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Dual descriptions of spin-two massive particles in $D = 2 + 1$ via master actions

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In the first part of this work we show the decoupling (up to contact terms) of redundant degrees of freedom which appear in the covariant description of spin-two massive particles in $D = 2 + 1$. We make use of a master action which interpolates, without solving any constraints, between a first-, second-, and third-order (in derivatives) self-dual model. An explicit dual map between those models is derived. In our approach the absence of ghosts in the third-order self-dual model, which corresponds to a quadratic truncation of topologically massive gravity, is due to the triviality (no particle content) of the Einstein-Hilbert action in $D = 2 + 1$. In the second part of the work, also in $D = 2 + 1$, we prove the quantum equivalence of the gauge invariant sector of a couple of self-dual models of opposite helicities (+ 2 and -2) and masses m_+ and m_- to a generalized self-dual model which contains a quadratic Einstein-Hilbert action, a Chern-Simons term of first order, and a Fierz-Pauli mass term. The use of a first-order Chern-Simons term instead of a third-order one avoids conflicts with the sign of the Einstein-Hilbert action.

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I. INTRODUCTION

In the last years there has been quite intense activity in the subject of higher spin theories in different dimensions and their dual formulations; see, for instance, [1–5] and references therein. One of the difficulties of a covariant description of higher spin fields is the amount of redundant degrees of freedom present in the higher rank tensor fields. This is a severe difficulty in constructing interacting theories for such fields; see comments in [3,4]. In the first part of our work (Sec. II) we address the issue of spurious degrees of freedom in $D = 2 + 1$ for massive fields of helicity ± 2. We show how duality can help us to prove the quantum decoupling of redundant degrees of freedom at the quadratic level (free theories). Our master action approach also leads us to a better understanding of the differences with the spin-one case where there are only first-order and second-order (in derivatives) self-dual models, unlike the spin-two case where we also have a third-order (ghost-free) self-dual model. In particular, based on the local symmetries of the dual models, we also explain why we do not expect a fourth-order (or higher) self-dual model for spin two and why we do not have a third-order (or higher) self-dual model for the spin-one case. Our approach makes it clear that the absence of ghosts in the third-order self-dual model is a consequence of the non-propagating nature of the Einstein-Hilbert (EH) action in $D = 2 + 1$.

In the second part of this work (Sec. III) we show that there exists a self-consistent quantum description of a couple of massive states of opposite helicities (+ 2 and -2) and different masses in general, by means of only one rank-two tensor field which we call a generalized self-dual (GSD) field in analogy with the spin-one case treated in [6,7]. We avoid the conflicts found in [8] with the sign of the Einstein-Hilbert term by working with a Chern-Simons (CS) term of first order instead of the gravitational Chern-Simons term of third order of [9]. The particle content of the GSD model is disentangled by showing its dual equivalence to the gauge invariant sector of a couple of non-interacting second-order self-dual models of opposite helicities.

II. FIRST, SECOND, AND THIRD-ORDER SELF-DUAL MODELS AND THEIR DUAL MAPS

Our starting point is the first-order self-dual model suggested in [10] which is the helicity $+2$ analogue of the helicity $+1$ self-dual model of [11],

$$S_{\mathrm{SD}}^{(1)} = \int d^3x \left[\frac{m}{2} \epsilon^{\mu\nu\lambda} f_{\mu}{}^{\alpha} \partial_{\nu} f_{\lambda\alpha} + \frac{m^2}{2} (f^2 - f_{\mu\nu} f^{\nu\mu}) \right], \tag{1}$$

where $f \equiv \eta^{\alpha\beta} f_{\alpha\beta}$. The metric is flat: $\eta_{\alpha\beta} = \mathrm{diag}(-, +, +)$. The upper index in $S_{\mathrm{SD}}^{(1)}$ indicates that we have a first-order model in the derivatives. In most of this work we use second rank tensor fields, like $f_{\alpha\beta}$ in (1), with no symmetry in their indices. Whenever symmetric and antisymmetric combinations show up, they will be denoted, respectively, by $f_{(\alpha\beta)} \equiv (f_{\alpha\beta} + f_{\beta\alpha})/2$ and $f_{[\alpha\beta]} \equiv (f_{\alpha\beta} - f_{\beta\alpha})/2$. Replacing m by $-m$ in $S_{\mathrm{SD}}^{(1)}$, we change the particle's helicity from $+2$ to -2. The first term in (1) reminds us of a spin-one topological Chern-Simons term which will henceforth be called a Chern-Simons term of first order (CS_1), to be distinguished from another (third-order) Chern-Simons term which appears later. The second term in (1) is the Fierz-Pauli (FP) mass term [12] which is the spin-two analogue of a spin-one Proca mass term. The FP term breaks the local invariance $\delta f_{\alpha\beta} = \partial_{\alpha} \xi_{\beta}$ of the CS_1 term.

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The equations of motion of (1),

$$\epsilon_\mu{}^{\nu\lambda}\partial_\nu f_{\lambda\alpha} = m(f_{\alpha\mu} - \eta_{\mu\alpha} f), \quad (2)$$

imply that $f_{\alpha\beta}$ is traceless, symmetric, and transverse, i.e.,

$$f = 0, \quad (3)$$

$$f_{[\alpha\beta]} = 0, \quad (4)$$

$$\partial^\alpha f_{\alpha\beta} = 0 = \partial^\beta f_{\alpha\beta}. \quad (5)$$

Furthermore, it follows that $f_{\alpha\beta}$ satisfies the Klein-Gordon equation $(\Box - m^2)f_{\alpha\beta} = 0$ and the helicity equation $(J^\mu P_\mu + 2m)^{\alpha\beta\gamma\delta} f_{\gamma\delta} = 0$, with $(2m)^{\alpha\beta\gamma\delta} = m(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma})$, and (see [4]) the quantities $(J^\mu)^{\alpha\beta\gamma\delta} = i(\eta^{\alpha\gamma}\epsilon^{\beta\mu\delta} + \eta^{\beta\gamma}\epsilon^{\alpha\mu\delta} + \eta^{\alpha\delta}\epsilon^{\beta\mu\gamma} + \eta^{\beta\delta}\epsilon^{\alpha\mu\gamma})/2$ satisfy the $2+1$ Lorentz algebra. In summary, all necessary equations to describe a helicity $+2$ massive particle in $D = 2+1$ are satisfied at the classical level.

