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Research Article

Design of a Takagi-Sugeno Fuzzy Regulator for a Set of Operation Points

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The paper proposes a new design method based on linear matrix inequalities (LMIs) for tracking constant signals (regulation) considering nonlinear plants described by the Takagi-Sugeno fuzzy models. The procedure consists in designing a single controller that stabilizes the system at operation points belonging to a certain range or region, without the need of remaking the design of the controller gains at each new chosen equilibrium point. The control system design of a magnetic levitator illustrates the proposed methodology.

1. Introduction

In recent years the design of tracking control systems for nonlinear plants described by Takagi-Sugeno fuzzy models [1] has been the subject of several studies [2–10]. For the tracking problem the goal is to make the tracking error (difference between the output and desired output) equal to zero, ensuring the asymptotic stability of the equilibrium point. The linear matrix inequality (LMI) formulation [11] has emerged recently as a useful tool for solving a great number of practical control problems [9, 12–17]. The advantage is that LMIs, when feasible, can be easily solved using available software [18, 19]. Furthermore, the procedure based on LMIs can also consider other design specifications regarding plant uncertainties, such as decay rate (related to the setting time) and output and input constraints [20].

An interesting method for the design of tracking control systems using LMIs was studied in [2]. The tracking process uses the concept of virtual desired variable, and the design is divided into two steps: first determine the virtual desired variables of the system; then determine the control gains based on LMIs, for the stabilization of the system. In [4] is proposed a design method for tracking system with disturbance rejection applied to a class of nonlinear systems using fuzzy control. The method is based on the minimization of the $\mathscr{H}_\infty$ norm between the reference signal and the tracking error signal, where the tracking error signal is the difference between the reference input signal and the output signal. In [6] is addressed the speed tracking control problem of permanent magnet synchronous motors with parameter uncertainties and load torque disturbance. Fuzzy logic systems are used to approximate the nonlinearities, and an adaptive backstepping technique is employed to construct the controllers. The proposed controller guarantees the convergence of the tracking error to a small neighborhood of the origin and achieves good tracking performance. A similar study is presented in [7], where a robust reference-tracking control problem for nonlinear distributed parameter systems with time delays, external disturbances, and measurement noises is studied; the nonlinear distributed parameter systems are measured at several sensor locations for output-feedback tracking control. A fuzzy-spatial state-space model derived via finite-difference approach was introduced to represent the nonlinear distributed parameter time-delayed system.

In this context there exist many other researches. In [21] a neural network-based approach was developed which combines $\mathscr{H}_\infty$ control performance with the Takagi-Sugeno fuzzy control for the purpose of stabilization and stability analysis of nonlinear systems. In [22] an analytical solution was derived to describe the wave-induced flow field and surge motion of a deformable platform structure controlled with fuzzy controllers in an oceanic environment. In the controller design procedure, a parallel distributed compensation scheme was utilized to construct a global fuzzy logic controller by blending all local state feedback controllers, and the Lyapunov method was used to carry out stability analysis of a real system structure.

This paper proposes a new control methodology for tracking constant signals for a class of nonlinear plants. This method is based on LMIs and uses the Takagi-Sugeno fuzzy models to accurately describe the nonlinear model of the plant. The main idea of the method was to add in the domain of the nonlinear functions of the plant the coordinate of the equilibrium point that we desire to track. An application of the methodology in the control of a magnetic levitator, given in [23], is presented.

The main advantage of this new procedure is its practical application because the designer chooses the desired region of the equilibrium points and designs a single set of gains of the regulator that guarantees asymptotic stability of the system at any equilibrium point previously chosen in the region. This region is flexible and can be specified by the designer. The project considers that the change from an operating point to another occurs after large time intervals, such that in the instants of the changes the system is practically in steady-state. In addition, this new methodology allows the use of well-known LMIs-based design methods, for the design of fuzzy regulators for plants described by the Takagi-Sugeno fuzzy models, for instance presented in [11, 14, 15, 24–28], which allows the inclusion of the specification of performance indices such as decay rate and constraints on the plant input and output.

2. Preliminary Results

2.1. The Takagi-Sugeno Fuzzy Regulator

As described in [1], the Takagi-Sugeno fuzzy model is as follows.

$$\text{Rule } i\text{: If } z_1(t) \text{ is } M_1^i, \dots, z_p(t) \text{ is } M_p^i, \text{ then } \dot{x}(t) = A_i x(t) + B_i u(t), \qquad y(t) = C_i x(t), \tag{2.1}$$

where $i = 1, 2, \dots, r$, M_j^i, $j = 1, 2, \dots, p$ is the fuzzy set j of Rule i, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^q$ is the output vector, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{q\times n}$, and $z_1(t), \dots, z_p(t)$ are premise variables, which in this paper are the state variables.

As in [24], $\dot{x}(t)$ given in (2.1) can be written as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} \alpha_i(x(t))(A_i x(t) + B_i u(t)), \tag{2.2}$$

where $\alpha_i(x(t))$ is the normalized weight of each local model system $A_i x(t) + B_i u(t)$ that satisfies the following properties:

$$\alpha_i(x(t)) \geq 0, \quad \text{for } i = 1, 2, \dots, r, \qquad \sum_{i=1}^{r} \alpha_i(x(t)) = 1. \tag{2.3}$$

Considering the Takagi-Sugeno fuzzy model (2.1), the control input of fuzzy regulators via parallel distributed compensation (PDC) has the following structure [24]:

$$\text{Rule } j\text{: If } z_1(t) \text{ is } M_1^j, \dots, z_p(t) \text{ is } M_p^j, \text{ then } u(t) = -F_j x(t). \tag{2.4}$$

Similar to (2.2), it can be concluded that

$$u(t) = -\sum_{j=1}^{r} \alpha_j(x(t)) F_j x(t). \tag{2.5}$$

From (2.5), (2.2) and observing that $\sum_{i=1}^{r} \alpha_i(x(t)) = 1$, we obtain that

$$\dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i(x(t)) \alpha_j(x(t)) \left[A_i - B_i F_j\right] x(t). \tag{2.6}$$

Defining

$$G_{ij} = A_i - B_i F_j, \tag{2.7}$$

then (2.6) can be written as follows:

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i(x(t))\alpha_j(x(t))G_{ij}x(t). \tag{2.8}$$

2.2. Stability of the Takagi-Sugeno Fuzzy Systems via LMIs

The following theorem, whose proof can be seen in [24], guarantees the asymptotic stability of the origin of the system (2.8).

