \mathcal{M}$;
(ii) Since $U_i \bigcap U_j \neq \varnothing$, then $\phi_i \circ \phi_j^{-1}$ is $C^k \ \ \forall\, i, j \ \in \ I$;
(iii) $\mathfrak{F}$ is maximal with respect to (ii), that is, if (U, ϕ) is a coordinate system such that, $\phi_i \circ \phi^{-1}$ and $\phi \circ \phi_i^{-1}$ are $C^k \ \forall\ i \ \in I$ then $(U, \phi) \in \mathfrak{F}$.

The pair $(\mathcal{M}, \mathfrak{F})$ is called *differentiable manifold.* When $k \to \infty$ (that is the case considered here), one just calls $(\mathcal{M}, \mathfrak{F})$ as smooth manifold or just manifold.

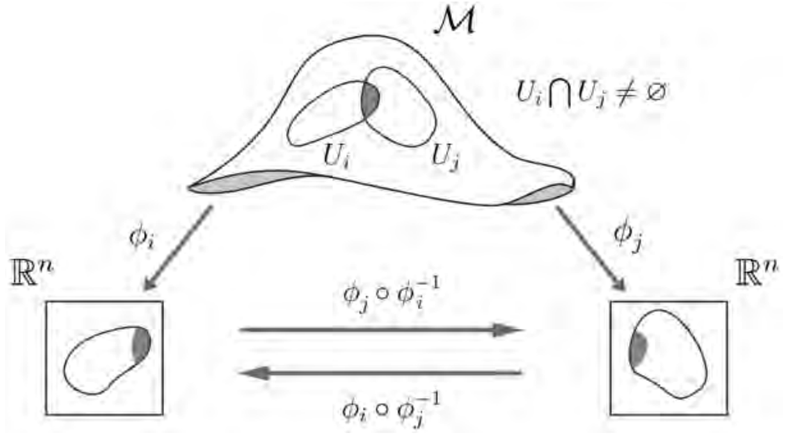

Figure A.1: A differentiable manifold $\mathcal{M}$.

Definition 12: Let $\mathcal{M}$ and $\mathcal{N}$ be smooth manifold, the map $f : \mathcal{M} \to \mathbb{R}$ is said to be *differentiable* of class C^∞ if $f \circ \phi^{-1}$ is of class C^∞, $\forall\ (U, \phi)$; An application $\Psi : \mathcal{M} \to \mathcal{N}$ is said to be a *differentiable application* of class C^∞, if $\theta \circ \Psi \circ \phi^{-1}$ is of class C^∞ $\forall\ (U, \phi)$ of $\mathcal{M}$ and (V, θ) of $\mathcal{N}$. A differentiable application $\Psi : \mathcal{M} \to \mathcal{N}$ bijective and with differentiable inverse is called *diffeomorphism.*

Tangent Spaces

Tangent Vector

In the euclidean space $\mathbb{R}^n$, there exists an injective correspondence between vectors and directional derivatives. A generic vector $v = (v^1, \ldots, v^n)$ defines a directional derivative operator

$$\sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i} \ ,$$

and vice versa. What we mean is that, if f is a C^∞ function on a neighborhood of a point $p \in \mathbb{R}^n$, then, the vector v assigns to f a real number $v(f)$, which is the directional derivative of f in the direction of v at p, so

$$v(f)|_p = v^1 \frac{\partial f}{\partial x^1}\bigg|_p + \cdots + v^n \frac{\partial f}{\partial x^n}\bigg|_p = \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i} := v^i \frac{\partial f}{\partial x^i}.$$

Where, the summation convention for repeated indices has been defined. Directional derivatives are linear and must satisfy the *Leibniz rule*: $v(fg) = g\,v(f) + f\,v(g)$.

Thus, on a manifold $\mathcal{M}$, let $\mathscr{F}$ be the collection of C^∞ functions f from $\mathcal{M}$ to $\mathbb{R}$, i.e. $f : \mathcal{M} \to \mathbb{R}$. A *tangent vector* v at a point $m \in \mathcal{M}$ is a function $v : \mathscr{F} \to \mathbb{R}$ that satisfies:

(i) *linearity*: $v(a\,f + b\,g) = a\,v(f) + b\,v(g), \;\; \forall\; f,\; g \;\in\; \mathscr{F}$ and $a,\; b \;\in\; \mathbb{R}$;

(ii) *Leibniz rule*: $v(fg) = g\,v(f) + f\,v(g), \;\forall\; f,\; g \;\in\; \mathscr{F}$.

Now, let $\mathcal{TM}_m$ be the set of tangent vectors to $\mathcal{M}$ at m, then, $\mathcal{TM}_m$ has a natural structure of a *linear vector space*, so

$(i)(v + w)(f) = v(f) + w(f),\; \forall\; f \in \mathscr{F}$ and $v,\; w \;\in \mathcal{TM}_m$;

$(ii)v(a\,f) = av(f),\; \forall\, a \in \mathbb{R},\; f \in \mathscr{F}$ and $v \in \mathcal{TM}_m$.

The vector space $\mathcal{TM}_m$ is called *tangent vector space* or just *tangent space.* Another important property of the tangent space $\mathcal{TM}_m$ is that, its dimension is the same of the manifold $\mathcal{M}$. This proof is far from trivial, but it was made in a amazing way by [50, 51]. Here, we receive that as a property:

$$\dim(\mathcal{TM}_m) = \dim(\mathcal{M}).$$

Definition 13: Let (U, ϕ) be a coordinate system with coordinate functions $(\varphi^1, \ldots, \varphi^n)$ and $m \in U$. For each index $i \in [1, n] \subset \mathbb{N}$ we define a tangent vector $X_i : \mathscr{F} \to \mathbb{R}$, as

$$X_i(f)|_m = \frac{\partial}{\partial \varphi^i}\bigg|_m (f) = \frac{\partial (f \circ \phi^{-1})}{\partial x^i}\bigg|_{\phi(m)},$$

where $f \in \mathscr{F} : \mathcal{M} \to \mathbb{R}$, and $f \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}$.

If $v \in \mathcal{TM}_m$, then, the old notion of directional derivative can be invoked, and we set down the tangent vector on $\mathcal{M}$ as

$$v|_m := \sum_{i=1}^{n} v(\varphi^i) \frac{\partial}{\partial \varphi^i}\bigg|_m := \sum_{i=1}^{n} v(\varphi^i) X_i\bigg|_m,$$

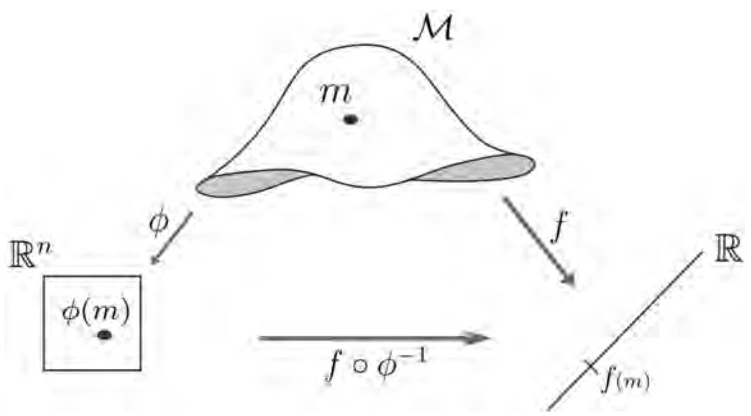

Figure A.2: A function from the Euclidean space $\mathbb{R}^n$ to the real line $\mathbb{R}$.

so, the set $\{X_i|_m := \frac{\partial}{\partial \varphi^i}\Big|_m | i = 1, \ldots, n\}$ defines a basis of $\mathcal{TM}_m$.

