

# Spin-1 duality in $D$ dimensions

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It is known that the Maxwell theory in  $D$  dimensions can be written in a first-order form (in derivatives) by introducing a totally antisymmetric field which leads to a  $(D - 3)$ -form dual theory. Remarkably, one can replace the antisymmetric field by a symmetric rank-two tensor ( $W_{\mu\nu} = W_{\nu\mu}$ ). Such master action establishes the duality between the Maxwell theory and a fourth-order higher-rank model in a  $D$ -dimensional flat space-time. A naive generalization to the curved space shows a connection between the recently found  $D = 4$  critical gravity and the Maxwell-theory plus a coupling term to the Ricci tensor ( $R_{\mu\nu}A^\mu A^\nu$ ). The mass of the spin-1 particle which appears in the  $D = 4$  critical gravity linearized around anti-de Sitter space is the same one obtained from the Ricci coupling term. We also work out, in flat space-time, the explicitly massive case (Maxwell-Proca) which is dual to a second-order theory for  $W_{\mu\nu}$ .

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## I. INTRODUCTION

The power of duality in field theory can hardly be underestimated, specially if we take into account the variety of applications of the AdS/CFT conjecture [1], see e.g. [2]. An earlier example where duality has also played an important role is the rigorous proof of confinement in a four dimensional (supersymmetric) field theory [3]. Those are examples of interacting theories.

In free (quadratic) field theories a typical approach to duality makes use of a master action [4] which depends on two different fields. Schematically, for massless fields, one starts from a Lagrangian density  $\mathcal{L}[A, B] = BB + 2A\hat{D}B$ , where  $\hat{D}$  is some differential operator. On one hand, the Gaussian integral over the  $B$  field furnishes  $\mathcal{L}[A] = A\hat{D}^\dagger\hat{D}A$  while the path integral over the  $A$  field leads to the functional constraint  $\hat{D}B = 0$  whose general solution  $B(C)$  introduces another field  $C$ . Back in the master action we get  $\mathcal{L}[C] = B(C)B(C)$  which is dual to  $\mathcal{L}[A]$ . Along those lines the authors of [5] have shown the duality between a massless scalar field (zero-form) and a massless 2-form in  $D = 3 + 1$  dimensions. Later this was generalized to  $p$ -form and  $(D - 2 - p)$ -form duality, see for instance [6].

For massive particles we have duality between a  $p$ -form and a  $(D - p - 1)$ -form. Now the master action is a bit different since we have also a quadratic term in the  $A$  field such that we can obtain the dual theories by Gaussian integrating either the  $B$  field or the  $A$  field. Usually, there is no constraint to be solved in the massive case. We review this procedure in both massless and massive cases in Secs. II A and III A respectively. There we also introduce sources and determine a local correspondence (dual map) between the dual theories which guarantees equivalence of correlation functions up to contact terms. Although the

above dualities involve free theories, they may suggest new interesting interacting theories as in [7].

For our purposes it is important to consider  $\mathcal{L}[A, B]$  as a lower order (in derivatives) version of  $\mathcal{L}[A]$  and notice that there is no need of using antisymmetric fields to decrease the order. Our starting point here is a first-order version of the Maxwell theory in  $D$  dimensions obtained in [8] with the help of a rank-two symmetric tensor. In Sec. II B we obtain, in flat space, the dual to the Maxwell theory by solving a functional constraint. We compare correlation functions of gauge invariants in both dual theories. We make some comments on a possible curved space version of this master action. In the curved space the Maxwell action is modified by a an interaction with the Ricci tensor:  $R_{\mu\nu}A^\mu A^\nu$ . On the dual side, a naive solution of the constraint equation leads to a dual gravitational theory in  $D = 4$  which has been recently considered [9] in the literature and known to possess spin-1 massive particles in the spectrum after linearization around Anti-de Sitter (AdS) spaces [10]. The mass of those spin-1 particles perfectly agrees with the one obtained in the dual vector theory when we consider the additional term  $R_{\mu\nu}A^\mu A^\nu$ .

In Sec. III B we work out the massive formulation of the master action of [8] and show the duality between the Maxwell-Proca model in a flat space with  $D$  dimensions and a second-order model for the rank-two symmetric field ( $S_W$ ). In the last section we draw some conclusions. In the Appendix we run the Dirac-Bergmann algorithm in the Hamiltonian approach as a double check on unitarity and the counting of degrees of freedom for the dual  $S_W$  theory.

## II. THE MASSLESS CASE

### A. $(D - 3)$ -form/1-form duality

In this section we recall the  $(D - 3)$ -form dual theory to the Maxwell theory and establish a local dual map between correlation functions in both theories.

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It is known that in  $D$  dimensions we can rewrite the Maxwell theory in a first-order form by using a totally antisymmetric tensor of rank  $D - 2$ , i.e., a  $(D - 2)$ -form. Namely,<sup>1</sup>

$$\begin{aligned}\mathcal{L}[A, B, J] = & -\frac{(D-2)!}{4}B_{\mu_1 \dots \mu_{D-2}}^2 \\ & + \frac{\epsilon^{\mu_1 \dots \mu_D}}{2}B_{\mu_1 \dots \mu_{D-2}}\partial_{\mu_{D-1}}A_{\mu_D} \\ & + B_{\mu_1 \dots \mu_{D-2}}J^{\mu_1 \dots \mu_{D-2}}.\end{aligned}\quad (1)$$

We have introduced a source term. Integrating over the  $(D - 2)$ -form  $B_{\mu_1 \dots \mu_{D-2}}$  in a path integral we have the Maxwell theory plus source terms:

$$\begin{aligned}\mathcal{L}[A, J] = & -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2(D-2)!}\epsilon_{\mu_1 \dots \mu_D}J^{\mu_1 \dots \mu_{D-2}}\partial^{\mu_{D-1}}A^{\mu_D} \\ & + \frac{J_{\mu_1 \dots \mu_{D-2}}^2}{2(D-2)!}.\end{aligned}\quad (2)$$

On the other hand, if we integrate over the vector field we obtain a functional delta function enforcing the vector constraint:

$$\epsilon^{\mu_1 \dots \mu_D}\partial_{\mu_{D-1}}B_{\mu_1 \dots \mu_{D-2}} = 0 \quad (3)$$

whose general solution introduces a  $(D - 3)$ -form:  $B_{\mu_1 \dots \mu_{D-2}} = \partial_{[\mu_{D-2}}C_{\mu_1 \dots \mu_{D-3}]}$ . Back in (1) we have a dual model to the Maxwell theory:

$$\begin{aligned}\mathcal{L}[C, J] = & -\frac{(D-2)!}{4}(\partial_{[\mu_{D-2}}C_{\mu_1 \dots \mu_{D-3}]} )^2 \\ & + J^{\mu_1 \dots \mu_{D-2}}\partial_{[\mu_{D-2}}C_{\mu_1 \dots \mu_{D-3}]}.\end{aligned}\quad (4)$$

Thus, we end up with the known duality between a 1-form and a  $(D - 3)$ -form which is a particular case of the  $p$ -form and  $(D - p - 2)$ -form duality for massless particles. Furthermore, by taking functional derivatives with respect to the source we have the local dual map:

$$\partial_{[\mu_{D-2}}C_{\mu_1 \dots \mu_{D-3}]} \leftrightarrow \frac{\epsilon_{\mu_1 \dots \mu_D}\partial^{\mu_{D-1}}A^{\mu_D}}{2(D-2)!}. \quad (5)$$

The map connect gauge invariant quantities in both theories  $\mathcal{L}[C, J]$  and  $\mathcal{L}[A, J]$ . It is such that the correlation functions of the left-hand side of (5) calculated in the theory (4) agree with the correlation functions of the right-hand side of (5) calculated in the theory (2) up to contact terms which have no particle content and stem from the quadratic term in the source in (2). The correspondence (5) also maps the equations of motion (in the absence of sources) of (2) and (4) into each other.