Next we combine the works [10,13] into one master action which takes us from the first-order self-dual model (1) to its second and third-order versions entirely within the path integral framework with no need of solving any constraint equation as in [10] or introducing any explicit gauge condition. Before we proceed, in order to keep the analogy with the spin-one case as close as possible and to avoid the profusion of indices, we use the shorthand notation

$$\int f\cdot df \equiv \int d^3x \epsilon_\mu{}^{\nu\lambda} f^{\mu\alpha}\partial_\nu f_{\lambda\alpha}, \quad (6)$$

$$\int (f^2)_{\rm FP} \equiv \int d^3x (f^2 - f_{\mu\nu}f^{\nu\mu}). \quad (7)$$

In the master action approach an important role will be played by the EH term. If we expand in the dreibein $e_{\mu\alpha} = \eta_{\mu\alpha} + h_{\mu\alpha}$ and keep only quadratic terms in the fluctuations, the EH action can be written [13]

$$\frac{1}{2}\int d^3x(\sqrt{-g}R)_{hh} = \int d^3x \frac{\epsilon^{\mu\nu\lambda}h_\mu{}^\alpha \partial_\nu \Omega_{\lambda\alpha}(h)}{4} = \frac{1}{4}\int h\cdot d\Omega(h), \quad (8)$$

where[1]

$$\Omega_\lambda{}^\alpha(h) \equiv \epsilon^{\alpha\beta\gamma}[\partial_\beta(h_{\gamma\lambda} + h_{\lambda\gamma}) - \partial_\lambda h_{\gamma\beta}]. \quad (9)$$

As explained in [7,14] with an explicit example, the existence of a master action does not guarantee *a priori* spectrum equivalence of the interpolated dual theories. It is crucial that the terms which mix the fields of the dual theories have no propagating degree of freedom like the spin-one CS term used in [15] or the BF-type mixing terms of [16]. Based on the works [10,13] we suggest the following master action:

$$S_M^S = \frac{m}{2}\int f\cdot df + \frac{m^2}{2}\int (f^2)_{\rm FP} - \frac{m}{2}\int (f-A)\cdot d(f-A) - a\int (h-A)\cdot d\Omega(h-A). \quad (10)$$

We have introduced two second rank tensor fields $A_{\alpha\beta}$ and $h_{\alpha\beta}$ with no symmetry in their indices. The upper index in S_M^S stands for a singlet (a parity singlet of helicity $+2$). The coefficient in front of the third term of (10) is such that the quadratic term of S_M^S in $f_{\alpha\beta}$ has no derivatives, which is important for deriving dual theories which are local. The constant "a" will be fixed later on for an analogous reason. If $a = 0$ we recover the intermediate master action of [10]. Let us introduce sources $j_{\alpha\beta}$ and define the generating function:

$$W^S[J] = \int \mathcal{D}A_{\alpha\beta}\mathcal{D}h_{\alpha\beta}\mathcal{D}f_{\alpha\beta}\exp i\left(S_M^S + \int d^3x f_{\alpha\beta}j^{\alpha\beta}\right). \quad (11)$$

After the trivial shift $h_{\alpha\beta} \to h_{\alpha\beta} + A_{\alpha\beta}$ followed by $A_{\alpha\beta} \to A_{\alpha\beta} + f_{\alpha\beta}$, the last two terms of (10) decouple, and since they have no particle content it is clear that S_M^S is equivalent to $S_{SD}^{(1)}$ and therefore describes a parity singlet of helicity $+2$. After those shifts and integrating over $h_{\alpha\beta}$ and $A_{\alpha\beta}$, we derive, up to an overall constant,

$$W^S[J] = \int \mathcal{D}f_{\alpha\beta}\exp i\left(S_{\rm SD}^{(1)} + \int d^3x f_{\alpha\beta}j^{\alpha\beta}\right). \quad (12)$$

On the other hand, since the linear term in the fields $f_{\alpha\beta}$ in the exponent in (11) is $f_{\alpha\beta}U^{\alpha\beta}$ with $U^{\alpha\beta} \equiv m\epsilon^{\alpha\nu\lambda}\partial_\nu A_\lambda{}^\beta + j^{\alpha\beta}$, after the shift $f_{\alpha\beta} \to f_{\alpha\beta} + (\eta_{\alpha\beta}U_\mu^\mu - 2U_{\alpha\beta})/(2m^2)$ we decouple $f_{\alpha\beta}$ completely. After integrating over $f_{\alpha\beta}$ we obtain, up to an overall constant,

$$W^S[J] = \int \mathcal{D}A_{\alpha\beta}\mathcal{D}h_{\alpha\beta}\exp iS_I[j], \quad (13)$$

where

$$S_I[j] = \int\left[\frac{A\cdot d\Omega(A)}{4} - \frac{m}{2}A\cdot dA\right] - a\int (h-A)\cdot d\Omega(h-A) + \int d^3x\left[j^{\alpha\beta}F_{\alpha\beta}(A) + \frac{j^{\alpha\beta}j_{\beta\alpha}}{2m^2} - \frac{(j_\mu^\mu)^2}{4m^2}\right]. \quad (14)$$

The sources are now coupled to the gauge invariant combination:

[1]Our definition of $\Omega_\lambda{}^\alpha(h)$ differs from [13] by an overall sign.