Theorem 2.1. *The equilibrium point of the continuous time fuzzy control system given in* (2.6) *is asymptotically stable in the large if a common symmetric positive definite matrix* $X \in \mathbb{R}^{n\times n}(X \succ 0)$ *and* $M_i \in \mathbb{R}^{n\times m}$, $i = 1, 2, \ldots, r$ *exists such that the following LMIs are satisfied:*

$$\begin{gathered} XA_i^T + A_iX - B_iM_i - M_i^TB_i^T \prec 0, \\ (A_i + A_j)X + X(A_i + A_j)^T - B_iM_j - B_jM_i - M_i^TB_j^T - M_j^TB_i^T \preccurlyeq 0, \quad i < j, \end{gathered} \tag{2.9}$$

for all $i, j = 1, 2, \ldots, r$, *excepting the pairs* (i, j) *such that* $\alpha_i(x(t))\alpha_j(x(t)) = 0$, *for all* $x(t)$. *If there exists such a solution, the controller gains are given by* $F_i = M_iX^{-1}$, $i = 1, 2, \ldots, r$.

In a control design it is important to assure stability and usually other indices of performance for the controlled system, such as the response speed, restrictions on input control, and output signals. The speed of the response is related to the decay rate of the system (2.6) or largest Lyapunov exponent, which is defined as the largest $\beta > 0$ such that

$$\lim_{t\to\infty} e^{\beta t}\|x(t)\| = 0 \tag{2.10}$$

holds for all trajectories $x(t)$.

As in [11, page 66], one can use a quadratic Lyapunov function $V(x(t)) = x(t)^T Px(t)$ to establish a lower bound for the decay rate of system (2.6). The condition $\dot{V}(x(t)) \leq -2\beta V(x(t))$ for all trajectories $x(t)$ assures that the system has a decay rate greater or equal to β. This condition is considered in Theorem 2.2, whose proof can be found, for instance, in [24].

Theorem 2.2. *The equilibrium point of the continuous time fuzzy control system given in* (2.8) *is globally asymptotically stable, with decay rate greater or equal to* β, *if there exists a positive definite symmetric matrix* $X \in \mathbb{R}^{n\times n}(X \succ 0)$ *and matrices* $M_i \in \mathbb{R}^{n\times m}$, $i = 1, 2, \ldots, r$, *such that the following LMIs are satisfied:*

$$\begin{gathered} XA_i^T + A_iX - B_iM_i - M_i^TB_i^T + 2\beta X \prec 0, \\ (A_i + A_j)X + X(A_i + A_j)^T - B_iM_j - B_jM_i - M_i^TB_j^T - M_j^TB_i^T + 4\beta X \preccurlyeq 0, \quad i < j, \end{gathered} \tag{2.11}$$

for all $i, j = 1, 2, \ldots, r$, *excepting the pairs* (i, j) *such that* $\alpha_i(x(t))\alpha_j(x(t)) = 0$, *for all* $x(t)$. *If there exists this solution, the controller gains are given by* $F_i = M_iX^{-1}$, $i = 1, 2, \ldots, r$.

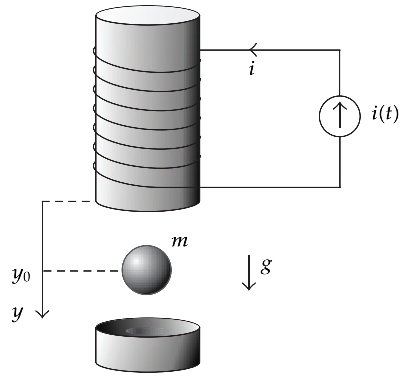


Figure 1: Magnetic levitator.

3. Magnetic Levitator

Currently, magnetic suspension systems are mainly used in applications where the reduction of friction force due to mechanical contact is essential. They are usually found in high-speed trains, gyroscopes, and accelerometers [23, page 23].

This paper considers the mathematical model of a magnetic levitator to illustrate the proposed control design method. Figure 1 shows the basic configuration of a magnetic levitator whose mathematical model [23, page 24] is given by

$$m\ddot{y} = -k\dot{y} + mg - \frac{\lambda\mu i^2}{2(1+\mu y)^2}, \tag{3.1}$$

where m is the mass of the ball; g is the gravity acceleration; μ and k are positive constants; i is the electric current; and y is the position of the ball.

Define the state variable $\overline{x}_1 = y$ and $\overline{x}_2 = \dot{y}$. Then, (3.1) can be written as follows [29]:

$$\dot{\overline{x}}_1 = \overline{x}_2, \qquad \dot{\overline{x}}_2 = g - \frac{k}{m}\overline{x}_2 - \frac{\lambda\mu i^2}{2m(1+\mu\overline{x}_1)^2}. \tag{3.2}$$

Consider that, during the required operation, $[\overline{x}_1 \ \ \overline{x}_2]^T \in D$, where

$$D = \left\{ [\overline{x}_1 \ \overline{x}_2]^T \in \mathbb{R}^2 : 0 \leq \overline{x}_1 \leq 0.15 \right\}. \tag{3.3}$$

The paper aims to design a controller that keeps the ball in a desired position $y = \overline{x}_1 = y_0$, after a transient response. Thus, the equilibrium point of the system (3.2) is $\overline{x}_e = [\overline{x}_{1_e} \ \ \overline{x}_{2_e}]^T = [y_0 \ \ 0]^T$.