<sup>1</sup>In this work we use  $\eta_{\mu\nu} = (-, +, \dots, +)$  and  $\epsilon_{\mu_1 \mu_2 \dots \mu_k \mu_{k+1} \dots \mu_D} \epsilon^{\mu_1 \mu_2 \dots \mu_k \nu_{k+1} \dots \nu_D} = -k!(D-k)! \det \delta_{\mu_i}^{\nu_j}$ . Moreover for totally antisymmetric indices we have  $[\alpha_1 \dots \alpha_N] = \sum_P (-1)^P P(\alpha_1, \dots, \alpha_N)/N!$  while  $(\alpha\beta) = (\alpha\beta + \beta\alpha)/2$ .

## B. Fourth-order Maxwell dual

Remarkably, one can use instead of a  $(D - 2)$ -form a symmetric tensor  $W_{\mu\nu} = W_{\nu\mu}$  to rewrite the Maxwell theory in a first-order form:

$$\begin{aligned}S[A, W, T] = & \int d^Dx \left( W^{\mu\nu}W_{\mu\nu} - \frac{W^2}{D-1} \right. \\ & \left. + 2W^{\mu\nu}\partial_{(\mu}A_{\nu)} + W_{\mu\nu}T^{\mu\nu} \right),\end{aligned}\quad (6)$$

where  $W = W_\mu^\mu$ . As far as we know, the above theory has first appeared in the appendix of [8] in the case of  $D = 4$  dimensions. Its  $D$  dimensional generalization is trivial from formulas of [8]. We have added an external source term  $W_{\mu\nu}T^{\mu\nu}$ . Despite of depending only on the symmetric combination  $\partial_{(\mu}A_{\nu)} = (\partial_\mu A_\nu + \partial_\nu A_\mu)/2$ , the action (6) is equivalent to the Maxwell theory. This can be made clear rewriting (6), after neglecting a surface term, as

$$\begin{aligned}S[A, W, T] = & \int d^Dx \left\{ \left[ W^{\mu\nu} + \partial^{(\mu}A^{\nu)} \right. \right. \\ & \left. - \eta^{\mu\nu}\partial \cdot A + \frac{T^{\mu\nu}}{2} - \frac{\eta^{\mu\nu}T}{2} \right]^2 \\ & - \frac{1}{D-1} \left[ W - (D-1) \left( \partial \cdot A + \frac{T}{2} \right) \right]^2 - \frac{1}{4}F_{\mu\nu}^2 \\ & \left. + T^{\mu\nu}[\eta_{\mu\nu}\partial \cdot A - \partial_{(\mu}A_{\nu)}] + \frac{T^2 - T_{\mu\nu}^2}{4} \right\}.\end{aligned}\quad (7)$$

After the shift  $W^{\mu\nu} = \tilde{W}^{\mu\nu} - \partial^{(\mu}A^{\nu)} - T^{\mu\nu}/2 + \eta^{\mu\nu}(\partial \cdot A + T/2)$  we have two decoupled mass terms without dynamics for the  $\tilde{W}_{\mu\nu}$  fields plus the Maxwell theory and source terms. After integrating over  $\tilde{W}_{\mu\nu}$  we have the Lagrangian density:

$$\mathcal{L}[A, T] = -\frac{1}{4}F_{\mu\nu}^2 + T^{\mu\nu}[\eta_{\mu\nu}\partial \cdot A - \partial_{(\mu}A_{\nu)}] + \frac{T^2 - T_{\mu\nu}^2}{4}. \quad (8)$$

Equation (7) makes patent the invariance of (6), in the absence of sources, under the gauge transformations [8]:

$$\begin{aligned}\delta A_\mu &= \partial_\mu \phi; \\ \delta W_{\mu\nu} &= \square \theta_{\mu\nu} \phi \Rightarrow \delta W = (D-1)\square \phi.\end{aligned}\quad (9)$$

Where we define the projection operators

$$\theta_{\alpha\beta} = (\eta_{\alpha\beta} - \omega_{\alpha\beta}), \quad \omega_{\alpha\beta} = \frac{\partial_\alpha \partial_\beta}{\square}. \quad (10)$$

On the other hand, if we start from the action (6) and integrate over the vector field we get the functional constraint below which plays the role of (3)

$$\partial^\mu W_{\mu\nu} = 0. \quad (11)$$

In  $D = 1 + 1$  the reader can check that the general solution, linear in fields, of (11), i.e.,  $W_{\mu\nu} = \epsilon_\mu^\delta \epsilon_\nu^\gamma \partial_\delta \partial_\gamma h$  is pure gauge  $W_{\mu\nu} = \square \theta_{\mu\nu} h$ . This is in agreement with the fact that we have  $D - 2$  degrees of freedom for a massless spin-1 particle in  $D$  dimensions.

In  $D = 2 + 1$ , the general solution of (11), linear in fields, is given by  $W_{\mu\nu} = \epsilon_\mu^{\alpha\delta} \epsilon_\nu^{\beta\gamma} \partial_\delta \partial_\gamma h_{\alpha\beta}$ , where  $h_{\alpha\beta} = h_{\beta\alpha}$ . Plugging it back in (6), at  $T^{\mu\nu} = 0$ , we have the linearized version of the massless limit [11] of the new massive gravity [7], the so called  $K$ -term

$$\begin{aligned} S^*[h] &= \int d^3x \left[ \sqrt{-g} \left( R_{\mu\nu}^2 - \frac{3}{8} R^2 \right) \right]_{hh} \\ &= \int d^3x h_{\alpha\beta} (2\square \theta^{\alpha\mu} \square \theta^{\beta\nu} - \square \theta^{\alpha\beta} \square \theta^{\mu\nu}) h_{\mu\nu}, \end{aligned} \quad (12)$$

where  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ . Although of fourth-order, the  $K$  model is unitary [11,12] and describes one massless mode in agreement with its dual theory (Maxwell) which is on its turn equivalent to a massless scalar theory in  $D = 2 + 1$ . The duality between the  $K$ -term and the Maxwell theory in  $D = 2 + 1$  is not new [13]. However, in [13] the  $K$  model appears upon integration over a symmetric rank-2 tensor while the Maxwell action is obtained via solution of a constraint equation, in this sense the master action (1) is dual to the corresponding one of [13]. We also notice that the gauge symmetry  $\delta W_{\mu\nu} = \square \theta_{\mu\nu} \phi$  follows from the linearized Weyl symmetry  $\delta h_{\mu\nu} = \eta_{\mu\nu} \phi$  of (12).