$$F_{\alpha\beta}(A) \equiv T_{\alpha\beta}(A) - \frac{T_\mu{}^\mu(A)}{2}\eta_{\alpha\beta} \tag{15}$$

where $T_{\beta\alpha}(A) \equiv (\frac{1}{m})\epsilon_\beta{}^{\nu\lambda}\partial_\nu A_{\lambda\alpha}$ is invariant under the gauge transformations $\delta A_{\alpha\beta} = \partial_\alpha \xi_\beta$. The shift $h_{\alpha\beta} \to h_{\alpha\beta} + A_{\alpha\beta}$ in (14) decouples $h_{\alpha\beta}$ for arbitrary values of the constant "a," which has played no role so far. Integrating $h_{\alpha\beta}$, up to an overall constant again, we obtain

$$W^S[J] = \int \mathcal{D}A_{\alpha\beta} \times \exp i\Big\{S_{\rm SD}^{(2)}(A) + \int d^3x\Big[j^{\alpha\beta}F_{\alpha\beta}(A) + \frac{j^{\alpha\beta}j_{\beta\alpha}}{2m^2} - \frac{(j_\mu^\mu)^2}{4m^2}\Big]\Big\}, \tag{16}$$

where the second-order self-dual model is given by

$$S_{\rm SD}^{(2)} = \int\Big[\frac{A\cdot d\Omega(A)}{4} - \frac{m}{2}A\cdot dA\Big]. \tag{17}$$

The model $S_{\rm SD}^{(2)}$ has appeared before in [10,13]. It looks very similar to the spin-one Maxwell-Chern-Simons (MCS) theory of [9]. In particular, $S_{\rm SD}^{(2)}$ is a gauge theory invariant under $\delta A_{\alpha\beta} = \partial_\alpha\xi_\beta$. The first term in (17) is the analogue of the Maxwell term in the MCS theory and corresponds exactly to the quadratic approximation of the Einstein-Hilbert action [see (8)], with its usual sign.

From the classical point of view, the equations of motion of $S_{\rm SD}^{(2)}$ can be cast in the same self-dual form (2) with the identification $f_{\alpha\beta} \leftrightarrow F_{\alpha\beta}(A)$. Therefore, it is clear that $S_{\rm SD}^{(2)}$ is a perfectly acceptable classical description of such a particle.

At the quantum level, deriving (12) and (16) with respect to the sources we demonstrate the following equivalence of correlation functions:

$$\langle f_{\mu_1\nu_1}(x_1)\cdots f_{\mu_N\nu_N}(x_N)\rangle_{S_{\rm SD}^{(1)}} = \langle F_{\mu_1\nu_1}[A(x_1)]\cdots F_{\mu_N\nu_N}[A(x_N)]\rangle_{S_{\rm SD}^{(2)}} + \text{contact terms}. \tag{18}$$

The contact terms appear due to the quadratic terms in the sources in (16). In conclusion, we have the dual map below at the classical and quantum levels,

$$f_{\alpha\beta} \leftrightarrow F_{\alpha\beta}(A) = T_{\alpha\beta}(A) - \frac{T_\mu^\mu(A)}{2}\eta_{\alpha\beta}. \tag{19}$$

Because of the gauge invariance of $T_{\alpha\beta}(A) = \epsilon_\alpha{}^{\nu\lambda}\partial_\nu A_{\lambda\beta}/m$ our dual map is gauge invariant as expected since $S_{\rm SD}^{(1)}$ is not a gauge theory. The map (19) is similar to the spin-one map $f_\mu \leftrightarrow \epsilon_{\mu\nu\alpha}\partial^\nu A^\alpha/m$ between the self-dual model of [11] and the MCS theory of [9].

Next we show that $S_{\rm SD}^{(1)}$ is also dual to a third-order self-dual model. Neglecting surface terms, after some integration by parts it is easy to prove the identities

$$\int h\cdot d\Omega(A) = \int A\cdot d\Omega(h) = \int \Omega(h)\cdot dA. \tag{20}$$

By using those identities in (14) and fixing $a = 1/4$, we can cancel the second-order term $\int A\cdot d\Omega(A)/4$, and the intermediate action (14) can be written as

$$\begin{aligned} S_I[j] = &-\frac{m}{2}\int\Big[A - \frac{\Omega(h)}{2m}\Big]\cdot d\Big[A - \frac{\Omega(h)}{2m}\Big] \\ &+ \frac{1}{8m}\int \Omega(h)\cdot d\Omega(h) - \frac{1}{4}\int h\cdot d\Omega(h) \\ &+ \int d^3x\Big[j^{\alpha\beta}F_{\alpha\beta}(A) + \frac{j^{\alpha\beta}j_{\beta\alpha}}{2m^2} - \frac{(j_\mu^\mu)^2}{4m^2}\Big]. \end{aligned} \tag{21}$$

It is clear that the shift $A_{\alpha\beta} \to A_{\alpha\beta} + \Omega_{\alpha\beta}(h)/2m$ will decouple $A_{\alpha\beta}$ from $h_{\alpha\beta}$ and produce the third-order action $\int \Omega d\Omega$ out of the second-order theory (14). Another, less obvious, shift $A_{\alpha\beta} \to A_{\alpha\beta} + (j_{\beta\alpha} - \eta_{\beta\alpha}j_\mu^\mu/2)/m^2$ decouples $A_{\alpha\beta}$ completely and gives rise to the CS_1 term $-(m/2)\int A\cdot dA$ with no particle content. After integrating over $A_{\alpha\beta}$ we derive from (13) and (21), up to an overall constant,

$$W^S[J] = \int \mathcal{D}h_{\alpha\beta} \times \exp i\Big\{S_{\rm SD}^{(3)}(h) + \int d^3x\Big[j^{\alpha\beta}F_{\alpha\beta}\Big(\frac{\Omega}{2m}\Big) + \mathcal{O}(j^2)\Big]\Big\}, \tag{22}$$