Consider now two coordinate systems (U, ϕ) and (U, θ), with coordinate functions $\varphi^1, \ldots, \varphi^n$ and $\vartheta^1, \ldots, \vartheta^n$, respectively, therefore

$$X_i|_m = \frac{\partial}{\partial \varphi^i}\bigg|_m = \sum_{j=1}^{n} \frac{\partial}{\partial \varphi^i}\bigg|_m (\vartheta^j) \frac{\partial}{\partial \vartheta^j}\bigg|_m := \sum_{j=1}^{n} \frac{\partial}{\partial \varphi^i}\bigg|_m (\vartheta^j)\ Y_j\bigg|_m,$$

so, by definition, we have

$$X_i|_m = \sum_{j=1}^{n} \frac{\partial(\vartheta^j \circ \phi^{-1})}{\partial x^i}\bigg|_{\phi(m)} Y_j\bigg|_m := \sum_{j=1}^{n} \frac{\partial y^j}{\partial x^i}\bigg|_{\phi(m)} Y_j\bigg|_m,$$

where we have defined the coordinate function $y^j := \vartheta^j \circ \phi^{-1} : \ \mathbb{R}^n \to \mathbb{R}$. This last equation express the relation between coordinate basis, while the relation between components of the vector can be derived from

$$\begin{aligned} v|_m &= \sum_{i=1}^{n} v(\varphi^i)\ X_i\bigg|_m = \sum_{j=1}^{n} v(\vartheta^j)\ Y_j\bigg|_m \\ &= \sum_{i=1}^{n} v(\varphi^i) \left(\sum_{j=1}^{n} \frac{\partial y^j}{\partial x^i}\bigg|_{\theta(m)} Y_j\bigg|_m \right) = \sum_{j=1}^{n} v(\vartheta^j)\ Y_j\bigg|_m, \end{aligned}$$

then,

$$v(y^j) = \sum_{i=1}^{n} v(x^i) \frac{\partial y^j}{\partial x^i}\bigg|_{\theta(m)}.$$

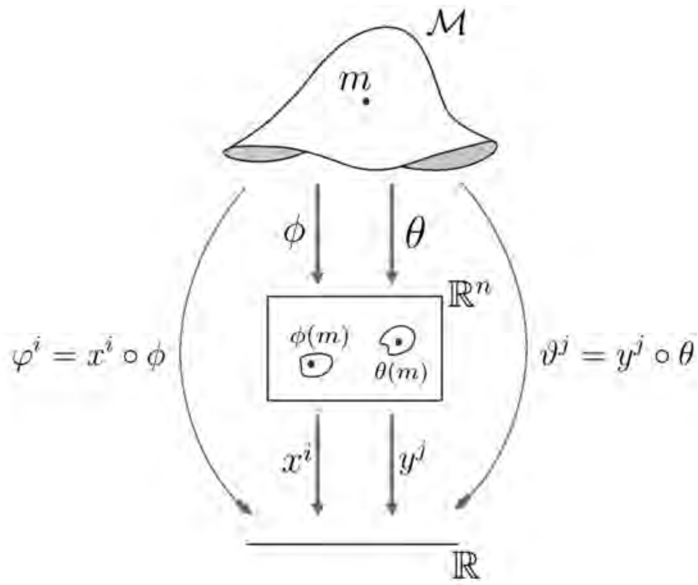

Figure A.3: An example of two different coordinate functions.

This last equation is known as *vector transformation law.*

A smooth curve γ, on a manifold $\mathcal{M}$ is a C^∞ application of $\mathbb{R}$ (or an interval $\mathbb{S}$) into $\mathcal{M}$, i.e. $\gamma : \mathbb{S} \subseteq \mathbb{R} \to \mathcal{M}$. At each point $p \in \mathcal{M}$ on the curve γ, the tangent vector $T \in \mathcal{TM}_m$ to the curve is defined as:

$$T(f) := \frac{d}{dx}(f \circ \gamma)\bigg|_t = \sum_{i=1}^{n} \frac{d\varphi^i}{dx}\bigg|_{\gamma(t)} \frac{\partial f}{\partial \varphi^i}\bigg|_{\gamma(t)} = \sum_{i=1}^{n} \frac{dx^i}{dx}\bigg|_{\phi(\gamma(t))} \frac{\partial f \circ \phi^{-1}}{\partial x^i}\bigg|_{\phi(\gamma(t))}$$

then

$$T(f) = \sum_{i=1}^{n} \frac{dx^i}{dx}\bigg|_{\phi(\gamma(t))} X_i(f)\bigg|_{\phi(\gamma(t))}.$$

Where, $x^i = x^i(x)$, $f \in \mathscr{F}$, $f \circ \gamma : \mathbb{R} \to \mathbb{R}$, $t \in \mathbb{S} \subseteq \mathbb{R}$ and x is the canonical coordinate function on $\mathbb{R}$. Hence, the components T^i of the tangent vector to the curve are

$$T^i = \frac{dx^i}{dx}\bigg|_{\phi(\gamma(t))}.$$

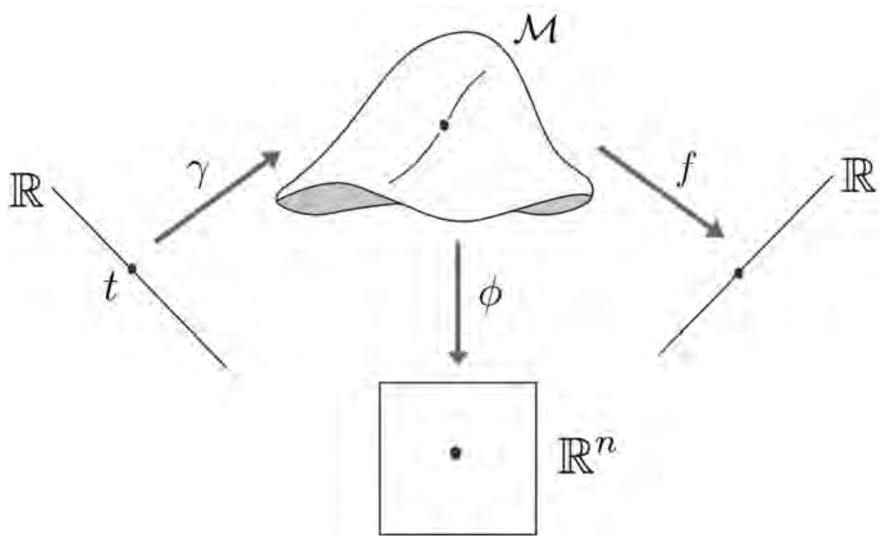

Figure A.4: A smooth curve γ.

The Differential

Let $\psi : \mathcal{M} \to \mathcal{N}$ be C^∞ and $m \in \mathcal{M}$. The differential of ψ at m is the application

$$d\psi : \mathcal{TM}_m \to \mathcal{TN}_{\psi(m)},$$

so, if $v \in \mathcal{TM}_m$, then $d\psi(v)$ is a tangent vector at $\psi(m) \in \mathcal{TN}_m$. The action of the tangent vector $d\psi(v)$ in a function $g \in \mathscr{G}$, that is the set of all C^∞ functions from $\mathcal{N}$ to $\mathbb{R}$, is defined as:

$$[d\psi(v)](g) := v(g \circ \psi), \tag{A.1}$$

where $g \in \mathscr{G} : \mathcal{N} \to \mathbb{R}$ and $g \circ \psi \in \mathscr{F} : \mathcal{M} \to \mathbb{R}$. In the special case of a function $f \in \mathscr{F}$, and $v \in \mathcal{TM}_m$ we have

$$df(v) := v(f) \left. \frac{d}{dx} \right|_{f(m)} . \tag{A.2}$$

In this case, we usually take $df : \mathcal{TM}_m \to \mathbb{R}$ as:

$$df(v) := v(f),$$

that is an element of the dual space $\mathcal{TM}_m^*$.

Let $(U, \phi) \equiv (U, \varphi^1, \ldots, \varphi^n)$ and $(V, \rho) \equiv (V, \varrho^1, \ldots, \varrho^n)$ be coordinate systems around $m \in \mathcal{M}$ and $\phi(m) \in \mathcal{N}$ respectively. Then,

$$d\psi(X_i|_m) := d\psi\left(\left.\frac{\partial}{\partial\varphi^i}\right|_m\right),$$

and using (A.1),

$$d\psi(X_i|_m)(g) := d\psi\left(\left.\frac{\partial}{\partial\varphi^i}\right|_m\right)(g) = \left.\frac{\partial}{\partial\varphi^i}\right|_m (g \circ \psi),$$

where $g \in \mathscr{G} : \mathcal{N} \to \mathbb{R}$ and $g \circ \psi \in \mathscr{F} : \mathcal{M} \to \mathbb{R}$, so

$$d\psi(X_i|_m)(g) = \left(\sum_{j=1}^{n} \left.\frac{\partial(\varrho^j \circ \psi)}{\partial\varphi^i}\right|_m \left.\frac{\partial}{\partial\varrho^i}\right|_{\psi(m)}\right)(g).$$

We finally conclude that

$$d\psi(X_i|_m) = \sum_{j=1}^{n} \left.\frac{\partial(\varrho^j \circ \psi)}{\partial\varphi^i}\right|_m \left.\frac{\partial}{\partial\varrho^i}\right|_{\psi(m)},$$

and $\left\{ \left.\frac{\partial(\varrho^j \circ \psi)}{\partial\varphi^i}\right|_m \right\}$ is called *Jacobian of the application* $\psi : \mathcal{M} \to \mathcal{N}$.