From the second equation $\dot{\overline{x}}_2$ in (3.2), observe that, in the equilibrium point, $\dot{\overline{x}}_2 = 0$ and $i = i_0$, where

$$i_0^2 = \frac{2mg}{\lambda\mu}(1+\mu y_0)^2. \tag{3.4}$$

Note that the equilibrium point is not in the origin $[\overline{x}_1\ \ \overline{x}_2]^T = [0\ \ 0]^T$. Thus, the following change of coordinates is necessary for the stability analysis:

$$\begin{cases} x_1 = \overline{x}_1 - y_0, \\ x_2 = \overline{x}_2, \\ u = i^2 - i_0^2, \end{cases} \Longrightarrow \begin{cases} \overline{x}_1 = x_1 + y_0, \\ \overline{x}_2 = x_2, \\ i^2 = u + i_0^2. \end{cases} \tag{3.5}$$

Therefore, $\dot{x}_1 = \dot{\overline{x}}_1$, $\dot{x}_2 = \dot{\overline{x}}_2$, and from (3.4), $i^2 = u + (2mg/\lambda\mu)(1 + \mu y_0)^2$.

Hence, the system (3.2) can be written as follows:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = g - \frac{k}{m}x_2 - \frac{\lambda\mu\left(u + (2mg/\lambda\mu)(1 + \mu y_0)^2\right)}{2m(1 + \mu(x_1 + y_0))^2}, \tag{3.6}$$

and also, after some simple calculations, by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \frac{g\mu(\mu x_1 + 2\mu y_0 + 2)}{(1 + \mu(x_1 + y_0))^2}x_1 - \frac{k}{m}x_2 - \frac{\lambda\mu}{2m(1 + \mu(x_1 + y_0))^2}u. \tag{3.7}$$

Finally, from (3.7) it follows that

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ f_{21}(x_1, y_0) & \dfrac{-k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g_{21}(x_1, y_0) \end{bmatrix} u, \tag{3.8}$$

where

$$f_{21}(x_1, y_0) = \frac{g\mu(\mu x_1 + 2\mu y_0 + 2)}{(1 + \mu(x_1 + y_0))^2}, \tag{3.9}$$

$$g_{21}(x_1, y_0) = \frac{-\lambda\mu}{2m(1 + \mu(x_1 + y_0))^2}. \tag{3.10}$$

4. Regulator Design for an Operating Point

The goal of the design in this subsection is to keep the ball in a given position $\overline{x}_1 = y_0$. In the first design, $y_0 = 0.1\,\text{m}$, and in the second, $y_0 = 0.05\,\text{m}$. Table 1 presents the parameters of the plant (3.8)–(3.10), for the controller design.

First, assume that $y_0 = 0.1\,\text{m}$ and consider the following domain during the operation

$$D_1 = \left\{[x_1\ \ x_2]^T \in \mathbb{R}^2 : -0.1 \le x_1 \le 0.05,\ \ y_0 = 0.1\right\}. \tag{4.1}$$

Table 1: Parameters of the plant (3.8)–(3.10) [29].

m	0.05 Kg
g	$9.8\,\mathrm{m/s^2}$
k	0.001 Ns/m
λ	0.460 H
μ	$2\,\mathrm{m^{-1}}$

Now, from the generalized form proposed in [26], it is necessary to obtain the maximum and minimum values of the functions f_{21} and g_{21} in the domain D_1. After the calculations, it follows that

$$\begin{aligned} a_{21_1} &= \max_{x_1 \in D_1}\{f_{21}(x_1)\} = 43.1200, \\ a_{21_2} &= \min_{x_1 \in D_1}\{f_{21}(x_1)\} = 28.9941, \\ b_{21_1} &= \max_{x_1 \in D_1}\{g_{21}(x_1)\} = -5.4438, \\ b_{21_2} &= \min_{x_1 \in D_1}\{g_{21}(x_1)\} = -9.2000. \end{aligned} \tag{4.2}$$

Thus, the nonlinear the function f_{21} can be represented by a Takagi-Sugeno fuzzy model, considering that there exists a convex combination with membership functions $\sigma_{21_1}(x_1)$ and $\sigma_{21_2}(x_1)$ and constant values a_{21_1} and a_{21_2} given in (4.2) such that [26]

$$f_{21}(x_1) = \sigma_{21_1}(x_1)a_{21_1} + \sigma_{21_2}(x_1)a_{21_2}, \tag{4.3}$$

with

$$0 \leq \sigma_{21_1}(x_1), \qquad \sigma_{21_2}(x_1) \leq 1, \qquad \sigma_{21_1}(x_1) + \sigma_{21_2}(x_1) = 1. \tag{4.4}$$

Therefore, from (4.3) and (4.4) note that

$$\sigma_{21_1}(x_1) = \frac{f_{21}(x_1) - a_{21_2}}{a_{21_1} - a_{21_2}}, \qquad \sigma_{21_2}(x_1) = 1 - \sigma_{21_1}(x_1). \tag{4.5}$$

Similarly, from (4.2) there exist $\xi_{21_1}(x_1)$ and $\xi_{21_2}(x_1)$ such that

$$g_{21}(x_1) = \xi_{21_1}(x_1)b_{21_1} + \xi_{21_2}(x_1)b_{21_2}, \tag{4.6}$$

with

$$0 \leq \xi_{21_1}(x_1), \qquad \xi_{21_2}(x_1) \leq 1, \qquad \xi_{21_1}(x_1) + \xi_{21_2}(x_1) = 1. \tag{4.7}$$