where $\mathcal{O}(j^2)$ stands for quadratic terms in the sources which lead only to contact terms in the correlation functions and therefore do not need to be specified. From (9) and (15) we have

$$F_{\alpha\beta}\Big(\frac{\Omega}{2m}\Big) = T_{\alpha\beta}\Big(\frac{\Omega}{2m}\Big) - \frac{T_\mu{}^\mu(\frac{\Omega}{2m})}{2}\eta_{\alpha\beta}, \tag{23}$$

$$T^{\alpha\beta}\Big(\frac{\Omega}{2m}\Big) = \frac{\epsilon^{\alpha\nu\lambda}\partial_\nu\Omega_\lambda{}^\beta}{2m^2} = -\frac{E^{\alpha\gamma}E^{\beta\lambda}h_{(\gamma\lambda)}}{m^2}, \tag{24}$$

with $E^{\lambda\mu} \equiv \epsilon^{\lambda\mu\nu}\partial_\nu$. The third-order self-dual model $S_{\rm SD}^{(3)}(h)$ is given by

$$\begin{aligned} S_{\rm SD}^{(3)}(h) &= \frac{1}{8m}\int \Omega(h)\cdot d\Omega(h) - \frac{1}{4}\int h\cdot d\Omega(h) \\ &= \int d^3x\Big[\frac{1}{2m}h_{(\lambda\mu)}(\eta^{\lambda\delta}\Box - \partial^\lambda\partial^\delta)E^{\mu\alpha}h_{(\alpha\delta)} \\ &\quad - \frac{1}{2}h_{(\lambda\mu)}E^{\lambda\delta}E^{\mu\alpha}h_{(\alpha\delta)}\Big]. \end{aligned} \tag{25}$$

The first term in $S_{\rm SD}^{(3)}(h)$ is the quadratic approximation in the fluctuations of the dreibein $e_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ of a gravitational Chern-Simons term (see [9,13]), while the second one is the EH term at the same approximation; [see

(8)]. Both terms form the quadratic approximation for the so-called topologically massive gravity of [9]. The action $S_{\rm SD}^{(3)}$ is invariant under the local transformations $\delta h_{\alpha\beta} = \partial_\alpha \xi_\beta + \epsilon_{\alpha\beta\gamma}\phi^\gamma$. Notice that the sign of the EH term is not the expected one. By construction, in passing from $S_{\rm SD}^{(2)}(h)$ to $S_{\rm SD}^{(3)}(h)$ there is a sign inversion. The unexpected sign, as explained in [9], is in fact necessary for the absence of ghosts, which is a surprising feature of the higher-order theory $S_{\rm SD}^{(3)}(h)$ that we now understand from another point of view, since we have shown directly that $S_{\rm SD}^{(3)}(h)$ can be derived from the first-order ghost-free theory $S_{\rm SD}^{(1)}(h)$ by the addition of two extra terms (mixing terms) [see (10)] with no particle content. Now it is clear why we do not have a third-order self-dual model in the spin-one case; the point is that when we derive $S_{\rm SD}^{(3)}(h)$ from a first-order theory a second-order mixing term is necessary. We have used the quadratic Einstein-Hilbert action as a mixing term since it has no particle content. However, its spin-one analogue is the Maxwell action which contains a scalar massless particle in the spectrum and cannot be used to mix dual fields without leading to a spectrum mismatch between the dual theories.

At the classical level, the equations of motion $\delta S_{\rm SD}^{(3)} = 0$ can be written in the first-order self-dual form (2) with the identification $f_{\alpha\beta} \leftrightarrow F_{\alpha\beta}(\frac{\Omega}{2m})$. Consequently, $S_{\rm SD}^{(3)}$ classically describes a parity singlet of helicity $+2$ just like $S_{\rm SD}^{(2)}$ or $S_{\rm SD}^{(1)}$.

From (12) and (22) we deduce

$$\langle f_{\mu_1\nu_1}(x_1)\cdots f_{\mu_N\nu_N}(x_N)\rangle_{S_{\rm SD}^{(1)}} = \left\langle F_{\mu_1\nu_1}\left[\frac{\Omega(x_1)}{2m}\right]\cdots F_{\mu_N\nu_N}\left[\frac{\Omega(x_N)}{2m}\right]\right\rangle_{S_{\rm SD}^{(3)}} + \text{contact terms.} \tag{26}$$

It is remarkable that now in the $S_{\rm SD}^{(3)}(h)$ theory we have $T_{\alpha\beta}(\frac{\Omega}{2m}) = T_{\beta\alpha}(\frac{\Omega}{2m})$ [see (24)], and consequently $F_{\alpha\beta}(\frac{\Omega}{2m}) = F_{\beta\alpha}(\frac{\Omega}{2m})$. Therefore the dual map $f_{\alpha\beta} \leftrightarrow F_{\alpha\beta}(\frac{\Omega}{2m})$ that we read from (26) now automatically assures the vanishing of correlation functions of the antisymmetric combinations $f_{[\alpha\beta]}$, up to contact terms, which is not obvious either in $S_{\rm SD}^{(1)}(f)$ or in $S_{\rm SD}^{(2)}(A)$. This is a typical advantage of having dual formulations of the same theory.