For arbitrary dimensions  $D \geq 2$  the general solution of (11), linear in fields, is given by

$$W_{\mu\nu} = \epsilon_\mu^{\alpha_1 \dots \alpha_{D-2}\delta} \epsilon_\nu^{\beta_1 \dots \beta_{D-2}\gamma} \partial_\delta \partial_\gamma h_{[\alpha_1 \dots \alpha_{D-2}][\beta_1 \dots \beta_{D-2}]}, \quad (13)$$

where  $h_{[\alpha_1 \dots \alpha_{D-2}][\beta_1 \dots \beta_{D-2}]} = h_{[\beta_1 \dots \beta_{D-2}][\alpha_1 \dots \alpha_{D-2}]}$ .

Alternatively, we can write, as in the  $D = 3 + 1$  case treated in [14], the general solution (13) as

$$W_{\mu\nu} = \partial^\delta \partial^\gamma B_{\mu\delta\nu\gamma}, \quad (14)$$

where

$$B_{\mu\delta\nu\gamma} = \epsilon_{\mu\delta}^{\alpha_1 \dots \alpha_{D-2}} \epsilon_{\nu\gamma}^{\beta_1 \dots \beta_{D-2}} h_{[\alpha_1 \dots \alpha_{D-2}][\beta_1 \dots \beta_{D-2}]} \quad (15)$$

Substituting the general solution in (1) we have a dual fourth-order description of the Maxwell theory in any dimension  $D \geq 3$ :

$$\begin{aligned} S^*[h] &= \int d^Dx \left[ (\partial^\delta \partial^\gamma B_{\mu\delta\nu\gamma})^2 - \frac{(\partial^\delta \partial^\gamma B_{\delta\mu\gamma}^\mu)^2}{D-1} \right. \\ &\quad \left. + \partial^\delta \partial^\gamma B_{\mu\delta\nu\gamma} T^{\mu\nu} \right], \end{aligned} \quad (16)$$

where  $B_{\mu\nu\gamma}(h)$  is given in (15). In  $D = 3 + 1$  the same theory was obtained before [14] in a different approach and shown to be unitary by means of a canonical analysis. In  $D = 2 + 1$ , as we have already mentioned, the theory is also unitary. Based on the master action (6) we believe that unitarity will hold in arbitrary dimensions as a consequence of the unitarity of the Maxwell theory and the fact that (13) is the general solution of (11).

As we have done in the last subsection, see (5), we can compare correlation functions of gauge invariants in both theories, i.e., Maxwell, see (8), and its dual  $S^*[h]$ . Perhaps, the most natural invariant under (9) in the dual theory  $S^*[h]$  is  $\partial^\mu W_{\mu\nu}$ . However, due to the functional constraint (11) its correlation functions are trivial (vanish). Later we will see that they correspond to correlation functions of  $-\partial^\mu F_{\mu\nu}(A)/2$  in the Maxwell theory up to contact terms. In searching for a nontrivial gauge invariant, we consider that on the Maxwell side the basic local gauge invariant is the antisymmetric tensor ( $F_{\alpha\beta}$ ) and on the dual side we must take at least one derivative of  $W_{\mu\nu}$  in order to have invariance under (9). So we can think of taking linear combinations of  $\partial_\alpha W_{\beta\nu} - \partial_\beta W_{\alpha\nu}$  and  $(\eta_{\alpha\nu} \partial_\beta - \eta_{\beta\nu} \partial_\alpha) W$ . Along this way we end up with the gauge invariant below which has nonvanishing correlation functions in the dual theory  $S^*[h]$

$$G_{[\alpha\beta]\nu} \equiv \partial_\alpha W_{\beta\nu} - \partial_\beta W_{\alpha\nu} + \frac{(\eta_{\alpha\nu} \partial_\beta - \eta_{\beta\nu} \partial_\alpha)}{D-1} W. \quad (17)$$

Replacing the source term  $T^{\mu\nu} W_{\mu\nu}$  by a new one  $T^{[\alpha\beta]\nu} G_{[\alpha\beta]\nu}$  in (6), it is clear that we can keep the gauge invariance of the action without requiring any constraint on the sources. Integrating by parts one derivative we can rewrite the source term once again in the form  $\tilde{T}^{\mu\nu} W_{\mu\nu}$ , where  $\tilde{T}^{\mu\nu}$  contain combinations of one derivative of  $T^{[\alpha\beta]\nu}$ , so we can still use the action (8) replacing  $T^{\mu\nu}$  by  $\tilde{T}^{\mu\nu}$ . This procedure leads to the dual map,

$$G_{[\alpha\beta]\nu} \leftrightarrow -\frac{1}{2} \partial_\nu F_{\alpha\beta}. \quad (18)$$

Thus, correlation functions of  $G_{[\alpha\beta]\nu}$  in the dual theory  $S^*[h]$  correspond to correlation functions of  $\partial_\nu F_{\alpha\beta}/2$  in the Maxwell theory up to contact terms which are due to quadratic terms in  $T^{\mu\nu}$  in (8). From those correlation functions we can infer the correlations of  $F_{\mu\nu}$ .

Now we finish this section commenting on a possible curved space generalization of the  $S^*[h]$ /Maxwell duality. The natural curved space version of the master action (6), in the absence of sources, is given by

$$\begin{aligned}
S[A, W, T] &= \int d^D x \sqrt{-g} \left( W^{\mu\nu} W_{\mu\nu} - \frac{W^2}{D-1} + 2W^{\mu\nu} \nabla_{(\mu} A_{\nu)} \right) \\
&= \int d^D x \sqrt{-g} \left\{ (W^{\mu\nu} + \nabla^{(\mu} A_{\nu)} - g^{\mu\nu} \nabla \cdot A)^2 - \frac{[W - (D-1) \nabla \cdot A]^2}{D-1} - \frac{1}{4} F_{\mu\nu}^2 + R_{\mu\nu} A^\mu A^\nu \right\} \\
&= \int d^D x \sqrt{-g} \left( \tilde{W}^{\mu\nu} \tilde{W}_{\mu\nu} - \frac{\tilde{W}^2}{D-1} - \frac{1}{4} F_{\mu\nu}^2 + R_{\mu\nu} A^\mu A^\nu \right). \tag{19}
\end{aligned}$$

Where  $\nabla_\mu$  is the curved space covariant derivative and  $\tilde{W}_{\mu\nu} = W_{\mu\nu} + \nabla_{(\mu} A_{\nu)} - g_{\mu\nu} \nabla \cdot A$ .

