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Black string corrections in variable tension braneworld scenarios

Roldão da Rocha*

Centro de Matemática, Computação e Cognição, Universidade Federal do ABC (UFABC) 09210-170, Santo André, SP, Brazil

J. M. Hoff da Silva†

Departamento de Física e Química, Universidade Estadual Paulista, Av. Dr. Ariberto Pereira da Cunha, 333, Guaratinguetá, SP, Brazil

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Braneworld models with variable tension are investigated, and the corrections on the black string horizon along the extra dimension are provided. Such corrections are encrypted in additional terms involving the covariant derivatives of the variable tension on the brane, providing profound consequences concerning the black string horizon variation along the extra dimension, near the brane. The black string horizon behavior is shown to be drastically modified by the terms corrected by the brane variable tension. In particular, a model motivated by the phenomenological interesting case regarding Eötvös branes is investigated. It forthwith provides further physical features regarding variable tension braneworld scenarios, heretofore concealed in all previous analysis in the literature. All precedent analysis considered uniquely the expansion of the metric up to the second order along the extra dimension, which is able to evince solely the brane variable tension absolute value. Notwithstanding, the expansion terms aftermath, further accomplished in this paper from the third order on, elicits the successive covariant derivatives of the brane variable tension, and their respective coupling with the extrinsic curvature, the Weyl tensor, and the Riemann and Ricci tensors, as well as the scalar curvature. Such additional terms are shown to provide sudden modifications in the black string horizon in a variable tension braneworld scenario.

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I. INTRODUCTION

Braneworld models perform a distinct branch of high-energy physics. Several of the main ideas concerning such models were inspired in formal advances in string theory [1]. It is possible to delineate the precursory works dealing with the possibility of a braneworld universe [2], and nowadays there is a great variety of alternative models dealing with several aspects regarding braneworlds. Motivated by a comprehensive program regarding gravity and black strings on braneworld scenarios, and subsequent generalizations probing extra-dimensional features [3–10], some gravitational aspects concerning black strings, arising from braneworld models with variable brane tension, are introduced and widely investigated. In order to analyze a conceivable gravitational effect, arising genuinely from the brane variable tension, we analyze the corrections on the black string time-dependent horizon, in such a scenario. This formalism provides manageable models and their possible ramifications into some aspects of gravity in this context, cognizable corrections, and physical effects as well. Besides the black string behavior being sharply modified by the variable tension brane localized at $y = 0$ (the extra dimension is denoted by y), unexpected additional terms in a Taylor expansion of the metric along the extra dimension are elicited. This Taylor expansion regards the black string horizon behavior along the extra dimension. Taking into account a variable tension brane, this expansion involves the covariant derivatives of the brane tension—besides its Lie derivative along the extra dimension likewise. Such expansion, for instance, was used in [11] to investigate the gravitational collapse of compact objects in braneworld scenarios.

Braneworld models, wherein a brane is endowed with variable tension $\lambda = \lambda(x^\mu)$, are a prominent approach as an attempt to ascertain new signatures coming from high-energy physics [12]. In fact, due to the drastic modification of the temperature of the Universe along its cosmological evolution, a variable tension braneworld scenario is indeed demanded. The full covariant variable tension brane dynamics was established in Ref. [13], and further explored in [14]. Moreover, the variable tension was implemented in the braneworld model consisting of two branes [15,16], also in the context of scalar tensor bulk gravity [17]. Furthermore, the cosmological evolution of the Universe was investigated in a particular model in which the brane tension has an exponential dependence with the scale factor [18].

In this paper, we are concerned with analyzing the information about the black string behavior, evinced by a time variable tension on the brane, delving into the Taylor expansion outside a black hole metric along the extra dimension, where the corrections in the area of the five-dimensional black string horizon are elicited. It is shown how the variable tension and its covariant derivatives determine the variation in the area of the black string

*roldao.rocha@ufabc.edu.br
†hoff@feg.unesp.br;hoff@ift.unesp.br

horizon along the extra dimension. It induces interesting physical effects, for instance, the preclusion of the horizon decreasing, when the brane tension varies in a model physically motivated by Eötvös law.

Our program throughout this paper explicitly consists of the following: the next section provides the black string behavior along the extra dimension, studying the variation in the black string horizon due to terms including the covariant derivatives of the variable brane tension. In particular, it is considered the Eötvös phenomenological brane case, and the Lie derivatives of λ along the extra dimension are identically zero and therefore concealed in the subsequent analysis. Some results [19] are extended in order to encompass this case. The fine character of the expansion along the extra dimension is crucial to analyze variable tension models. The Taylor expansion along the extra dimension, concerning variable tension braneworld models, provides terms of superior order and brings some more precise information about the behavior of the black string than in models with brane constant tension. Furthermore, when variable tension braneworld models are taken into account, the extra terms—from the third order on—probe physical features concerning the variable tension. For instance, such terms induce the horizon to decrease in a different rate along the extra dimension, when we analyze the physical concrete example of Eötvös branes. In Sec. III, we delve into the corrections for the Schwarzschild case analysis, and to this point the framework is completely general. In Sec. IV, the specific and physically motivated example, regarding Eötvös branes endowed with a de Sitter-like scale factor variable tension, is investigated.

II. BLACK STRING BEHAVIOR ALONG THE EXTRA DIMENSION

In this section, the comprehensive formalism in [19] is briefly introduced and reviewed. Hereon, $\{\theta_\mu\}$, $\mu = 0, 1, 2, 3$ $[\{\theta_A\}, A = 0, 1, 2, 3, 4]$ denotes a basis for the cotangent space T_x^*M at a point x in a 3-brane M embedded in a bulk. If a local coordinate chart is chosen, it is possible to represent $\theta^A = dx^A$. Take now $n = n^A e_A$, a timelike vector orthogonal to T_x^*M, and let y be the associated Gaussian coordinate, indicating how an observer upheavals out the brane into the bulk. In particular, $n_A dx^A = dy$ in the hyper-surface defined by $y = 0$. A vector field $v = x^A e_A$ in the bulk is split into components in the brane and orthogonal to the brane, respectively, as $v = x^\mu e_\mu + y e_4 = (x^\mu, y)$. The bulk is endowed with a metric $\mathring{g}_{AB} dx^A dx^B = g_{\mu\nu}(x^\alpha, y) dx^\mu dx^\nu + dy^2$. The brane metric components $g_{\mu\nu}$ and the bulk metric are related by $\mathring{g}_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$, since according to the notation in [19], as $g_{44} = 1$ and $g_{i4} = 0$, the bulk indices A, B effectively run from 0 to 3.