Hence, from (4.6) and (4.7) observe that

$$\xi_{21_1}(x_1) = \frac{g_{21}(x_1) \ - b_{21_2}}{b_{21_1} - b_{21_2}}, \qquad \xi_{21_2}(x_1) = 1 - \xi_{21_1}(x_1). \tag{4.8}$$

Recall that $\xi_{21_1}(x_1) + \xi_{21_2}(x_1) = 1$. Therefore, from (4.3) it follows that

$$\begin{aligned} f_{21}(x_1) &= (\xi_{21_1}(x_1) + \xi_{21_2}(x_1))(\sigma_{21_1}(x_1)a_{21_1} + \sigma_{21_2}(x_1)a_{21_2}) \\ &= \sigma_{21_1}(x_1)\xi_{21_1}(x_1)a_{21_1} + \sigma_{21_1}(x_1)\xi_{21_2}(x_1)a_{21_1} + \sigma_{21_2}(x_1)\xi_{21_1}(x_1)a_{21_2} + \sigma_{21_2}(x_1)\xi_{21_2}(x_1)a_{21_2}. \end{aligned} \tag{4.9}$$

Similarly, from (4.6) and $\sigma_{21_1}(x_1) + \sigma_{21_2}(x_1) = 1$, we obtain

$$\begin{aligned} g_{21}(x_1) &= (\sigma_{21_1}(x_1) + \sigma_{21_2}(x_1))(\xi_{21_1}(x_1)b_{21_1} + \xi_{21_2}(x_1)b_{21_2}) \\ &= \sigma_{21_1}(x_1)\xi_{21_1}(x_1)b_{21_1} + \sigma_{21_1}(x_1)\xi_{21_2}(x_1)b_{21_2} + \sigma_{21_2}(x_1)\xi_{21_1}(x_1)b_{21_1} + \sigma_{21_2}(x_1)\xi_{21_2}(x_1)b_{21_2}. \end{aligned} \tag{4.10}$$

Now, define

$$\begin{aligned} \alpha_1(x_1) &= \sigma_{21_1}(x_1)\xi_{21_1}(x_1), \\ \alpha_2(x_1) &= \sigma_{21_1}(x_1)\xi_{21_2}(x_1), \\ \alpha_3(x_1) &= \sigma_{21_2}(x_1)\xi_{21_1}(x_1), \\ \alpha_4(x_1) &= \sigma_{21_2}(x_1)\xi_{21_2}(x_1), \end{aligned} \tag{4.11}$$

as the membership functions of the system (3.8)–(3.10), and their local models

$$\begin{gathered} A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ a_{21_1} & -0.02 \end{bmatrix}, \qquad A_3 = A_4 = \begin{bmatrix} 0 & 1 \\ a_{21_2} & -0.02 \end{bmatrix}, \\ B_1 = B_3 = [0 \ b_{21_1}]^T, \qquad B_2 = B_4 = [0 \ b_{21_2}]^T, \end{gathered} \tag{4.12}$$

where a_{21_1} and a_{21_2}, b_{21_1} and b_{21_2} are the maximum and minimum values of the functions $f_{21}(x_1)$ and $g_{21}(x_1)$, respectively, as described in (4.2).

Therefore, the system (3.8)–(3.10), with the control law (2.5), can be represented as a Takagi-Sugeno fuzzy model, given in (2.6) and (2.8), with $r = 4$:

$$\dot{x}(t) = \sum_{i=1}^{4}\sum_{j=1}^{4}\alpha_i(x(t))\alpha_j(x(t))G_{ij}x(t), \quad \text{where } G_{ij} = A_i - B_iF_j. \tag{4.13}$$

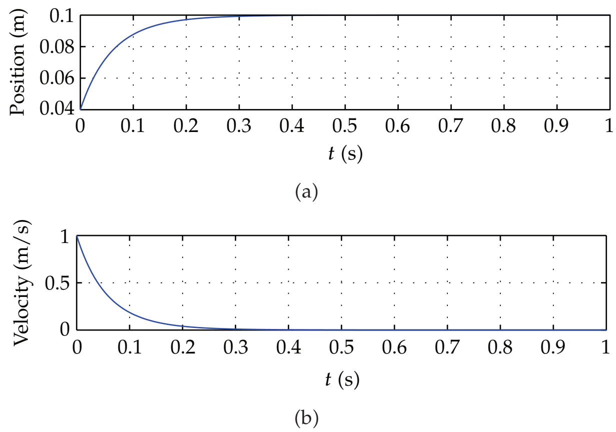


Figure 2: Position $y(t) = \overline{x}_1(t)$ and velocity $(\overline{x}_2(t))$ of the controlled system for $y_0 = 0.1\,\mathrm{m}$.

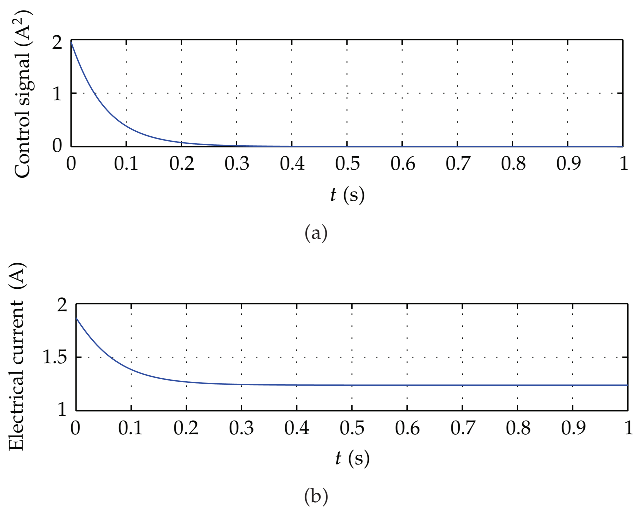


Figure 3: Control signal $(u(t) = i(t)^2 - i_0^2)$ and electrical current $(i(t))$ of the controlled system for $y_0 = 0.1\,\mathrm{m}$.