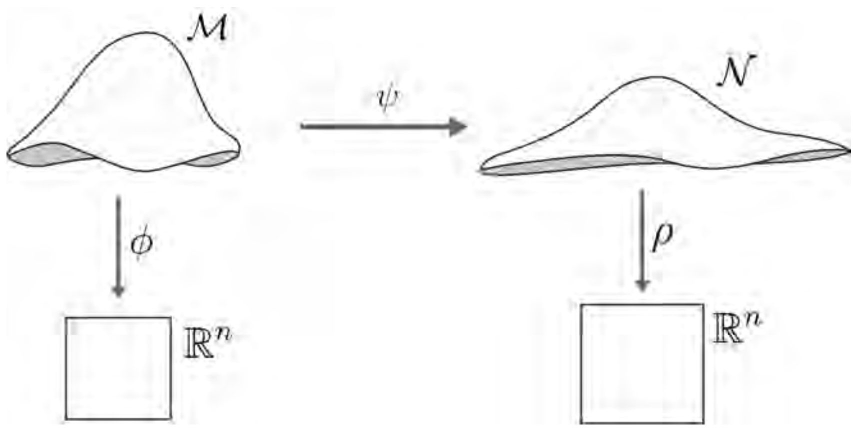

Figure A.5: An application ψ between two manifolds $\mathcal{M}$ and $\mathcal{N}$.

If $(U, \varphi^1, \ldots, \varphi^n)$ is a coordinate system on $\mathcal{M}$, $m \in U$ and $\left\{\frac{\partial}{\partial\varphi^i}\right\}$ is the basis of $\mathcal{TM}_m$, we can define a basis $d\varphi^i$ of the dual space $\mathcal{TM}^*_m$, writing the differential of $f \in \mathscr{F} : \mathcal{M} \to \mathbb{R}$ as

$$df_m := \sum_{i=1}^{n} \left.\frac{\partial f}{\partial\varphi^i}\right|_m d\varphi^i|_m.$$

The dual vector space $\mathcal{TM}_m^*$ of $\mathcal{M}$ at m will be called *Cotangent Space* and its elements *covectors*, i.e., the cotangent space is

$$\mathcal{TM}_m^* := \{w : \mathcal{TM}_m \to \mathbb{R} |\ w \text{ linear}\}.$$

The dual basis $\{d\varphi^i|_m\}$ of $\left\{\frac{\partial}{\partial\varphi^i}\right\}$, defined above can be chosen, without loss of generality, by

$$d\varphi^i\Big|_m \left(\frac{\partial}{\partial\varphi^j}\right) := \delta^i_j.$$

Tangent and Cotangent Bundles

Let $\mathcal{M}$ be a C^∞ manifold, the complete family of tangent and cotangent spaces at $\mathcal{M}$ define the tangent and cotangent bundles respectively, i.e.

$$\mathcal{TM} \equiv \bigcup_{m\in\mathcal{M}} \mathcal{TM}_m,$$

and

$$\mathcal{TM}^* \equiv \bigcup_{m\in\mathcal{M}} \mathcal{TM}_m^*.$$

There are projections such that, to the vector $v \in \mathcal{TM}_m$ and to the covector $w \in \mathcal{TM}_m^*$, we have

$$\pi : \mathcal{TM} \to \mathcal{M},\ \pi(v) = m, \quad \text{if} \quad v \in \mathcal{TM}_m,$$

$$\pi^* : \mathcal{TM}^* \to \mathcal{M},\ \pi^*(w) = m, \quad \text{if} \quad w \in \mathcal{TM}_m^*.$$

Vector Fields

A *tangent vector field* along the curve $\gamma : \mathbb{S} \subseteq \mathbb{R} \to \mathcal{M}$ is a mapping $X : \mathbb{S} \subseteq \mathbb{R} \to \mathcal{TM}$ which assigns γ, that is, $\pi \circ X \equiv \gamma$

The vector field is called *smooth vector field* along γ if the application $X : \mathbb{S} \subseteq \mathbb{R} \to \mathcal{TM}$ is C^∞.

A vector field X on a open set $U \subset \mathcal{M}$ is an assignment of U into $\mathcal{TM}$, in other words, an application $X : U \to \mathcal{TM}$, such that $\pi \circ X \equiv$ identity map on U. Again, the notion of smooth vectors fields is obvious.

The set of vector fields over U forms a vector space over $\mathbb{R}$. If X is a vector field on U and $m \in U \subseteq \mathcal{M}$, then $X(m) \equiv X|_m$ is an element of $\mathcal{TM}_m$. Consider now,

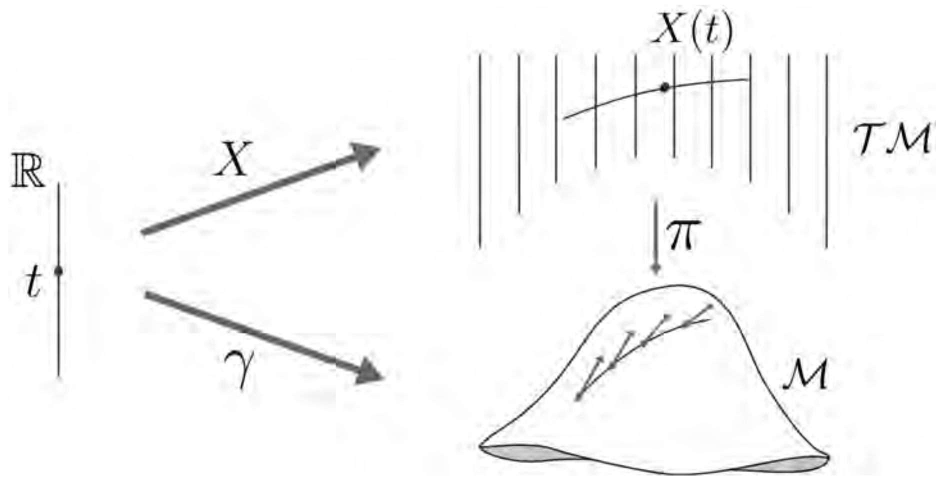

Figure A.6: A smooth vector field along the curve γ.

the function $f \in \mathscr{F}$, then, $X(f)$ is the function on U whose value at m is $X|_m(f)$. Loosely speaking, a vector field X on a manifold $\mathcal{M}$ is an assignment of a tangent vector $v \in \mathcal{TM}_m$ to each point $m \in \mathcal{M}$ [51].

Figure A.7: A vector field on a generic manifold.

Proposition: Let X be a vector field on $\mathcal{M}$. If $(U, \varphi^1, \ldots, \varphi^n)$ is a coordinate system on $\mathcal{M}$ and $\{a^i\}$ a collection of C^∞ functions on U, then

$$X := \sum_{i=1}^{n} a^i \frac{\partial}{\partial \varphi^i}.$$

If $f \in \mathscr{F}$, the composition of two vector fields X and Y can be evaluated as:

$$X(Y(f))|_m = X\left(b(\varphi^j)\frac{\partial f}{\partial \varphi^j}\right)\Bigg|_m = a(\varphi^i)\frac{\partial}{\partial \varphi^i}\left(b(\varphi^j)\frac{\partial f}{\partial \varphi^j}\right)\Bigg|_m$$

by Leibniz, we can write

$$X(Y(f))|_m = \left(a(\varphi^i)\frac{\partial b(\varphi^j)}{\partial \varphi^i}\frac{\partial}{\partial \varphi^j}\right)\Bigg|_m (f) + a(\varphi^i)b(\varphi^j)\frac{\partial}{\partial \varphi^i}\left(\frac{\partial f}{\partial \varphi^j}\right)\Bigg|_m .$$

Due the second term on the right-hand side, this composition is not a vector field, however if we define

$$[X,Y]\Bigg|_m (f) := X(Y(f))|m - Y(X(f))|m,$$

then

$$[X,Y]\Bigg|_m (f) = \left(a(\varphi^i)\frac{\partial b(\varphi^j)}{\partial \varphi^i} - a(\varphi^i)\frac{\partial b(\varphi^j)}{\partial \varphi^i}\right)\Bigg|_m (f).$$

Thus, $[X,Y]$ is again a vector field and it is called *commutator* or *Lie bracket* (mainly by mathematicians). The commutator satisfies:
(i) $[X,Y] = -[Y,X]$, (*skew-symmetry*);
(ii) $[aX + bY, Z] = a[X,Z] + b[Y,Z]$, (*bi-linearity*);
(iii)$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$, $\forall a, b \in \mathbb{R}$, (*Jacobi Identity*);
(iv) $[fX, gY] = fg[X,Y] + f(X(g))Y - g(Y(f))X$, $\forall f, g \in \mathscr{F}$, (*Derivation*).