The decoupling of the trace $f = \eta^{\alpha\beta} f_{\alpha\beta}$ is not obvious in any of the three self-dual formulations given here. In what follows we take advantage of the second-order formulation to prove it. First, suppose we had defined the sources from the very beginning as $j^{\alpha\beta} \equiv \phi\eta^{\alpha\beta} + j_S^{\alpha\beta} + j_A^{\alpha\beta}$, such that $f_{\alpha\beta} j^{\alpha\beta} = f\phi + j_S^{\alpha\beta} f_{(\alpha\beta)} + j_A^{\alpha\beta} f_{[\alpha\beta]}$ where $j_S^{\alpha\beta} = j_S^{\beta\alpha}$ and $j_A^{\alpha\beta} = -j_A^{\beta\alpha}$. Back in (16) and using (15) we can write down the action in the exponent of (16) as follows:

$$S[j] = \int d^3x \bigg[-\frac{A_{\mu\alpha} E^{\mu\lambda} E^{\alpha\gamma}(A_{\gamma\lambda} + A_{\lambda\gamma})}{4} - m^2 \frac{A_{\mu\alpha} T^{\mu\alpha}(A)}{2} + j_A^{\mu\alpha} T_{\mu\alpha}(A) + j_S^{\mu\alpha} T_{\mu\alpha}(A) - \frac{[\phi + (j_S)^\nu_\nu] T^\mu_\mu}{2} + \mathcal{O}(j^2_{\alpha\beta}) \bigg]. \tag{27}$$

Since the first term in (27), which is the quadratic Einstein-Hilbert action, only depends on $A_{(\mu\alpha)}$ it is clear that we get rid of $j_A^{\mu\alpha} T_{\mu\alpha}(A)$ through the shift $A^{\mu\alpha} \to A^{\mu\alpha} + j_A^{\mu\alpha}/m^2$. So we can see the decoupling of $f_{[\alpha\beta]}$ directly in the $S_{\rm SD}^{(2)}$ formulation. After $A^{\mu\alpha} \to A^{\mu\alpha} + (\frac{E^{\mu\alpha}}{m} - \eta^{\mu\alpha})\frac{\phi}{2m^2}$ we cancel out $-\phi T^\mu_\mu/2$ in (27). Consequently, all correlation functions of $f_{[\alpha\beta]}$ or the trace f will vanish, up to contact terms, in agreement with the classical results (3) and (4).

Regarding the transverse condition (5), from the trace of the dual map (19) we have the correspondence $f \leftrightarrow -T^\mu_\mu(A)/2$. So, the decoupling of the trace f implies that correlation functions in the $S_{\rm SD}^{(2)}(A)$ theory involving $T^\mu_\mu(A)$ must vanish (up to contact terms). Classically, $T^\mu_\mu(A) = 0$ follows from the equations of motion of $S_{\rm SD}^{(2)}(A)$. Thus, we can reduce the dual map (19) to $f_{\alpha\beta} \leftrightarrow T_{\alpha\beta}(A)$. Because of the trivial (nondynamical) identity $\partial_\alpha T^{\alpha\beta} = 0$ it follows that $\partial_\alpha f^{\alpha\beta} = 0$, and since $f^{[\alpha\beta]}$ decouples we have $\partial_\alpha f^{\alpha\beta} = 0 = \partial_\alpha f^{\beta\alpha} = 0$ inside correlation functions up to contact terms. Therefore all constraints (3)–(5) are satisfied. We can use the dual maps between correlation functions (18) and (26) and the detailed studies (including the pole structure of the propagator) made in [9] (see also [10]) to finally establish that the three models $S_{\rm SD}^{(1)}(f)$, $S_{\rm SD}^{(2)}(A)$, and $S_{\rm SD}^{(3)}(h)$ correctly describe a parity singlet of helicity $+2$ and mass m.

The fact that (4) and (5) are consequences of trivial (nondynamical) identities is relevant for a consistent coupling to other fields. In the spin-one case the transverse condition on the self-dual field $\partial_\mu f^\mu = 0$ is traded, in the Maxwell-Chern-Simons theory, in the Bianchi identity $\partial_\mu F^\mu(A) = \partial_\mu(\epsilon^{\mu\nu\alpha}\partial_\nu A_\alpha) = 0$. Since this is trivially satisfied it will hold also after coupling to other fields. In particular, in [17], we have coupled the self-dual model to charged scalar fields by using an arbitrary constant "a" as follows: $\partial_\mu \phi^* \partial^\mu \phi \to (D_\mu\phi)^* D^\mu \phi + e^2(a-1)f^2\phi^*\phi$, where "$e$" is the charge and $D_\mu\phi = (\partial_\mu + ief_\mu)\phi$. We have shown in [17] that the Bianchi identity $\partial_\mu F^\mu(A) = 0$ gives rise via a dual map to the constraint $\partial_\mu\{[m^2 + 2e^2(a-1)\phi^*\phi] f^\mu\} = 0$. Although we only have a "minimal coupling" for $a = 1$, the correct counting of degrees of freedom is guaranteed for any value of "a." In the spin-two case the traceless condition $f = 0$ does not correspond to a trivial identity in the dual gauge theories. Therefore we expect restrictions on the possible couplings of the spin-two self-dual model to other fields.

Concerning the local symmetries of the models $S_{\rm SD}^{(2)}$ and $S_{\rm SD}^{(3)}$, a comment is in order. Namely, the first term in $S_{\rm SD}^{(1)}$ is invariant under the local transformations $\delta_\xi f_{\alpha\beta} = \partial_\alpha \xi_\beta$. This symmetry is broken by the Fierz-Pauli mass term. However, in the dual theory $S_{\rm SD}^{(2)}$ such symmetry is restored. Analogously, the first term in $S_{\rm SD}^{(2)}$ is invariant under antisymmetric local shifts $\delta_\Lambda A_{\alpha\beta} = \Lambda_{\alpha\beta}$, where $\Lambda_{\alpha\beta} = -\Lambda_{\beta\alpha}$, and that symmetry is broken by the mass term of $S_{\rm SD}^{(2)}$ (the CS_1 term). Once again the symmetry is restored in $S_{\rm SD}^{(3)}$, which depends only on $h_{(\alpha\beta)}$. Since both the quadratic Einstein-Hilbert action and the mass term (quadratic third-order Chern-Simons term) of $S_{\rm SD}^{(3)}$ are invariant under the same set of local symmetries, there will be no local symmetry to be restored by a higher-order (fourth) self-dual model. So we claim that $S_{\rm SD}^{(3)}$ is the highest-order spin-two self-dual model. Likewise, in the spin-one case both the Maxwell and Chern-Simons terms are invariant under the same gauge symmetry and we have no third-order self-dual model of spin one.