The Ricci tensor  $R_{\mu\nu}$  has appeared due to the non-commutativity of the covariant derivatives:  $\nabla_\mu \nabla_\nu A^\nu = \nabla_\nu \nabla_\mu A^\nu - R_{\mu\nu} A^\nu$ . In (19) we have two trivial (nondynamic) terms for  $\tilde{W}_{\mu\nu}$  decoupled from the vector field. After neglecting those trivial terms we end up with the Maxwell theory plus a Ricci “mass term” in the curved space. So, the minimal coupling to gravity in the master action (6) originates a nonminimal coupling for the massless vector field which breaks the  $U(1)$  gauge symmetry.

On the other hand, integrating over the vector field  $A_\mu$  in the first line of (19) we have the curved space version of the constraint (3):

$$\nabla^\mu W_{\mu\nu} = 0. \tag{20}$$

We do not know its general solution but we can certainly begin with the lowest order terms in derivatives of the metric  $W_{\mu\nu}(g) = a_0 g_{\mu\nu} + a_2 (R_{\mu\nu} - g_{\mu\nu} R/2) + a_4 K_{\mu\nu} + \dots$ , where  $a_0, a_2, a_4, \dots$  are arbitrary constant coefficients. The tensor  $K_{\mu\nu} = (1/\sqrt{-g}) \delta S_4 / \delta g^{\mu\nu}$  is obtained from a general fourth-order Lagrangian density which can be written as  $\mathcal{L}_4 = \alpha R^2 + \beta R_{\mu\nu}^2 + \gamma \mathcal{L}_{\text{GB}}$ , where  $\alpha, \beta, \gamma$  are arbitrary constants and the Gauss-Bonnet term is  $\mathcal{L}_{\text{GB}} = R_{\mu\nu\alpha\beta}^2 - 4R_{\mu\nu}^2 + R^2$ . Plugging the solution  $W_{\mu\nu}(g)$  back in the first line of (19) we get

$$\begin{aligned}
S[W_{\mu\nu}(g)] &= \int d^D x \sqrt{-g} \left\{ \frac{a_0^2 D}{D-1} + a_0 a_2 \frac{D-2}{D-1} R \right. \\
&\quad \left. + a_2^2 \left[ R_{\mu\nu}^2 - \frac{D}{4(D-1)} R^2 \right] - \frac{2a_0 a_4}{(D-1)} K + \dots \right\}, \tag{21}
\end{aligned}$$

where the dots stand for terms of sixth or higher order in derivatives of the metric and

$$\begin{aligned}
K &= g^{\mu\nu} \cancel{K}_{\mu\nu}^{\frac{4-D}{2}} (\alpha R^2 + \beta R_{\mu\nu}^2 + \gamma \mathcal{L}_{\text{GB}}) + \left[ 2\alpha(D-1) \right. \\
&\quad \left. + \frac{\beta D}{2} + 4\gamma \right] \nabla_\mu \nabla^\mu R - 4\gamma \nabla_\mu \nabla_\nu R^{\mu\nu}. \tag{22}
\end{aligned}$$

The last two terms of (22) are total derivatives which can be neglected in (21) for arbitrary  $D$  dimensions. In  $D = 4$  we can discard  $K$  completely and (21) becomes exactly, dropping the dots, the critical gravity theory recently found in [9].

Upon linearization around an AdS background  $R_{\mu\nu} = \Lambda g_{\mu\nu}$  the authors of [10] have shown the existence of “Proca-log-modes” in the critical gravity theory in  $D = 4$ . Those spin-1 massive modes can be derived from a curved space Lagrangian density, in the notation<sup>2</sup> of [10], of the Maxwell-Proca form  $\mathcal{L}_{MP} = -F_{\mu\nu}^2/4 + 3m^2 \sigma A^\mu A_\mu$ . This is in full agreement with (19) since the criticality condition [9] requires  $\Lambda = 3\sigma m^2$  in the notation of [10]. A unitary theory requires the unusual sign  $\sigma < 0$  for the Einstein-Hilbert term as in the  $D = 3$  case [13] which is an earlier example of a critical gravity.

Regarding the general case of critical gravity in  $D$  dimensions (see [15]) we must mention that the mass of the Proca modes predicted by (19) is still in agreement with the results obtained for the critical gravity but in the absence of the Gauss-Bonnet term where the criticality condition becomes, in the notation of [10],  $\Lambda = (D-1)\sigma m^2$ . From the point of view of (21) we must set  $a_4 = a_6 = \dots = 0$ . Apparently, the key point is to make sure that upon linearization the solution  $W_{\mu\nu}(g)$  is in fact a general solution to the constraint (20) without redundancies. We believe that possible redundancies can be eliminated by field redefinitions.

We finish this section by mentioning that, alternatively, in order to keep the  $U(1)$  gauge invariance in the curved space we could have added the nonminimal coupling term with negative sign  $-R_{\mu\nu} A^\mu A^\nu$  to the first line of (19) such that we end up with the pure Maxwell theory in the curved space after the shifts in the  $W_{\mu\nu}$  fields. At the level of master action the  $U(1)$  gauge invariance would be restored in the curved space. However, due to this new term there would be no functional constraint equation for the  $W_{\mu\nu}$  fields, and we could in principle Gaussian integrate over the vector field and obtain a dual theory containing the exotic term  $\nabla^\mu W_{\mu\nu} (R^{-1})^{\nu\beta} \nabla^\alpha W_{\alpha\beta}$  which involves the inverse of the Ricci tensor. It is not yet clear if this a consistent solution, even for special backgrounds, to the

<sup>2</sup>The notation of [10] can be recovered by matching the coefficients of the terms proportional to  $a_0 a_2$  and  $a_2^2$ . This leads to  $a_2 = 1/(\kappa m \sqrt{D-2})$  and  $a_0 = m\sigma(D-1)/(\kappa \sqrt{D-2})$ , up to an overall sign. The coefficient of the  $a_0^2$  term (cosmological term) in (21) comes out correctly without any fit just like the relative coefficient between  $R_{\mu\nu}^2$  and  $R^2$  inside the term proportional to  $a_2^2$ .

gravitational coupling problem mentioned in [14]. We are still investigating this possibility.