Evidently, at $m \in \mathcal{M}$, the commutator is an operation between two elements of an vector space $\mathcal{TM}_m$ and it leads to another element of that vector space. So, the vector space is endowed with a vector *multiplication*, and then, forms an algebra. An algebra which satisfies *skew-symmetry* and *Jacobi Identity* is called *Lie Algebra.*

Let $\mathcal{M}$ and $\mathcal{N}$ be two smooth manifolds and $\psi : \mathcal{M} \to \mathcal{N}$ a differentiable application. Consider now, the vector X in $\mathcal{M}$ and the vector field Y in $\mathcal{N}$. We say that X and Y are *ψ-related* if

$$d\psi \circ X = Y \circ \psi.$$

Proposition Still considering this application we can take the vector fields X_1 and X_2 on $\mathcal{M}$ together with Y_1 and Y_2 on $\mathcal{N}$ and consider that X_1 is ψ-related to Y_1 and X_2 is ψ-related to Y_2. With these suppositions, we conclude that $[X_1, X_2]$ is

ψ-related to $[Y_1, Y_2]$.
Proof. Since X_1 is *ψ-related* to Y_1 and X_2 is *ψ-related* to Y_2, then

$$d\psi \circ X_1 = Y_1 \circ \psi$$

and

$$d\psi \circ X_2 = Y_2 \circ \psi.$$

Considering the function $g \in \mathscr{G} : \mathcal{N} \to \mathbb{R}$, we can write:

$$\begin{aligned}
d\psi([X_1, X_2](g) &= [X_1, X_2]|_m(g \circ \psi) \\
&= X_1|_m(X_2(g \circ \psi)) - X_2|_m(X_1(g \circ \psi)) \\
&= X_1|_m[(d\psi \circ X_2)(g)] - X_2|_m[(d\psi \circ X_1)(g)] \\
&= X_1|_m[Y_2(g) \circ \psi] - X_2|_m[Y_1(g) \circ \psi] \\
&= (d\psi \circ X_1|_m)(Y_2(g)) - (d\psi \circ X_2|_m)(Y_1(g)) \\
&= Y_1|_{\psi(m)}(Y_2(g)) - Y_2|_{\psi(m)}(Y_1(g)) \\
&= [Y_1, Y_2]_{\psi(m)}(g), \ \forall \ g \in \mathscr{G}.
\end{aligned}$$

Then, we conclude that,

$$d\psi \circ [X_1, X_2] = [Y_1, Y_2] \circ \psi.$$

Q.E.D.

■

As a last comment about vector fields, let us talk a bit about an important concept that mediates the relation between Lie Groups and Lie Algebras, as we will see soon. The *one-parameter group of diffeomorphism* η_t is a C^∞ application $\mathbb{R} \times \mathcal{M} \to \mathcal{M}$, such that, if $t \in \mathbb{R}$ is fixed, then $\eta_t : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism and $\forall\, t,\ s \in \mathbb{R} \quad \eta_t \circ \eta_s \equiv \eta_{t+s}$, with this definition, we realize that $\eta_0 := e$ is the identity. Considering now, $m \in \mathcal{M}$ fixed, $\eta_t(m) : \mathbb{R} \to \mathcal{M}$ is a curve called *orbit* of η_t which passes through m at $t = 0$. We associate to m a tangent vector $v|_m$ to this curve and to a one-parameter group, we associate a vector field X.

Appendix B

Lie Groups and Lie Algebras

Definitions

Lie Groups

A *Lie Group* is a differentiable manifold $\mathcal{G}$ which is also, a group such that, the application $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$, defined by $(\sigma, \tau) \mapsto \sigma\tau^{-1}$ is C^{∞}, i.e. its group structure is a differentiable application.
Example$_\mathbf{1}$: The euclidean space $\mathbb{R}^n$ is a Lie group under vector addition;
Example$_\mathbf{2}$: The *non-zero* real numbers $\mathbb{R}^*$ and complex numbers $\mathbb{C}^*$ are Lie groups under the multiplication, respectively defined;
Example$_\mathbf{3}$: The unit circle $\mathbb{S}^1 := \{z = a + ib \mid ||z|| := \sqrt{a^2 + b^2} = 1\} \subset \mathbb{C}^*$ is a Lie group under $\mathbb{C}$ multiplication;
Example$_\mathbf{4}$: If $\mathcal{G}$ and $\mathcal{H}$ are Lie groups, the Cartesian product $\mathcal{G} \times \mathcal{H}$ is also a Lie group. Then a *torus-2* $\mathbf{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$ is a Lie group;
Example$_\mathbf{5}$: Let $\mathbf{V}$ be a vector space, the group of linear maps $T : \mathbf{V} \to \mathbf{V}$ forms a Lie group denoted by $Gl(V)$, and it is called *General Linear Group*. If $\mathbf{V} = \mathbb{R}^n$, this group can be identified as the $n \times n$ matrices and it is a Lie Group under the matrix multiplication and it is signed by $Gl(n, \mathbb{R})$.
Example$_\mathbf{6}$: In the same way, the general linear group $Gl(n, \mathbb{C})$, but now, with $n \times n$ complex matrices.

Lie Algebras

A *Lie algebra* $\mathfrak{g}$ over the körper $\mathbb{R}$ is a vector space $\mathfrak{g}$ endowed with an application $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, such that $\forall\ x, y, z \in \mathfrak{g}$ this operation satisfies:

1. $[x, y] = -[y, x]$, (*anti-commutativity*)

2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$. (*Jacobi identity*)

Example$_{\mathbf{1}}$: The vector space of all smooth vector field on the manifold $\mathcal{M}$ is a Lie algebra under the Lie bracket on vector fields;

Example$_{\mathbf{2}}$: The euclidean space $\mathbb{R}^n$ is an abelian Lie algebra;

Example$_{\mathbf{3}}$: Let $\mathbf{V}$ be a vector space, the set of linear maps $T : \mathbf{V} \to \mathbf{V}$ forms a Lie algebra denoted by $\mathfrak{gl}(\mathbf{V})$, with the algebra structure defined by the commutator of two linear maps $A,\ B \in \mathfrak{gl}(\mathbf{V})$

$$[A, B] := A \circ B - B \circ A \ \in \mathfrak{gl}(n, \mathbb{R}).$$

If $\mathbf{V} = \mathbb{R}^n$, this algebra is signed by $\mathfrak{gl}(n, \mathbb{R})$ and can be identified (as proved by [50] and suggested by the exponential map, that we going to consider soon) as the Lie algebra of the matrix elements of the group of $n \times n$ real matrices in $Gl(n, \mathbb{R})$.

Example$_{\mathbf{4}}$: In the same way, $\mathfrak{gl}(n, \mathbb{R})$ can be identified as the Lie algebra of the matrix elements of the group of $n \times n$ complex matrices in $Gl(n, \mathbb{C})$.

Example$_{\mathbf{5}}$: The euclidean space $\mathbb{R}^3$ with the cross product of vectors $\vec{v} \times \vec{u}$ forms a Lie algebra.

We can define in the Lie algebra $\mathfrak{g}$, a derivative map, i.e. a map $\delta : \mathfrak{g} \to \mathfrak{g}$, that satisfies the Leibniz rule

$$[x, y] \in \mathfrak{g} \mapsto \delta([x, y]) := [\delta(x), y] + [x, \delta(y)].$$

As an example, we can define, for some $x \in \mathfrak{g}$ the derivative map $ad_x : \mathfrak{g} \to \mathfrak{g}$ setting

$$y \mapsto ad_x(y) := [x, y],\ \forall\ y \in \mathfrak{g}.$$

Using the Jacobi identity, we can show that the Leibniz rule is valid

$$\begin{aligned} ad_x([y, z]) :&= [x, [y, z]] \\ &= -[y, [z, x]] - [z, [x, y]] = [y, [x, z]] + [[x, y], z] \\ &= [ad_x(y), z] + [y, ad_x(z)]. \end{aligned}$$

Definition: Let $\mathbf{V}$ be a vector space and $\mathfrak{gl}(\mathbf{V})$ the Lie algebra of linear maps $T : \mathbf{V} \to \mathbf{V}$. A representation of a Lie algebra $\mathfrak{g}$ into $\mathbf{V}$ is the homomorphism

$$\mathscr{R} : \mathfrak{g} \to \mathfrak{gl}(\mathbf{V}),$$

with

$$[x, y] \mapsto \mathscr{R}([x, y]) = [\mathscr{R}(x), \mathscr{R}(y)], \quad \forall\ x, y \in \mathfrak{g}.$$

The importance of this definition is too obvious, however, let us be redundant and reinforce that with this concept in mind, we can easily work and operate with the group elements.