It is well know that the four-dimensional gravitational constant is an effective coupling constant inherited from the fundamental coupling constant, and the four-dimensional cosmological constant is nonzero when the balance between the bulk cosmological constant and the brane tension—provided by the Randall-Sundrum braneworld model [20]—is broken [19]:

$$\kappa_4^2 = \frac{1}{6}\lambda\kappa_5^4, \qquad \Lambda_4 = \frac{\kappa_5^2}{2}\left(\Lambda_5 + \frac{1}{6}\kappa_5^2\lambda^2\right), \tag{1}$$

where Λ_4 is the effective brane cosmological constant, κ_5 $[\kappa_4]$ denotes the five-dimensional [four-dimensional] gravitational coupling, and λ is the brane tension. The extrinsic curvature $K_{\mu\nu} = \frac{1}{2}\mathcal{L}_{\mathbf{n}} g_{\mu\nu}$ (hereon $\mathcal{L}_{\mathbf{n}}$ denotes the Lie derivative, which in Gaussian normal coordinates reads $\mathcal{L}_{\mathbf{n}} = \partial/\partial y$). The junction condition determines the extrinsic curvature on the brane, as

$$K_{\mu\nu} = -\tfrac{1}{2}\kappa_5^2[T_{\mu\nu} + \tfrac{1}{3}(\lambda - T)g_{\mu\nu}], \tag{2}$$

where $T = T^\mu{}_\mu$. In what follows, we denote $K = K_\mu{}^\mu$ and $K^2 = K_{\alpha\beta}K^{\alpha\beta}$.

Given the five-dimensional Weyl tensor

$$C_{\mu\nu\sigma\rho} = R_{\mu\nu\sigma\rho} - \tfrac{2}{3}(\mathring{g}_{[\mu\sigma}R_{\nu]\rho} + \mathring{g}_{[\nu\rho}gR_{\mu]\sigma}) - \tfrac{1}{6}R(g_{\mu[\sigma}g_{\nu\rho]}), \tag{3}$$

where $R_{\mu\nu\sigma\rho}$ denotes the components of the bulk Riemann tensor ($R_{\mu\nu}$ and R obviously are the associated Ricci tensor and the scalar curvature), the symmetric and trace-free components, respectively, denoted by $\mathcal{E}_{\mu\nu}$ and $\mathcal{B}_{\mu\nu\alpha}$ are known as the electric and magnetic Weyl tensor components—given by $\mathcal{E}_{\mu\nu} = C_{\mu\nu\sigma\rho}n^\sigma n^\rho$ and $\mathcal{B}_{\mu\nu\alpha} = g_\mu^\rho g_\nu^\sigma C_{\rho\sigma\alpha\beta}n^\beta$. The Weyl tensor represents the part of the curvature that is not determined locally by matter, and the field equations for the Weyl tensor are simulated by Bianchi identities, determining the part of the spacetime curvature that depends on the matter distribution at other points [21].

The effective field equations are complemented by a set of equations that are forthwith obtained from the five-dimensional Einstein and Bianchi equations [22]. Those equations are also obtained in [19,23], as well as in a brane with variable tension context [13]. All approaches are shown to be completely similar and hereon the following effective field equations are considered:

$$\mathcal{L}_{\mathbf{n}}K_{\mu\nu} = K_{\mu\alpha}K^\alpha{}_\nu - \mathcal{E}_{\mu\nu} - \tfrac{1}{6}\Lambda_5 g_{\mu\nu}, \tag{4}$$

$$\begin{aligned}\mathcal{L}_{\mathbf{n}}\mathcal{E}_{\mu\nu} = {}& \nabla^\alpha\mathcal{B}_{\alpha(\mu\nu)} + \tfrac{1}{6}\Lambda_5(K_{\mu\nu} - g_{\mu\nu}K) \\ &+ K^{\alpha\beta}R_{\mu\alpha\nu\beta} + 3K^\alpha{}_{(\mu}\mathcal{E}_{\nu)\alpha} - K\mathcal{E}_{\mu\nu} \\ &+ (K_{\mu\alpha}K_{\nu\beta} - K_{\alpha\beta}K_{\mu\nu})K^{\alpha\beta},\end{aligned} \tag{5}$$

$$\mathcal{L}_{\mathbf{n}}\mathcal{B}_{\mu\nu\alpha} = -2\nabla_{[\mu}\mathcal{E}_{\nu]\alpha} + K_\alpha{}^\beta\mathcal{B}_{\mu\nu\beta} - 2\mathcal{B}_{\alpha\beta[\mu}K_{\nu]}{}^\beta, \tag{6}$$

$$\mathcal{L}_{\mathbf{n}}R_{\mu\nu\alpha\beta} = -2R_{\mu\nu\gamma[\alpha}K_{\beta]}{}^\gamma - \nabla_\mu\mathcal{B}_{\alpha\beta\nu} + \nabla_\mu\mathcal{B}_{\beta\alpha\nu}. \tag{7}$$

These equations are to be solved subject to the boundary condition $\mathcal{B}_{\mu\nu\alpha} = 2\nabla_{[\mu}K_{\nu]\alpha}$ at the brane [19].

Those expressions are subsequently used in order to explicitly calculate the terms of the Taylor expansion of the metric along the extra dimension. Such expansion provides the black string profile and some physical consequences. The equations above were used to develop a covariant analysis of the weak field [22], and are used to develop a Taylor expansion of the metric along the extra dimension, which predicts the black string behavior. We intend here to perform the calculations beyond second-order term in the expansion along the extra dimension. Solely in this way, it is possible to realize how the variable tension generates additional terms concerning the covariant derivatives of the brane variable tension.