### III. THE MASSIVE CASE

#### A. ( $D-2$ )-form/1-form duality

This subsection parallels the Sec. II A. Adding a mass term and a source term for the vector field in (1) we have the master action, see [16],

$$\begin{aligned} \mathcal{L}_m(A, B) = & -\frac{(D-2)!}{4} B_{\mu_1 \dots \mu_{D-2}}^2 \\ & + \frac{1}{2} \epsilon^{\mu_1 \dots \mu_D} B_{\mu_1 \dots \mu_{D-2}} \partial_{\mu_{D-1}} A_{\mu_D} \\ & + B_{\mu_1 \dots \mu_{D-2}} J^{\mu_1 \dots \mu_{D-2}} - \frac{m^2}{2} A_\mu A^\mu + J_\mu A^\mu. \end{aligned} \quad (23)$$

Integrating over  $B_{\mu_1 \dots \mu_{D-2}}$  in the path integral we have the Maxwell-Proca theory plus source dependent terms:

$$\begin{aligned} \mathcal{L}_m(A) = & -\frac{1}{4} F^2 - \frac{m^2}{2} A_\mu A^\mu + J_\mu A^\mu \\ & + \frac{\epsilon^{\mu_1 \dots \mu_D} J_{\mu_1 \dots \mu_{D-2}} \partial_{\mu_{D-1}} A_{\mu_D}}{(D-2)!} + \frac{J_{\mu_1 \dots \mu_{D-2}}^2}{(D-2)!}. \end{aligned} \quad (24)$$

On the other hand, integrating over the vector field in (23) we have the Kalb-Ramond ( $D-2$ )-form dual model:

$$\begin{aligned} \mathcal{L}_m(B) = & -\frac{[(D-1)!]^3}{4m^2} (\partial_{[\mu_1} B_{\mu_2 \dots \mu_{D-1}]} )^2 \\ & - \frac{(D-2)!}{4} B_{\mu_1 \dots \mu_{D-2}}^2 + B_{\mu_1 \dots \mu_{D-2}} J^{\mu_1 \dots \mu_{D-2}} \\ & - \frac{\epsilon^{\mu_1 \dots \mu_D} J_{\mu_D} \partial_{\mu_{D-1}} B_{\mu_1 \dots \mu_{D-2}}}{m^2} + \frac{J_\mu J^\mu}{m^2}. \end{aligned} \quad (25)$$

Comparing (24) with (25) we can calculate correlation functions in the Kalb-Ramond model in terms of correlations in the Maxwell-Proca theory and vice-versa by using the dual maps below, respectively:

$$B_{\mu_1 \dots \mu_{D-2}} \leftrightarrow \frac{\epsilon_{\mu_1 \dots \mu_{D-2}}^{\mu_{D-1} \mu_D} \partial_{\mu_{D-1}} A_{\mu_D}}{(D-2)!}, \quad (26)$$

$$A^\mu \leftrightarrow -\frac{\epsilon^{\mu_1 \dots \mu_{D-1} \mu} \partial_{\mu_{D-1}} B_{\mu_1 \dots \mu_{D-2}}}{m^2}. \quad (27)$$

Notice that in the massive case we do not need to worry about gauge invariance. The maps (26) and (27) are consistent with each other up to contact terms as expected. This completes the 1-form/ $(D-2)$ -form duality which is a special case of the  $p$ -form/ $(D-1-p)$ -form duality for massive theories.

#### B. $S_W$ /Maxwell-Proca duality

Similarly we can define the massive version of (6),

$$\begin{aligned} S_m[W, A] = & \int d^D x \left( W^{\mu\nu} W_{\mu\nu} - \frac{W^2}{D-1} + 2W^{\mu\nu} \partial_{(\mu} A_{\nu)} \right. \\ & \left. - \frac{m^2}{2} A^\mu A_\mu + J \cdot A \right), \end{aligned} \quad (28)$$

which can be written as

$$\begin{aligned} S_m = & \int d^D x \left\{ (W^{\mu\nu} + \partial^{(\mu} A^{\nu)} - \eta^{\mu\nu} \partial \cdot A)^2 \right. \\ & \left. - \frac{[W - (D-1)\partial \cdot A]^2}{D-1} - \frac{1}{4} F_{\mu\nu}^2 - m^2 \frac{A^2}{2} + J \cdot A \right\}. \end{aligned} \quad (29)$$

Thus, after a shift in  $W_{\mu\nu}$ , we have the Proca action plus some decoupled mass terms for  $W_{\mu\nu}$  which can be dropped; whereas integrating over the vector field and rescaling  $W_{\mu\nu} \rightarrow mW_{\mu\nu}/\sqrt{2}$  we get

$$\begin{aligned} S_m = S_W + & \int d^D x \left[ \frac{J^2}{2m^2} - \frac{\sqrt{2} J^\mu \partial^\nu W_{\mu\nu}}{m^2} \right. \\ & \left. - \frac{m^2}{2} \left( A_\mu + \frac{2\partial^\nu W_{\mu\nu}}{m} - \frac{J_\mu}{m^2} \right)^2 \right], \end{aligned} \quad (30)$$

where

$$S_W = \int d^D x \left[ (\partial^\nu W_{\mu\nu})^2 + \frac{m^2}{2} \left( W_{\mu\nu}^2 - \frac{W^2}{D-1} \right) \right]. \quad (31)$$

After a shift in  $A_\mu$  in (30) it is clear that  $S_W$  must be dual to the Maxwell-Proca theory. Indeed, from (31) we can read off the propagator

$$\begin{aligned} \langle W^{\lambda\mu}(x) W_{\alpha\beta}(y) \rangle = & \left\{ \frac{P_{SS}^{(2)}}{m^2} - \frac{P_{SS}^{(1)}}{\square - m^2} \right. \\ & + \left[ \frac{2\square(D-1)}{m^2} + 2 - D \right] \frac{P_{SS}^{(0)}}{m^2} \\ & \left. - \frac{\sqrt{D-1}}{m^2} (P_{SW}^{(0)} + P_{WS}^{(0)}) \right\}_{\alpha\beta}^{\lambda\mu}, \end{aligned} \quad (32)$$

where the differential operators are defined as

$$(P_{SS}^{(2)})^{\lambda\mu}_{\alpha\beta} = \frac{1}{2} (\theta^\lambda{}_\alpha \theta^\mu{}_\beta + \theta^\mu{}_\alpha \theta^\lambda{}_\beta) - \frac{\theta^{\lambda\mu} \theta_{\alpha\beta}}{D-1}, \quad (33)$$

$$\begin{aligned} (P_{SS}^{(1)})^{\lambda\mu}_{\alpha\beta} = & \frac{1}{2} (\theta^\lambda{}_\alpha \omega^\mu{}_\beta + \theta^\mu{}_\alpha \omega^\lambda{}_\beta + \theta^\lambda{}_\beta \omega^\mu{}_\alpha \\ & + \theta^\mu{}_\beta \omega^\lambda{}_\alpha), \end{aligned} \quad (34)$$

$$\begin{aligned}(P_{SS}^{(0)})^{\lambda\mu}_{\alpha\beta} &= \frac{1}{D-1}\theta^{\lambda\mu}\theta_{\alpha\beta}, \\ (P_{WW}^{(0)})^{\lambda\mu}_{\alpha\beta} &= \omega^{\lambda\mu}\omega_{\alpha\beta},\end{aligned}\quad (35)$$

$$\begin{aligned}(P_{SW}^{(0)})^{\lambda\mu}_{\alpha\beta} &= \frac{1}{\sqrt{D-1}}\theta^{\lambda\mu}\omega_{\alpha\beta}, \\ (P_{WS}^{(0)})^{\lambda\mu}_{\alpha\beta} &= \frac{1}{\sqrt{D-1}}\omega^{\lambda\mu}\theta_{\alpha\beta}.\end{aligned}\quad (36)$$

