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The Lyra-Schwarzschild Spacetime and Free Particle Motion

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I dedicate this thesis to my parents, Edgar Paquiyauri and Isabel Ruiz, who have always supported me in every professional and personal endeavor.

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"Great are the works of the LORD,
studied by all who delight in them."

Psalm 111:2

Resumo

Este trabalho apresenta um estudo abrangente da solução esfericamente simétrica mais geral no âmbito da gravidade escalar–tensorial baseada na geometria de Lyra (LyST). Essa formulação estende a estrutura geométrica da relatividade geral ao introduzir, além do tensor métrico, uma função de escala que modifica a conexão afim e outras quantidades geométricas fundamentais. Ao definir um volume invariante consistente e uma ação gravitacional, constrói-se um princípio variacional que conduz a equações de campo generalizadas compatíveis com as simetrias da teoria. A partir dessas equações, demonstra-se uma versão do teorema de Birkhoff, garantindo que a solução seja estática e possa ser descrita por um potencial efetivo caracterizado por uma "massa geométrica" e por uma constante de integração associada à escala de Lyra, denominada raio de Lyra. As trajetórias de partículas e fótons são analisadas por meio do formalismo de Hamilton–Jacobi, incluindo órbitas circulares, precessão do periélio, entre outros aspectos. Além disso, discutem-se os efeitos de desvio gravitacional para o vermelho, bem como uma transformação de coordenadas que relaciona explicitamente esse espaço-tempo à métrica clássica de Schwarzschild, fornecendo novas perspectivas sobre as implicações físicas da geometria de Lyra.

Palavras-chave: Relatividade Geral; Geometria de Lyra; Gravidade Escalar-Tensorial.

Áreas do conhecimento: Física; Gravitação; Geometria Diferencial.

Abstract

This work presents a comprehensive study of the most general spherically symmetric solution within the scalar–tensor gravity framework based on Lyra geometry (LyST). This formulation extends the geometric structure of general relativity by introducing, alongside the metric tensor, a scale function that modifies the affine connection and other fundamental geometric quantities. By defining a consistent invariant volume and gravitational action, a variational principle is constructed, leading to generalized field equations compatible with the symmetries of the theory. From these equations, a version of Birkhoff's theorem is demonstrated, ensuring that the solution is static and can be described by an effective potential characterized by a "geometric mass" and an integration constant associated with the Lyra scale, referred to as the Lyra radius. Particle and photon trajectories are analyzed using the Hamilton–Jacobi formalism, including circular orbits, perihelion precession, etc. Additionally, gravitational redshift effects are discussed, along with a coordinate transformation that explicitly relates this spacetime to the classical Schwarzschild metric, providing new insights into the physical implications of Lyra geometry.

Keywords: General Relativity; Lyra Geometry; Scalar-Tensor Gravity.

Knowledge Areas: Physics; Gravitation; Differential Geometry.

Contents

Chapter 1

Introduction

General Relativity (GR) is the most successful theory available for describing gravity [1, 2]. Its predictions have been thoroughly tested and confirmed at solar-system scales, especially through precise experiments like Gravity Probe B [3]. Additionally, GR supports several modern technologies like GPS [4]. Beyond weak-field areas, the theory predicts the emission of gravitational waves from merging compact objects, which the LVK Collaboration has directly observed [5]. Black holes, another significant prediction of GR, have also been captured in images by the EHT Collaboration [6, 7]. On a cosmic scale, GR serves as the foundation of the standard cosmological model. Within this framework, the observed expansion of the universe and the formation of large-scale structures [8] are naturally explained. The theory also explains primordial nucleosynthesis and the resulting amounts of light atomic nuclei [9], along with the existence of Cosmic Microwave Background (CMB) radiation [10, 11]. A variety of observational programs have confirmed these predictions with great precision [12, 13, 14, 15, 16, 17].

Despite its strong empirical success, GR has some conceptual and practical limitations. The current description of the universe depends on a dark sector, constituted by dark matter and dark energy. Dark energy is used to explain the observed late-time accelerated expansion of the universe [18], while dark matter explains galaxy rotation curves [19], the stability of galaxy clusters [20], and the formation of large-scale structures seeded by CMB anisotropies [21]. In addition to these unresolved issues, GR predicts spacetime singularities where its classical description fails [22, 23]. These singularities appear both in cosmology at the Big Bang and at the centers of black holes. These limitations point to the need for a consistent quantum theory of gravity, which makes it hard to describe regions of spacetime with extremely high curvature [24].

These challenges have led to efforts to explore extensions and modifications of GR [25, 26, 27]. One of the most studied alternatives is scalar–tensor theories of gravity [28, 26], where an additional scalar degree of freedom complements the metric field. Notable examples include the Kaluza–Klein framework [29, 30] and the Jordan–Brans–Dicke theory [31]. From a geometrical perspective, Lyra proposed an alternative approach to scalar–tensor gravity in the early 1950s [32]. In Lyra geometry, a scalar scale function ϕ is incorporated directly into the definition of the reference frame, generalizing Riemannian geometry. As a result, spacetime transformations involve not only diffeomorphisms but also scale transformations, which modifies connection, curvature, and torsion.

The first studies of Lyra gravity were conducted by Sen and collaborators [33, 34, 35, 36]; see also [37]. In this formulation, a gauge potential $A_\mu := \phi^{-1}\partial_\mu \ln \phi^2$ was introduced, similar to Weyl's attempt to unify gravitation and electromagnetism [38]. However, this approach faced significant formal and physical inconsistencies. Specifically, the resulting field equations were not covariant under Lyra transformations or derivable from a clear variational principle. Furthermore, Weyl's theory struggles with the non-integrability of vector lengths [39], leading to observable effects that do not match astrophysical observations [40, 41]. Attempts to fix these issues through gauge fixing, such as setting $\phi = 1$, end up reducing the theory to standard GR [42]. A later proposal by Sen and Dunn [35] introduced a scalar field as a true dynamical degree of freedom, but this theory was later shown to be dynamically equivalent to Brans–Dicke gravity for a specific coupling parameter, leaving the basic problems unresolved.

Due to these shortcomings and building on Sen's original ideas, a new scalar–tensor theory on Lyra's manifold was recently proposed in Ref. [42]. This construction is based on the fundamental symmetries of Lyra geometry—diffeomorphisms and scale transformations—and produces well-defined field equations derived from an action principle. The resulting theory, known as Lyra scalar–tensor gravity (LyST), treats the metric tensor $g_{\mu\nu}$ and the scale function ϕ as fundamental dynamical fields, providing an accurate Newtonian limit in the weak-field, low-velocity regime. In the same work, researchers identified the simplest static and spherically symmetric solution of the LyST field equations. By applying a suitable ansatz for the line element and solving the corresponding differential equations for the metric functions and the scale field, they derived a generalization of the Schwarzschild solution, known as the Lyra–Schwarzschild spacetime, introducing the Lyra radius r_L. In addition, they examined the motion of massive and massless test particles in this background [43], providing insights into its geodesic structure.

This thesis revisits and expands on these findings using a more fundamental and systematic approach. Chapter 2 presents the defining elements of Lyra geometry, stressing the scale function's role in reference frames, modifying the transformation properties of tensors, and the form of geometrical quantities such as the line element and the volume element. In the same way, the scale function affects the affine structure by generalizing, with respect to Riemannian manifolds, the form of connection coefficients and related quantities like curvature and torsion. The construction of Cartan's formalism [44] and symmetries of tensor fields are also presented.

In Chapter 3, LyST theory is derived from a well-defined action principle, leading to the corresponding field equations and their Newtonian limit. The energy-momentum tensor and its conservation are also discussed. Chapter 4 focuses on constructing the Lyra–Schwarzschild solution under the assumption of spherical symmetry, without setting a priori relations between the temporal and radial metric components. This analysis shows that the resulting solution must be static, consistent with a Birkhoff-type theorem [45].

Chapter 5 deals with Hamilton-Jacobi formalism on Lyra geometry by following an approach similar to that of Hagihara [46]. This formalism is applied to the study of the motion of massive and massless particles in Lyra-Schwarzschild spacetime. The corresponding effective potentials are derived and analyzed, facilitating the classification of different types of orbits. Some physical implications regarding spacetime properties are briefly discussed, including the periastron advance of Mercury, the innermost stable circular orbit, a lower bound for the Lyra radius r_L, gravitational redshift, and causal structure. Finally, Chapter 6 summarizes the main findings and outlines potential directions for future research.

Chapter 2

Lyra Spacetime

When the metric itself acquires physical meaning, as in General Relativity, geometry becomes inseparable from physics. Yet, the possibility that scale might also play a fundamental geometric role has inspired alternative formulations of spacetime. Among these alternatives, Lyra's proposal introduces a local scale field into the geometric framework without leading to the non-integrable behavior of the length, characteristic of Weyl's theory.

This perspective suggests that the local properties of spacetime may not be entirely captured by coordinates and metric relations alone, but also by how scale varies smoothly from point to point. Such an idea invites a reformulation of the basic geometric notions from the ground up, starting at the level of reference systems and extending to all derived quantities. The present chapter undertakes this foundational task, establishing the structures upon which a consistent Lyra geometry can be developed.

2.1 Lyra Manifolds

The study of Lyra manifolds begins with the basic concept of a topological manifold, which formalizes the idea of a space that is locally similar to Euclidean space.

Definition 2.1. A topological space $\mathcal{M}$ is called a *topological manifold* if it is Hausdorff, second-countable, and locally homeomorphic to $\mathbb{R}^n$, for some $n \in \mathbb{N}$. Here, n is referred to as the dimension of the manifold.

The Hausdorff property ensures that any two distinct points in $\mathcal{M}$ can be separated by disjoint open sets, while being second-countable means that the topology on $\mathcal{M}$ admits a countable basis. These conditions are not merely mathematical conveniences; they were introduced because they lead to physically desirable features, such as the uniqueness of limits and the possibility of endowing the space with a metric structure. Although we will not include the proof or discuss further implications, their role is fundamental in ensuring a well-behaved framework for physical theories formulated on manifolds.

On the other hand, being locally homeomorphic to $\mathbb{R}^n$ means that for each $p \in \mathcal{M}$ there is a pair $(\mathcal{U}, \chi)$, called a *chart*, where $\mathcal{U} \subset \mathcal{M}$ is an open set or neighborhood containing p, and $\chi : \mathcal{U} \to \mathbb{R}^n$ is a homeomorphism—that is, a bijective map whose inverse and itself are both continuous.

Consider two Lyra reference systems $(\mathcal{U}_1, \chi_1, \Phi_1)$ and $(\mathcal{U}_2, \chi_2, \Phi_2)$ with intersected regions, i.e., such that $\mathcal{U}_1 \cap \mathcal{U}_2 \neq \emptyset$. Then for each point $p \in \mathcal{U}_1 \cap \mathcal{U}_2$, its coordinates in both systems are related through the *transition maps*:

$$\chi_{1\to 2} := \chi_2 \circ \chi_1^{-1} \quad \text{and} \quad \chi_{2\to 1} := \chi_1 \circ \chi_2^{-1},$$

with domains given by $\chi_1(\mathcal{U}_1 \cap \mathcal{U}_2)$ and $\chi_2(\mathcal{U}_1 \cap \mathcal{U}_2)$, respectively. Since chart maps are homeomorphisms, it follows that transition maps are also homeomorphisms.

Definition 2.2. A *Lyra reference system* is a triple $(\mathcal{U}, \chi, \Phi)$, where $(\mathcal{U}, \chi)$ is a chart of the manifold and $\Phi : \mathcal{U} \to \mathbb{R}^*$ is a continuous map, called *scale map*. Here, $\mathbb{R}^*$ denotes the set of all non-zero real numbers.

It is important to note that this does not imply a one-to-one relationship between charts and scale maps. Multiple scale maps can be defined for a single chart, and the same scale map can be associated with multiple charts, thereby defining different Lyra reference systems. In this sense, each scale map has a continuous local representation in every chart, called *scale function*, and defined by $\phi := \Phi \circ \chi^{-1} : \chi(\mathcal{U}) \to \mathbb{R}^*$.

In order to develop a suitable mathematical framework, it is necessary to go one step further in the study of the relationship between Lyra reference systems, extending the concept of smoothness from calculus in Euclidean spaces to manifolds.

Definition 2.3. Two charts are said to be *smoothly compatible* if the corresponding transition maps are smooth as functions between open subsets in $\mathbb{R}^n$. The same definition applies to Lyra reference systems.