The standard Taylor expansion along the extra dimension y is given by the expression

$$\begin{aligned} g_{\mu\nu}(x,y) &= g_{\mu\nu}(x,0) + (\mathcal{L}_{\mathbf{n}} g_{\mu\nu}(x,y))|_{y=0}|y| \\ &\quad + (\mathcal{L}_{\mathbf{n}}(\mathcal{L}_{\mathbf{n}} g_{\mu\nu}(x,y)))|_{y=0}\frac{|y|^2}{2!} \\ &\quad + (\mathcal{L}_{\mathbf{n}}(\mathcal{L}_{\mathbf{n}}(\mathcal{L}_{\mathbf{n}} g_{\mu\nu}(x,y))))|_{y=0}\frac{|y|^3}{3!} + \cdots \\ &\quad + (\mathcal{L}_{\mathbf{n}}^k(g_{\mu\nu}(x,y))|_{y=0}\frac{|y|^k}{k!} + \cdots. \end{aligned} \tag{8}$$

The term in the first-order $|y|$ above is immediately calculated by the definition of the extrinsic curvature $K_{\mu\nu} = \frac{1}{2}\mathcal{L}_{\mathbf{n}} g_{\mu\nu}$ and by the junction condition (2). The term in y^2 is proportional to $\mathcal{L}_{\mathbf{n}} K_{\mu\nu}$, which is given by Eq. (4), and in order to calculate the term $K_{\mu\alpha}K^\alpha{}_\nu$ in Eq. (4), the junction condition (2) is used forthwith. Furthermore, the coefficient term of $|y|^3$ in Eq. (8) can be expressed as

$$\begin{aligned} &2\mathcal{L}_{\mathbf{n}}(\mathcal{L}_{\mathbf{n}} K_{\mu\nu})\frac{|y|^3}{3!} \\ &\quad = \left(\mathcal{L}_{\mathbf{n}}(K_{\mu\alpha}K^\alpha{}_\nu) - \mathcal{L}_{\mathbf{n}}\mathcal{E}_{\mu\nu} - \frac{\Lambda_5}{6}\mathcal{L}_{\mathbf{n}} g_{\mu\nu}\right)\frac{|y|^3}{3}, \end{aligned} \tag{9}$$

where Eq. (4) was used. As the Lie derivatives terms in the right-hand side of this last expression are, respectively, given by Eq. (4) [where the Leibniz rule $\mathcal{L}_{\mathbf{n}}(K_{\mu\alpha}K^\alpha{}_\nu) = \mathcal{L}_{\mathbf{n}}(K_{\mu\alpha})K^\alpha{}_\nu + K_{\mu\alpha}\mathcal{L}_{\mathbf{n}}(K^\alpha{}_\nu)$ is employed], by Eq. (5), and by the definition of the extrinsic curvature, one arrives further at the expression for $|y|^3$ in Eq. (8). Finally, the term in y^4 is obtained when the Lie derivative of the right-hand side of Eq. (9) is taken into account, as well as Eqs. (4)–(7) again.

Explicitly, up to fourth order in the extra dimension, the Taylor expansion is given by [hereon we denote $g_{\mu\nu}(x,0) = g_{\mu\nu}$]:

$$\begin{aligned} g_{\mu\nu}(x,y) &= g_{\mu\nu}(x,0) - \kappa_5^2\left[T_{\mu\nu} + \frac{1}{3}(\lambda - T)g_{\mu\nu}\right]|y| + \left[-\mathcal{E}_{\mu\nu} + \frac{1}{4}\kappa_5^4\left(T_{\mu\alpha}T^\alpha{}_\nu + \frac{2}{3}(\lambda - T)T_{\mu\nu}\right)\right. \\ &\quad \left. + \frac{1}{6}\left(\frac{1}{6}\kappa_5^4(\lambda - T)^2 - \Lambda_5\right)g_{\mu\nu}\right]y^2 + [2K_{\mu\beta}K^\beta{}_\alpha K^\alpha{}_\nu - (\mathcal{E}_{\mu\alpha}K^\alpha{}_\nu + K_{\mu\alpha}\mathcal{E}^\alpha{}_\nu) - \frac{1}{3}\Lambda_5 K_{\mu\nu} - \nabla^\alpha\mathcal{B}_{\alpha(\mu\nu)} \\ &\quad + \frac{1}{6}\Lambda_5(K_{\mu\nu} - g_{\mu\nu}K) + K^{\alpha\beta}R_{\mu\alpha\nu\beta} + 3K^\alpha{}_{(\mu}\mathcal{E}_{\nu)\alpha} - K\mathcal{E}_{\mu\nu} + (K_{\mu\alpha}K_{\nu\beta} - K_{\alpha\beta}K_{\mu\nu})K^{\alpha\beta} - \frac{\Lambda_5}{3}K_{\mu\nu}\Big]\frac{|y|^3}{3!} \\ &\quad + \left[\frac{\Lambda_5}{6}\left(R - \frac{\Lambda_5}{3} + K^2\right)g_{\mu\nu} + \left(\frac{K^2}{3} - \Lambda_5\right)K_{\mu\alpha}K^\alpha{}_\nu + (R - \Lambda_5 + 2K^2)\mathcal{E}_{\mu\nu} + (K^\alpha{}_\sigma K^{\sigma\beta} + \mathcal{E}^{\alpha\beta} + KK^{\alpha\beta})R_{\mu\alpha\nu\beta}\right. \\ &\quad - \frac{1}{6}\Lambda_5 R_{\mu\nu} + 2K_{\mu\beta}K^\beta{}_\sigma K^\sigma{}_\alpha K^\alpha{}_\nu + K_{\sigma\rho}K^{\sigma\rho}KK_{\mu\nu} + \mathcal{E}_{\mu\alpha}\left(K_{\nu\beta}K^{\alpha\beta} - 3K^\alpha{}_\sigma K^\sigma{}_\nu + \frac{1}{2}KK^\alpha{}_\nu\right) \\ &\quad + \left(\frac{7}{2}KK^\alpha{}_\mu - 3K^\alpha{}_\sigma K^\sigma{}_\mu\right)\mathcal{E}_{\nu\alpha} - \frac{13}{2}K_{\mu\beta}\mathcal{E}^\beta{}_\alpha K^\alpha{}_\nu + (3K^\alpha{}_\mu K^\beta{}_\alpha - K_{\mu\alpha}K^{\alpha\beta})\mathcal{E}_{\nu\beta} \\ &\quad - K_{\mu\alpha}K_{\nu\beta}\mathcal{E}^{\alpha\beta} - 4K^{\alpha\beta}R_{\mu\nu\gamma\alpha}K^\gamma_\beta - \frac{7}{6}K^{\sigma\beta}K^\alpha_\mu R_{\nu\sigma\alpha\beta}]\frac{y^4}{4!} + \cdots. \end{aligned} \tag{10}$$

Such expansion was analyzed in [19] only up to the second order. This procedure does not suffice to evince the additional terms, arising from the variable tension in the brane. For an alternative method that does not take into account the $\mathbb{Z}_2$ symmetry, and some subsequent applications, see [24].