An important representation that we can consider is the *adjoint representation*

$$\mathscr{R}_{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),$$

which is defined by the elements transformation

$$a \mapsto ad_a.$$

In this representation, the Lie bracket is defined by

$$[ad_x, ad_y] := ad_x \circ ad_y - ad_y \circ ad_x,$$

with this definition, we get

$$\begin{aligned}
[ad_x, ad_y])(z) &= (ad_x \circ ad_y)(z) - (ad_y \circ ad_x)(z) \\
&= ad_x(ad_y z) - ad_y(ad_x z) \\
&= ad_x([y, z]) - ad_y([x, z]) \\
&= [x, [y, z]] - [y, [x, z]] \\
&= -[z, [x, y]] = [[x, y], z] := ad_{[x,y]}(z),
\end{aligned}$$

for some $z \in \mathfrak{g}$. We conclude so, that the adjoint representation carries Lie brackets from $\mathfrak{g}$ to Lie brackets in $\mathfrak{gl}(\mathfrak{g})$.

We said before in the page (70) that, the concept of homomorphism would be best understood, when we work with a well defined algebraic structure, now we are work on it, then notice that, a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$, from the Lie algebra $\mathfrak{g}$ to the Lie algebra $\mathfrak{h}$, is a homomorphism if it carries the Lie bracket, i.e.

$$\varphi : [x, y] \mapsto \varphi([x, y]) = [\varphi(x), \varphi(y)],$$

$\forall\, x, y \in \mathfrak{g}$. When the map φ is bijective, we say that it is an isomorphism. Evidently, the adjoint representation $\mathscr{R}_{ad}$ is a homomorphism between Lie algebras.

Let σ be an element of the Lie group $\mathcal{G}$. We define the *left translation* and *right translation* as the diffeomorphisms l_σ and r_σ of $\mathcal{G}$ respectively,

$$l_\sigma(\tau) = \sigma\tau,$$

$$r_\sigma(\tau) = \tau\sigma,$$

$\forall \tau \in \mathcal{M}$. Consider the subset $\mathcal{H} \subset \mathcal{G}$, the left translation of this complete subset is $l_\sigma(\mathcal{H}) = \sigma\mathcal{H}$, analogously, the right translation is $r_\sigma(\mathcal{H}) = \mathcal{H}\sigma$.

A left invariant vector field X on $\mathcal{G}$ is the only one that for each $\sigma \in \mathcal{G}$, X is l_σ-related to itself, that is

$$dl_\sigma \circ X = X \circ l_\sigma.$$

Since a $dl_\sigma(X|_\tau)$ is defined to be a tangent vector at $l_\sigma(\tau) \in \mathcal{G}$, where $\sigma,\ \tau \in \mathcal{G}$, we can consider too, the coordinate systems (U_1, ϕ) around $\tau \in \mathcal{G}$ and (U_2, θ) around $l_\sigma(\tau) \in \mathcal{G}$. Then we can set

$$\begin{aligned} dl_\sigma(X|_\tau)(g) :&= X|_\tau(g \circ l_\sigma) = X|_\tau(f)|_{l_\sigma(\tau)} \\ &= a(\phi^i)\ \frac{\partial}{\partial \phi^i}\bigg|_\tau (f)\bigg|_{l_\sigma(\tau)} = a(\phi^i)\ \frac{\partial \theta^j}{\partial \phi^i}\frac{\partial}{\partial \theta^j}\bigg|_\tau (f)\bigg|_{l_\sigma(\tau)}, \end{aligned}$$

where $a^i \equiv a(\phi^i)$ are C^∞ and $g \in \mathscr{G} : \mathcal{G} \to \mathbb{R}$. Evidently, this vector does not have necessarily the same value that the vector field X would have at the point $\tau \in \mathcal{G}$. When this happen, we say that this vector field is left invariant. Last chapter, when we considered the vector fields X_i and Y_i, $i = 1, 2$ ψ-related (see page 88), we have proved that the bracket $[X_1, X_2]$ is ψ-related to $[Y_1, Y_2]$. Naturally we can use this concept and realize that, since two vector fields are left invariant, i.e. each vector field is l_σ-related to itself, the Lie bracket is also, l_σ-related to itself, then if $dl_\sigma \circ X = X \circ l_\sigma$ and $dl_\sigma \circ Y = Y \circ l_\sigma$ we have

$$dl_\sigma \circ [X, Y] = [X, Y] \circ l_\sigma.$$

The set of all left invariant vector fields on a Lie group $\mathcal{G}$ will be denoted by $\mathfrak{g}$ and we easily conclude that $\mathfrak{g}$ forms a Lie algebra. We define $\mathfrak{g}$, to be the Lie algebra of the Lie group $\mathcal{G}$.

For any Lie group $\mathcal{G}$, we can take a set of $n \equiv \dim(\mathcal{G})$ linearly independent left invariant vector fields. This set, that we denote by $\{\mathbf{T}_a|\ (a = 1 \dots n)\}$, expand the tangent space $\mathcal{TG}_m$ at any point $m \in \mathcal{G}$, then it is a basis of this space and must satisfy the relation

$$[\mathbf{T}_a, \mathbf{T}_b] := \sum_{c=1}^{n} i f_{a\,b}^{c} \mathbf{T}_c = i f_{a\,b}^{c} \mathbf{T}_c,$$

where $f_{a\,b}^{c}$ are constants and the factor i was chosen because in the case when the vector fields satisfies $\mathbf{T}_a^\dagger = \mathbf{T}_a$, the constants $f_{a\,b}^{c}$ must be real. Evidently, this relation remains unchanged, whatever the point $m \in \mathcal{G}$ that we are interested in, for this reason, the constants $f_{a\,b}^{c}$ are called *structure constants* of the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$.

We can work with the adjoint representation now,

$$ad_{\mathbf{T}_a}(\mathbf{T}_b) = [\mathbf{T}_a, \mathbf{T}_b] = i f_{a\ b}^{c} \mathbf{T}_c.$$

This last expression shows an amazing result: in terms of the basis, the adjoint representation is given by the structure constants.

When we were talking about vector spaces, we have considered the metric, that was a map from the Cartesian product $\mathbf{V} \times \mathbf{V}$ into the körper K, that without loss of generality we considered the real numbers $\mathbb{R}$, i.e

$$g(\ ,\ ) : \mathbf{V} \times \mathbf{V} \to \mathbb{R}.$$

We define now this metric, that has a special name here, the *Cartan-Killing form*:

Definition: The Cartan-Killing form is the linear map κ from the Cartesian product $\mathfrak{g} \times \mathfrak{g}$ into the real numbers,

$$\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R},$$

defined by

$$(x, y) \mapsto \kappa(x, y) := trace(ad_x \circ ad_y).$$

We can operate now with the basis $\{\mathbf{T}_a\}$, so

$$\kappa_{ab} = trace(ad_{\mathbf{T}_a} \circ ad_{\mathbf{T}_b}).$$

Let us consider the case

$$\begin{aligned}
ad_{\mathbf{T}_a} \circ ad_{\mathbf{T}_b}(\mathbf{T}_c) &= ad_{\mathbf{T}_a}(ad_{\mathbf{T}_b}(\mathbf{T}_c)) \\
&= ad_{\mathbf{T}_a}([\mathbf{T}_b, \mathbf{T}_c] \\
&= ad_{\mathbf{T}_a}(if^d_{bc}\mathbf{T}_d) = ad_{\mathbf{T}_a}(\mathbf{T}_d)if^d_{bc} \\
&= [\mathbf{T}_a, \mathbf{T}_d]if^d_{bc} = -f^e_{ad}f^d_{bc}\mathbf{T}_e,
\end{aligned}$$

we can obtain the trace, doing a summation over $c = e$, then

$$\kappa_{ab} := trace(ad_{\mathbf{T}_a} \circ ad_{\mathbf{T}_b}) = -f^c_{ad}f^d_{bc}.$$

Exponential Map

We call the *one-parameter subgroup*[1] of $\mathcal{G}$ the homomorphism between the Lie groups $\mathbb{R}$ and $\mathcal{G}$, the map $\varphi : \mathbb{R} \to \mathcal{G}$. Since $\mathbb{R}$ is an abelian group under scalar addition, we have

$$\varphi(t) \circ \varphi(s) = \varphi(t + s) = \varphi(s + t) = \varphi(s) \circ \varphi(t),$$

$\forall\, t,\ s \in \mathbb{R}$, we conclude that the one parameter subgroup is an abelian group, even when the Lie group $\mathcal{G}$ is non-abelian. Besides, we have

$$\varphi(0) \circ \varphi(t) = \varphi(0 + t) = \varphi(t),$$

then $\varphi(0) := \mathfrak{e} \equiv$ identity element of $\mathcal{G}$ and

$$\varphi(t) \circ \varphi(-t) = \varphi(t - t) = \varphi(0) = \mathfrak{e},$$

where t, s and 0 are elements of $\mathbb{R}$.