Now consider a map $F : \mathcal{M} \to \mathbb{R}$ and two overlapping charts $(\mathcal{U}_1, \chi_1)$ and $(\mathcal{U}_2, \chi_2)$ with a point $p \in \mathcal{U}_1 \cap \mathcal{U}_2$. The map has a local representation on each chart, namely $f_1 := F \circ \chi_1^{-1}$ and $f_2 := F \circ \chi_2^{-1}$, which satisfy

$$f_2 = f_1 \circ \chi_{2\to 1} \quad \text{and} \quad f_1 = f_2 \circ \chi_{1\to 2}.$$

Since the transition maps are smooth, it follows that if one local representation is smooth, then all the others are as well. This property allows the following definition:

Definition 2.4. A map $F : \mathcal{M} \to \mathbb{R}$ is said to be *smooth* if, for every point $p \in \mathcal{M}$, there exists a chart $(\mathcal{U}, \chi)$ containing p such that the local representation $f := F \circ \chi^{-1}$ is a smooth function on $\chi(\mathcal{U})$. The set of all smooth real-valued maps over $\mathcal{M}$ is denoted by $C^\infty(\mathcal{M})$.

From now on, every scale map $\Phi : \mathcal{U} \to \mathbb{R}^*$ will be assumed to be smooth in the sense of Definition 2.4. This assumption ensures the consistency of the Lyra structure under smooth compatibility, as required in the following construction.

Definition 2.5. A *Lyra atlas* on a topological manifold $\mathcal{M}$ is a collection of Lyra reference systems $\mathcal{A} = \{(\mathcal{U}_\alpha, \chi_\alpha, \Phi_\alpha)\}_{\alpha \in I}$ that covers $\mathcal{M}$, i.e.

$$\mathcal{M} = \bigcup_{\alpha \in I} \mathcal{U}_\alpha.$$

Additionally:

- The collection $\mathcal{A}$ is said to be a *smooth Lyra atlas* if any two Lyra reference systems in $\mathcal{A}$ are smoothly compatible.

- The *maximal extension* of a smooth Lyra atlas $\mathcal{A}$ is the collection of all Lyra reference systems smoothly compatible with every element of $\mathcal{A}$. This is called a *maximal smooth Lyra atlas*.

The notion of a smooth Lyra atlas allows one to endow a topological manifold with a differentiable structure that extends the usual one by including smooth scale maps. When such an atlas is maximal, it completely determines the differentiable structure.

Definition 2.6. A *smooth Lyra manifold* of dimension n is a pair $(\mathcal{M}, \mathcal{A})$, where $\mathcal{M}$ is an n-dimensional topological manifold and $\mathcal{A}$ is a maximal smooth Lyra atlas on $\mathcal{M}$.

It is possible to define a Lyra manifold without the requirement of smoothness, by considering a Lyra atlas that does not satisfy the additional conditions introduced in Definition 2.5. However, smoothness is an essential property in the mathematical modeling of spacetime, whether in the presence of dynamical systems or as one of them.

2.2 Lyra Tensors

2.2.1 Vectors

The geometric and physical implications of scale maps will be analyzed in this and the following sections. From this point onward, it will be assumed that $\mathcal{M}$ denotes the underlying space of a smooth Lyra manifold, and the abbreviation LRS will be used to refer to Lyra reference systems.

Definition 2.7. A *curve* on $\mathcal{M}$ is a continuous map $\gamma : I \to \mathcal{M}$, where I is an interval. Also, γ is said to be a *smooth curve* at $t_0 \in I$ if there exists a chart $(\mathcal{U}, \chi)$ containing the point $p = \gamma(t_0)$, such that the local representation $\chi_\gamma := \chi \circ \gamma$ is smooth at t_0, as a function between subsets of $\mathbb{R}$ and $\mathbb{R}^n$.

It is worth noting that this definition is independent of the chosen chart. Indeed, if $(\mathcal{U}', \chi')$ is another chart containing p, then the corresponding local representation $\chi'_\gamma := (\chi' \circ \chi^{-1}) \circ \chi_\gamma$ is also smooth, since the transition map $\chi' \circ \chi^{-1}$ is smooth by definition.

Definition 2.8. Let $\gamma : I \to \mathcal{M}$ be a smooth curve. The *tangent vector* to γ at $t_0 \in I$ is the differential operator $\dot{\gamma}(t_0) : C^\infty(\mathcal{M}) \to \mathbb{R}$ defined by

$$\dot{\gamma}(t_0)F = \left.\frac{d(F \circ \gamma)}{dt}\right|_{t_0} \tag{2.1}$$

The set of all tangent vectors defined at $p = \gamma(t_0)$, obtained from smooth curves passing through p, forms a vector space called the *tangent space* of $\mathcal{M}$ at p, denoted by $T_p\mathcal{M}$.

The proof that $T_p\mathcal{M}$ forms a vector space is conceptually straightforward, though somewhat lengthy, and will not be included here.

Consider a LRS $(\mathcal{U}, \chi, \Phi)$ containing $p = \gamma(t_0)$ and denote $\boldsymbol{v} := \dot{\gamma}(t_0)$. Then the action of $\boldsymbol{v}$ upon a map $F \in C^\infty(\mathcal{M})$ can be written as

$$\boldsymbol{v}F = \left.\frac{d(F \circ \chi^{-1} \circ \chi \circ \gamma)}{dt}\right|_{t_0} = \left.\frac{d(f \circ \chi_\gamma)}{dt}\right|_{t_0} = \left.\frac{\partial f}{\partial x^\mu}\right|_{\chi(p)} \left.\frac{d\chi^\mu_\gamma}{dt}\right|_{t_0}, \tag{2.2}$$

where x^μ are the coordinates defined by χ, and χ^μ_γ are the components of the map χ_γ. The last equality shows that the partial derivatives of f are independent of the particular tangent vector or the associated curve; they only depend on the map χ of the LRS. Hence, these derivatives naturally correspond to certain vectors that can serve as a basis for the tangent space. Now, the Lyra trick is to also include the scale map in the definition of a basis associated with the LRS. In Lyra geometry, however, this basis is modified by incorporating the scale map Φ, which introduces an additional geometric degree of freedom associated with local scaling.

Proposition 2.1. Let $(\mathcal{U}, \chi, \Phi)$ be a LRS containing a point $p \in \mathcal{M}$, and let $F : \mathcal{M} \to \mathbb{R}$ be an arbitrary smooth map. Then the operators $\boldsymbol{e}_\mu$ defined by

$$\boldsymbol{e}_\mu F = \frac{1}{\phi(x_p)} \left.\frac{\partial f}{\partial x^\mu}\right|_{\chi(p)}, \tag{2.3}$$

form a basis for $T_p\mathcal{M}$, called the *Lyra canonical basis* of $(\mathcal{U}, \chi, \Phi)$.

Proof. First, prove that they are tangent vectors. Eq.(2.3) can be rewritten:

$$\boldsymbol{e}_\mu F = \left.\frac{\partial f}{\partial x^\nu}\right|_{\chi(p)} \frac{1}{\phi(x_p)} \delta^\nu_\mu.$$

Comparing with Eq. (2.2), it is clear that the associated curves γ_μ satisfy

$$\left.\frac{d(\chi \circ \gamma_\mu)^\nu}{dt}\right|_{t_0} = \frac{\delta^\nu_\mu}{\phi(x_p)},$$

or equivalently,

$$\chi \circ \gamma_\mu(t) = \left(c_1, \ldots, c_{\mu-1}, \frac{t}{\phi(x_p)}, c_{\mu+1}, \ldots, c_n\right),$$

where $c_{\nu \neq \mu}$ are integration constants.

Now, it must be proved that the vectors $\boldsymbol{e}_\mu$ are linearly independent. Let k^μ be real scalars such that $k^\mu \boldsymbol{e}_\mu = 0$. Evaluating this expression on $\chi^\nu : \mathcal{U} \to \mathbb{R}$ yields

$$\frac{k^\nu}{\phi(x_p)} = 0, \quad \forall \nu \in \{1, 2, \ldots, n\}.$$

Since $\phi(x_p) \neq 0$ by definition, all the coefficients k^ν must vanish. Finally, from Eq. (2.2) and Eq. (2.3), it follows that any tangent vector can be decomposed in terms of $\boldsymbol{e}_\mu$, with components

$$v^\mu = \phi(x_p)\left.\frac{d\chi^\mu_\gamma}{dt}\right|_{t_0} = \phi(x_p)\left.\frac{d(\chi\circ\gamma)^\mu}{dt}\right|_{t_0}. \tag{2.4}$$

□

Consider now another LRS $(\mathcal{U}', \chi', \Phi')$ defined around the same point $p \in \mathcal{M}$, with canonical vector basis $\{\boldsymbol{e}'_\mu\}$. Using the previous definitions and applying the chain rule, one obtains the transformation laws:

$$\boldsymbol{e}'_\mu = \frac{\phi(x_p)}{\phi'(x'_p)}\left.\frac{\partial x^\nu}{\partial x'^\mu}\right|_p \boldsymbol{e}_\nu, \tag{2.5}$$

for the canonical vector basis, and

$$v'^\mu = \frac{\phi'(x'_p)}{\phi(x_p)}\left.\frac{\partial x'^\mu}{\partial x^\nu}\right|_p v^\nu, \tag{2.6}$$

for the components of a tangent vector.

These expressions show that, unlike in the Riemannian case, the transformation of the canonical vector basis and components in Lyra geometry involves not only the coordinate transition maps but also the ratio of the corresponding scale maps.

2.2.2 Covectors

Having defined the tangent space and its associated Lyra basis, it is natural to introduce its dual structure, which acts linearly on tangent vectors.

Definition 2.9. The dual vector space of $T_p\mathcal{M}$, consisting of all the linear maps $\boldsymbol{\omega} : T_p\mathcal{M} \to \mathbb{R}$, is called the *cotangent space* of $\mathcal{M}$ at p and it is denoted by $T_p\mathcal{M}^*$. The elements of this space will be referred to as covectors.

A particularly important class of covectors arises from smooth scalar functions defined on the Lyra manifold. These covectors encode the infinitesimal variation of a scalar function along all tangent directions.

Definition 2.10. Let $F \in C^\infty(\mathcal{M})$ be a smooth scalar map and $p \in \mathcal{M}$ a point. The *differential* of F at p is the covector $d_pF \in T_p\mathcal{M}^*$ defined by

$$\begin{aligned} d_pF : \quad T_p\mathcal{M} &\to \mathbb{R} \\ \boldsymbol{v} &\mapsto \boldsymbol{v}F. \end{aligned} \tag{2.7}$$

This notation will make sense later when exterior calculus on $\mathcal{M}$ is introduced.

Similarly to the tangent space, one may define a basis for the cotangent space that is naturally associated with each LRS.

Proposition 2.2. Let $(\mathcal{U}, \chi, \Phi)$ be a LRS containing a point $p \in \mathcal{M}$, and let $\{\boldsymbol{e}_\mu\}$ be the corresponding canonical vector basis. The covectors $\boldsymbol{e}^\mu$ defined by

$$\boldsymbol{e}^\mu = \phi(x_p) d_p \chi^\mu, \tag{2.8}$$

form a basis for $T_p\mathcal{M}^*$, called the *Lyra canonical dual basis* of $(\mathcal{U}, \chi, \Phi)$.

Proof. First, note that the covectors defined in Eq. (2.8) satisfy:

$$\boldsymbol{e}^\mu(\boldsymbol{e}_\nu) = \delta^\mu_\nu, \tag{2.9}$$

which is called *duality relation*. Now, let k_μ be real scalars such that $k_\mu \boldsymbol{e}^\mu = 0$. Evaluating this expression on $\boldsymbol{e}_\nu$ yields

$$k_\nu = 0, \quad \forall \nu \in \{1, 2, \ldots, n\}.$$

Finally, let $\boldsymbol{v} \in T_p\mathcal{M}$ and $\boldsymbol{\omega} \in T_p\mathcal{M}^*$ be arbitrary. Then:

$$\boldsymbol{\omega}(\boldsymbol{v}) = \boldsymbol{\omega}(v^\mu \boldsymbol{e}_\mu) = \boldsymbol{\omega}(\boldsymbol{e}_\mu) v^\nu \delta^\mu_\nu = \boldsymbol{\omega}(\boldsymbol{e}_\mu) v^\nu \boldsymbol{e}^\mu(\boldsymbol{e}_\nu) = \boldsymbol{\omega}(\boldsymbol{e}_\mu) \boldsymbol{e}^\mu(\boldsymbol{v}),$$

showing that every covector $\boldsymbol{\omega}$ can be expressed as a linear combination of the $\boldsymbol{e}^\mu$, with components given by

$$\omega_\mu = \boldsymbol{\omega}(\boldsymbol{e}_\mu). \tag{2.10}$$

□

The transformation laws of the corresponding dual basis and covector components under a change of LRS $(\mathcal{U}, \chi, \Phi) \to (\mathcal{U}', \chi', \Phi')$ follow directly from the previous transformation laws. Specifically,

$$\boldsymbol{e}'^\mu = \frac{\phi'(x'_p)}{\phi(x_p)} \left. \frac{\partial x'^\mu}{\partial x^\nu} \right|_p \boldsymbol{e}^\nu, \tag{2.11}$$

for the canonical dual basis, and

$$\omega'_\mu = \frac{\phi(x_p)}{\phi'(x'_p)} \left. \frac{\partial x^\nu}{\partial x'^\mu} \right|_p \omega_\nu, \tag{2.12}$$

for the components of a covector.