It turns out that single and double poles of (32) at  $\square = 0$  cancel out exactly such that  $\langle W^{\mu\nu}(x)W_{\alpha\beta}(y)\rangle$  is analytic at  $\square = 0$  and contains only one single pole at  $\square = m^2$  in the spin-1 sector. In order to check the particle content of  $S_W$  we look at the saturated two-point amplitude

$$\begin{aligned}A_W(k) &= T_{\mu\nu}^*(k)\langle W^{\mu\nu}(-k)W_{\gamma\beta}(k)\rangle T^{\gamma\beta}(k) \\ &= -\frac{2i}{k^2(k^2+m^2)}(k^\epsilon T_{\epsilon\mu}^*\theta^{\mu\nu}k^\lambda T_{\lambda\nu} + \dots),\end{aligned}\quad (37)$$

where  $T_{\mu\nu}(k)$  are symmetric sources which are analytic functions of the momentum. The dots stand for analytic functions at  $k^2 = -m^2$ . It is instructive to compare (37) with the corresponding amplitude of the Maxwell-Proca theory:

$$\begin{aligned}A_{MP}(k) &= J_\mu^*(k)\langle A^\mu(-k)A^\nu(k)\rangle J_\nu(k) \\ &= i\left(\frac{J_\mu^*(k)\theta^{\mu\nu}J_\nu(k)}{k^2+m^2} + \frac{J_\mu^*(k)\omega^{\mu\nu}J_\nu}{m^2}\right).\end{aligned}\quad (38)$$

As in (37) the pole at  $k^2 = 0$  cancels out in (38). The imaginary part of the residue at  $k^2 = -m^2$  is positive as expected:

$$R_m = \Im \lim_{k^2 \rightarrow -m^2} (k^2 + m^2)A_{Proca}(k) = J_t^* \cdot J_t,\quad (39)$$

where  $J_t$  is the transverse part of the source,  $k \cdot J_t = 0$ . Choosing  $k_\mu = (m, 0, 0)$  it becomes clear that  $J_t^* \cdot J_t = |J_t|^2 + |J_t^2|^2 > 0$ . Analogously, identifying  $k^\lambda T_{\lambda\mu} \leftrightarrow J_\mu$  it is easy to see that  $R_m = \Im \lim_{k^2 \rightarrow -m^2} (k^2 + m^2)A_w(k) > 0$ . Therefore, both  $S_W$  and  $S_{Proca}$  have exactly the same particle content, one massive spin-1 mode in a  $D$ -dimensional space-time ( $D \geq 2$ ). In  $D = 3 + 1$  the action  $S_W$  has appeared before in [17] as a special case of a general expression for a second-order action quadratic in symmetric rank-2 tensors and restricted to be unitary. Here we are showing that  $S_W$  is dual to the Proca theory for any  $D$ . The linear terms in the source  $J_\mu$ , after the trivial shifts in  $W_{\mu\nu}$  and  $A_\mu$  in (29) and (30), reveal the dual map

$$A_\mu \leftrightarrow -\frac{\sqrt{2}}{m}\partial^\nu W_{\nu\mu}\quad (40)$$

which allows us to compute correlation functions in the Maxwell-Proca theory from the  $S_W$  action. Conversely, the correlations of the fundamental field  $W_{\mu\nu}$  can be obtained from the Maxwell-Proca theory via the map

$$W_{\mu\nu} \leftrightarrow \frac{\sqrt{2}}{m}[\eta_{\mu\nu}\partial \cdot A - \partial_{(\mu}A_{\nu)}]\quad (41)$$

which could be derived by introducing sources for  $W_{\mu\nu}$ , see (8). Those maps are similar to (26) and (27). As before, the correlation functions in the dual theories must match up to contact terms. We can give a simple argument to show the consistency of (40) and (41) as follows. Substituting the left-hand side of (41) in (40) we get the Maxwell-Proca equation of motion (45). From an integral of a total (functional) derivative

$$\int \mathcal{D}A \frac{\delta}{\delta A(x_1)}[e^{iS}A(x_2) \cdots A(x_N)] = 0,\quad (42)$$

where symbolically  $S = \int d^Dx A \hat{O}A/2$ , we can derive  $\langle \hat{O}A(x_1)A(x_2) \cdots A(x_N) \rangle = 0$  whenever  $x_1 \neq x_j$  for all  $j = 2, \dots, N$ . So the equation of motion  $\hat{O}A = 0$  is enforced in the correlation functions if we neglect coinciding points (contact terms), which is exactly when the dual maps are supposed to hold.

Regarding the classical equivalence between the Maxwell-Proca theory and the  $S_W$  model, from the equations of motion

$$\begin{aligned}\frac{\delta S_W}{\delta W^{\mu\nu}} &= \partial_\nu \partial^\alpha W_{\mu\alpha} + \partial_\mu \partial^\alpha W_{\nu\alpha} \\ &+ m^2 \left( \eta_{\mu\nu} \frac{W}{D-1} - W_{\mu\nu} \right) = 0,\end{aligned}\quad (43)$$

we can derive  $\partial^\mu \partial^\nu W_{\mu\nu} = -m^2 W/(2(D-1))$  and

$$\square(\partial^\alpha W_{\nu\alpha}) - \partial_\nu(\partial^\alpha \partial^\beta W_{\alpha\beta}) - m^2 \partial^\alpha W_{\alpha\nu} = 0\quad (44)$$

which is equivalent to the Maxwell-Proca equation with the identification (40):

$$\square A_\nu - \partial_\nu(\partial \cdot A) - m^2 A_\nu = 0.\quad (45)$$

From (45) we can derive the transverse condition  $\partial \cdot A = 0$  and the Klein-Gordon equation  $(\square - m^2)A_\mu = 0$  which describe a spin-1 massive particle. Since (44) has been derived from (43) by applying a derivative one might wonder whether the general solution of (43) contains more information than the transverse condition and the Klein-Gordon equation. In order to answer that question we start with a general Ansatz for a symmetric rank-2 tensor:  $W_{\mu\nu} = \partial_\mu f_\nu + \partial_\nu f_\mu + W_{\mu\nu}^{(T)}$ , where  $W_{\mu\nu}^{(T)}$  is given in (13) and  $f_\mu$  is arbitrary. From (44), which follows from (43), we deduce  $\partial^\mu \partial^\nu W_{\mu\nu} = 0$ , consequently due to  $\partial^\mu \partial^\nu W_{\mu\nu} = -m^2 W/(2(D-1))$ , which also follows from (43), we have  $W = 0$ . So, back in (43) we conclude that  $W_{\mu\nu}$  is purely longitudinal, i.e.,  $W_{\mu\nu}^{(T)} = 0$ . Substituting  $W_{\mu\nu} = \partial_\mu f_\nu + \partial_\nu f_\mu$  in (43) we obtain the third order equation

$$(\square - m^2)\partial_{(\mu}f_{\nu)} + \partial_\mu \partial_\nu(\partial \cdot f) = 0.\quad (46)$$

From  $W = 0$  we deduce  $\partial \cdot f = 0$  and from (46)  $\partial_\mu g_\nu + \partial_\nu g_\mu = 0$ , where  $g_\mu \equiv (\square - m^2)f_\mu$ . Assuming that the fields vanish at infinity, the solution is  $g_\mu = 0$ . Consequently, we conclude that (43) describes a massive spin-1 particle and nothing else in agreement with our pole analysis of (32). In the Appendix we perform the Dirac-Bergman constraints analysis for  $S_W$ . We end up with  $D - 1$  degrees of freedom and a positive definite Hamiltonian as required for a unitary massive spin-1 particle.