Consider again the Lie group $\mathcal{G}$ and its Lie algebra $\mathfrak{g}$. Consider too, the vector field $X \in \mathfrak{g}$, then, we can define a homomorphism between the Lie algebra of $\mathbb{R}$ and $\mathfrak{g}$ by

$$\lambda \frac{d}{dx} \mapsto \lambda X.$$

[1] A set $\mathcal{H} \subset \mathcal{G}$ is said to be a subgroup of $\mathcal{G}$ if satisfies the group requirements induced by the multiplication rule of the group $\mathcal{G}$. Evidently, the identity element and the group $\mathcal{G}$ itself is a subgroup, called *improper subgroups*, any other subgroup is called *proper subgroup*.

Also, one could show [50] that there is a unique one-parameter subgroup

$$\varphi_X : \mathbb{R} \to \mathcal{G}$$

such that

$$d\varphi_X\left(\lambda\frac{d}{dx}\right) = \lambda X,$$

since $d\varphi_X : \mathcal{T}\mathbb{R}_t \to \mathcal{T}\mathcal{G}_{\varphi_X(t)}$, i.e. there is a unique one-parameter subgroup, such that, the tangent vector at $t = 0$ is $X(\mathfrak{e})$:

$$d\varphi_X\left(\lambda\;\frac{d}{dx}\right)\bigg|_{t=0} = \lambda X\bigg|_{\varphi_X(t=0))}.$$

We are able now, to define the *exponential map*

$$exp : \mathfrak{g} \to \mathcal{G}$$

by

$$exp(X) := \varphi_X(1).$$

Consider the element of the algebra $X \in \mathfrak{g}$, then

$$exp(tX) = \varphi_X(t),$$

where $t \in \mathbb{R}$. As φ_X is a homomorphism between the groups $\mathbb{R}$ and $\mathcal{G}$, we have

$$exp[(t_1 + t_2)X] = \varphi_X(t_1)\varphi_X(t_2), \quad t_1\,t_2\; \in \mathbb{R},$$

$$exp[tX]|_{t=0} = \varphi_X(0) = \mathfrak{e}, \quad t = 0\; \in \mathbb{R},$$

$$exp[-tX] = (exp[tX])^{-1}, \quad t \in \mathbb{R}.$$

Then, we conclude that, the exponential map *exp* provides a diffeomorphism of a neighborhood of the zero element of $\mathcal{T}\mathcal{G}_{\mathfrak{e}}$ which belongs to the Lie algebra $\mathfrak{g}$ at $t = 0$, onto a neighborhood of the identity element $\mathfrak{e}$ of $\mathcal{G}$. Sometimes, this diffeomorphism can be extended to the entire Lie group $\mathcal{G}$.

As an example, we could consider the exponential map

$$exp : \mathfrak{gl}(n, \mathbb{C}) \to Gl(n, \mathbb{C}),$$

that is

$$e^{\mathbf{A}} = \mathbb{I} + \mathbf{A} + \frac{1}{2}\mathbf{A}^2 + \cdots + \frac{1}{n!}\mathbf{A}^n + \ldots \quad \in Gl(n, \mathbb{C}),$$

where $\mathbf{A} \in \mathfrak{gl}(n, \mathbb{C})$ and $\mathbb{I}$ is the identity element (rather than e) of $Gl(n, \mathbb{C})$. Also we can show[2] that

$$\det(e^{\mathbf{A}}) = e^{trace(\mathbf{A})}.$$

Remark Evidently, the algebra element $\mathbf{A}$ does not always commute with the element $\mathbf{B}$, i.e. $\mathbf{AB} \neq \mathbf{BA}$, so, the map $e^{\mathbf{A}+\mathbf{B}}$ is in general different from $e^{\mathbf{A}}e^{\mathbf{B}}$ and the equality is valid just when $\mathbf{A}$ and $\mathbf{B}$ commute.

Generalities on Lie algebras

Up to now, we have seen that a generic element of the Lie algebra $\mathfrak{g}$, can be expanded in terms of a basis

$$\mathcal{B} = \{\mathbf{T}_a \mid a = 1, \ldots, n = dim\mathcal{G}\},$$

and we usually call the elements $\mathbf{T}_a$ *generators*. Also, we know that, the Lie bracket $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, can be written in terms of these generators

$$[\mathbf{T}_a, \mathbf{T}_b] = i f_{a\,b}^{c} \mathbf{T}_c,$$

where $f_{a\,b}^{c}$ are the structure constants. From the antisymmetry property of the Lie bracket,

$$f_{a\,b}^{c} = -f_{b\,a}^{c},$$

and from the Jacobi identity, we get

$$f_{a\,d}^{e} f_{b\,c}^{d} + f_{b\,d}^{e} f_{c\,a}^{d} + f_{c\,d}^{e} f_{c\,a}^{d} = 0.$$

Also, we talked about the exponential map, which allows us to set an element g of the Lie group $\mathcal{G}$ as

$$g = exp(i\mathbf{T}) \equiv e^{i\mathbf{T}}$$

and using the generators $\mathcal{B}$, we have

$$g = exp(i\zeta^a \mathbf{T}_a),$$

[2]Heuristically, we can consider $\lambda_1 \ldots \lambda_i$ the eigenvalues of the matrix $\mathbf{A}$, since the determinant of a matrix is the product of its eigenvalues and the trace is the sum of its eigenvalues, the determinant of $e^{\mathbf{A}}$ is $e^{\lambda_1} \cdots e^{\lambda_i}$.

with the parameters $\zeta^a \in \mathbb{R}$. The imaginary unity i is present, because, when the generators are hermitian, i.e. $\mathbf{T}_a^\dagger = \mathbf{T}_a$, the group element g is unitary. Evidently, since $[\mathbf{T}_a, \mathbf{T}_a] = 0$, then we have

$$g^\dagger g = exp(-i\zeta^a \mathbf{T}_a^\dagger) exp(i\zeta^a \mathbf{T}_a) = exp(i\zeta^a(-\mathbf{T}_a^\dagger + \mathbf{T}_a)) = 1.$$

Definition: A vector subspace $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$, which is itself a Lie algebra, i.e. $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, is called *Lie subalgebra.*

Definition: We say that the Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is an *ideal* or *invariant subalgebra,* when

$$[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}.$$

The lie algebra $\mathfrak{g}$ and the neutral element $\mathfrak{e}$ (let us call this element as *zero*) are evidently ideals, they are called *improper ideals.*

Definition: The Lie algebra that has only itself and zero as ideals and that has dimension $\dim > 1$ are called *simple Lie algebras*[3]. We can analyze the structure of the Lie algebra, looking through the ideals. Let us derive a Lie algebra $\mathfrak{g}'$ from $\mathfrak{g}$ setting

$$\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}].$$

Naturally, $\mathfrak{g}'$ is a Lie subalgebra of $\mathfrak{g}$, beside, notice that in a obvious way, $\mathfrak{g}'$ is also an ideal, in fact

$$[\mathfrak{g}, \mathfrak{g}'] = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \subset \mathfrak{g}'.$$

Wisely, we define the sequence

$$\begin{aligned}
\mathfrak{g}^{(1)} &:= [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}' \\
\mathfrak{g}^{(2)} &:= [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] \\
&\vdots \\
\mathfrak{g}^{(i)} &:= [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}].
\end{aligned}$$

Each term of this series, called *derived series*, is an ideal of $\mathfrak{g}$,

Definition: A *soluble algebra* $\mathfrak{g}$ is the one that its derivative series vanishes for some $i_0 \geq 1$, i.e.

$$\mathfrak{g}^{(i_0)} = 0.$$

Evidently, this equality remains for all $i \geq i_0$.

[3]This is a requirement that makes simple and semi-simple Lie algebras compatible [55].

Definition: The Lie algebra $\mathfrak{g}$ such that, besides $\{0\}$ has no soluble ideal is called *semi-simple* Lie algebra. Actually, these are the Lie algebras that we are interested in, then, they are the kind of algebras that we will consider hereafter.

Definition: When the Cartan-Killing form is positive defined[4], we say that a semi-simple Lie algebra is *compact*.

Definition: Let $\mathfrak{g}$ be a semi-simple Lie algebra, the maximal abelian subalgebra $\mathfrak{g}_0$ is called *Cartan subalgebra*[5]. Also, we define the *rank* of the Lie algebra $\mathfrak{g}$, as the dimension of the Cartan subalgebra $\mathfrak{g}_0$. Let us call the elements of the Cartan subalgebra $\mathfrak{g}_0$ by $\mathbf{H}_i,\ \ i = 1, \ldots, r = rank(\mathfrak{g})$.