Thus, using the LMIs (2.9) from Theorem 2.1, we obtain the following controller gains:

$$\begin{aligned} F_1 &= \begin{bmatrix} -38.3098 & -4.7610 \end{bmatrix}, \\ F_2 &= \begin{bmatrix} -26.0150 & -3.1937 \end{bmatrix}, \\ F_3 &= \begin{bmatrix} -37.2796 & -4.8423 \end{bmatrix}, \\ F_4 &= \begin{bmatrix} -23.3273 & -3.0979 \end{bmatrix}. \end{aligned} \tag{4.14}$$

Considering the initial condition $\overline{x}_0 = [0.04\ 1]^T$ and $y_0 = 0.1\,\mathrm{m}$ for the system (3.2) (for the system (3.8)–(3.10), the initial condition is $x_0 = \overline{x}_0 - [y_0\ 0]^T = [-0.06\ 1]^T$), the simulation of the controlled system (3.8)–(3.10), (2.5), and (4.14) presented the responses shown in Figures 2 and 3. Note that $y(\infty) = y_0$, as desired.

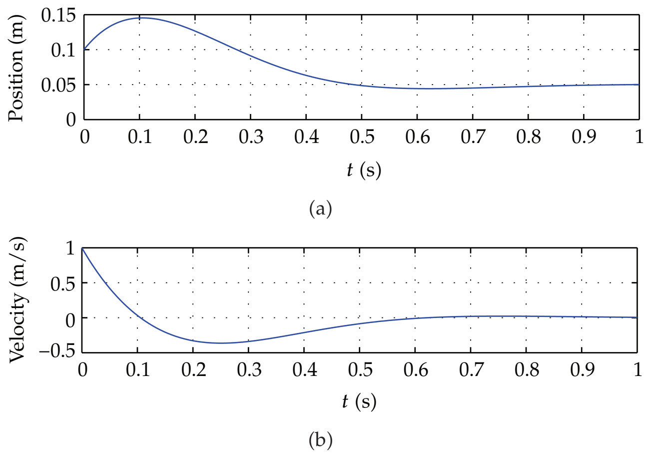


Figure 4: Position $y(t) = \overline{x}_1(t)$ and velocity $(\overline{x}_2(t))$ of the controlled system for $y_0 = 0.05\,\mathrm{m}$.

Now, suppose that the wanted position of the magnetic levitator is $y_0 = 0.05\,\mathrm{m}$. Thus, for the design of the control law, consider that in the required operation the domain is

$$D_2 = \left\{ [x_1 \ x_2]^T \in \mathbb{R}^2 : -0.05 \le x_1 \le 0.1,\ y_0 = 0.05 \right\}. \tag{4.15}$$

For the generalized form, as proposed in [26], it is necessary to find the maximum and minimum values of the functions f_{21} and g_{21} in the domain D_2. The obtained values were the following:

$$\begin{aligned} a_{21_1} &= \max_{x_1 \in D_2} \{ f_{21}(x_1) \} = 41.1600, \\ a_{21_2} &= \min_{x_1 \in D_2} \{ f_{21}(x_1) \} = 27.8343, \\ b_{21_1} &= \max_{x_1 \in D_2} \{ g_{21}(x_1) \} = -5.4438, \\ b_{21_2} &= \min_{x_1 \in D_2} \{ g_{21}(x_1) \} = -9.2000. \end{aligned} \tag{4.16}$$

Considering the same procedure adopted in (4.3)–(4.12) and from the condition given in Theorem 2.1, we obtain the following controller gains:

$$\begin{aligned} F_1 &= [-17.5514 \quad -1.7747], \\ F_2 &= [-12.0856 \quad -1.1922], \\ F_3 &= [-16.2911 \quad -1.8098], \\ F_4 &= [-10.0255 \quad -1.1717]. \end{aligned} \tag{4.17}$$

For the initial condition $\overline{x}_0 = [0.1 \ 1]$ and $y_0 = 0.05\,\mathrm{m}$ (for the system (3.8)–(3.10), the initial condition is $x_0 = [0.05 \ 1]^T$), the simulation of the controlled system (3.8)–(3.10), (2.5), and (4.17) presented the responses given in Figures 4 and 5.

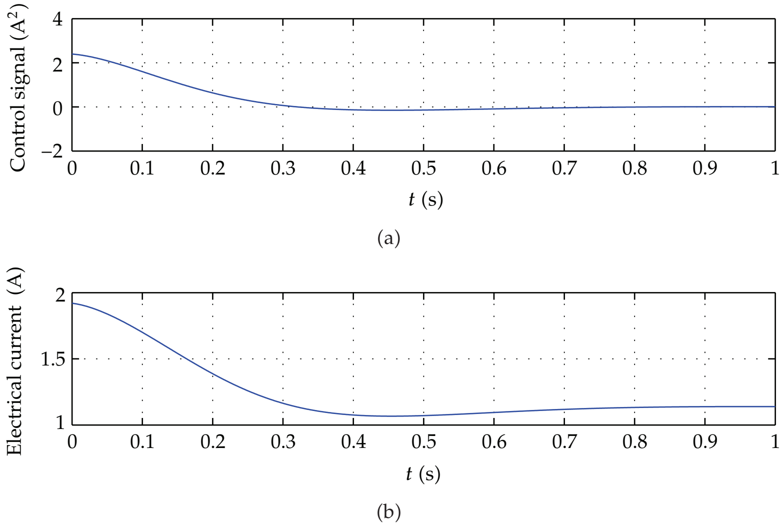


Figure 5: Control signal ($u(t) = i(t)^2 - i_0^2$) and electrical current ($i(t)$) of the controlled system for $y_0 = 0.05$ m.