III. GENERALIZED SELF-DUAL MODEL OF SPIN TWO AND ITS DUAL

In the last section we learned that there are at least three different consistent ways of giving mass to a parity singlet of spin two in $D = 2 + 1$ without using extra fields. We can use the Fierz-Pauli mass term, the CS_1 term, or the Chern-Simons term of third order which is a quadratic truncation of a gravitational Chern-Simons term; see (1), (17), and (25), respectively. In the spin-one case (parity singlet) we have two possible mass terms, i.e., the first-order Chern-Simons term and the Proca term which appears in the first-order self-dual model of [11]. Both terms can coexist in a generalized self-dual model (Maxwell-Chern-Simons-Proca theory) which contains two massive parity singlets of spin one in the spectrum. It is natural[2] to ask whether we could combine different mass terms also in the spin-two case. Indeed, this question has been addressed in [8]. As we have seen here, in passing from $S_{\rm SD}^{(2)}$ to $S_{\rm SD}^{(3)}$ the usual sign of the Einstein-Hilbert term must be reversed, which poses a problem when both the first and the third-order Chern-Simons terms are present since they require opposite signs for the Einstein-Hilbert action. It is also known that in the usual FP massive gravity the sign of the Einstein-Hilbert theory is the usual one. In fact, due to this problem the authors of [8] have concluded that the theory consisting of an Einstein-Hilbert action plus a topological Chern-Simons term of third order and a FP mass term does not have a physical spectrum. On the other hand, we could have two massive physical particles in the spectrum by combining both the first-order CS term and the FP mass term. In analogy with the spin-one case [7] we define a generalized self-dual model of spin two by adding a quadratic Einstein-Hilbert term to the $S_{\rm SD}^{(1)}$ self-dual model defined with arbitrary coefficients a_0, a_1:

$$S_{\rm GSD} = \int \left[\frac{a_0}{2} (f^2)_{\rm FP} + \frac{a_1}{2} f \cdot df + \frac{f \cdot d\Omega(f)}{4} \right]. \qquad (28)$$

We could ask, what is the gauge theory dual to $S_{\rm GSD}$ which generalizes $S_{\rm SD}^{(2)}$? Following [7], in order to avoid ghosts, it is appropriate to introduce auxiliary fields $(\lambda_{\alpha\beta})$ and rewrite the quadratic EH term of (28) in a first-order form with the help of a Fierz-Pauli mass term. Next we add two terms, with no particle content, to mix the initial fields $(f_{\alpha\beta}, \lambda_{\alpha\beta})$ with the new dual fields $(\tilde{A}_{\alpha\beta}, \tilde{B}_{\alpha\beta})$. Introducing a source term we have the generating function

$$W[j] = \int \mathcal{D}\tilde{A}\mathcal{D}\tilde{B}\mathcal{D}f\mathcal{D}\lambda \exp iS_M(j), \qquad (29)$$

where the source-dependent master action is given by

$$\begin{aligned} S_M(j) = {}& \frac{a_0}{2}\int (f^2)_{\rm FP} + \frac{a_1}{2}\int f\cdot df + \int d^3x j^{\mu\nu} f_{\mu\nu} \\ & + \frac{1}{2}\int (\lambda^2)_{\rm FP} + \int \lambda \cdot df - \int (\lambda - \tilde{B})\cdot d(f - \tilde{A}) \\ & - \frac{a_1}{2}\int (f - \tilde{A})\cdot d(f - \tilde{A}). \end{aligned} \qquad (30)$$

After the shifts $\tilde{B}_{\alpha\beta} \to \tilde{B}_{\alpha\beta} + \lambda_{\alpha\beta}$ and $\tilde{A}_{\alpha\beta} \to \tilde{A}_{\alpha\beta} + f_{\alpha\beta}$ in S_M, the last two terms decouple, and since they have no propagating mode, the particle content of S_M is the same as that of the generalized self-dual model $S_{\rm GSD}$. Integrating over $\tilde{A}$, $\tilde{B}$, and $\lambda_{\alpha\beta}$, we obtain the generating function of the GSD model up to an overall constant:

$$W[j] = \int \mathcal{D}f e^{i[S_{\rm GSD}(f) + \int d^3x j^{\mu\nu} f_{\mu\nu}]}. \qquad (31)$$

On the other hand, we can write

$$\begin{aligned} S_M(j) = {}& -\int \tilde{B}\cdot d\tilde{A} - \frac{a_1}{2}\int \tilde{A}\cdot d\tilde{A} + \int d^3x j^{\mu\nu} f_{\mu\nu} \\ & + \frac{1}{2}\int (\lambda^2)_{\rm FP} + \int \lambda\cdot d\tilde{A} + \frac{a_0}{2}\int (f^2)_{\rm FP} \\ & + \int f\cdot d(\tilde{B} + a_1\tilde{A}). \end{aligned} \qquad (32)$$

The integrals $\int \mathcal{D}\lambda$ and $\int \mathcal{D}f$ will produce two Einstein-Hilbert terms quadratic in the fields $\tilde{A}_{\alpha\beta}$ and $\tilde{B}_{\alpha\beta}$, including a mixing term involving both fields. A field redefinition can decouple $\tilde{A}_{\alpha\beta}$ from $\tilde{B}_{\alpha\beta}$. Guided by the spin-one case [7] we use the convenient notation

$$a_0 = m_+ m_-; \qquad a_1 = m_+ - m_-. \qquad (33)$$

[2]In a more general situation we might try to combine the three different spin-two mass terms altogether [18].