Cartan-Weyl basis

We can choose a well defined basis for the Lie algebra. In the case of a compact semi-simple Lie algebra, the best choice is the one called *Cartan-Weyl basis*. We already know that the Cartan subalgebra elements can be expanded through

$$\mathbf{H}_i,\ \ i = 1, \ldots, r = rank(\mathfrak{g}),$$

in a way that

$$[\mathbf{H}_i, \mathbf{H}_j] = 0. \tag{B.1}$$

One can show [52, 55, 56] that the semi-simple Lie algebra $\mathfrak{g}$ can be write[6] as

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p},$$

where the Cartan subalgebra is $\mathfrak{h}$ and $\mathfrak{p}$ is the complement of $\mathfrak{h}$ in $\mathfrak{g}$. In fact, we can set the trace in a given representation by

$$trace(\mathfrak{h}\mathfrak{p}) = 0,$$

that makes $\mathfrak{p}$ an orthogonal complement of $\mathfrak{h}$.

[4]Notice that, if we had taken the commutator of the generators without the imaginary factor i, we would require that the Cartan-Killing form was negative defined, compare with [55, 56].

[5]When we drop the semi-simple requirement, the definition of Cartan subalgebra involves some additional definitions that are beyond the scope of this work, see [55] for more informations.

[6]The symbol $\oplus$ is a formal summation of the pairs of elements in the algebra, which the precise definition escape of our aim.

The orthogonal complement $\mathfrak{p}$ contains all elements of $\mathfrak{g}$ which are not in $\mathfrak{h}$, then we can evaluate the trace

$$trace(\mathfrak{h}[\mathfrak{h},\mathfrak{p}]),$$

and using the cyclic property of the trace we get

$$trace(\mathfrak{h}[\mathfrak{h},\mathfrak{p}]) = trace(\mathfrak{p}[\mathfrak{h},\mathfrak{h}]) = trace(\mathfrak{h}\mathfrak{p}) = 0,$$

hence, we conclude that

$$[\mathfrak{h},\mathfrak{p}] \subset \mathfrak{p}.$$

In this sense, we can chose the remaining generators, that we will denote by $\{\mathbf{E}_\alpha \in \mathfrak{p}\}$, and this set satisfies an "eigenvalue equation" such that

$$[\mathbf{H}_i, \mathbf{E}_\alpha] := \alpha_i \mathbf{E}_\alpha, \quad i = 1, \ldots, r, \tag{B.2}$$

where we usually call the elements $\mathbf{E}_\alpha$ *step or ladder operators*. The eigenvalues α_i are components of an r-dimensional vector, this vector is called a *root* of the Lie algebra $\mathfrak{g}$. Let us denote the set of all roots of a semi-simple Lie algebra by Φ.

Definition: A basis of a compact semi-simple Lie algebra $\mathfrak{g}$ of the form

$$\mathcal{B} := \{\mathbf{H}_i \mid i = 1, \ldots, r\} \cup \{\mathbf{E}_\alpha \mid \alpha \in \Phi\},$$

with $\mathbf{H}_i$ and $\mathbf{E}_\alpha$ satisfying (B.1) and (B.2) respectively, is called *Cartan-Weyl basis*.

Finally, we can consider the commutator between two elements from the orthogonal complement $[\mathbf{E}_\alpha, \mathbf{E}_\beta]$. Let us evaluate this commutator through the Jacobi identity, then

$$\begin{aligned}
[\mathbf{H}_i, [\mathbf{E}_\alpha, \mathbf{E}_\beta]] &= -[\mathbf{E}_\alpha, [\mathbf{E}_\beta, \mathbf{H}_i]] - [\mathbf{E}_\beta, [\mathbf{H}_i, \mathbf{E}_\alpha]] \\
&= \alpha_i [\mathbf{E}_\alpha, \mathbf{E}_\beta] - \beta_i [\mathbf{E}_\beta, \mathbf{E}_\alpha] \\
&= (\alpha_i + \beta_i)[\mathbf{E}_\alpha, \mathbf{E}_\beta].
\end{aligned}$$

Since the Lie algebra is closed, there are three possibilities for this expression,

1. $\alpha + \beta$ root of the Lie algebra, so $[\mathbf{E}_\alpha, \mathbf{E}_\beta] \propto \mathbf{E}_{\alpha+\beta}$;

2. $\alpha + \beta$ is not a root of the Lie algebra, so $[\mathbf{E}_\alpha, \mathbf{E}_\beta] = 0$;

3. $\alpha + \beta = 0$, so $[\mathbf{E}_\alpha, \mathbf{E}_\beta]$ must be an element of the Cartan subalgebra.

We can sum up the results of the generators for the semi-simple Lie algebra as

$$
\begin{aligned}
&[\mathbf{H}_i, \mathbf{H}_j] = 0, \\
&[\mathbf{H}_i, \mathbf{E}_\alpha] = \alpha_i \mathbf{E}_\alpha, \\
&[\mathbf{E}_\alpha, \mathbf{E}_\beta] = \begin{cases} n_{\alpha\beta}\mathbf{E}_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \\ 0 & \text{if } \alpha+\beta \text{ is not a root} \\ \mathbf{H}_\alpha & \text{if } \alpha+\beta = 0. \end{cases}
\end{aligned}
$$

Lie Algebra $A^{(2)}$

The algebra that we are interested in, is the algebra $\mathfrak{su}(3)$, which is the Lie algebra of the Lie group $SU(3)$, that is a subgroup of t$Gl(3, \mathbb{C})$. Formally, we define this group as

$$SU(3) := \{A \in Gl(3, \mathbb{C}) \mid A^\dagger A = \mathbb{I},\ \det A = 1\},$$

where the S stands for *special*, because the determinant of these matrices is equal to the unity and U means *unitary*. Therefore, the group $SU(3)$ is the subgroup of $Gl(3, \mathbb{C})$ composed by unitary matrices with determinant equal to unity. To this group corresponds the Lie algebra $\mathfrak{su}(3)$. Using that, we can write an element of the Lie group as an exponential of the algebra element, i.e. $A = exp(i\mathbf{T})$, and from $AA^\dagger = 1$, we conclude that $\mathbf{T}$ must be hermitian. Also, one must remember that

$$\det(exp\{i\mathbf{T}\}) = exp\{trace(i\mathbf{T})\};$$

but, we know that $\det(exp\{i\mathbf{T}\}) = 1$, then we conclude that $trace(\mathbf{T}) = 0$. Therefore, the Lie algebra $\mathfrak{su}(3)$ is generated by the 3×3 hermitian traceless matrices. The Cartan subalgebra of this algebra has rank $r = 2$, then mathematicians usually call this algebra $A^{(2)}$.

The usual basis for this algebra is composed by the *Gell-Mann matrices*; and with them, we can write the following matrices. By abuse of terminology, let us call these matrices as *generators* of the Lie algebra $A^{(2)}$.

Generators

$$\mathbf{T_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \mathbf{T}_+ = \sqrt{2}\, i \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{T}_- = -\sqrt{2}\, i \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{L_0} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{L_1} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{L_{-1}} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{L_2} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{L_{-2}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Commutation Relations

$$[\mathbf{T_3}, \mathbf{T_3}] = 0 \qquad [\mathbf{T_+}, \mathbf{T_-}] = 2\,\mathbf{T_3} \qquad [\mathbf{T_3}, \mathbf{T_\pm}] = \pm\,\mathbf{T_\pm} \qquad [\mathbf{T_3}, \mathbf{L_k}] = k\,\mathbf{L_k}$$

$$[\mathbf{T_\pm}, \mathbf{L_k}] = \sqrt{6 - k(k \pm 1)}\;\mathbf{L_{k\pm1}} \qquad [\mathbf{L_k}, \mathbf{L_{-k}}] = (-1)^k\,\frac{k}{2}\,\mathbf{T_3}$$

$$[\mathbf{L_0}, \mathbf{L_{\pm1}}] = -\frac{\sqrt{6}}{4}\,\mathbf{T_\pm} \quad [\mathbf{L_0}, \mathbf{L_{\pm2}}] = 0 \quad [\mathbf{L_1}, \mathbf{L_{-2}}] = \frac{1}{2}\,\mathbf{T_-} \quad [\mathbf{L_{-1}}, \mathbf{L_2}] = \frac{1}{2}\,\mathbf{T_+}$$

$$[\mathbf{L_1}, \mathbf{L_2}] = 0 \quad [\mathbf{L_{-1}}, \mathbf{L_{-2}}] = 0$$

This particular set of matrices is the best choice to write the Lax pair of the Tzitzéica model.