Note that the gains of the controllers (4.14) and (4.17) change according to the change of y_0. It happens because each time we change the value of y_0, the operation point changes. Thus, the local models and membership functions also change. Therefore, following the presented control design method, it is necessary to design a new regulator when the value of y_0 changes, which makes difficult the practical implementation in cases where the system can work in different operating points. To solve this problem, in the next section we present a method for designing a single Takagi-Sugeno fuzzy controller via LMIs, for all of the range of known values of y_0, related to the operation of the system.

5. Regulator Design for a Set of Operation Points

Before presenting the method, it is necessary to understand the following property.

Property 1. Let $I_1 \subset \mathbb{R}^{n_1}$ and $I_0 \subset \mathbb{R}^{n_0}$ be compact subsets such that $I = I_1 \times I_0$, $f : I \subset \mathbb{R}^{n_t} \to \mathbb{R}$ a continuous function and $n_t = n_1 + n_0$. If for some given $y_0 \in I_0$, $M = \max_{y \in I_1}\{f(y, y_0)\}$ and $m = \min_{y \in I_1}\{f(y, y_0)\}$, then $M \le \max_{(y,y_0)\in I}\{f(y, y_0)\}$ and $m \ge \min_{(y,y_0)\in I}\{f(y, y_0)\}$.

Proof. Suppose, by contradiction, that $M > \max_{(y,y_0)\in I}\{f(y, y_0)\}$. Then, this implies that $f(y, y_0) < M$ for all $(y, y_0) \in I$ which is an absurd because I is compact. Thus, there exists $(y^*, y_0^*) \in I$ such that $M \le f(y^*, y_0^*)$.

Similarly it is shown that $m \ge \min_{(y,y_0)\in I}\{f(y, y_0)\}$. □

Property 1 is important to justify the proposed methodology. For instance, suppose that the plant can work in the region $\overline{x}_1 \in [0, 0.15]$ and that we want the asymptotic stability of operating points $[\overline{x}_1\ \overline{x}_2]^T = [y_0\ 0]^T$, where y_0 is a known constant and $y_0 \in I_0 = [0.04, 0.11]$. Thus the range of $x_1 = \overline{x}_1 - y_0$ for all $y_0 \in I_0$ is $I_1 = [-0.11, 0.11]$. So we could get the gains of regulator for all $y_0 \in I_0$, where y_0 will be considered as a new variable for the specification of the domain D_3 of the nonlinear functions f_{21} and g_{21}:

$$D_3 = \left\{ \left[x_1 \ x_2 \ y_0\right]^T \in \mathbb{R}^3 : -0.11 \le x_1 \le 0.11, \ 0.04 \le y_0 \le 0.11 \right\}. \tag{5.1}$$

From Property 1, (4.1), (4.15), and (5.1) note that

$$\begin{aligned}
\max_{(x_1,y_0)\in D_1,D_2} \{f_{21}(x_1,y_0)\} &\le \max_{(x_1,y_0)\in D_3} \{f_{21}(x_1,y_0)\},\\
\min_{(x_1,y_0)\in D_1,D_2} \{f_{21}(x_1,y_0)\} &\ge \min_{(x_1,y_0)\in D_3} \{f_{21}(x_1,y_0)\},\\
\max_{(x_1,y_0)\in D_1,D_2} \{g_{21}(x_1,y_0)\} &\le \max_{(x_1,y_0)\in D_3} \{g_{21}(x_1,y_0)\},\\
\min_{(x_1,y_0)\in D_1,D_2} \{g_{21}(x_1,y_0)\} &\ge \min_{(x_1,y_0)\in D_3} \{g_{21}(x_1,y_0)\}.
\end{aligned} \tag{5.2}$$

Indeed, after the calculations, considering (3.9), (3.10), Table 1, and (5.1), we obtain

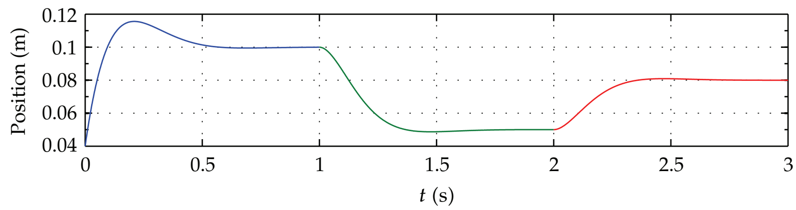
$$\begin{aligned}
a_{21_1} &= \max_{(x_1,y_0)\in D_3} \{f_{21}(x_1,y_0)\} = 51.4116,\\
a_{21_2} &= \min_{(x_1,y_0)\in D_3} \{f_{21}(x_1,y_0)\} = 25.1427,\\
b_{21_1} &= \max_{(x_1,y_0)\in D_3} \{g_{21}(x_1,y_0)\} = -4.4367,\\
b_{21_2} &= \min_{(x_1,y_0)\in D_3} \{g_{21}(x_1,y_0)\} = -12.4392.
\end{aligned} \tag{5.3}$$

Based on the same procedure adopted in (4.3)–(4.12) (now for the domain D_3) and from the LMIs of Theorem 2.1, the controller gains are the following:

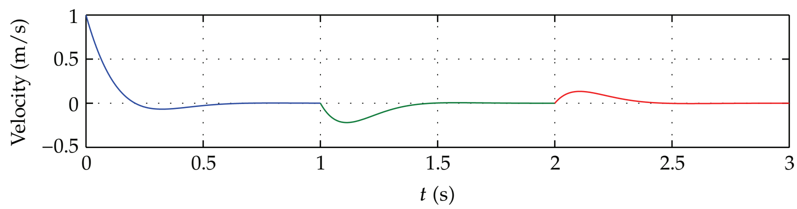
$$\begin{aligned}
F_1 &= \left[-23.5425 \ \ -2.4447\right],\\
F_2 &= \left[-14.1328 \ \ -1.3935\right],\\
F_3 &= \left[-22.2383 \ \ -2.5629\right],\\
F_4 &= \left[-10.8040 \ \ -1.2964\right].
\end{aligned} \tag{5.4}$$