There are still some additional terms that were concealed in the expression above, that are going to be emphatically focused on the next sections, concerning the derivatives of the variable tension. The additional terms coming from the variable brane tension are shown to be essential for the subsequent analysis on the brane tension influence on the black string behavior along the extra dimension.

A. Additional terms elicited from the variable tension

Although the brane tension is variable, up to terms in y^2 at the expansion (10) there are no additional terms, which appear only from the order $|y|^3$ on, regarding (10). The unique term to contribute to the derivatives of the variable tension λ comes from the coefficient $\mathcal{L}_{\mathbf{n}}\mathcal{E}_{\mu\nu}$ of $|y|^3$ in (10), given by $\nabla^\alpha\mathcal{B}_{\alpha(\mu\nu)}$, wherein one can substitute the boundary condition $\mathcal{B}_{\mu\nu\alpha} = 2\nabla_{[\mu}K_{\nu]\alpha}$ [19,22]. Apart from the term related to the energy-momentum tensor in $K_{\mu\nu} = -\frac{1}{2}\kappa_5^2[T_{\mu\nu} + \frac{1}{3}(\lambda - T)g_{\mu\nu}]$, since we are concerned only about the extra terms coming from the variable tension, such extra terms read

$$\nabla^\alpha\mathcal{B}_{\alpha(\mu\nu)} = -\tfrac{2}{3}\kappa_5^2((\nabla^\alpha\nabla_\alpha\lambda)g_{\mu\nu} - (\nabla_{(\nu}\nabla_{\mu)}\lambda)). \tag{11}$$

Terms in order y^4 can be obtained immediately, besides the additional terms coming from the variable tension brane, since we are interested on the further effects of the variable brane tension on the black string character. Such extra terms in the order y^4 of expansion (10) are given by

$$\begin{aligned}
&-\tfrac{1}{3}\kappa_5^2[\Box(\Box\lambda)g_{\mu\nu} - \nabla_{(\nu}\nabla_{\mu)}(\Box\lambda)] + \left(\tfrac{1}{3}\kappa_5^2 + 2K\right)[(\Box\lambda)\mathcal{E}_{(\mu\nu)} - \nabla^\alpha((\nabla_{(\mu}\lambda)\mathcal{E}_{\nu)\alpha})] + 6[(\Box\lambda)K_{(\mu\sigma}\mathcal{E}^\sigma_{\nu)} - \nabla^\alpha((\nabla_{(\mu}\lambda)\mathcal{E}_{\nu)\alpha})]\\
&\quad + 2\left(K + \tfrac{7}{3}\kappa_5^2\right)[(\Box\lambda)KK_{\mu\nu} - \nabla^\alpha((\nabla_{(\mu}\lambda)KK_{\alpha\nu)})] + \tfrac{1}{3}\kappa_5^2[(\Box\lambda)R_{\mu\nu} - \nabla^\alpha((\nabla_{(\mu}\lambda)R_{\alpha\nu)})]\\
&\quad - 2K^{\sigma\beta}[(\Box\lambda)R_{(\mu\sigma\nu)\beta} - \nabla^\alpha((\nabla_{(\mu}\lambda)R_{\alpha\sigma\nu)\beta})] + \left(2K_{\sigma\rho}K^{\sigma\rho} - \tfrac{1}{3}\Lambda_5\right)[(\Box\lambda)g_{\mu\nu} - \nabla_{(\nu}\nabla_{\mu)}\lambda]\\
&\quad + \tfrac{1}{3}\kappa_5^2[(\Box\lambda)(K_{(\mu\sigma}K_{\nu)\beta}K^{\sigma\beta} - (K_{\sigma\rho}K^{\sigma\rho})K_{(\mu\nu)}) - \nabla^\alpha((\nabla_{(\mu}\lambda)(K_{\alpha\sigma}K_{\nu)}{}^{\sigma} - KK_{\alpha\nu)}))]
\end{aligned} \tag{12}$$

and are obtained when one substitutes the Lie derivative of each term of such equation in the expression for $\mathcal{L}_{\mathbf{n}}(\mathcal{L}_{\mathbf{n}}\mathcal{E}_{\mu\nu})$, taking into account once more Eqs. (4)–(7). We shall consider in this paper the brane tension as a time-dependent function $\lambda = \lambda(t)$.

B. Black string corrections for the vacuum case

Now, we focus on the situation where Eqs. (11) and (12) provide the further extra terms in the expansion given by Eq. (10) in the vacuum on the brane. In this case, $T_{\mu\nu} = 0$, and consequently Eq. (2), is led to

$$K_{\mu\nu} = -\tfrac{1}{6}\kappa_5^2\lambda g_{\mu\nu}, \tag{13}$$

which provides the expansion at Eq. (10) to be written straightforwardly as

$$\begin{aligned}
g_{\mu\nu}(x,y) = {}& g_{\mu\nu} - \frac{1}{3}\kappa_5^2\lambda g_{\mu\nu}|y| + \left[-\mathcal{E}_{\mu\nu} + \left(\frac{1}{36}\kappa_5^4\lambda^2 - \frac{1}{6}\Lambda_5\right)g_{\mu\nu}\right]y^2 + \left(\left(-\frac{193}{216}\lambda^3\kappa_5^6 - \frac{5}{18}\Lambda_5\kappa_5^2\lambda\right)g_{\mu\nu}\right.\\
&\left. + \frac{1}{6}\kappa_5^2\mathcal{E}_{\mu\nu} + \frac{1}{3}\kappa_5^2(\mathcal{E}_{\mu\nu} + R_{\mu\nu})\right)\frac{|y|^3}{3!} + \left[\frac{1}{6}\Lambda_5\left(\left(R - \frac{1}{3}\Lambda_5 - \frac{1}{18}\lambda^2\kappa_5^4\right) + \frac{7}{324}\lambda^4\kappa_5^8\right)g_{\mu\nu}\right.\\
&\left. + \left(R - \Lambda_5 + \frac{19}{36}\lambda^2\kappa_5^4\right)\mathcal{E}_{\mu\nu} + \left(\frac{37}{216}\lambda^2\kappa_5^4 - \frac{1}{6}\Lambda_5\right)R_{\mu\nu} + \mathcal{E}^{\alpha\beta}R_{\mu\alpha\nu\beta}\right]\frac{y^4}{4!} + \cdots.
\end{aligned} \tag{14}$$