2.2.3 Tensors

Having introduced tangent vectors and covectors as the fundamental linear objects associated with a differentiable manifold, it is natural to consider more general quantities that act multilinearly on combinations of these objects. Such entities encompass scalars, vectors, and covectors as particular cases within a unified algebraic framework.

Definition 2.11. A *tensor of rank* (r,s) at a point $p \in \mathcal{M}$ is a multilinear map

$$\boldsymbol{T} : \underbrace{T_p\mathcal{M}^* \times \cdots \times T_p\mathcal{M}^*}_{r\text{ times}} \times \underbrace{T_p\mathcal{M} \times \cdots \times T_p\mathcal{M}}_{s\text{ times}} \to \mathbb{R},$$

where r and s are non-negative integers. Here, r and s are referred to as contravariant and covariant ranks, respectively. The collection of all such tensors forms a vector space denoted by $T_p\mathcal{M}^{(r,s)}$. The notations $T_p\mathcal{M} := T_p\mathcal{M}^{(1,0)}$ and $T_p\mathcal{M}^* := T_p\mathcal{M}^{(0,1)}$ will be considered.

In many geometric and physical contexts, one needs to construct higher-order tensors from simpler ones, preserving multilinearity in each argument. The following construction provides a rigorous way to do so.

Definition 2.12. Let $\boldsymbol{T} \in T_p\mathcal{M}^{(r,s)}$ and $\boldsymbol{S} \in T_p\mathcal{M}^{(k,l)}$ be two arbitrary tensors. The tensor $\boldsymbol{T} \otimes \boldsymbol{S} \in T_p\mathcal{M}^{(r+k,s+l)}$, called *tensor product*, is defined by

$$\begin{aligned}&\boldsymbol{T} \otimes \boldsymbol{S}\big(\boldsymbol{\omega}^{(1)}, \ldots, \boldsymbol{\omega}^{(r+k)}, \boldsymbol{v}_{(1)}, \ldots, \boldsymbol{v}_{(s+l)}\big) := \\ &\boldsymbol{T}\big(\boldsymbol{\omega}^{(1)}, \ldots, \boldsymbol{\omega}^{(r)}, \boldsymbol{v}_{(1)}, \ldots, \boldsymbol{v}_{(s)}\big)\boldsymbol{S}\big(\boldsymbol{\omega}^{(r+1)}, \ldots, \boldsymbol{\omega}^{(r+k)}, \boldsymbol{v}_{(s+1)}, \ldots, \boldsymbol{v}_{(s+l)}\big)\end{aligned}$$

for all $\boldsymbol{\omega}^{(1)}, \ldots, \boldsymbol{\omega}^{(r+k)} \in T_p\mathcal{M}^*$ and $\boldsymbol{v}_{(1)}, \ldots, \boldsymbol{v}_{(s+l)} \in T_p\mathcal{M}$.

Immediately, some properties of this operation are observed:

i. Bilinearity.
$$(a_1\boldsymbol{T}_1 + b_2\boldsymbol{T}_2) \otimes \boldsymbol{S} = a_1(\boldsymbol{T}_1 \otimes \boldsymbol{S}) + a_2(\boldsymbol{T}_2 \otimes \boldsymbol{S}),$$
and similarly on the right.

ii. Associativity.
$$(\boldsymbol{T} \otimes \boldsymbol{S}) \otimes \boldsymbol{R} = \boldsymbol{T} \otimes (\boldsymbol{S} \otimes \boldsymbol{R}),$$
so one simply writes $\boldsymbol{T} \otimes \boldsymbol{S} \otimes \boldsymbol{R}$.

iii. Non-commutativity. In general,
$$\boldsymbol{T} \otimes \boldsymbol{S} \neq \boldsymbol{S} \otimes \boldsymbol{T},$$
since the order of arguments in the multilinear evaluation changes.

Proposition 2.3. Let $(\mathcal{U}, \chi, \Phi)$ be LRS containing a point $p \in \mathcal{M}$, and let $\{\boldsymbol{e}_\mu\}$ and $\{\boldsymbol{e}^\mu\}$ denote, respectively, the Lyra canonical basis for $T_p\mathcal{M}$ and its corresponding Lyra dual basis for $T_p\mathcal{M}^*$. Then, the set of tensor products

$$\{\boldsymbol{e}_{\mu_1}, \otimes \cdots \otimes \boldsymbol{e}_{\mu_r} \otimes \boldsymbol{e}^{\nu_1} \otimes \cdots \otimes \boldsymbol{e}^{\nu_s}\}, \tag{2.13}$$

forms a basis for the vector space $T_p\mathcal{M}^{(r,s)}$.

The proof of this proposition follows the same reasoning as those of Propositions 2.1 and 2.2. Furthermore, by analogy with these cases, the components of any tensor $\boldsymbol{T} \in T_p\mathcal{M}^{(r,s)}$ with respect to the basis (2.3) are given by

$$T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} = \boldsymbol{T}\big(\boldsymbol{e}^{\mu_1}, \ldots, \boldsymbol{e}^{\mu_r}, \boldsymbol{e}_{\nu_1}, \ldots, \boldsymbol{e}_{\nu_s}\big), \tag{2.14}$$

and consequently, the components of the tensor product defined above are

$$(\boldsymbol{T} \otimes \boldsymbol{S})^{\mu_1\cdots\mu_{r+k}}{}_{\nu_1\cdots\nu_{s+l}} = T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} S^{\mu_{r+1}\cdots\mu_{r+k}}{}_{\nu_{s+1}\cdots\nu_{s+l}}. \tag{2.15}$$

While the tensor product allows the construction of higher-rank tensors from simpler ones, it is also useful to have an operation that performs the opposite task—reducing the rank of a tensor by pairing one contravariant and one covariant index.

Definition 2.13. Let $\boldsymbol{T} \in T_p\mathcal{M}^{(r,s)}$ a tensor such that $r, s \geq 1$. Given one contravariant index $i \in \{1, \ldots, r\}$ and one covariant index $j \in \{1, \ldots, s\}$, the *contraction* of $\boldsymbol{T}$ with respect to these indices is the tensor $C^i_j[\boldsymbol{T}] \in T_p\mathcal{M}^{(r-1,s-1)}$ defined by

$$\begin{aligned} &C^i_j[\boldsymbol{T}]\big(\boldsymbol{\omega}^{(1)}, \ldots, \boldsymbol{\omega}^{(r-1)}, \boldsymbol{v}_{(1)}, \ldots, \boldsymbol{v}_{(s-1)}\big) := \\ &\boldsymbol{T}\big(\boldsymbol{\omega}^{(1)}, \ldots, \boldsymbol{\omega}^{(i-1)}, \boldsymbol{e}^{\mu}, \boldsymbol{\omega}^{(i)}, \ldots \boldsymbol{\omega}^{(r-1)}, \boldsymbol{v}_{(1)}, \ldots, \boldsymbol{v}_{(j-1)}, \boldsymbol{e}_{\mu}, \boldsymbol{v}_{(j)}, \ldots, \boldsymbol{v}_{(s-1)}\big), \end{aligned}$$

where $\{\boldsymbol{e}_\mu\}$ is some Lyra canonical basis for $T_p\mathcal{M}$ and $\{\boldsymbol{e}^\mu\}$ is the corresponding Lyra dual basis for $T_p\mathcal{M}^*$.

Clearly, the contraction defined above is a linear operation, and it is straightforward to verify that the components of the contracted tensor are given by

$$\big(C^i_j[\boldsymbol{T}]\big)^{\mu_1\cdots\mu_{r-1}}{}_{\nu_1\cdots\nu_{s-1}} = T^{\mu_1\cdots\mu_{i-1}\alpha\mu_i\cdots\mu_{r-1}}{}_{\nu_1\cdots\nu_{j-1}\alpha\nu_j\cdots\nu_{s-1}}. \tag{2.16}$$

It is also common to consider contractions involving two distinct tensors. However, such an operation can be understood as a contraction applied to their tensor product. For instance, for a vector $\boldsymbol{v} \in T_p\mathcal{M}$ and a covector $\boldsymbol{\omega} \in T_p\mathcal{M}^*$,

$$C^1_1[\boldsymbol{v} \otimes \boldsymbol{\omega}] = \boldsymbol{v} \otimes \boldsymbol{\omega}(\boldsymbol{e}^\mu, \boldsymbol{e}_\mu) = \boldsymbol{v}(\boldsymbol{e}^\mu)\boldsymbol{\omega}(\boldsymbol{e}_\mu) = v^\mu \omega_\mu \in \mathbb{R}.$$

Finally, consider a change of LRS $(\mathcal{U}, \chi, \Phi) \to (\mathcal{U}', \chi', \Phi')$. By applying the linearity of the tensor product, along with the transformation rules for vectors and covectors, the resulting change for tensors follows directly:

$$\begin{aligned} &\boldsymbol{e}'_{\mu_1}, \otimes \cdots \otimes \boldsymbol{e}'_{\mu_r} \otimes \boldsymbol{e}'^{\nu_1} \otimes \cdots \otimes \boldsymbol{e}'^{\nu_s} = \\ &\left(\frac{\phi(x_p)}{\phi'(x'_p)}\right)^{r-s} \left(\frac{\partial x^{\alpha_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial x^{\alpha_r}}{\partial x'^{\mu_r}} \frac{\partial x'^{\nu_1}}{\partial x^{\beta_1}} \cdots \frac{\partial x'^{\nu_s}}{\partial x^{\beta_s}}\right)_p \boldsymbol{e}_{\mu_1}, \otimes \cdots \otimes \boldsymbol{e}_{\mu_r} \otimes \boldsymbol{e}^{\nu_1} \otimes \cdots \otimes \boldsymbol{e}^{\nu_s}, \end{aligned} \tag{2.17}$$

for the Lyra tensor basis, and

$$T'^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} = \left(\frac{\phi'(x'_p)}{\phi(x_p)}\right)^{r-s} \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_r}}{\partial x^{\alpha_r}} \frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_s}}{\partial x'^{\nu_s}}\right)_p T^{\alpha_1\cdots\alpha_r}{}_{\beta_1\cdots\beta_s}, \tag{2.18}$$

for the components of any tensor.

2.2.4 Tensor Fields

At each point of the manifold, tensors of a fixed rank belong to their own vector space. In order to describe quantities that extend over the entire manifold, it is useful to combine these separate spaces into a single structure that encompasses all tensors of the same rank.

Definition 2.14. The *bundle* of tensors of rank (r,s) of the Lyra manifold $\mathcal{M}$ is defined as the disjoint union

$$T\mathcal{M}^{(r,s)} := \bigcup_{p\in\mathcal{M}} T_p\mathcal{M}^{(r,s)}, \tag{2.19}$$

equipped with the projection map

$$\begin{aligned} \pi^{(r,s)}: \quad T\mathcal{M}^{(r,s)} &\to \mathcal{M} \\ \boldsymbol{T} &\mapsto p, \end{aligned} \tag{2.20}$$

where p is the unique point such that $\boldsymbol{T} \in T_p\mathcal{M}^{(r,s)}$. Here, each $T_p\mathcal{M}^{(r,s)}$ is called a *fiber* of the corresponding bundle.

As particular cases, there are the *tangent bundle* $T\mathcal{M} := T\mathcal{M}^{(1,0)}$ and the *cotangent bundle* $T\mathcal{M}^* := T\mathcal{M}^{(0,1)}$, with projection maps $\pi := \pi^{(1,0)}$ and $\pi^* := \pi^{(0,1)}$, respectively.

The construction of bundles does not specify any particular relation between the tensors in different fibers. The following definition associates to each position on the manifold a tensor from the corresponding fiber, allowing the description of quantities that vary coherently from point to point.