For numerical simulation, at $t = 0$ s, the initial conditions are $\overline{x}_0 = [0.04 \ 1]^T$ and $y_0 = 0.1$ m. Thus, $x(0) = \overline{x}(0) - [y_0 \ \ 0]^T = [-0.06 \ \ 1]^T$. In $t = 1$ s, from Figure 6, the system is practically at the point $\overline{x}(1) = [\overline{x}_1(1) \ \ \overline{x}_2(1)]^T = [0.1 \ \ 0]^T$. After changing y_0 from 0.1 m to 0.05 m at $t = 1$ s, we can see that the system is practically at the point $\overline{x}(2) = [0.05 \ 0]^T$ at $t = 2$ s from Figure 6. After changing again y_0 from 0.05 m to 0.08 m at $t = 2$ s, we can see from Figure 6 that $\overline{x}(\infty) = [0.08 \ \ 0]^T$. Figures 6 and 7 illustrate the system response.

Note that the control law, given in (2.5) with $r = 4$, uses a single set of gains presented in (5.4). However, the membership functions $\alpha_i(x(t))$, $i = 1,2,3,4$, specified in (3.9), (3.10), (4.3)–(4.11), and (5.3), are functions of y_0, and so they must be updated each time that there is a change in the value of y_0. Finally, observe that $x = [\overline{x}_1 \ \overline{x}_2]^T - [y_0 \ 0]^T$ also must be changed in the control law (2.5), when y_0 is modified.

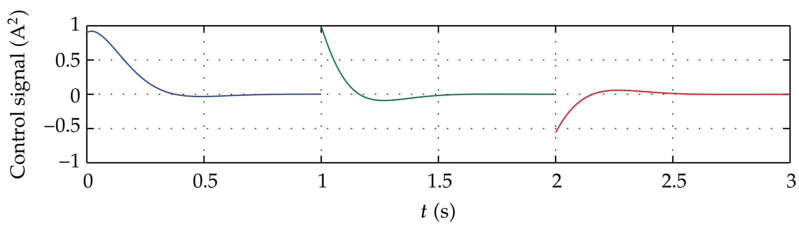


(a)

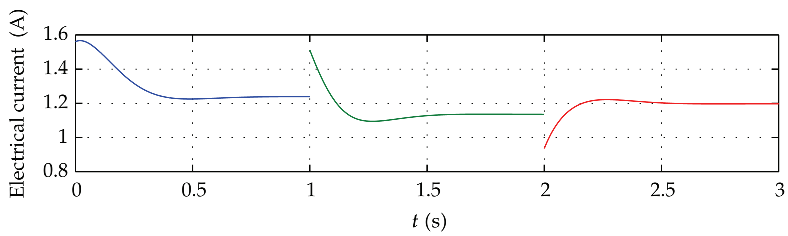


(b)

Figure 6: Position ($y(t) = \overline{x}_1(t)$) and velocity ($\overline{x}_2(t)$) of the controlled system for $y_0 \in [0.04, 0.11]$, considering $y_0 = 0.1$ m, 0.05 m, and 0.08 m for $t \in [0, 1), t \in [1, 2)$, and $t \geq 2$ s, respectively.



(a)



(b)

Figure 7: Control signal ($u(t) = i(t)^2 - i_0^2$) and electrical current ($i(t)$) for $y_0 \in [0.04, 0.11]$, considering $y_0 = 0.1$ m, 0.05 m, and 0.08 m for $t \in [0, 1)$, $t \in [1, 2)$, and $t \geq 2$ s, respectively.

Remark 5.1. In the example of the levitator, the range considered for the desired position point y_0 was $[0.04, 0.11]$ and the domain of $\overline{x}_1$ was $[0, 0.15]$. Thus, in general, the restriction for the proposed method is only that the region containing the desired equilibrium points must be contained in the domain of the state variables of the system. This region is flexible and can be chosen by the designer.

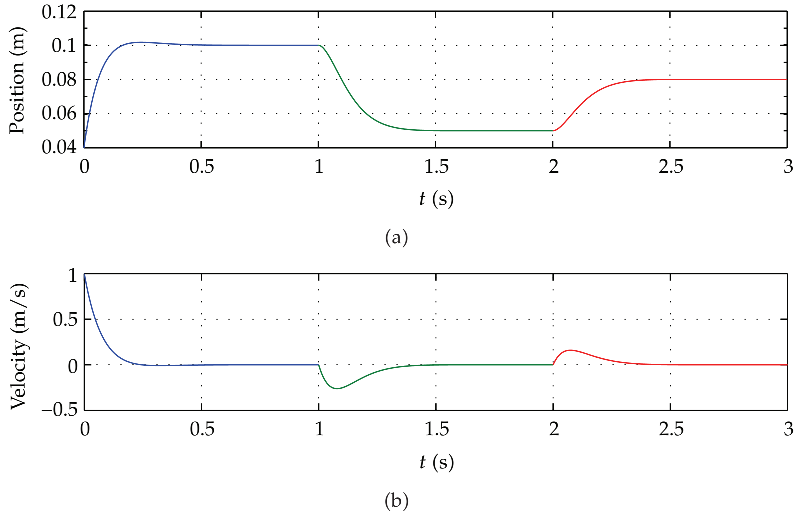


Figure 8: Position ($y(t) = \overline{x}_1(t)$) and velocity ($\overline{x}_2(t)$) of the controlled system for $y_0 \in [0.04, 0.11]$, considering $y_0 = 0.1\,\text{m}$, $0.05\,\text{m}$, and $0.08\,\text{m}$ for $t \in [0, 1)$, $t \in [1, 2)$, and $t \geq 2\,\text{s}$, respectively, and decay rate $\beta = 0.8$.