As the coefficients above concern uniquely quantities in the brane, we evince the property that when there is vacuum on the brane the brane field equations

$$R_{\mu\nu} = -\mathcal{E}_{\mu\nu}, \qquad R^\mu_\mu = 0 = \mathcal{E}^\mu{}_\mu, \qquad \nabla^\nu\mathcal{E}_{\mu\nu} = 0 \tag{15}$$

hold. It induces the last term of $|y|^3$ in Eq. (14) equals zero. Hence, Eq. (14) is written as

$$\begin{aligned}
g_{\mu\nu}(x,y) = {}& g_{\mu\nu} - \frac{1}{3}\kappa_5^2\lambda g_{\mu\nu}|y| + \left[-\mathcal{E}_{\mu\nu} + \left(\frac{1}{36}\kappa_5^4\lambda^2 - \frac{1}{6}\Lambda_5\right)g_{\mu\nu}\right]y^2 - \left(\left(\frac{193}{216}\lambda^3\kappa_5^6 + \frac{5}{18}\Lambda_5\kappa_5^2\lambda\right)g_{\mu\nu} + \frac{1}{6}\kappa_5^2R_{\mu\nu}\right)\frac{|y|^3}{3!}\\
& + \left[\frac{1}{6}\Lambda_5\left(\left(R - \frac{1}{3}\Lambda_5 - \frac{1}{18}\lambda^2\kappa_5^4\right) + \frac{7}{324}\lambda^4\kappa_5^8\right)g_{\mu\nu} + \left(R + \frac{5}{6}\Lambda_5 - \frac{77}{216}\lambda^2\kappa_5^4\right)R_{\mu\nu} - R^{\alpha\beta}R_{\mu\alpha\nu\beta}\right]\frac{y^4}{4!} + \cdots.
\end{aligned} \tag{16}$$

Furthermore, in the vacuum Eq. (12)—corresponding to the additional terms arising from the derivatives of variable brane tension λ—is led to

$$-\tfrac{1}{3}\kappa_5^2(\Box(\Box\lambda)g_{\mu\nu} - \nabla_{(\nu}\nabla_{\mu)}\Box\lambda) + \left(-\tfrac{1}{3}\Lambda_5 + \tfrac{8}{9}\lambda^2\kappa_5^4\right)((\Box\lambda)g_{\mu\nu} - \nabla_{(\nu}\nabla_{\mu)}\lambda) + \tfrac{1}{9}\lambda^2\kappa_5^4(\lambda g_{\mu\nu} + \nabla_{(\nu}\nabla_{\mu)}\lambda)$$
$$-\tfrac{4}{3}\kappa_5^2[(\Box\lambda)\mathcal{E}_{(\mu\nu)} - \nabla^\alpha((\nabla_{(\mu}\lambda)\mathcal{E}_{\nu)\alpha})] + \tfrac{1}{3}\kappa_5^2(\Box\lambda)\left(\tfrac{5}{216}\lambda^3\kappa_5^6 g_{\mu\nu}\right) + \tfrac{1}{3}\lambda\kappa_5^2[(\Box\lambda)R_{(\mu\nu)}$$
$$-\nabla^\alpha((\nabla_{(\mu}\lambda)R_{\alpha\nu)})] + 6[(\Box\lambda)K_{(\mu\sigma}\mathcal{E}^{\sigma}_{\nu)} - \nabla^\alpha(\nabla_{(\mu}\lambda)\mathcal{E}_{\nu)\alpha}],$$

where on the brane $R_{\mu\nu} = -\mathcal{E}_{\mu\nu}$ holds as one of the field equations in (15).

III. BLACK STRING SCHWARZSCHILD CORRECTIONS

As the case of interest to be investigated is exactly the corrections on the black string horizon along the extra dimension, we shall focus on the term $g_{\theta\theta}$—corresponding to the square of the black string horizon—of the expansion at Eq. (10). Clearly, a time-dependent brane tension shall modify the black string Schwarzschild background. The complete solution is, however, hugely difficult to accomplish. Therefore, we adopt an effective approach, studying the horizon variation with tension variation corrections only in the Taylor expansion. As shall be shown, even in this approximative case interesting results are accomplished.

A static spherical metric on the brane can be expressed as

$$g_{\mu\nu}dx^\mu dx^\nu = -F(r)dt^2 + (H(r))^{-1}dr^2 + r^2 d\Omega^2, \quad (18)$$

where $d\Omega^2$ denotes the line element of a two-dimensional unit sphere. The projected Weyl term on the brane is given by [19]

$$\mathcal{E}_{\theta\theta} = -1 + H + \frac{r}{2}H\left(\frac{F'}{F} + \frac{H'}{H}\right) = 0, \quad (19)$$

for the Schwarzschild metric $F(r) = H(r) = (1 - \frac{2M}{r})$, where we denote $M \mapsto GM/c^2$. As the black string horizon variation along the extra dimension is analyzed, the term $g_{\theta\theta}(x, y)$ in (14) above is given by

$$\begin{aligned} g_{\theta\theta}(x, y) = {} & r^2 - \frac{r^2}{3}\kappa_5^2\lambda|y| + \left(\frac{1}{36}\kappa_5^4\lambda^2 - \frac{1}{6}\Lambda_5\right)r^2y^2 \\ & - \left(\frac{193}{216}\lambda^3\kappa_5^6 + \frac{5}{18}\Lambda_5\kappa_5^2\lambda\right)\frac{r^2|y|^3}{3!} \\ & - \frac{1}{18}\Lambda_5\left(\left(\Lambda_5 + \frac{1}{6}\lambda^2\kappa_5^4\right) + \frac{7}{324}\lambda^4\kappa_5^8\right) \\ & \times \frac{r^2y^4}{4!} + \cdots. \end{aligned} \quad (20)$$