Definition 2.15. A *tensor field* of rank (r,s) is a section of the bundle $T\mathcal{M}^{(r,s)}$; that is, a map

$$\boldsymbol{T} : \mathcal{M} \to T\mathcal{M}^{(r,s)},$$

such that $\pi^{(r,s)} \circ \boldsymbol{T}$ equals the identity map on $\mathcal{M}$. The collection of all smooth tensor fields of rank (r,s) is denoted by $\Gamma\big(T\mathcal{M}^{(r,s)}\big)$.

Before establishing what it means for a tensor field to be smooth, it is important to recall that $\Gamma\big(T\mathcal{M}^{(r,s)}\big)$ forms a vector space under pointwise addition and multiplication by scalars in $\mathbb{R}$. However, its algebraic structure is richer: it is also a module over the commutative ring $C^\infty(\mathcal{M})$. In this setting, scalar multiplication can be generalized so that a tensor field $\boldsymbol{T}$ may be multiplied by a smooth function $F \in C^\infty(\mathcal{M})$, yielding another tensor field $F\boldsymbol{T}$ of the same rank. The key difference is that not all smooth functions have a multiplicative inverse, as there may be some points where the function becomes zero.

Now, consider a LRS $(\mathcal{U}, \chi, \Phi)$. This LRS determines, for each point $p \in \mathcal{U}$, a Lyra canonical basis $\{\boldsymbol{e}_\mu(p)\}$ in the corresponding tangent space, thereby defining a collection of local vector fields $\boldsymbol{e}_\mu$ that preserve smoothness when acting upon maps in $C^\infty(\mathcal{M})$:

$$\begin{aligned} \boldsymbol{e}_\mu : \quad C^\infty(\mathcal{M}) \quad &\rightarrow \quad C^\infty(\mathcal{M}) \\ F \quad &\mapsto \quad \boldsymbol{e}_\mu F : \quad \mathcal{M} \rightarrow \mathbb{R} \\ &\qquad\qquad\quad p \mapsto \boldsymbol{e}_\mu(p) F = \frac{1}{\phi(x_p)} \left.\frac{\partial f}{\partial x^\mu}\right|_{\chi(p)} . \end{aligned}$$

Therefore, these vector fields are said to be smooth. Moreover, any vector field $\boldsymbol{v}$ can be written as a linear combination $v^\mu \boldsymbol{e}_\mu$, where the components v^μ are functions on $\mathcal{M}$. If these components are smooth, then the vector field is said to be smooth as well, and it belongs to $\Gamma(T\mathcal{M})$.

In a similar way, the LRS defines a collection of local covector fields $\boldsymbol{e}^\mu$ that naturally preserve smoothness when acting upon vector fields in $\Gamma(T\mathcal{M})$:

$$\begin{aligned} \boldsymbol{e}^\mu : \quad \Gamma(T\mathcal{M}) \quad &\rightarrow \quad C^\infty(\mathcal{M}) \\ \boldsymbol{v} \quad &\mapsto \quad \boldsymbol{e}^\mu(\boldsymbol{v}) = v^\mu . \end{aligned}$$

Again, any covector field $\boldsymbol{\omega}$ can be written as a linear combination $\omega_\mu \boldsymbol{e}^\mu$, where the components ω_μ are functions on $\mathcal{M}$. If these components are smooth, then the covector field is said to be smooth, and it belongs to $\Gamma(T\mathcal{M}^*)$.

In addition to Def. 2.15, and in light of the recent discussion, a tensor field $\boldsymbol{T}$ can also be described as a map

$$\begin{aligned} \boldsymbol{T} : \ \Gamma(T\mathcal{M}^*) \times \cdots \times \Gamma(T\mathcal{M}^*) \times \Gamma(T\mathcal{M}) \times \cdots \times \Gamma(T\mathcal{M}) \quad &\rightarrow \quad C^\infty(\mathcal{M}) \\ \left(\boldsymbol{\omega^1}, \ldots, \boldsymbol{\omega^r}, \boldsymbol{v_1}, \ldots, \boldsymbol{v_s}\right) \quad &\mapsto \quad \boldsymbol{T}\left(\boldsymbol{\omega^1}, \ldots, \boldsymbol{\omega^r}, \boldsymbol{v_1}, \ldots, \boldsymbol{v_s}\right), \end{aligned}$$

with pointwise evaluation given by

$$\begin{aligned} \boldsymbol{T}\left(\boldsymbol{\omega^1}, \ldots, \boldsymbol{\omega^r}, \boldsymbol{v_1}, \ldots, \boldsymbol{v_s}\right) : \mathcal{M} \quad &\rightarrow \quad \mathbb{R} \\ p \quad &\mapsto \quad \boldsymbol{T}(p)\left(\boldsymbol{\omega^1}(p), \ldots, \boldsymbol{\omega^r}(p), \boldsymbol{v_1}(p), \ldots, \boldsymbol{v_s}(p)\right). \end{aligned}$$

From this perspective, the tensor field $\boldsymbol{T}$ can be written as a linear combination

$$\boldsymbol{T} = T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s} \boldsymbol{e}_{\mu_1} \otimes \cdots \otimes \boldsymbol{e}_{\mu_r} \otimes \boldsymbol{e}^{\nu_1} \otimes \cdots \otimes \boldsymbol{e}^{\nu_s}, \tag{2.21}$$

where $\boldsymbol{e}_{\mu_1} \otimes \cdots \otimes \boldsymbol{e}_{\mu_r} \otimes \boldsymbol{e}^{\nu_1} \otimes \cdots \otimes \boldsymbol{e}^{\nu_s}$ are tensor fields that preserve smoothness, and the components $T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s}$ are functions on $\mathcal{M}$. If these components are smooth, then the tensor field is said to be smooth as well, and it belongs to $\Gamma\left(T\mathcal{M}^{(r,s)}\right)$.

One can also find transformation rules for tensor fields, which are generalizations of the rules for simple tensors. Specifically:

$$\begin{aligned} &\boldsymbol{e'}_{\mu_1} \otimes \cdots \otimes \boldsymbol{e'}_{\mu_r} \otimes \boldsymbol{e'}^{\nu_1} \otimes \cdots \otimes \boldsymbol{e'}^{\nu_s} = \\ &\left(\frac{\phi(x)}{\phi'(x')}\right)^{r-s} \left(\frac{\partial x^{\alpha_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial x^{\alpha_r}}{\partial x'^{\mu_r}} \frac{\partial x'^{\nu_1}}{\partial x^{\beta_1}} \cdots \frac{\partial x'^{\nu_s}}{\partial x^{\beta_s}}\right) \boldsymbol{e}_{\alpha_1} \otimes \cdots \otimes \boldsymbol{e}_{\alpha_r} \otimes \boldsymbol{e}^{\beta_1} \otimes \cdots \otimes \boldsymbol{e}^{\beta_s}, \end{aligned} \tag{2.22}$$

for the tensor field basis, and

$$T'^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}(x') = \left(\frac{\phi'(x')}{\phi(x)}\right)^{r-s}\left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial x'^{\mu_r}}{\partial x^{\alpha_r}}\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial x^{\beta_s}}{\partial x'^{\nu_s}}\right)T^{\alpha_1\cdots\alpha_r}{}_{\beta_1\cdots\beta_s}(x), \tag{2.23}$$

for the tensor field components.

Since both tensors and tensor fields have very simple and similar notations, it will be assumed, in cases of ambiguity, that the object under consideration is a tensor field, unless otherwise specified. Specifically, no distinction will be made between the notation of a tensor field and that of its value at a given point when it is clear from the context which one is being referred to.

Structure coefficients

Since the action of a vector field on a smooth map is another smooth map, it is possible to apply many vector fields consecutively. Thus, one can define the commutator of two vector fields $\boldsymbol{u}, \boldsymbol{v} \in \Gamma(T\mathcal{M})$ in the following way:

$$[\boldsymbol{u}, \boldsymbol{v}] := \boldsymbol{u}\boldsymbol{v} - \boldsymbol{v}\boldsymbol{u}, \tag{2.24}$$

A very special case is the commutator of two elements of a vector field basis $\{\boldsymbol{e}_\mu\}$. Its action upon a smooth map $F \in C^\infty(\mathcal{M})$ yields:

$$[\boldsymbol{e}_\mu, \boldsymbol{e}_\nu]F = \frac{1}{\phi^2(x)}\left(\delta^\lambda_\mu\frac{\partial\phi}{\partial x^\nu} - \delta^\lambda_\nu\frac{\partial\phi}{\partial x^\mu}\right)\frac{1}{\phi(x)}\frac{\partial f}{\partial x^\lambda}.$$

Then, one obtains the following relation:

$$[\boldsymbol{e}_\mu, \boldsymbol{e}_\nu] = \gamma^\lambda{}_{\mu\nu}\boldsymbol{e}_\lambda, \tag{2.25}$$

where the quantities $\gamma^\lambda{}_{\mu\nu}$ are not tensor fields. They are called the *structure coefficients* related to the basis $\{\boldsymbol{e}_\mu\}$:

$$\boxed{\gamma^\lambda{}_{\mu\nu} = \phi^{-2}(\delta^\lambda_\mu\partial_\nu\phi - \delta^\lambda_\nu\partial_\mu\phi).} \tag{2.26}$$

Finally, by using Eq. (2.26), one obtains an alternative expression for Eq. (2.24):

$$[\boldsymbol{u}, \boldsymbol{v}] = (\boldsymbol{u}v^\lambda - \boldsymbol{v}u^\lambda + u^\mu v^\nu\gamma^\lambda{}_{\mu\nu})\boldsymbol{e}_\lambda. \tag{2.27}$$

Therefore, the commutator of any two vector fields is a vector field. Also, it is worth noting that the commutator operation is bilinear with respect to real scalars but not with respect to functions in $C^\infty(\mathcal{M})$. Commutators are very important, and they will be seen again when other quantities are introduced in Section 2.4.

2.3 Geometrical Structure

In Lyra geometry, the presence of a gauge or scale function introduces an additional degree of freedom in the local structure of the manifold, affecting how quantities defined at different points are compared. To provide a consistent way of characterizing geometric relations among vectors in each tangent space—while preserving compatibility with the underlying scale transformations—it becomes necessary to introduce an additional entity that encodes these relations in a coordinate-independent manner. This object serves as the fundamental ingredient for defining geometric notions such as length, angle, and separation between nearby points, thereby endowing the manifold with a well-defined geometrical structure.

Definition 2.16. A *metric tensor* on $\mathcal{M}$ is a tensor field $\boldsymbol{g} \in \Gamma\big(T\mathcal{M}^{(0,2)}\big)$ that defines a scalar product on each tangent space. This means that, for each $p \in \mathcal{M}$, the tensor field $\boldsymbol{g}$ satisfies:

(i) Symmetry:

$$\forall \boldsymbol{u}, \boldsymbol{v} \in T_p\mathcal{M} : \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{u}).$$

(ii) Non-degeneracy:

$$\big\{\exists \boldsymbol{u} \in T_p\mathcal{M} : \forall \boldsymbol{v} \in T_p\mathcal{M} : \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = 0\big\} \quad \Longrightarrow \quad \boldsymbol{u} = \boldsymbol{0}.$$

The non-degeneracy property guarantees the existence of a non-degenerate symmetric matrix for each Lyra basis $\{\boldsymbol{e}_\mu\}$, whose matrix elements are the tensor components $g_{\mu\nu}$. The elements of the corresponding inverse matrix will be denoted by $g^{\alpha\beta}$. Therefore,

$$g^{\mu\lambda} g_{\lambda\alpha} = \delta^\mu_\alpha. \tag{2.28}$$

It is worth mentioning that the metric tensor induces a one-to-one correspondence between tensor fields. In general, for any $\boldsymbol{T} \in \Gamma\big(T\mathcal{M}^{(r,s)}\big)$, there is a unique $\widetilde{\boldsymbol{T}} \in \Gamma\big(T\mathcal{M}^{(r-1,s+1)}\big)$, whose components are related by:

$$T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} = g^{\mu_r\alpha}\widetilde{T}^{\mu_1\cdots\mu_{r-1}}{}_{\alpha\nu_1\cdots\nu_s} \quad \text{and} \quad \widetilde{T}^{\mu_1\cdots\mu_{r-1}}{}_{\alpha\nu_1\cdots\nu_s} = g_{\alpha\mu_r} T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}. \tag{2.29}$$

From now on, all tensors related by this kind of correspondence will have the same notation, unless stated otherwise when confusion might arise.