Appendix C

Calculations of the Darboux matrix elements

We can take the gauge

$$\mathbf{K} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \beta_1 \\ \beta_2 & 0 & 0 \\ 0 & \beta_3 & 0 \end{pmatrix} \frac{1}{\lambda} + \begin{pmatrix} 0 & \delta_1 & 0 \\ 0 & 0 & \delta_2 \\ \delta_3 & 0 & 0 \end{pmatrix} \frac{1}{\lambda^2} + \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix} \frac{1}{\lambda^3}.$$

First Part Consider the equation

$$\partial_+ \mathbf{K} = \mathbf{K} A_+ - \bar{A}_+ \mathbf{K},$$

where A_+ involve fields v and $\bar{A}_+$ with fields $\bar{v}$. Therefore, we find the system

λ	λ^{-3}
$\alpha_1 = \frac{\bar{v}}{v}\alpha_2$	$\partial_+\gamma_1 = 0 \Rightarrow \ \gamma_1 = constant_1$
$\alpha_3 = \frac{v}{\bar{v}}\alpha_2$	$\partial_+\gamma_2 = 0 \Rightarrow \ \gamma_2 = constant_2$
$\alpha_3 = \frac{v^2}{\bar{v}^2}\alpha_1$	$\partial_+\gamma_3 = 0 \Rightarrow \ \gamma_3 = constant_3$

(C.1)

λ^0	λ^{-1}	λ^{-2}
$\partial_+\alpha_1 - \frac{\beta_1}{v^2} - i\bar{v}\beta_2 = 0$	$\partial_+\beta_1 + iv\delta_1 - i\bar{v}\delta_2 = 0$	$\partial_+\delta_1 + iv\gamma_1 - i\bar{v}\gamma_2 = 0$
$\partial_+\alpha_2 + iv\beta_2 - i\bar{v}\beta_3 = 0$	$\partial_+\beta_2 - \frac{\delta_2}{v^2} - i\bar{v}\delta_3 = 0$	$\partial_+\delta_2 + iv\gamma_2 - i\bar{v}\gamma_3 = 0$
$\partial_+\alpha_3 + \frac{\beta_1}{\bar{v}^2} + iv\beta_3 = 0$	$\partial_+\beta_3 + \frac{\delta_1}{\bar{v}^2} + iv\delta_3 = 0$	$\partial_+\delta_3 + \frac{\gamma_1}{\bar{v}^2} - \frac{\gamma_3}{v^2} = 0$

In the same way for

$$\partial_- \mathbf{K} = \mathbf{K} A_- - \bar{A}_- \mathbf{K},$$

then

$$
\begin{array}{|c|}
\hline
\lambda^{0} \\
\hline
\alpha_1 \frac{\partial_- v}{v} - \alpha_1 \frac{\partial_- \bar{v}}{\bar{v}} + \partial_- \alpha_1 = 0 \\
\partial_+ \alpha_2 = 0 \Rightarrow \;\; \alpha_2 = constant_4 := \xi \\
-\alpha_3 \frac{\partial_- v}{v} + \alpha_3 \frac{\partial_- \bar{v}}{\bar{v}} + \partial_+ \alpha_3 = 0 \\
\hline
\end{array}
\quad
\begin{array}{|c|}
\hline
\lambda^{-1} \\
\hline
\partial_- \beta_1 + \alpha_1 - \alpha_3 - \beta_1 \left(\frac{\partial_- v}{v} + \frac{\partial_- \bar{v}}{\bar{v}} \right) = 0 \\
\partial_- \beta_2 - i\alpha_1 + i\alpha_2 + \beta_2 \frac{\partial_- v}{v} = 0 \\
\partial_- \beta_3 - i\alpha_2 + i\alpha_3 + \beta_3 \frac{\partial_- \bar{v}}{\bar{v}} = 0 \\
\hline
\end{array}
$$

$$
\begin{array}{|c|}
\hline
\lambda^{-2} \\
\hline
\partial_- \delta_1 + i\beta_1 - \beta_3 - \delta_1 \frac{\partial_- \bar{v}}{\bar{v}} = 0 \\
\partial_- \delta_2 - i\beta_1 + \beta_2 - \delta_2 \frac{\partial_- v}{v} = 0 \\
\partial_- \delta_3 - i\beta_2 + i\beta_3 + \delta_3 \left(\frac{\partial_- v}{v} + \frac{\partial_- \bar{v}}{\bar{v}} \right) = 0 \\
\hline
\end{array}
\quad
\begin{array}{|c|}
\hline
\lambda^{-4} \\
\hline
\gamma_1 = \nu \\
\gamma_2 = \nu \\
\gamma_3 = \nu \\
\hline
\end{array}
\tag{C.2}
$$

$$
\begin{array}{|c|}
\hline
\lambda^{-3} \\
\hline
\partial_- \gamma_1 + i\delta_1 - \delta_3 + \gamma_1 \left(\frac{\partial_- v}{v} - \frac{\partial_- \bar{v}}{\bar{v}} \right) = 0 \\
\partial_- \gamma_2 - i\delta_1 + i\delta_2 = 0 \\
\partial_- \gamma_3 - i\delta_2 + \delta_3 - \gamma_3 \left(\frac{\partial_- v}{v} - \frac{\partial_- \bar{v}}{\bar{v}} \right) = 0 \\
\hline
\end{array}
$$

Second Part From equations (C.1; λ) and (C.2; λ^0):

$$\boxed{\alpha 1 := \alpha = \xi \tfrac{\bar{v}}{v}, \qquad \alpha_2 = \xi, \qquad \alpha_3 = \tfrac{\xi^2}{\alpha}.} \tag{C.3}$$

From equations (C.1; λ^{-3}) and (C.2; λ^{-4}):

$$\boxed{\gamma_1 = \gamma_2 = \gamma_3 = \nu.} \tag{C.4}$$

Also, from equation (C.2; λ^{-3}):

$$\boxed{\delta_1 = \delta_2, \qquad i\delta_1 - \delta_3 - \nu \frac{\partial_- \alpha}{\alpha} = 0.} \tag{C.5}$$

From (C.1; λ^0)

$$\boxed{\beta_2 = \tfrac{\alpha \beta_3}{\xi}, \qquad \partial_+ \alpha - \tfrac{\beta_1}{v^2} - i\bar{v}\beta_2 = 0.} \tag{C.6}$$

Third Part Now we use a little trick, since Borisov [30] have used a Lax pair without imaginary factor $i = \sqrt{-1}$, naturally, he found a Darboux matrix without explicit factor i, even that an implicit dependence he doesn't deny, in fact, imaginary terms can appear in the fields ϕ and $\bar{\phi}$. Here, we do the same considerations, then, in our Darboux matrix, we allow that the components have imaginary terms just decoded in the fields. From equations (C.1; λ^{-1}) and (C.6)

$$\boxed{\beta_1 = \tfrac{\delta_1 \xi}{\alpha \beta_3}(\alpha + \xi) \qquad \delta_3 = \tfrac{\alpha}{2\xi^2}\beta_3^2} \tag{C.7}$$

Fourth Part Now, consider the equations (C.2; λ^0 and λ^{-2}), with some algebraic manipulations we can write

$$\boxed{\partial_- \alpha + \tfrac{\alpha}{\delta_1}(\beta_2 + \beta_3) - \tfrac{2i\alpha}{\delta_1}\beta_1 = 0,} \tag{C.8}$$

and using (C.6) and (C.7)

$$\boxed{\partial_- \alpha + \tfrac{\alpha}{\delta_1}(\tfrac{\alpha}{\xi} + 1)\beta_3 - \tfrac{2i\xi}{\beta_3}(\alpha + \xi) = 0,} \tag{C.9}$$

which we can compare with (C.5) and find

$$\boxed{\delta_1 = \tfrac{2\nu\xi}{\alpha\beta_3}(\alpha + \xi) = \delta_2,} \tag{C.10}$$

finally, we call

$$\boxed{\beta_3 := \beta.} \tag{C.11}$$

We can join every result and write the system of equations

$$\partial_+ \alpha - i\,\frac{\alpha\,\gamma}{\xi} - \frac{2\,\nu}{\gamma^2}\,(\alpha + \xi\,)^2 = 0 \tag{C.12}$$

$$\frac{1}{\bar{v}}\partial_+ \gamma - \frac{\gamma}{\bar{v}^2}\partial_+ \bar{v} + \frac{2\,\nu\,\xi}{\alpha\,\bar{v}\,\gamma}\,(\alpha + \xi\,) + i\,\frac{\gamma^2}{2\,\bar{v}\,\xi} = 0 \tag{C.13}$$

$$\partial_- \alpha - i\,\frac{2\,\bar{v}\,\xi}{\gamma}\,(\alpha + \xi\,) + \frac{\alpha^2\gamma^2}{2\,\nu\,\bar{v}^2\,\xi^2} = 0 \tag{C.14}$$

$$\frac{\partial_- \gamma}{\bar{v}} + i\,\xi\left(\frac{\xi}{\alpha} - 1\right) = 0 \tag{C.15}$$

where we have defined

$$\boxed{\gamma := \bar{v}\beta.} \tag{C.16}$$

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