Note that $g_{\theta\theta} = g_{\theta\theta}(x, y)$ coming from Eq. (14) is a component of the Taylor expanded metric along the extra dimension, while $g_{\mu\nu}$ in Eq. (16) depends only on the brane variables, as usual [19]. Now, as $\lambda = \lambda(t)$, the terms $\nabla^\alpha\nabla_\alpha\lambda$ for the Schwarzschild metric are computed and the additional terms in $|y|^3$ for $g_{\theta\theta}$ in Eq. (20) are given by

$$-\tfrac{2}{3}\kappa_5^2\lambda'' r^2, \quad (21)$$

where λ' denotes the derivative with respect to the time t—and the additional terms in y^4 for $g_{\theta\theta}$ are given by

$$\begin{aligned} & \left(1 - \frac{2M}{r}\right)\left[-\frac{1}{3}\kappa_5^2\left(1 - \frac{2M}{r}\right)\lambda'''' r^2 \right. \\ & \left. + \lambda'' r^2\left(\left(\frac{1}{3}\Lambda_5 - \frac{8}{9}\lambda^2\kappa_5^4\right) + \frac{5}{648}\lambda^3\kappa_5^8\right)\right] \\ & + \frac{1}{9}\lambda^2\kappa_5^4\lambda r^2. \end{aligned} \quad (22)$$

All the expressions obtained are so far the most general. In order to better understand the physical implications of a variable tension in the event horizon along the extra dimension, let us particularize our analysis to a specific physical motivated case.

A specific example regarding a brane variable tension

In general, there are two distinct approaches to implement a variable tension brane. On the one hand, one may realize the brane tension as a (fundamental) scalar field appearing in the Lagrangian. This comprehensive picture is widely assumed in the context of string theory [25] and supersymmetric branes [26]. Instead, to emulate such approach the brane tension may be understood as an intrinsic property of the brane as in, e.g., [13–16,18]. We delve into this point of view in our subsequent analysis.

From the braneworld picture, the functional form of the variable tension on the brane is an open issue. However, taking into account the huge variation of the Universe temperature during its cosmological evolution, it is indeed plausible to implement the brane tension as a variable function of spacetime coordinates. In particular, in such case as a function of time coordinate. Although it lacks a complete scenario still, the phenomenological interesting case regarding Eötvös standard fluid membranes [27] is useful to extract deep physical results.

The phenomenological Eötvös law asserts that the fluid membrane depends on the temperature as

$$\lambda = \chi(T_c - T), \quad (23)$$

where χ is a constant and T_c represents a critical temperature denoting the highest temperature for which the membrane exists. We heuristically depict in what follows how the brane tension varies, in full compliance with Eötvös membranes models.

If there are no stresses in the bulk—apart the cosmological constant—there is no exchange of energy momentum between the bulk and the brane [19]. Herewith, it is possible to assert that there is no exchange of heat between the brane and the bulk—the regarded interaction is purely gravitational. Therefore, the well-known expression $dQ = dE + pdV = 0$ holds for the brane, and concerning photons from the cosmological microwave background the explicit formulas $E = E_\gamma = \sigma T^4 V$ and $p = E/3 = \frac{\sigma T^4}{3} V$ accrue. Using these relations, it is straightforwardly verified that

$$\frac{1}{T}\frac{dT}{dt} = -\frac{1}{3V}\frac{dV}{dT}. \tag{24}$$

Now, relating the volume to the Friedmann-Robertson-Walker scale factor—$V = a^3(t)$—it follows that $T \sim \frac{1}{a(t)}$, and therefore one may use a physically motived input regarding the tension variable functions.

A complete cosmological setup must be forthwith obtained from the solution of the full cosmological brane equations taking into account variable tension, which is in general obviously a hard task. Notwithstanding, we are assuming that the variability of tension may be outlined by considering the relation $T \sim a^{-1}(t)$ valid in some approximation [14]. This conservative approach, although not the most complete one, is certainly in full compliance with the standard cosmological model. Therefore, considering the analogy to Eötvös membranes and the expression $T \sim a^{-1}(t)$ we have $\lambda = \lambda_0(1 - a_{\rm min}/a)$, where λ_0 is a constant related to the four-dimensional coupling constants [14], and $a_{\rm min}$ denotes the minimum scale factor under which the brane cannot exist, since its tension would become negative. For the qualitative analysis that we develop, it is sufficient to consider the general shape of λ given by

$$\lambda = 1 - \frac{1}{a(t)}, \tag{25}$$

with normalized tension and scale factor. In order to supplement our assumption in (25), we stress that this type of tension variation may be useful to reconcile supersymmetry and inflationary cosmology. Indeed, as first noticed in Ref. [14], the time-dependent brane tension gives rise to a time-dependent four-dimensional effective cosmological constant. In particular for a tension varying as Eq. (25), it is possible to show that

$$\Lambda_{4D} \sim 1 - \frac{1}{a(t)}\left(1 - \frac{1}{a(t)}\right). \tag{26}$$

Hence, the cosmological constant starts at high negative values and as the brane universe expands it converges to a positive small value. It is indeed a remarkable characteristic of this type of tension.[1]

To finalize, a de Sitter-like brane behavior is assumed, by setting $a \sim e^{\alpha t}$ with positive α [29], in such a way that

$$\lambda(t) = 1 - e^{-\alpha t}. \tag{27}$$

It is important, in view of the assumption (27), to assert few remarks concerning its phenomenological viability. It is well known that the projected gravitational constant depends linearly on the brane tension [19,22]. Therefore, a time variable tension engenders a variation on the Newtonian constant $G \sim \lambda(t)$. The variation of a dimensional constant may always be incorporated in a suitable redefinition of length, time, and energy [30]. Nevertheless, it is possible to pick up some arbitrary value defining it as the standard one, and study its possible fractional variation. The recent astrophysical data indicate the constancy of the Newtonian constant (for an up-to-date review see [31]). In fact, the best model-independent bound on $\dot{G}/G$ is given by lunar laser ranging measurements whose upper limit is $(4 \pm 9) \times 10^{-13}$ yr^{-1}. Hence, taking into account Eq. (27), the following condition,

$$e^{\alpha t} > 1 + 10^{13}\alpha, \tag{28}$$

must hold. In this qualitative analysis, we shall not fix the value of α, but as it is not desirable to have a huge departure from the standard scenario, it is expected to have $\alpha \sim H_0$, the Hubble parameter [19]. Thus, in principle, $10^{13}\alpha$ is not so far from 1. An important characteristic of the constraint (28) is that there is a particular time, say, $\bar{t}$, given by $\bar{t} = \ln(1 + 10^{13}\alpha)/\alpha$ below which this constraint is violated. This apparent difficulty may be circumvented by the fact that it is possible to introduce free parameters on the functional form of λ such that $\bar{t}$ is small enough, making the condition $t < \bar{t}$ to belong to an early time range, when the assumption $T \sim 1/a(t)$—possibly—no longer holds anymore. Besides, it is possible to interpret the violation of (28) as a change variation faster than 10^{-13} yr^{-1}, which could in principle be scrutinized at intermediate redshifts [32].