One of the fundamental concepts that can be defined thanks to the metric tensor is the length of a curve. Let $\gamma : I \to \mathcal{M}$ be a smooth curve with tangent vector $\boldsymbol{v}(t)$, with $I = [t_1, t_2]$. In order to consistently define its length, one must first guarantee that the scalar product of the tangent vector with itself has a constant sign on each point of the curve. In this case, the length will be given by

$$\ell = \int_{t_1}^{t_2} \sqrt{\big|\boldsymbol{g}\big(\boldsymbol{v}(t), \boldsymbol{v}(t)\big)\big|}\, dt. \tag{2.30}$$

Clearly, this definition does not depend on the LRS or the parameterization used. Given a LRS $(\mathcal{U}, \chi, \Phi)$, Eq. (2.30) can be written as:

$$\ell = \int_{t_1}^{t_2} \sqrt{\left|\phi^2(x) g_{\mu\nu}(x) \frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right|}\, dt. \tag{2.31}$$

and, for two infinitesimally close points,

$$d\ell^2 = \big|\phi^2(x) g_{\mu\nu}(x) dx^\mu dx^\nu\big|. \tag{2.32}$$

From this expression, one defines the *line element* of the manifold:

$$\boxed{ds^2 = \phi^2(x) g_{\mu\nu}(x) dx^\mu dx^\nu,} \tag{2.33}$$

which, unlike Eq. (2.32), specifies the type of tangent vector of the curve according to the sign of its scalar product with itself. Eq. (2.33) shows that the line element is invariant not only under diffeomorphisms but also under Lyra scale transformations.

2.3.1 Geodesics

This subsection focuses on a special type of curve, defined below.

Definition 2.17. A *geodesic* is a smooth curve whose length is stationary under infinitesimal variations, with the condition of fixed end-points. Mathematically, this means $\delta\ell = 0$, with $\delta x^\mu(t_1) = \delta x^\mu(t_2) = 0$.

Applying this to Eq. (2.31), one obtains the Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\nu} - \frac{\partial L}{\partial x^\nu} = 0, \quad L := \frac{d\ell}{dt} = \sqrt{\left|\phi^2(x) g_{\mu\nu}(x) \frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right|}, \tag{2.34}$$

where $\dot{x}^\nu := \frac{dx^\nu}{dt}$. Multiplying by $2L$, Eq. (2.34) becomes

$$\frac{d}{dt}\frac{\partial L^2}{\partial \dot{x}^\nu} - 2\frac{dL}{dt}\frac{\partial L}{\partial \dot{x}^\nu} - \frac{\partial L^2}{\partial x^\nu} = 0.$$

Now, by considering a parameter such that $t = c_1\ell + c_2$, where c_1 and c_2 are constants, the last term vanishes:

$$\frac{d}{dt}\frac{\partial L^2}{\partial \dot{x}^\nu} - \frac{\partial L^2}{\partial x^\nu} = 0,$$

Moreover, the absolute value in L^2 can be ignored, as the sign of its argument is constant along the curve. Then,

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{x}^\nu} - \frac{\partial L^2}{\partial x^\nu} = 0, \quad L' = \phi^2(x) g_{\alpha\beta}(x)\dot{x}^\alpha\dot{x}^\beta.$$

By performing the corresponding calculations, one obtains the geodesic equation:

$$\boxed{\frac{d^2x^\mu}{dt^2} + \left\{ {\mu \atop \alpha\beta} \right\} \frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} + \left(\delta^\mu_\alpha \nabla_\beta\phi + \delta^\mu_\beta \nabla_\alpha\phi - g_{\alpha\beta} g^{\mu\nu}\nabla_\nu\phi\right)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0,} \tag{2.35}$$

where $\left\{ {\mu \atop \alpha\beta} \right\}$ are the well known *Christoffel symbols*:

$$\left\{ {\mu \atop \alpha\beta} \right\} := \frac{1}{2} g^{\mu\nu}\left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}\right). \tag{2.36}$$

Note that there are terms within the parentheses of Eq. (2.35) that do not appear in the geodesic equation of Riemannian geometry. Remember that this equation is valid only for affine parameters.

2.3.2 Volume Element

The metric tensor allows the introduction of the concept of volume. As it is well known, in Euclidean geometry, the volume element with respect to orthonormal coordinates x^μ is given by

$$d^n x = |dx^1 \wedge \cdots \wedge dx^n|. \tag{2.37}$$

Now, for general coordinates related to some LRS on the Lyra manifold:

$$\begin{aligned} dx'^1 \wedge \cdots \wedge dx'^n &= \left(\frac{\partial x'^1}{\partial x^{\mu_1}} \cdots \frac{\partial x'^n}{\partial x^{\mu_n}} \right) dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \\ &= \left(\frac{\partial x'^1}{\partial x^{\mu_1}} \cdots \frac{\partial x'^n}{\partial x^{\mu_n}} \right) \epsilon^{\mu_1 \cdots \mu_n} dx^1 \wedge \cdots \wedge dx^n \\ &= \det\left(\frac{\partial x'}{\partial x} \right) dx^1 \wedge \cdots \wedge dx^n, \end{aligned} \tag{2.38}$$

where $\epsilon^{\mu_1 \cdots \mu_n} = \epsilon_{\mu_1 \cdots \mu_n}$ is the Levi-Civita symbol with convention $\epsilon_{1 \cdots n} = +1$. Here, one observes the Jacobian matrix $\frac{\partial x'}{\partial x}$ of the coordinate transformation, and its corresponding determinant:

$$\det\left(\frac{\partial x'}{\partial x} \right) = \epsilon^{\mu_1 \cdots \mu_n} \left(\frac{\partial x'^1}{\partial x^{\mu_1}} \cdots \frac{\partial x'^n}{\partial x^{\mu_n}} \right). \tag{2.39}$$

From Eqs. (2.37) and (2.38), one obtains:

$$d^n x' = \left| \det\left(\frac{\partial x'}{\partial x} \right) \right| d^n x. \tag{2.40}$$

One the other hand, the determinant of the matrix associated to the metric tensor, denoted by $\det(g)$, transforms like

$$\begin{aligned} \det(g') &= \epsilon^{\mu_1 \cdots \mu_n} g'_{\mu_1 1} \cdots g'_{\mu_n n} \\ &= \epsilon^{\mu_1 \cdots \mu_n} \left(\frac{\phi^2(x)}{\phi'^2(x')} \frac{\partial x^{\alpha_1}}{\partial x'^{\mu_1}} \frac{\partial x^{\beta_1}}{\partial x'^1} g_{\alpha_1 \beta_1} \right) \cdots \left(\frac{\phi^2(x)}{\phi'^2(x')} \frac{\partial x^{\alpha_n}}{\partial x'^{\mu_n}} \frac{\partial x^{\beta_n}}{\partial x'^n} g_{\alpha_n \beta_n} \right) \\ &= \left(\frac{\phi(x)}{\phi'(x')} \right)^{2n} \left(\epsilon^{\mu_1 \cdots \mu_n} \frac{\partial x^{\alpha_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial x^{\alpha_n}}{\partial x'^{\mu_n}} \right) \frac{\partial x^{\beta_1}}{\partial x'^1} \cdots \frac{\partial x^{\beta_n}}{\partial x'^n} g_{\alpha_1 \beta_1} \cdots g_{\alpha_n \beta_n} \\ &= \left(\frac{\phi(x)}{\phi'(x')} \right)^{2n} \det\left(\frac{\partial x}{\partial x'} \right) \frac{\partial x^{\beta_1}}{\partial x'^1} \cdots \frac{\partial x^{\beta_n}}{\partial x'^n} \left(\epsilon^{\alpha_1 \cdots \alpha_n} g_{\alpha_1 \beta_1} \cdots g_{\alpha_n \beta_n} \right) \\ &= \left(\frac{\phi(x)}{\phi'(x')} \right)^{2n} \det\left(\frac{\partial x}{\partial x'} \right) \det(g) \left(\epsilon_{\beta_1 \cdots \beta_n} \frac{\partial x^{\beta_1}}{\partial x'^1} \cdots \frac{\partial x^{\beta_n}}{\partial x'^n} \right) \\ &= \left(\frac{\phi(x)}{\phi'(x')} \right)^{2n} \det\left(\frac{\partial x}{\partial x'} \right)^2 \det(g), \end{aligned}$$

or equivalently,

$$\sqrt{|\det(g')|} = \left(\frac{\phi(x)}{\phi'(x')} \right)^{n} \left| \det\left(\frac{\partial x}{\partial x'} \right) \right| \sqrt{|\det(g)|}. \tag{2.41}$$

By considering Eqs. (2.38) and (2.41), it is clear that the appropriate volume element on the Lyra manifold, invariant under LRS transformations, is defined by

$$\boxed{dV = \phi^n(x)\sqrt{\left|\det(g)\right|}\, d^n x.} \tag{2.42}$$

2.4 Affine Structure

Tensor fields assign geometric quantities to each point of the manifold, yet these quantities cannot be directly compared across different points. To describe their variation throughout the manifold in a consistent way, it is necessary to introduce an additional rule linking neighboring tangent spaces. This rule defines the affine structure of the manifold.

2.4.1 Connection

The affine structure is encoded in an operator that generalizes the notion of directional differentiation to curved and scaled spaces, determining how tensor fields vary along the manifold while preserving the local scale defined by the Lyra scale function.

Definition 2.18. A *connection* on the manifold $\mathcal{M}$ is an operation

$$\begin{aligned} \boldsymbol{\nabla} : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) &\to \Gamma(T\mathcal{M}) \\ (\boldsymbol{u}, \boldsymbol{v}) &\mapsto \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}, \end{aligned}$$

such that, for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \Gamma(T\mathcal{M})$, $F \in C^\infty(\mathcal{M})$ and $\lambda \in \mathbb{R}$, satisfies the following properties:

(i) Additive on the left: $\boldsymbol{\nabla}_{(\boldsymbol{u}+\boldsymbol{v})}\boldsymbol{w} = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w} + \boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{w}$.

(ii) Additive on the right: $\boldsymbol{\nabla}_{\boldsymbol{u}}(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} + \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w}$.

(iii) Linear on the left with respect to $C^\infty(\mathcal{M})$: $\boldsymbol{\nabla}_{(F\boldsymbol{u})}\boldsymbol{v} = F\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}$.

(iv) Linear on the right with respect to $\mathbb{R}$: $\boldsymbol{\nabla}_{\boldsymbol{u}}(\lambda\boldsymbol{v}) = \lambda\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}$.

(v) Leibniz rule: $\boldsymbol{\nabla}_{\boldsymbol{u}}(F\boldsymbol{v}) = (\boldsymbol{u}F)\boldsymbol{v} + F\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}$.

A connection $\boldsymbol{\nabla}$ is completely determined by its action upon the elements of a Lyra basis $\{\boldsymbol{e}_\mu\}$. By definition, $\boldsymbol{\nabla}_{\boldsymbol{e}_\mu}\boldsymbol{e}_\nu$ is a vector field, and consequently it can be decomposed in the same basis:

$$\boldsymbol{\nabla}_{\boldsymbol{e}_\mu}\boldsymbol{e}_\nu = \Gamma^\lambda{}_{\nu\mu}\boldsymbol{e}_\lambda, \tag{2.43}$$

where $\Gamma^\lambda{}_{\mu\nu}$ are the *connection coefficients* in the given basis. These are not tensor fields.