Since we have rescaled  $W_{\mu\nu} \rightarrow \sqrt{2}W_{\mu\nu}/m$  it is worth looking at the massless limit of  $S_W$ . At  $m = 0$  we have the local gauge invariance  $\delta_\Lambda W^{\mu\nu} = \epsilon^{\mu\alpha_1 \dots \alpha_{D-2}\delta} \epsilon^{\nu\beta_1 \dots \beta_{D-2}\gamma} \partial_\delta \partial_\gamma \Lambda_{[\alpha_1 \dots \alpha_{D-2}][\beta_1 \dots \beta_{D-2}]}$ . By adding a symmetry breaking term with an arbitrary coefficient  $\lambda$  we have

$$\mathcal{L}_W^{m=0} = (\partial^\nu W_{\mu\nu})^2 + \frac{\lambda}{2}(\partial^\mu W)^2 \quad (47)$$

which allows us to obtain

$$\langle W^{\mu\nu}(x)W_{\alpha\beta}(y) \rangle_{m=0} = - \left\{ \frac{2P_{SS}^{(1)}}{\square} + \frac{(P_{SW}^{(0)} + P_{WS}^{(0)})}{\square\sqrt{D-1}} - \frac{P_{WW}^{(0)}}{\square} \right. \\ \left. - \left( 2 + \frac{\lambda}{D-1} \right) \frac{P_{SS}^{(0)}}{\lambda\square} \right\}_{\alpha\beta}^{\mu\nu}. \quad (48)$$

It contains single and double poles at  $\square = 0$ . However, if we add a source term and require gauge invariance  $\delta_\Lambda(W_{\mu\nu}T^{\mu\nu}) = 0$ , the source (in momentum space) must be of the form  $T^{\mu\nu}(k) = k^\mu J^\nu(k) + k^\nu J^\mu(k)$  where  $J_\mu$  does not need to be conserved. The saturated (gauge invariant) two-point amplitude gets contribution only from the  $P_{SS}^{(1)}$  and  $P_{WW}^{(0)}$  pieces (which are  $\lambda$  independent). At the end the poles cancel out and we are left with an analytic function:

$$A_W^{m=0}(k) = \frac{i}{2} T_{\mu\nu}^*(k) \langle W^{\mu\nu}(-k)W_{\alpha\beta}(k) \rangle_{S_W(m=0)} T^{\alpha\beta}(k) \\ = -2iJ_\mu^*(k)J^\mu(k). \quad (49)$$

Thus, the  $m = 0$  limit of  $S_W$  has no particle content in any dimension  $D$ . This is in agreement with the study of [17] for the case  $D = 3 + 1$ . In the Hamiltonian formalism one can show that  $S_W$  at  $m = 0$  has enough first class constraints to gauge away all degrees of freedom.

#### IV. CONCLUSION

We have reviewed the usual duality between a 1-form and a  $(D-3)$ -form (massless case) and between a 1-form and a  $(D-2)$ -form (massive case) in Secs. II A and III A respectively. We have established a local dual map in both cases, see (5) and the pair (26) and (27), respectively. They allow us to calculate correlation functions of local physical quantities in one theory in terms of the dual quantities in the dual theory, up to contact terms.

In Sec. II B, starting from a master action recently obtained in [8], we have derived a fourth-order (in derivatives) dual model to the Maxwell theory in arbitrary  $D$  dimensions. In particular, in  $D = 2 + 1$  the corresponding dual theory is the massless limit, see [11], of the linearized new massive gravity of [7]. Once again we have determined a dual map between gauge invariants in both theories.

Remarkably, a naive curved space version of the master action of [8] leads, on one hand, to the Maxwell action plus an interacting term with the Ricci tensor ( $R_{\mu\nu}A^\mu A^\nu$ ). On the other hand, the “dual theory” (21) (neglecting higher than fourth-order terms) corresponds in  $D = 4$  to a critical gravity theory which was recently found in [9]. It contains curvature square terms with fine-tuned coefficients plus a fine-tuned cosmological term and the usual Einstein-Hilbert action. The linearized theory around an AdS background contains [10] Proca modes (spin-1) whose mass matches exactly the one obtained here via master action. However, a remark is in order. Namely, the linearized AdS critical gravity of [9] contains also spin-2 modes [10]. It is not clear how our naive generalization of the master action of [8] to curved spaces could be improved in order to encompass the spin-2 modes appropriately on both sides of the duality.

In Sec. III B we have generalized the flat space master action of [8] by adding an explicit mass term for the vector field (Proca term). Integrating over the vector field we have obtained a second-order dual theory to the Maxwell-Proca action in terms of a symmetric rank-2 tensor. We have found the dual map between those models in arbitrary  $D$ -dimensional flat space-time with  $D \geq 2$  and checked that the dual theory indeed shares the same particle content of the Maxwell-Proca theory by analyzing the analytic structure of the symmetric tensor propagator. In the Appendix we confirm that one has  $(D-1)$  degrees of freedom in the Hamiltonian approach by running the Dirac-Bergmann algorithm. We also show that the Hamiltonian is definite positive. The massless limit of this higher-rank description of spin-1 particles has no particle content.

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#### APPENDIX

Here, we analyze the Hamiltonian constraints generated by the massive action (31), dual to the Maxwell-Proca model in  $D$  dimensions.