Having explicitly presented the type of variable tension, let us particularize our analysis taking into account Eq. (27). For the Eötvös brane scenario, the following figures show the black string behavior along the extra dimension y in the Schwarzschild picture, as predicted by Eqs. (20)–(22). The graphics below for each figure illustrate how the black string horizon varies along the extra dimension. Equation (20) is the landmark for all graphics hereon, which we imposed $r = 1 = \Lambda_5 = \kappa_5$, in order to make our analysis straightforward. It is clear that such rescaling does not affect the shape of the graphics below.

In Figs. 1–3 the black string horizon behavior is obviously different for differing values of αt. In Fig. 1, for the sake of completeness we depict the black string horizon behavior along the extra dimension, but now considering only terms up to the order y^2 in Eq. (10), commonly

[1] For another model presenting such a behavior of the effective cosmological constant see [28].

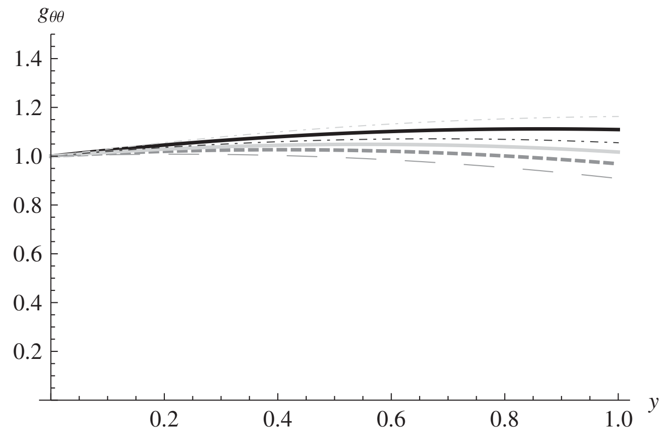

FIG. 1. Graphic of the brane effect-corrected black string horizon $g_{\theta\theta}$ for the Schwarzschild metric with variable tension, along the extra dimension y and also as function of the time t. The brane tension is given by $\lambda(t) = 1 - \exp(-\alpha t)$. We included, for comparison criteria, merely terms up to the order y^2 in the expansion (20). For the long-dashed gray line (lower curve): $\alpha t = 0.25$; for the dashed thick dark-gray line: $\alpha t = 0.5$; for the thick gray line: $\alpha t = 0.75$; for the dash-dotted line: $\alpha t = 1$; for the black line: $\alpha t = 1.5$; for the dash-dotted light-gray line: $\alpha t = 2.5$.

approached in the literature, for instance, in [19]. Figures 2 and 3 show the variable tension brane correction including all terms up to order y^4, respectively, without and with the terms elicited in Eqs. (21) and (22), regarding the derivatives of the brane variable tension. Those graphics show the paramount importance of considering more terms in

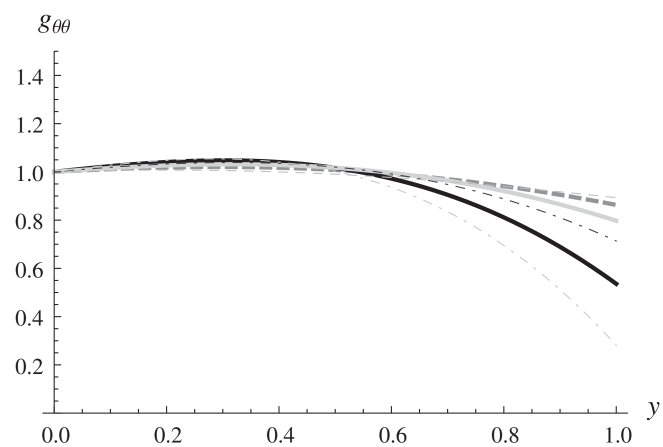

FIG. 2. Graphic of the brane effect-corrected black string horizon $g_{\theta\theta}$ for the Schwarzschild metric with variable tension, along the extra dimension y and also as function of the time t. The brane tension is given by $\lambda(t) = 1 - \exp(-\alpha t)$. This graphic *does not* take into account the extra terms given by Eqs. (21) and (22). For the dashed light-gray line: $\alpha t = 0.25$; for the dashed thick dark-gray line: $\alpha t = 0.5$; for the gray thick line: $\alpha t = 0.75$; for the dash-dotted line: $\alpha t = 1$; for the black line: $\alpha t = 1.5$; for the dash-dotted light-gray line: $\alpha t = 2.5$.

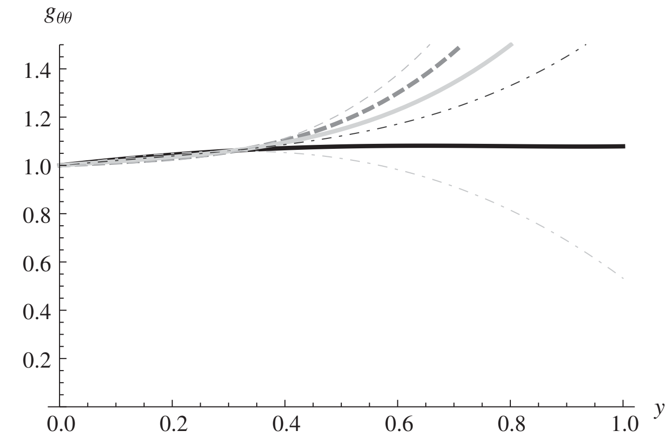

FIG. 3. Graphic of the brane effect-corrected black string horizon $g_{\theta\theta}$ for the Schwarzschild metric with variable tension, along the extra dimension y and also as function of the time t. The brane tension is given by $\lambda(t) = 1 - \exp(-\alpha t)$. This graphic *does* take into account the extra terms given by Eqs. (21) and (22). For the dashed light-gray line: $\alpha t = 0.25$; for the dashed thick dark-gray line: $\alpha t = 0.5$; for the gray thick line: $\alpha t = 0.75$; for the dash-dotted line: $\alpha t = 1$; for the black line: $\alpha t = 1.5$; for the dash-dotted light-gray line: $\alpha t = 2.5$.

the metric expansion given by Eq. (8), as accomplished heretofore. It also robustly illustrates that those terms drastically modify the black string horizon along the extra dimension, in the model here presented.