By using the connection properties and Eq. (2.43), the expression for $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}$ in the Lyra basis $\{\boldsymbol{e}_\mu\}$ can be written as

$$\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} = u^\alpha(\nabla_\mu v^\alpha)\boldsymbol{e}_\alpha, \tag{2.44}$$

where

$$\nabla_\mu v^\alpha := (\boldsymbol{\nabla}_{\boldsymbol{e}_\mu}\boldsymbol{v})^\alpha = \boldsymbol{e}_\mu v^\alpha + \Gamma^\alpha{}_{\nu\mu}v^\nu. \tag{2.45}$$

The quantity $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}$ is also called the *covariant derivative* of $\boldsymbol{v}$ with respect to $\boldsymbol{u}$. The argument $\boldsymbol{v}$ of this derivative can be extended to tensor fields of any rank, including smooth maps. Accordingly, the first four connection properties extend naturally when the second argument is replaced by a general tensor field. Property (v) is generalized by considering the tensor product of two tensor fields in the second argument, as follows:

(v′) $\boldsymbol{\nabla}_{\boldsymbol{u}}(\boldsymbol{T}\otimes\boldsymbol{S}) = (\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{T})\otimes\boldsymbol{S} + \boldsymbol{T}\otimes(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{S})$.

and comparing with (v), one obtains the expression for the covariant derivative of a smooth map:

$$\boldsymbol{\nabla}_{\boldsymbol{u}}F = \boldsymbol{u}F. \tag{2.46}$$

In addition, the notation used in Eq. (2.45) can be generalized for tensor fields:

$$\nabla_\mu F := \boldsymbol{e}_\mu F \quad \text{and} \quad \nabla_\lambda T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} := (\boldsymbol{\nabla}_{\boldsymbol{e}_\lambda}\boldsymbol{T})^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}. \tag{2.47}$$

Thus, property (iv′) expressed in a Lyra basis can be written as

$$\nabla_\lambda\big(T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}S^{\alpha_1\cdots\alpha_m}{}_{\beta_1\cdots\beta_n}\big) = \nabla_\lambda T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}S^{\alpha_1\cdots\alpha_m}{}_{\beta_1\cdots\beta_n} + T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}\nabla_\lambda S^{\alpha_1\cdots\alpha_m}{}_{\beta_1\cdots\beta_n}.$$

In particular, for the contraction of a vector field $\boldsymbol{v}$ with a covector field $\boldsymbol{\omega}$,

$$\boldsymbol{e}_\mu(v^\nu\omega_\nu) = \nabla_\mu(v^\nu\omega_\nu) = (\nabla_\mu v^\nu)\omega_\nu + v^\nu(\nabla_\mu\omega_\nu).$$

From this, one directly obtains:

$$\nabla_\mu\omega_\nu = \boldsymbol{e}_\mu\omega_\nu - \Gamma^\lambda{}_{\nu\mu}\omega_\lambda, \tag{2.48}$$

and, particularly:

$$\boldsymbol{\nabla}_{\boldsymbol{e}_\mu}\boldsymbol{e}^\nu = -\Gamma^\nu{}_{\lambda\mu}\boldsymbol{e}^\lambda. \tag{2.49}$$

All these results lead to the corresponding expression for a tensor field:

$$\begin{aligned}\nabla_\lambda T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} = {} & \boldsymbol{e}_\lambda T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} \\ & + \Gamma^{\mu_1}{}_{\sigma\lambda}T^{\sigma\cdots\mu_r}{}_{\nu_1\cdots\nu_s} + \cdots + \Gamma^{\mu_r}{}_{\sigma\lambda}T^{\mu_1\cdots\sigma}{}_{\nu_1\cdots\nu_s} \\ & - \Gamma^\sigma{}_{\nu_1\lambda}T^{\mu_1\cdots\mu_r}{}_{\sigma\cdots\nu_s} - \cdots - \Gamma^\sigma{}_{\nu_s\lambda}T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\sigma}.\end{aligned} \tag{2.50}$$

The explicit form of the connection coefficients are, in principle, arbitrary. At the end of the chapter, a specific choice will be made according to some conditions.

2.4.2 Parallel Transport

When moving along a curve on a manifold, a tensor can be carried so that it remains "unchanged" relative to the path itself. This captures the intuitive idea of transporting the object without twisting or turning it unnecessarily, preserving its orientation and magnitude in a way that is intrinsic to the manifold.

Definition 2.19. Let $\gamma : I \to \mathcal{M}$ be a smooth curve with tangent vector $\boldsymbol{v}(t)$. A tensor field $\boldsymbol{T} \in \Gamma\big(T\mathcal{M}^{(r,s)}\big)$ is said to be *parallel transported* along this curve if

$$\boldsymbol{\nabla}_{\boldsymbol{v}} \boldsymbol{T} = \boldsymbol{0}. \tag{2.51}$$

In this sense, the curve is said to be *auto-parallel* if its own tangent vector is parallel transported along the curve, i.e., if $\boldsymbol{\nabla}_{\boldsymbol{v}} \boldsymbol{v} = \boldsymbol{0}$.

According to Eq. (2.43), the covariant derivative of the tangent vector $\boldsymbol{v}(t)$ with respect to itself can be written as

$$\boldsymbol{\nabla}_{\boldsymbol{v}} \boldsymbol{v} = v^{\alpha}\big(\phi^{-1}\partial_{\alpha} v^{\mu} + \Gamma^{\mu}{}_{\beta\alpha} v^{\beta}\big)\boldsymbol{e}_{\mu} = \big[\boldsymbol{v}(v^{\mu}) + \Gamma^{\mu}{}_{\alpha\beta} v^{\alpha} v^{\beta}\big]\boldsymbol{e}_{\mu},$$

where, because of the t-parameterization, one can write:

$$v^{\mu} = \frac{dx^{\mu}}{dt} \quad \text{and} \quad \boldsymbol{v}(v^{\mu}) = \frac{d}{dt}\left(\phi \frac{dx^{\mu}}{dt}\right). \tag{2.52}$$

Thus, the resulting expression for the characteristic equation of an auto-parallel curve is:

$$\boxed{\frac{d^2 x^{\mu}}{dt^2} + \left(\phi \Gamma^{\mu}{}_{\alpha\beta} + \delta^{\mu}_{\alpha} \nabla_{\beta}\phi\right) \frac{dx^{\alpha}}{dt}\frac{dx^{\beta}}{dt} = 0.} \tag{2.53}$$

Note that this equation is different from the geodesic equation (2.35), as they represent different types of curves. By considering $\phi = 1$, both equations they take quite similar forms. However, connection coefficients are not necessarily equal to Christoffel symbols, and it is also important to remember that Eq. (2.35) is valid only for linear parameterizations with respect to the curve length.

2.4.3 Torsion, Curvature and Non-Metricity

To describe how a manifold twists and bends, it is useful to quantify the failure of objects to commute under infinitesimal transport and the deviation of nearby paths from one another. With this purpose, two important objects are introduce.

Definition 2.20. The *torsion tensor* on the Lyra manifold $\mathcal{M}$ with connection $\boldsymbol{\nabla}$ is a tensor field $\boldsymbol{\tau} \in T\mathcal{M}^{(1,2)}$, defined as

$$\boldsymbol{\tau}(\boldsymbol{\omega}, \boldsymbol{u}, \boldsymbol{v}) := \boldsymbol{\omega}\big(\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{v} - \boldsymbol{\nabla}_{\boldsymbol{v}} \boldsymbol{u} - [\boldsymbol{u}, \boldsymbol{v}]\big), \tag{2.54}$$

where $\boldsymbol{\omega} \in T\mathcal{M}^*$ and $\boldsymbol{u}, \boldsymbol{v} \in T\mathcal{M}$. Note that the torsion is anti-symmetric with respect to the last two arguments.

By considering a Lyra basis $\{\boldsymbol{e}_{\mu}\}$, one easily obtains the torsion components:

$$\boxed{\tau^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu} - \Gamma^{\alpha}{}_{\mu\nu} - \gamma^{\alpha}{}_{\mu\nu}.} \tag{2.55}$$

Now, in order to see the effect of torsion on the manifold, consider a smooth function $F \in C^{\infty}(\mathcal{M})$, with Euclidean representation equally denoted, and calculate:

$$[\nabla_\mu, \nabla_\nu]F = \nabla_{[\mu}\nabla_{\nu]}F = \boldsymbol{e}_{[\mu}\boldsymbol{e}_{\nu]}F - \Gamma^\alpha{}_{[\nu\mu]}\boldsymbol{e}_\alpha F = \gamma^\alpha{}_{\mu\nu}\boldsymbol{e}_\alpha F - \left(\tau^\alpha{}_{\mu\nu} + \gamma^\alpha{}_{\mu\nu}\right)\boldsymbol{e}_\alpha F.$$

Here, $[\quad]$ denotes unnormalized anti-symmetrization (without factorial). So when a smooth function is transported along two successive directions, the order matters. In fact, the discrepancy arises from the torsion:

$$[\nabla_\mu, \nabla_\nu]F = -\tau^\alpha{}_{\mu\nu}\boldsymbol{e}_\alpha F. \tag{2.56}$$

Definition 2.21. The *curvature tensor* on the Lyra manifold $\mathcal{M}$ with connection $\boldsymbol{\nabla}$ is a tensor field $\boldsymbol{R} \in T\mathcal{M}^{(1,3)}$, defined as

$$\boldsymbol{R}(\boldsymbol{\omega}, \boldsymbol{X}, \boldsymbol{u}, \boldsymbol{v}) := \boldsymbol{\omega}\big([\boldsymbol{\nabla}_{\boldsymbol{u}}, \boldsymbol{\nabla}_{\boldsymbol{v}}]\boldsymbol{X} - \boldsymbol{\nabla}_{[\boldsymbol{u},\boldsymbol{v}]}\boldsymbol{X}\big), \tag{2.57}$$

where $\boldsymbol{\omega} \in T\mathcal{M}^*$ and $\boldsymbol{X}, \boldsymbol{u}, \boldsymbol{v} \in T\mathcal{M}$. Note that the curvature tensor is anti-symmetric with respect to the last two arguments.

By considering a Lyra basis $\{\boldsymbol{e}_\mu\}$, one easily obtains the curvature components:

$$\boxed{R^\alpha{}_{\beta\mu\nu} = \boldsymbol{e}_\mu\Gamma^\alpha{}_{\beta\nu} + \Gamma^\alpha{}_{\lambda\mu}\Gamma^\lambda{}_{\beta\nu} - \boldsymbol{e}_\nu\Gamma^\alpha{}_{\beta\mu} - \Gamma^\alpha{}_{\lambda\nu}\Gamma^\lambda{}_{\beta\mu} - \Gamma^\alpha{}_{\beta\lambda}\gamma^\lambda{}_{\mu\nu}.} \tag{2.58}$$

Now, in order to see the effect of curvature on the manifold, consider a tensor field, for instance $\boldsymbol{A} \in T\mathcal{M}^{(1,1)}$, and calculate:

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu]A^\alpha{}_\beta &= \boldsymbol{e}_{[\mu}\nabla_{\nu]}A^\alpha{}_\beta + \Gamma^\alpha{}_{\lambda[\mu}\nabla_{\nu]}A^\lambda{}_\beta - \Gamma^\lambda{}_{\beta[\mu}\nabla_{\nu]}A^\alpha{}_\lambda - \Gamma^\lambda{}_{[\nu\mu]}\nabla_\lambda A^\alpha{}_\beta \\
&= \boldsymbol{e}_{[\mu}\left(\boldsymbol{e}_{\nu]}A^\alpha{}_\beta + \Gamma^\alpha{}_{\lambda\nu]}A^\lambda{}_\beta - \Gamma^\lambda{}_{\beta\nu]}A^\alpha{}_\lambda\right) \\
&\quad + \Gamma^\alpha{}_{\lambda[\mu}\left(\boldsymbol{e}_{\nu]}A^\lambda{}_\beta + \Gamma^\lambda{}_{\sigma\nu]}A^\sigma{}_\beta - \Gamma^\sigma{}_{\beta\nu]}A^\lambda{}_\sigma\right) \\
&\quad - \Gamma^\lambda{}_{\beta[\mu}\left(\boldsymbol{e}_{\nu]}A^\alpha{}_\lambda + \Gamma^\alpha{}_{\sigma\nu]}A^\sigma{}_\lambda - \Gamma^\sigma{}_{\lambda\nu]}A^\alpha{}_\sigma\right) \\
&\quad - \Gamma^\lambda{}_{[\nu\mu]}\left(\boldsymbol{e}_\lambda A^\alpha{}_\beta + \Gamma^\alpha{}_{\sigma\lambda}A^\sigma{}_\beta - \Gamma^\sigma{}_{\beta\lambda}A^\alpha{}_\sigma\right) \\
&= \gamma^\lambda{}_{\mu\nu}\boldsymbol{e}_\lambda A^\alpha{}_\beta + \boldsymbol{e}_{[\mu}\Gamma^\alpha{}_{\lambda\nu]}A^\lambda{}_\beta + \Gamma^\alpha{}_{\lambda[\nu}\boldsymbol{e}_{\mu]}A^\lambda{}_\beta - \boldsymbol{e}_{[\mu}\Gamma^\lambda{}_{\beta\nu]}A^\alpha{}_\lambda - \Gamma^\lambda{}_{\beta[\nu}\boldsymbol{e}_{\mu]}A^\alpha{}_\lambda \\
&\quad + \Gamma^\alpha{}_{\lambda[\mu}\boldsymbol{e}_{\nu]}A^\lambda{}_\beta + \Gamma^\alpha{}_{\lambda[\mu}\Gamma^\lambda{}_{\sigma\nu]}A^\sigma{}_\beta - \Gamma^\lambda{}_{\beta[\mu}\boldsymbol{e}_{\nu]}A^\alpha{}_\lambda + \Gamma^\lambda{}_{\beta[\mu}\Gamma^\sigma{}_{\lambda\nu]}A^\alpha{}_\sigma \\
&\quad - \gamma^\lambda{}_{\mu\nu}\boldsymbol{e}_\lambda A^\alpha{}_\beta - \gamma^\lambda{}_{\mu\nu}\Gamma^\alpha{}_{\sigma\lambda}A^\sigma{}_\beta + \gamma^\lambda{}_{\mu\nu}\Gamma^\sigma{}_{\beta\lambda}A^\alpha{}_\sigma - \tau^\lambda{}_{\mu\nu}\nabla_\lambda A^\alpha{}_\beta \\
&= \left(\boldsymbol{e}_{[\mu}\Gamma^\alpha{}_{\lambda\nu]} + \Gamma^\alpha{}_{\sigma[\mu}\Gamma^\sigma{}_{\lambda\nu]} - \Gamma^\alpha{}_{\lambda\sigma}\gamma^\sigma{}_{\mu\nu}\right)A^\lambda{}_\beta \\
&\quad - \left(\boldsymbol{e}_{[\mu}\Gamma^\lambda{}_{\beta\nu]} + \Gamma^\lambda{}_{\sigma[\mu}\Gamma^\sigma{}_{\beta\nu]} - \Gamma^\lambda{}_{\beta\sigma}\gamma^\sigma{}_{\mu\nu}\right)A^\alpha{}_\lambda - \tau^\lambda{}_{\mu\nu}\nabla_\lambda A^\alpha{}_\beta \\
&= R^\alpha{}_{\lambda\mu\nu}A^\lambda{}_\beta - R^\lambda{}_{\beta\mu\nu}A^\alpha{}_\lambda - \tau^\lambda{}_{\mu\nu}\nabla_\lambda A^\alpha{}_\beta.
\end{aligned}$$