After the redefinitions

$$\tilde{A}_{\alpha\beta} = \frac{\sqrt{m_+}A_{\alpha\beta} - \sqrt{m_-}B_{\alpha\beta}}{\sqrt{m_+ + m_-}}, \quad (34)$$

$$\tilde{B}_{\alpha\beta} = -\frac{m_+^{3/2}A_{\alpha\beta} + m_-^{3/2}B_{\alpha\beta}}{\sqrt{m_+ + m_-}}, \quad (35)$$

we deduce, up to an overall constant,

$$W[j] = \int \mathcal{D}A\mathcal{D}B e^{iS[j,m_+,m_-]} \quad (36)$$

where

$$S[j, m_+, m_-] = S_{\rm SD}^{(2)}(A, m_+) + S_{\rm SD}^{(2)}(B, -m_-) + \int d^3x\Big[j^{\alpha\nu}F_{\alpha\nu}(A, B) + \frac{j^{\alpha\nu}j_{\nu\alpha}}{2m_+m_-} - \frac{j_\mu^\mu j_\alpha^\alpha}{4m_+m_-}\Big]. \quad (37)$$

The tensor $F_{\alpha\nu}(A, B)$ is invariant under independent gauge transformations $\delta A_{\alpha\beta} = \partial_\alpha\xi_\beta$ and $\delta B_{\alpha\beta} = \partial_\alpha\zeta_\beta$, explicitly:

$$F_{\alpha\nu}(A, B) = \epsilon_{\alpha\beta\gamma}\partial^\beta C^\gamma{}_\nu - \frac{\eta_{\alpha\nu}}{2}\epsilon^{\mu\gamma\lambda}\partial_\mu C_{\gamma\lambda}, \quad (38)$$

$$C_{\alpha\beta} = -\frac{1}{\sqrt{m_+ + m_-}}\Big(\frac{A_{\alpha\beta}}{\sqrt{m_+}} + \frac{B_{\alpha\beta}}{\sqrt{m_-}}\Big). \quad (39)$$

For $m_+ = m_-$ parity symmetry is restored in both (28) and (37). Using the physical interpretation of $S_{\rm SD}^{(2)}$ from the last section, it is now clear that $S_{\rm GSD}$ describes two massive particles of masses m_+ and m_- and helicities $+2$ and -2. Comparing correlation functions from (31) and (36) we derive

$$\langle f_{\mu_1\nu_1}(x_1)\cdots f_{\mu_N\nu_N}(x_N)\rangle_{S_{\rm GSD}(f,m_+,m_-)} = \langle F_{\mu_1\nu_1}[C(x_1)]\cdots F_{\mu_N\nu_N}[C(x_N)]\rangle_{S_{\rm SD}^{(2)}(A,m_+)+S_{\rm SD}^{(2)}(B,-m_-)} + \text{contact terms}. \quad (40)$$

So we have the map $f_{\alpha\beta} \leftrightarrow F_{\alpha\beta}(C)$. For a complete proof of equivalence between $S_{\rm GSD}(f, m_+, m_-)$ and the gauge invariant sector of $S_{\rm SD}^{(2)}(A, m_+) + S_{\rm SD}^{(2)}(B, -m_-)$, it is rather puzzling that $f_{\alpha\beta}$ is mapped into a gauge invariant function of one specific linear combination of the fields $A_{\alpha\beta}$ and $B_{\alpha\beta}$, while on the other side we have two independent and local gauge invariant objects, namely, $T_{\mu\alpha}(A) = \epsilon_\mu{}^{\nu\lambda}\partial_\nu A_{\lambda\alpha}/m$ and $T_{\mu\alpha}(B)$. We should be able to compute any correlation function of $T_{\mu\alpha}(A)$ and $T_{\mu\alpha}(B)$ in terms of the generalized self-dual field $f_{\alpha\beta}$. Indeed, as in the spin-one case [7], this is possible, as we show next. We first suppress the source term $f_{\alpha\beta}j^{\alpha\beta}$ in (30) and add sources for $T_{\mu\alpha}(A)$ and $T_{\mu\alpha}(B)$. So we define the generating function

$$\tilde{W}[\tilde{j}_+, \tilde{j}_-] = \int \mathcal{D}f\mathcal{D}\lambda\mathcal{D}\tilde{A}\mathcal{D}\tilde{B}\exp i\tilde{S}_M[\tilde{j}_+, \tilde{j}_-] \quad (41)$$

where

$$\tilde{S}_M[\tilde{j}_+, \tilde{j}_-] = S_M(j = 0) + \int d^3x[\tilde{j}_+^{\mu\alpha}T_{\mu\alpha}(\tilde{A}) + \tilde{j}_-^{\mu\alpha}T_{\mu\alpha}(\tilde{B})]. \quad (42)$$

We have introduced the sources

$$\tilde{j}_+ \equiv \frac{1}{\sqrt{m_+ + m_-}}\Big(\frac{m_-j_+}{\sqrt{m_+}} - \frac{m_+j_-}{\sqrt{m_-}}\Big), \quad (43)$$

$$\tilde{j}_- \equiv -\frac{1}{\sqrt{m_+ + m_-}}\Big(\frac{j_+}{\sqrt{m_+}} + \frac{j_-}{\sqrt{m_-}}\Big), \quad (44)$$

in such a way that, after integrating over $f_{\alpha\beta}$ and λ_{ab} and redefining the fields according to (34) and (35), we obtain, up to an overall constant,

$$\begin{aligned} W[j_+, j_-] &= \tilde{W}[\tilde{j}_+, \tilde{j}_-] \\ &= \int \mathcal{D}f\mathcal{D}\lambda\mathcal{D}A\mathcal{D}B\exp i\Big\{S_{\rm SD}^{(2)}(A, m_+) \\ &\quad + S_{\rm SD}^{(2)}(B, -m_-) + \int d^3x[j_+^{\mu\alpha}T_{\mu\alpha}(A) \\ &\quad + j_-^{\mu\alpha}T_{\mu\alpha}(B)]\Big\}. \end{aligned} \quad (45)$$