5.1. Regulator Design for a Set of Points of Operation with Rate of Decay

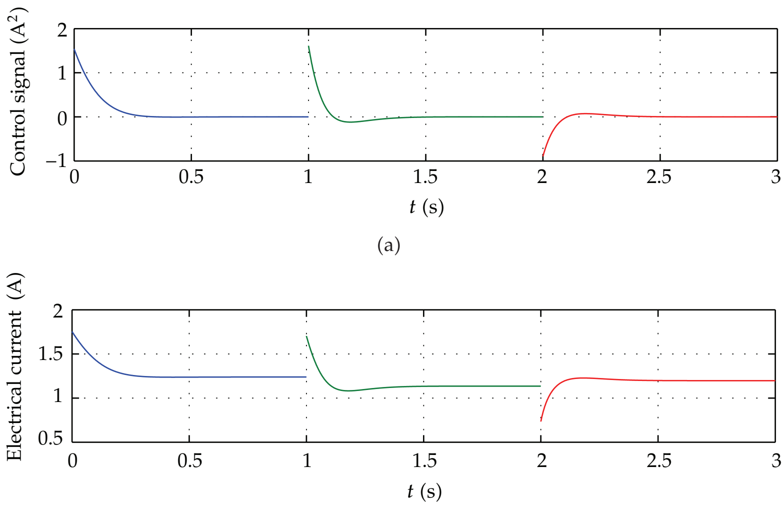
Usually, in control system designs, it is important to consider the stability and other performance indices for the controlled system, such as response speed, input constraint, and output constraint. The proposed methodology allows the specification of these performance indices, without changing the LMIs given in [11] (the same presented in Theorems 2.1 and 2.2), or their relaxations presented, for instance, in [14, 24], by adding a new set of LMIs.

Now, a decay rate will be specified for designing the new control gains for the magnetic levitator. Thus, it was considered (4.3)–(4.12), (5.3), and LMIs (2.11) from Theorem 2.2, with decay rate $\beta = 0.8$, and the obtained controller gains are the following:

$$\begin{aligned} F_1 &= \begin{bmatrix} -36.2829 & -3.9509 \end{bmatrix}, \\ F_2 &= \begin{bmatrix} -21.8487 & -2.3121 \end{bmatrix}, \\ F_3 &= \begin{bmatrix} -36.6258 & -4.2662 \end{bmatrix}, \\ F_4 &= \begin{bmatrix} -17.9936 & -2.1555 \end{bmatrix}. \end{aligned} \tag{5.5}$$

The numerical simulation supposes the same condition of the last simulation; that is, initially it was considered the initial condition $\overline{x}_0 = [0.04\ 1]^T$ and $y_0 = 0.1\,\text{m}$. In $t = 1\,\text{s}$ the system is practically at the point $\overline{x}_0 = [0.1\ 0]^T$, and then, y_0 is changed from 0.1 m to 0.05 m. At $t = 2\,\text{s}$ the system is almost at the point $\overline{x}_0 = [0.05\ 0]^T$, and now y_0 is changed from 0.05 m to 0.08 m. Figures 8 and 9 illustrate the response of the system.

As can be seen in Figure 8, using a decay rate greater than or equal to $\beta = 0.8$, the response of the system was adequate and faster, when compared with Figure 6. However, from (5.4) and (5.5), note that the controller gains are now greater and consequently the control signal and the electric current are also greater, as can be seen by comparing Figures 7 and 9.



(a)

(b)

Figure 9: Control signal ($u(t) = i(t)^2 - i_0^2$) and electrical current ($i(t)$) for $y_0 \in [0.04,\ 0.11]$, considering $y_0 = 0.1$ m, 0.05 m, and 0.08 m for $t \in [0, 1)$, $t \in [1, 2)$, and $t \geq 2$ s, respectively, and decay rate $\beta = 0.8$.

Remark 5.2. The proposed methodology can also be applied when the plant has known parameters belonging to a given region. In this case, one must consider these parameters as new variables in the domain of the nonlinearities and obtain the maximum and minimum values of the nonlinearities, in the region of operation. In the example of the levitator, we can consider, for instance, that the mass m is a known constant parameter, belonging to the range $m \in [m_{\min}, m_{\max}]$ with $m_{\min}$ and $m_{\max}$ known constants. Thus, the nonlinearities (3.9) and (3.10) are now given by $f_{21}(x_1, y_0, m)$ and $g_{21}(x_1, y_0, m)$, respectively. Note that, from (3.4), in this case i_0 also depends on the mass m and therefore must be updated when the mass changes. The design considers that the change of the mass occurs after large time intervals, such that in the instants of the changes the system is practically in steady state.

6. Conclusions

In this paper we proposed a new design method of regulators with operating points belonging to a given region, which allows the tracking of constant signals for nonlinear plants described by the Takagi-Sugeno fuzzy models. The design is based on LMIs, and an application in the control design of a magnetic levitator illustrated the proposed procedure.

An advantage of the proposed methodology is that it does not change the LMIs given in the control design methods usually adopted for plants described by the Takagi-Sugeno fuzzy models, for instance, as proposed in [4, 11, 14, 15, 24–28, 30–32]. Furthermore, it allows to choose an equilibrium point of the system in a region of values previously established without needing of remaking the design of the controller gains for each new chosen equilibrium point. Moreover, the simulation of the application of this new control design method in a magnetic levitator presented an appropriate transient response, as can be seen in Figures 8 and 9. Thus, the authors think that the proposed method can be useful in practical applications of nonlinear control systems.

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