Here, the anti-symmetrization was performed only upon the indices μ and ν, and groups of terms highlighted with the same color yield a total contribution equal to zero. Furthermore, one can generalize this last result for a tensor of higher rank:

$$\begin{aligned}[\nabla_\mu, \nabla_\nu] T^{\alpha_1\cdots\alpha_r}{}_{\beta_1\cdots\beta_s} = {} & R^{\alpha_1}{}_{\lambda\mu\nu} T^{\lambda\cdots\alpha_r}{}_{\beta_1\cdots\beta_s} + \cdots + R^{\alpha_r}{}_{\lambda\mu\nu} T^{\alpha_1\cdots\lambda}{}_{\beta_1\cdots\beta_s} \\ & - R^{\lambda}{}_{\beta_1\mu\nu} T^{\alpha_1\cdots\alpha_r}{}_{\lambda\cdots\beta_s} - \cdots - R^{\lambda}{}_{\beta_s\mu\nu} T^{\alpha_1\cdots\alpha_r}{}_{\beta_1\cdots\lambda} \\ & - \tau^{\lambda}{}_{\mu\nu} \nabla_\lambda T^{\alpha_1\cdots\alpha_r}{}_{\beta_1\cdots\beta_s}. \end{aligned} \tag{2.59}$$

Therefore, transporting a tensor along two successive directions yields a result that depends on the order in which those directions are taken. The discrepancy arises from both torsion and curvature: the torsion contributes a term analogous to that found in the previous case, while the curvature contributes an additional term for each tensor index.

There are two important mathematical relationships between torsion and curvature tensors, known as the Bianchi identities. The first one is obtained through the following calculation:

$$\begin{aligned}\nabla_{[\mu}\tau^{\alpha}{}_{\nu\beta]} + \tau^{\alpha}{}_{\lambda[\beta}\tau^{\lambda}{}_{\mu\nu]} = {} & \boldsymbol{e}_{[\mu}\tau^{\alpha}{}_{\nu\beta]} + \Gamma^{\alpha}{}_{\lambda[\mu}\tau^{\lambda}{}_{\nu\beta]} - \Gamma^{\lambda}{}_{[\nu\mu}\tau^{\alpha}{}_{\lambda\beta]} - \Gamma^{\lambda}{}_{[\beta\mu}\tau^{\alpha}{}_{\nu]\lambda} \\ & + \tau^{\alpha}{}_{\lambda[\beta}\Gamma^{\lambda}{}_{\nu\mu]} - \tau^{\alpha}{}_{\lambda[\beta}\Gamma^{\lambda}{}_{\mu\nu]} - \tau^{\alpha}{}_{\lambda[\beta}\gamma^{\lambda}{}_{\mu\nu]} \\ = {} & \boldsymbol{e}_{[\mu}\Gamma^{\alpha}{}_{\beta\nu]} - \boldsymbol{e}_{[\mu}\Gamma^{\alpha}{}_{\nu\beta]} - \boldsymbol{e}_{[\mu}\gamma^{\alpha}{}_{\nu\beta]} \\ & + \Gamma^{\alpha}{}_{\lambda[\mu}\Gamma^{\lambda}{}_{\beta\nu]} - \Gamma^{\alpha}{}_{\lambda[\mu}\Gamma^{\lambda}{}_{\nu\beta]} - \Gamma^{\alpha}{}_{\lambda[\mu}\gamma^{\lambda}{}_{\nu\beta]} \\ & - \Gamma^{\alpha}{}_{[\beta\lambda}\gamma^{\lambda}{}_{\mu\nu]} + \Gamma^{\alpha}{}_{\lambda[\beta}\gamma^{\lambda}{}_{\mu\nu]} + \gamma^{\alpha}{}_{\lambda[\beta}\gamma^{\lambda}{}_{\mu\nu]} \end{aligned}$$

Here, the anti-symmetrization was performed only upon the indices β, μ and ν, and groups of terms highlighted with the same color yield a total contribution equal to zero. The sum of the remaining terms yield the anti-symmetrization of the curvature tensor as a result. Thus, one obtains the *first Bianchi identity*:

$$\boxed{R^{\alpha}{}_{[\beta\mu\nu]} = \nabla_{[\beta}\tau^{\alpha}{}_{\mu\nu]} + \tau^{\alpha}{}_{\lambda[\beta}\tau^{\lambda}{}_{\mu\nu]}.} \tag{2.60}$$

For the second relation, calculate:

$$\nabla_{[\rho}R^{\alpha}{}_{\beta\mu\nu]} = \boldsymbol{e}_{[\rho}R^{\alpha}{}_{\beta\mu\nu]} + \Gamma^{\alpha}{}_{\lambda[\rho}R^{\lambda}{}_{\beta\mu\nu]} - \Gamma^{\lambda}{}_{\beta[\rho}R^{\alpha}{}_{\lambda\mu\nu]} - \Gamma^{\lambda}{}_{[\mu\rho}R^{\alpha}{}_{\beta\lambda\nu]} - \Gamma^{\lambda}{}_{[\nu\rho}R^{\alpha}{}_{\beta\mu]\lambda},$$

where the anti-symmetrization is performed only upon the indices ρ, μ and ν. Now, expand the three first terms on the right hand side:

$$\begin{aligned}\boldsymbol{e}_{[\rho}R^{\alpha}{}_{\beta\mu\nu]} = {} & \boldsymbol{e}_{[\rho}\left(\boldsymbol{e}_{\mu}\Gamma^{\alpha}{}_{\beta\nu]} - \boldsymbol{e}_{\nu}\Gamma^{\alpha}{}_{\beta\mu]} + \Gamma^{\alpha}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\beta\nu]} - \Gamma^{\alpha}{}_{\lambda\nu]}\Gamma^{\lambda}{}_{\beta\mu]} - \Gamma^{\alpha}{}_{\beta\lambda}\gamma^{\lambda}{}_{\mu\nu]}\right) \\ = {} & \left(\boldsymbol{e}_{[\rho}\boldsymbol{e}_{\mu}\Gamma^{\alpha}{}_{\beta\nu]} - \boldsymbol{e}_{[\mu}\boldsymbol{e}_{\rho}\Gamma^{\alpha}{}_{\beta\nu]}\right) + \boldsymbol{e}_{[\rho}\Gamma^{\alpha}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\beta\nu]} + \Gamma^{\alpha}{}_{\lambda[\mu}\boldsymbol{e}_{\rho}\Gamma^{\lambda}{}_{\beta\nu]} \\ & - \boldsymbol{e}_{[\rho}\Gamma^{\alpha}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\beta\mu]} - \Gamma^{\alpha}{}_{\lambda[\nu}\boldsymbol{e}_{\rho}\Gamma^{\lambda}{}_{\beta\mu]} - \boldsymbol{e}_{[\rho}\Gamma^{\alpha}{}_{\beta\lambda}\gamma^{\lambda}{}_{\mu\nu]} - \Gamma^{\alpha}{}_{\beta\lambda}\cancel{\boldsymbol{e}_{[\rho}\gamma^{\lambda}{}_{\mu\nu]}} \\ = {} & \gamma^{\lambda}{}_{[\rho\mu}\boldsymbol{e}_{\lambda}\Gamma^{\alpha}{}_{\beta\nu]} - \boldsymbol{e}_{[\rho}\Gamma^{\alpha}{}_{\beta\lambda}\gamma^{\lambda}{}_{\mu\nu]} + \boldsymbol{e}_{[\rho}\Gamma^{\alpha}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\beta\nu]} \\ & + \Gamma^{\alpha}{}_{\lambda[\mu}\boldsymbol{e}_{\rho}\Gamma^{\lambda}{}_{\beta\nu]} - \boldsymbol{e}_{[\rho}\Gamma^{\alpha}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\beta\mu]} - \Gamma^{\alpha}{}_{\lambda[\nu}\boldsymbol{e}_{\rho}\Gamma^{\lambda}{}_{\beta\mu]} \end{aligned}$$

$$\begin{aligned}\Gamma^{\alpha}{}_{\lambda[\rho}R^{\lambda}{}_{\beta\mu\nu]} = \Gamma^{\alpha}{}_{\lambda\rho}\boldsymbol{e}_{\mu}\Gamma^{\lambda}{}_{\beta\nu]} - \Gamma^{\alpha}{}_{\lambda[\rho}\boldsymbol{e}_{\nu}\Gamma^{\lambda}{}_{\beta\mu]} + \Gamma^{\alpha}{}_{\lambda[\rho}\Gamma^{\lambda}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\beta\nu]} \\ - \Gamma^{\alpha}{}_{\lambda[\rho}\Gamma^{\lambda}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\beta\mu]} - \Gamma^{\alpha}{}_{\lambda[\rho}\Gamma^{\lambda}{}_{\beta\sigma}\gamma^{\sigma}{}_{\mu\nu]}\end{aligned}$$

$$\begin{aligned}-\Gamma^{\lambda}{}_{\beta[\rho}R^{\alpha}{}_{\lambda\mu\nu]} = -\Gamma^{\lambda}{}_{\beta[\rho}\boldsymbol{e}_{\mu}\Gamma^{\alpha}{}_{\lambda\nu]} + \Gamma^{\lambda}{}_{\beta[\rho}\boldsymbol{e}_{\nu}\Gamma^{\alpha}{}_{\lambda\mu]} - \Gamma^{\lambda}{}_{\beta[\rho}\Gamma^{\alpha}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\lambda\nu]} \\ + \Gamma^{\lambda}{}_{\beta[\rho}\Gamma^{\alpha}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\lambda\mu]} + \Gamma^{\lambda}{}_{\beta[\rho}\Gamma^{\alpha}{}_{\lambda\sigma}\gamma^{\sigma}{}_{\mu\nu]}\end{aligned}$$

Here, groups of terms highlighted with the same color yield a total contribution equal to zero. The remaining terms yield:

$$\begin{aligned}\nabla_{[\rho}R^{\alpha}{}_{\beta\mu\nu]} &= \gamma^{\lambda}{}_{[\rho\mu}\big(\boldsymbol{e}_{\lambda}\Gamma^{\alpha}{}_{\beta\nu]} - \boldsymbol{e}_{\nu]}\Gamma^{\alpha}{}_{\beta\lambda} + \Gamma^{\alpha}{}_{\sigma\lambda}\Gamma^{\sigma}{}_{\beta\nu]} - \Gamma^{\alpha}{}_{\sigma\nu]}\Gamma^{\sigma}{}_{\beta\lambda}\big) \\ &\quad + \Gamma^{\lambda}{}_{[\rho\mu}R^{\alpha}{}_{\beta\lambda\nu]} - \Gamma^{\lambda}{}_{[\mu\rho}R^{\alpha}{}_{\beta\lambda\nu]} \\ &= -\big(\Gamma^{\lambda}{}_{[\mu\rho} - \Gamma^{\lambda}{}_{[\rho\mu} - \gamma^{\lambda}{}_{[\rho\mu}\big)R^{\alpha}{}_{\beta\lambda\nu]} - \Gamma^{\alpha}{}_{\beta\sigma}\cancel{\gamma^{\sigma}{}_{\lambda[\nu}\gamma^{\lambda}{}_{\rho\mu]}}.\end{aligned}$$

Then, one obtains the *second Bianchi identity*:

$$\boxed{R^{\alpha}{}_{\beta[\mu\nu;\rho]} + R^{\alpha}{}_{\beta\lambda[\rho}\tau^{\lambda}{}_{\mu\nu]} = 0,} \tag{2.61}$$

where the notation $R^{\alpha}{}_{\beta\mu\nu;\rho} := \nabla_{\rho}R^{\alpha}{}_{\beta\mu\nu}$ was used in order to emphasize that β is not involved in the anti-symmetrization.