Since  $W_{\mu\nu} = W_{\nu\mu}$ , in order to avoid unnecessary constraints we work only with independent phase space variables  $W_{\mu\nu}$  and  $\pi^{\mu\nu}$  with  $\mu \leq \nu$ . From the Lagrangian density,

$$\begin{aligned}\mathcal{L} &= \partial^\nu W_{\mu\nu} \partial_\alpha W^{\mu\alpha} + \frac{m^2}{2} \left( W_{\mu\nu} W^{\mu\nu} - \frac{W^2}{D-1} \right) \\ &= -(-\partial_0 W_{00} + \partial_i W_{0i})^2 + (-\partial_0 W_{0i} + \partial_j W_{ij})^2 \\ &\quad + \frac{m^2}{2} \left( W_{\mu\nu} W^{\mu\nu} - \frac{W^2}{D-1} \right),\end{aligned}\quad (\text{A1})$$

we can calculate the conjugated momenta:

$$\begin{aligned}\pi^{00} &= \frac{\partial \mathcal{L}}{\partial(\partial_0 W_{00})} \approx 2(-\partial_0 W_{00} + \partial^i W_{0i}) \longrightarrow \partial_0 W_{00} \\ &= -\frac{\pi^{00}}{2} + \partial^i W_{0i},\end{aligned}\quad (\text{A2})$$

$$\begin{aligned}\pi^{0i} &= \frac{\partial \mathcal{L}}{\partial(\partial_0 W_{0i})} \approx -2(-\partial_0 W_{0i} + \partial^j W_{ij}) \longrightarrow \partial_0 W_{0i} \\ &= \frac{\pi^{0i}}{2} + \partial^j W_{ij},\end{aligned}\quad (\text{A3})$$

$$\pi^{ij} = \frac{\partial \mathcal{L}}{\partial(\partial_0 W_{ij})} \approx 0; \quad i \leq j. \quad (\text{A4})$$

In (A4) we have  $\frac{D(D-1)}{2}$  primary constraints. Now, we can calculate the canonical Hamiltonian:

$$H_c = \int d^{D-1}x [\pi^{00} \partial_0 W_{00} + \pi^{0i} \partial_0 W_{0i} + \pi^{ij} \partial_0 W_{ij} - \mathcal{L}], \quad (\text{A5})$$

$$\begin{aligned}H_c &= \int d^{D-1}x \left[ -\frac{(\pi^{00})^2}{4} + \pi^{00} \partial^i W_{0i} + \frac{(\pi^{0i})^2}{4} \right. \\ &\quad \left. + \pi^{0i} \partial^j W_{ij} - \frac{m^2}{2} \left( W_{\mu\nu} W^{\mu\nu} - \frac{W^2}{D-1} \right) \right].\end{aligned}\quad (\text{A6})$$

The total Hamiltonian is

$$H_t = H_c + \int d^{D-1}x \lambda_{ij} \pi^{ij}, \quad (\text{A7})$$

where the sums above exist only for  $i \leq j$ . Using the canonical Poisson brackets for the  $D(D-1)$  phase space variables,

$$\begin{aligned}\{W_{\mu\nu}(x), W^{\rho\lambda}(y)\} &= 0; \quad \{\pi_{\mu\nu}(x), \pi^{\rho\lambda}(y)\} = 0; \\ \{W_{\mu\nu}(x), \pi^{\rho\lambda}(y)\} &= \delta_\mu^\rho \delta_\nu^\lambda \delta(x-y).\end{aligned}\quad (\text{A8})$$

We have the secondary constraints:

$$\chi^{lk} \equiv \dot{\pi}^{lk} = \partial^k \pi^{0l} + \partial^l \pi^{0k} + 2m^2 W_{lk} \approx 0; \quad l < k, \quad (\text{A9})$$

$$\begin{aligned}\chi^k &\equiv \dot{\pi}^{kk} = \partial^k \pi^{0k} + m^2 \left( W_{kk} - \frac{W^2}{D-1} \right) \approx 0; \\ k &= 1, 2, \dots, D-1.\end{aligned}\quad (\text{A10})$$

In (A10) there is no sum over  $k$ . The consistency conditions  $\dot{\chi}^{lk} = \{\chi^{lk}, H_t\} \approx 0$  fix the coefficients  $\lambda_{lk}$ ,  $l < k$  and do not generate new constraints, while  $\dot{\chi}^k = 0$  determines all  $\lambda_{kk}$  (no sum) except the sum  $\lambda_{jj}$ . This can be easily seen from the combination

$$\chi \equiv \sum_{k=1}^{D-1} \chi^k = \partial^j \pi^{0j} + m^2 W_{00} \approx 0. \quad (\text{A11})$$

The consistency equation  $\dot{\chi} \approx 0$  leads to another (tertiary) constraint:

$$\dot{\chi} \approx \left( \nabla^2 - \frac{m^2}{2} \right) \pi^{00} - m^2 \partial^i W_{0i} \equiv \phi \approx 0, \quad (\text{A12})$$

where  $\nabla^2 = \partial_j \partial^j$ . Finally, rewriting  $\dot{\phi} \approx 0$  with the help of (A10) and (A11) we have

$$W = -W_{00} + W_{jj} \approx 0. \quad (\text{A13})$$

From  $W \approx 0$  the constraint  $\chi^k \approx 0$  can be written as (no sum)

$$\partial^k \pi^{0k} + m^2 W_{kk} \approx 0, \quad k = 1, 2, \dots, D-1. \quad (\text{A14})$$

In summary, adding up (A4), (A9), (A10), (A12), and (A14) we have  $D(D-1) + 2$  independent second class constraints. Thus, we end up with  $2(D-1)$  unconstrained phase space variables which correspond to  $(D-1)$  degrees of freedom as expected for a massive spin-1 particle in  $D$  dimensions. In particular, we can eliminate all variables in term of  $(W_{0k}, \pi^{0j})$ . Back in the total Hamiltonian, after some cancellations we have

$$\begin{aligned}H_t &= \int d^{D-1}x \left[ m^2 \frac{\partial^j W_{0j} (\nabla^2 - 3m^2/4) \partial^j W_{0j}}{(\nabla^2 - \frac{m^2}{2})^2} \right. \\ &\quad \left. + m^2 W_{0j} W_{0j} + \frac{(\pi^{0j})^2}{4} + \frac{\sum_{j < k} (\partial^j \pi^{0k} - \partial^k \pi^{0j})^2}{4m^2} \right].\end{aligned}\quad (\text{A15})$$

Decomposing  $W_{0j} = W_j^T + W_j^L$ , where

$$W_j^T = \theta_{jk} W_{0k}; \quad W_j^L = \omega_{jk} W_{0k}, \quad (\text{A16})$$

we can rewrite  $H_t$  in a explicitly definite positive form as expected from our proof of unitarity

$$H_t = \int d^{D-1}x \left[ \frac{m^4}{4} \frac{W_j^L(m^2 - \nabla^2)W_j^L}{(\nabla^2 - \frac{m^2}{2})^2} + m^2 W_j^T W_j^T \right. \\ \left. + \frac{(\pi^{0j})^2}{4} + \frac{\sum_{j < k} (\partial^j \pi^{0k} - \partial^k \pi^{0j})^2}{4m^2} \right]. \quad (\text{A17})$$

The positiveness of the first term in (A17) can be checked by integrating by parts the factor  $\nabla^2$  in the numerator or by going to the momentum space (Fourier transform). Notice that the poles at  $\nabla^2 = 0$  present in the projection operators  $\theta_{ij}$  and  $\omega_{ij}$  cancel out in (A17) in agreement with (A15).

We have checked that, by using the appropriate Dirac brackets, the reduced phase space Hamiltonian (A17) leads indeed to the equations of motion (43).

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