On the other hand, it is not difficult to convince ourselves that after some shifts of $\tilde{B}_{\alpha\beta}$ and $\tilde{A}_{\alpha\beta}$ in (41), we can decouple those fields completely. Their integration leads to a constant. By further integrating over the auxiliary fields $\lambda_{\alpha\beta}$ we obtain from (41), up to an overall constant, the dual version of (45),

$$\begin{aligned} W[j_+, j_-] &= \tilde{W}[\tilde{j}_+, \tilde{j}_-] \\ &= \int \mathcal{D}f\exp i\Big\{S_{\rm GSD}(f) \\ &\quad + \int d^3x[j_+^{\lambda\alpha}D_{\lambda\alpha}{}^{\mu\nu}(x, -m_-)f_{\mu\nu} \\ &\quad + j_-^{\lambda\alpha}D_{\lambda\alpha}{}^{\mu\nu}(x, m_+)f_{\mu\nu}] + \mathcal{O}(j^2)\Big\} \end{aligned} \quad (46)$$

where $\mathcal{O}(j^2)$ stand for quadratic terms in the sources j_+ and j_-. We have introduced the differential operator

$$D^{\lambda\alpha\mu\nu}(x, m) = \frac{1}{|m|\sqrt{m_+ + m_-}}[mE_x^{\lambda\mu}\eta^{\alpha\nu} - E_x^{\lambda(\mu}E_x^{\nu)\alpha}]. \quad (47)$$

Note that (45) and (46) are both symmetric under $(m_+, m_-, j_+, j_-) \to (-m_-, -m_+, j_-, j_+)$ as expected. Correlation functions of $T_{\mu\alpha}(A)$ and $T_{\mu\alpha}(B)$ can now be calculated from the GSD model. For instance, from (45) and (46) we derive

$$\langle T^{\alpha_1\beta_1}[A(x_1)]\cdots T^{\alpha_N\beta_N}[A(x_N)]\rangle_{S^{(2)}_{\rm SD}(A,m_+)+S^{(2)}_{\rm SD}(B,-m_-)}$$
$$= D^{\alpha_1\beta_1\mu_1\nu_1}(x_1,m_+)\cdots D^{\alpha_N\beta_N\mu_N\nu_N}(x_N,m_+)$$
$$\times\langle f_{\mu_1\nu_1}(x_1)\cdots f_{\mu_N\nu_N}(x_N)\rangle_{S_{\rm GSD}} + \text{contact terms.} \tag{48}$$

Of course, we can also calculate correlation functions of $T_{\mu\alpha}(B)$, and mixed correlation functions involving both $T_{\mu\alpha}(A)$ and $T_{\mu\alpha}(B)$ from the GSD model (28). So we prove the quantum equivalence between the gauge invariant sector of $S^{(2)}_{\rm SD}(A,m_+)+S^{(2)}_{\rm SD}(B,-m_-)$ and the GSD model, up to contact terms. The classical equivalence between those models can also be established in a fashion analogous to what has been done in the spin-one case in [7].

IV. CONCLUSION

We have shown in the master action approach how duality can help us to prove the decoupling of redundant degrees of freedom at the quantum level. We have compared correlation functions and derived a dual map between the first-, second-, and third-order self-dual models which describe parity singlets of helicity $+2$ (or -2) in $D=2+1$. In particular, the decoupling of the antisymmetric combinations $f_{[\alpha\beta]}$ and the transverse conditions $\partial_\alpha f^{\alpha\beta}=0=\partial_\beta f^{\alpha\beta}$ have been shown to be related via dual maps to the trivial (nondynamical) identities $T_{\alpha\beta}(\Omega)-T_{\beta\alpha}(\Omega)=0$ and $\partial_\alpha T^{\alpha\beta}(\Omega)=\partial_\alpha(\epsilon^{\alpha\nu\gamma}\partial_\nu\Omega_\gamma{}^\beta)=0$, respectively, which indicates that those constraints will not be obstacles for the inclusion of interactions, contrary to the traceless condition $f^\mu_\mu=0$. Furthermore, we have seen that the spectrum equivalence of the three self-dual models follows from the nonpropagating (pure gauge) nature of the mixing terms in the master action, namely, the Chern-Simons term of first order and the Einstein-Hilbert action. Based on the local symmetries of the self-dual models we have argued that we should not expect a fourth- or higher-order self-dual model of spin two and that there is no third-order (or higher) self-dual model in the spin-one case.

In Sec. III we defined a GSD model by adding a quadratic Einstein-Hilbert term to the first-order self-dual model of [10] and showed its equivalence to the gauge invariant sector of a couple of noninteracting free particles of opposite helicities ($+2$ and -2) and different masses, i.e., $S^{(2)}_{\rm SD}(A,m_+)+S^{(2)}_{\rm SD}(B,-m_-)$. This generalizes previous works [19–21]. We have identified the gauge invariant field of the GSD model with a gauge invariant function of one specific linear combination of the opposite helicity gauge fields; see (39). In the opposite direction we have also shown how to compute correlation functions of gauge invariant objects of $S^{(2)}_{\rm SD}(A,m_+)+S^{(2)}_{\rm SD}(B,-m_-)$ from the dual GSD theory. No specific gauge condition has been used.

The decoupling of spurious degrees of freedom after the inclusion of interactions is under investigation. It is also of interest to formulate consistent self-dual models for higher spin ($s\geq 3$) massive particles in $D=2+1$ since the cases $s=1$ and $s=2$ seem to indicate, as we have seen here, a connection between topological actions and self-dual models. Finally, since there are dimensional reductions from massless particles in $D+1$ to massive particles in D dimensions, one might wonder which mechanisms or which dual massless spin-two models in $D=4$ give rise to the three self-dual models described here in a unified way.

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