For the three-dimensional graphics, Fig. 4 does not take into account the time derivative terms, while Fig. 5 does. We shall draw some remarks in detail on the general behavior encoded in the figures in the next section.

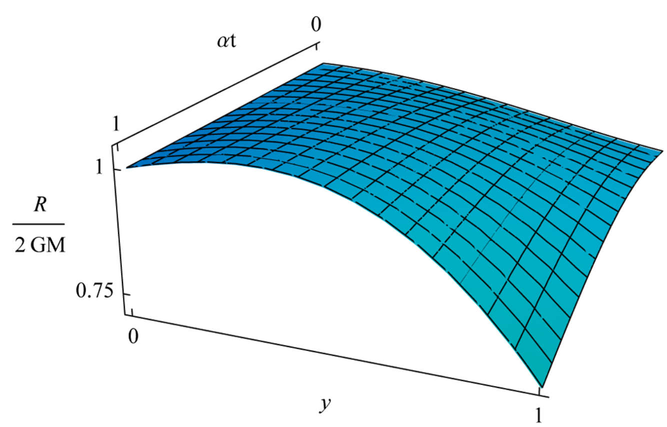

FIG. 4 (color online). Graphic of the brane effect-corrected black string horizon $g_{\theta\theta}$ for the Schwarzschild metric with variable tension, along the extra dimension y and also as function of the time t. The brane tension is given by $\lambda(t) = 1 - \exp(-\alpha t)$. This graphic *does not* take into account the extra terms given by Eq. (12) and (21), that for the case considered are encrypted in Eq. (22).

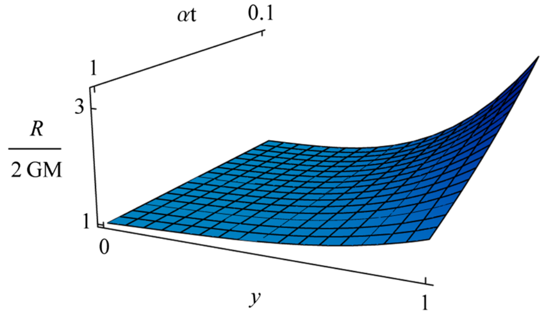

FIG. 5 (color online). Graphic of the brane effect-corrected black string horizon $g_{\theta\theta}$ for the Schwarzschild metric with variable tension, along the extra dimension y and also as function of the time t. The brane tension is given by $\lambda(t) = 1 - \exp(-\alpha t)$. This graphic *does* take into account the extra terms given by Eq. (12) and (21), that for the case considered are encrypted in Eq. (22). The black string normalized horizon varies from $R/(2GM)$ to $R = 3/(2GM)$ approximately.

IV. CONCLUDING REMARKS

In this paper, branes whose variable tension is only dependent on the time are focused. The Taylor expansion of the metric along the extra dimension for this case is provided.

Albeit terms up to the second order already evince the variable tension character, when terms from the third order on are investigated, it is thoroughly possible to unveil the real character of the variable tension concerning the changing of the event horizon. It accrues additional and unexpected properties, since only in this case the covariant derivatives of the variable tension appear, in the metric expansion along the extra dimension. Furthermore, it is prominently necessary to consider such terms, since in some regions of spacetime described by a brane the value of the tension can be very small, and at the same time the brane tension can bluntly vary along the spacetime coordinates. Additional terms in the expansion analyzed modify the rate of the horizon variation along the extra dimension, when the brane tension varies in a braneworld model based upon the Eötvös law.

This unexpected feature on this effective model is explicitly shown in Figs. 1–5. In the first figure, the brane effect-corrected black string horizon $g_{\theta\theta}$ for the Schwarzschild metric with variable tension is depicted along the extra dimension y. In Fig. 1, the black string horizon behavior along the extra dimension is illustrated, regarding only terms up to the order y^2, commonly approached in the literature, for instance, in [19]. Figure 2 shows that the black string behavior is obviously different for different fixed moments without taking into account terms of time derivative of the tension, while Fig. 3 does take into account such terms.

By comparing Figs. 4 and 5 , it is possible to see that in Fig. 5, for all $t \geq 0$, the black string presents no singularities. It is precluded by the extra terms in the Taylor expansion of the metric in the brane-corrected black string horizon in our model. Those terms do not appear in models with constant brane tension, since they encode the covariant derivatives of the brane variable tension λ.

The general result encoded in the figures is exhaustive: whenever the time derivative terms are taken into account the variation of the horizon along the extra dimension is such that the horizon does not tend to zero. This effect is, presumable, naively interpreted within this (eminently) classical framework. Still, one could speculate that it would be interpreted in terms of fluctuations around the brane. In fact, combing the fact that a completely rigid object cannot exist in the general relativity framework with the presence of a scalar field representing the brane position into the bulk, one arrives at the possibility of a spontaneous symmetry breaking of the bulk diffeomorphism. In this way, a perturbative spectrum of scalar particles, the so-called branons, may appear if the tension scale is much smaller than the higher-dimensional mass scale [33].

Now, the brane tension being a variable quantity, a nontrivial contribution to the branons production is expected. Besides, the tension derivatives (computing the rate of tension variation) may also have an important role in the branons production. Following this reasoning, one could guess that such fluctuations could (in principle) supply the black hole horizon along the extra dimension horizon, which makes its approach to the singularity difficult. Obviously, this interpretation must be enforced by a critical analysis of the precise influence of a variable tension on the branons production, as well as the branons influence on the black string behavior. These issues together with possible quantum effects [34] are currently under investigation.

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