To complete the set of relevant object, the standard curvature contractions are defined: First, the *Ricci tensor*, whose components are given by

$$R_{\mu\nu} := R^{\alpha}{}_{\mu\alpha\nu}; \tag{2.62}$$

and, subsequently, the *Ricci scalar*, obtained through the trace of the Ricci tensor:

$$R := R^{\alpha}{}_{\alpha}. \tag{2.63}$$

One may also consider the *Kretschmann scalar*, built from the full contraction of the Riemann tensor with itself:

$$K = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}. \tag{2.64}$$

While both R and K encode curvature information, the Ricci scalar measures the trace part of curvature and can vanish even in strongly curved spacetimes, whereas the Kretschmann scalar captures the total curvature strength and is sensitive to all components of the Riemann tensor, including tidal and singular contributions.

Definition 2.22. The *non-metricity tensor* on the Lyra manifold $\mathcal{M}$ with connection $\boldsymbol{\nabla}$ is a tensor field $\boldsymbol{M} \in T\mathcal{M}^{(0,3)}$, defined as

$$\boldsymbol{M}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) := \big(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{g}\big)(\boldsymbol{v},\boldsymbol{w}) - \big(\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{g}\big)(\boldsymbol{u},\boldsymbol{w}) - \big(\boldsymbol{\nabla}_{\boldsymbol{w}}\boldsymbol{g}\big)(\boldsymbol{v},\boldsymbol{u}), \tag{2.65}$$

where $\boldsymbol{g}$ is the metric tensor and $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in T\mathcal{M}$.

In a Lyra basis $\{\boldsymbol{e}_\mu\}$, the components of the non-metricity tensor are

$$M_{\alpha\mu\nu} = \nabla_\alpha g_{\mu\nu} - \nabla_\mu g_{\alpha\nu} - \nabla_\nu g_{\mu\alpha}. \tag{2.66}$$

Equivalently,

$$\begin{aligned}
M_{\alpha\mu\nu} &= (\phi^{-1}\partial_\alpha g_{\mu\nu} - \Gamma^\lambda{}_{\mu\alpha} g_{\lambda\nu} - \Gamma^\lambda{}_{\nu\alpha} g_{\mu\lambda}) - (\phi^{-1}\partial_\mu g_{\alpha\nu} - \Gamma^\lambda{}_{\alpha\mu} g_{\lambda\nu} - \Gamma^\lambda{}_{\nu\mu} g_{\alpha\lambda}) \\
&\quad - (\phi^{-1}\partial_\nu g_{\mu\alpha} - \Gamma^\lambda{}_{\mu\nu} g_{\lambda\alpha} - \Gamma^\lambda{}_{\alpha\nu} g_{\mu\lambda}) \\
&= \phi^{-1}(\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\alpha\nu} - \partial_\nu g_{\mu\alpha}) - (\Gamma^\lambda{}_{\mu\alpha} - \Gamma^\lambda{}_{\alpha\mu}) g_{\lambda\nu} \\
&\quad - (\Gamma^\lambda{}_{\nu\alpha} - \Gamma^\lambda{}_{\alpha\nu}) g_{\mu\lambda} - (\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}) g_{\lambda\alpha} + 2\Gamma^\lambda{}_{\mu\nu} g_{\lambda\alpha} \\
&= -2\phi^{-1}\{\alpha\mu\nu\} - (\tau_{\nu\alpha\mu} + \tau_{\mu\alpha\nu} + \tau_{\alpha\nu\mu}) - (\gamma_{\nu\alpha\mu} + \gamma_{\mu\alpha\nu} + \gamma_{\alpha\nu\mu}) + 2\Gamma_{\alpha\mu\nu},
\end{aligned} \tag{2.67}$$

where the following notations have been introduced:

$$\{\alpha\mu\nu\} := g_{\alpha\lambda}\left\{ {\lambda \atop \mu\nu} \right\}, \quad \gamma_{\alpha\nu\mu} := g_{\alpha\lambda}\gamma^\lambda{}_{\nu\mu} \quad \text{and} \quad \Gamma_{\alpha\mu\nu} := g_{\alpha\lambda}\Gamma^\lambda{}_{\mu\nu}. \tag{2.68}$$

Solving Eq. (2.67) for $\Gamma_{\alpha\mu\nu}$ and contracting with $g^{\lambda\alpha}$ yields the most general expression of the connection coefficients:

$$\begin{aligned}
\Gamma^\lambda{}_{\mu\nu} &= \phi^{-1}\left\{ {\lambda \atop \mu\nu} \right\} + \frac{1}{2} g^{\lambda\alpha}(\gamma_{\mu\alpha\nu} + \gamma_{\nu\alpha\mu} - \gamma_{\alpha\mu\nu}) \\
&\quad + \frac{1}{2} M^\lambda{}_{\mu\nu} + \frac{1}{2} g^{\lambda\alpha}(\tau_{\mu\alpha\nu} + \tau_{\nu\alpha\mu} - \tau_{\alpha\mu\nu}).
\end{aligned} \tag{2.69}$$

A change of the Lyra reference system produces the following transformation of the connection coefficients. Starting from

$$\Gamma'^{\alpha}{}_{\mu\nu} \boldsymbol{e}'_\alpha = \boldsymbol{\nabla}_{\boldsymbol{e}'_\nu} \boldsymbol{e}'_\mu = \boldsymbol{\nabla}_{\boldsymbol{e}'_\nu} \left(\frac{\phi}{\phi'} \frac{\partial x^\rho}{\partial x'^\mu} \boldsymbol{e}_\rho \right),$$

one finds

$$\begin{aligned}
\Gamma'^{\alpha}{}_{\mu\nu} \boldsymbol{e}'_\alpha &= \left(\frac{1}{\phi'} \nabla'_\nu \phi \frac{\partial x^\rho}{\partial x'^\mu} - \frac{\phi}{\phi'^2} \nabla'_\nu \phi' \frac{\partial x^\rho}{\partial x'^\mu} + \frac{\phi}{\phi'^2} \frac{\partial^2 x^\rho}{\partial x'^\nu \partial x'^\mu} \right) \boldsymbol{e}_\rho + \frac{\phi}{\phi'} \frac{\partial x^\rho}{\partial x'^\mu} \boldsymbol{\nabla}_{\boldsymbol{e}'_\nu} \boldsymbol{e}_\rho \\
&= \left(\frac{1}{\phi} \nabla'_\nu \phi - \frac{1}{\phi'} \nabla'_\nu \phi' \right) \boldsymbol{e}'_\mu + \frac{1}{\phi'} \frac{\partial x'^\alpha}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} \boldsymbol{e}'_\alpha + \frac{\phi^2}{\phi'^2} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma^\lambda{}_{\rho\sigma} \boldsymbol{e}_\lambda \\
&= \delta^\alpha_\mu \left(\frac{1}{\phi} \nabla'_\nu \phi - \frac{1}{\phi'} \nabla'_\nu \phi' \right) \boldsymbol{e}'_\alpha + \frac{1}{\phi'} \frac{\partial x'^\alpha}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} \boldsymbol{e}'_\alpha + \frac{\phi'}{\phi} \frac{\partial x'^\alpha}{\partial x^\lambda} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma^\lambda{}_{\rho\sigma} \boldsymbol{e}'_\alpha.
\end{aligned}$$

Thus the general transformation law for the connection coefficients in Lyra geometry is

$$\Gamma'^{\alpha}{}_{\mu\nu} = \frac{\phi'}{\phi} \frac{\partial x'^\alpha}{\partial x^\lambda} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma^\lambda{}_{\rho\sigma} + \frac{1}{\phi'} \frac{\partial x'^\alpha}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} + \delta^\alpha_\mu \left(\frac{1}{\phi} \nabla'_\nu \phi - \frac{1}{\phi'} \nabla'_\nu \phi' \right), \tag{2.70}$$

which explicitly shows that connection coefficients do not transform as tensor components.

2.4.4 Cartan Structure Equations

In this part, the formalism of exterior calculus developed by Cartan is employed (see Ref. [44]). To begin with, consider the exterior derivative of the elements of dual Lyra bases:

$$de^\mu = \partial_\nu \phi \, dx^\nu \wedge dx^\mu = \phi^{-2} \delta^\mu_\beta \partial_\nu \phi \, e^\nu \wedge e^\beta = -\frac{1}{2} \gamma^\mu{}_{\nu\beta} \, e^\nu \wedge e^\beta = \frac{1}{2} (\tau^\mu{}_{\nu\beta} + \Gamma^\mu{}_{\nu\beta} - \Gamma^\mu{}_{\beta\nu}) \, e^\nu \wedge e^\beta.$$

By defining the *connection one-form*

$$\boldsymbol{\Omega}^\mu{}_\nu := \Gamma^\mu{}_{\nu\beta} \, \boldsymbol{e}^\beta, \tag{2.71}$$

and the *torsion two-form*

$$\boldsymbol{\tau}^\mu := \frac{1}{2} \tau^\mu{}_{\nu\beta} \, \boldsymbol{e}^\nu \wedge \boldsymbol{e}^\beta, \tag{2.72}$$

one immediately obtains the *first Cartan structure equation*:

$$\boxed{\boldsymbol{\tau}^\mu = d\boldsymbol{e}^\mu - \boldsymbol{e}^\nu \wedge \boldsymbol{\Omega}^\mu{}_\nu.} \tag{2.73}$$

Now, compute the exterior derivative of this result using the fact that $d^2 = 0$:

$$\begin{aligned}
0 &= d\boldsymbol{\tau}^\mu + d\boldsymbol{e}^\nu \wedge \boldsymbol{\Omega}^\mu{}_\nu - \boldsymbol{e}^\nu \wedge d\boldsymbol{\Omega}^\mu{}_\nu \\
&= \frac{1}{2} d\tau^\mu{}_{\alpha\beta} \, \boldsymbol{e}^\alpha \wedge \boldsymbol{e}^\beta + \frac{1}{2} \tau^\mu{}_{\alpha\beta} \, d\boldsymbol{e}^\alpha \wedge \boldsymbol{e}^\beta - \frac{1}{2} \tau^\mu{}_{\alpha\beta} \, \boldsymbol{e}^\alpha \wedge d\boldsymbol{e}^\beta \\
&\quad + \left(\boldsymbol{\tau}^\nu + \boldsymbol{e}^\alpha \wedge \boldsymbol{\Omega}^\nu{}_\alpha \right) \wedge \boldsymbol{\Omega}^\mu{}_\nu - \boldsymbol{e}^\nu \wedge d\boldsymbol{\Omega}^\mu{}_\nu \\
&= \frac{1}{2} e_\nu \tau^\mu{}_{\alpha\beta} \, \boldsymbol{e}^\nu \wedge \boldsymbol{e}^\alpha \wedge \boldsymbol{e}^\beta + \frac{1}{2} \tau^\mu{}_{\alpha\beta} \left(\boldsymbol{\tau}^\alpha + \boldsymbol{e}^\lambda \wedge \boldsymbol{\Omega}^\alpha{}_\lambda \right) \wedge \boldsymbol{e}^\beta - \frac{1}{2} \tau^\mu{}_{\alpha\beta} \, \boldsymbol{e}^\alpha \wedge \left(\boldsymbol{\tau}^\beta + \boldsymbol{e}^\lambda \wedge \boldsymbol{\Omega}^\beta{}_\lambda \right) \\
&\quad + \frac{1}{2} \tau^\nu{}_{\alpha\beta} \, \boldsymbol{e}^\alpha \wedge \boldsymbol{e}^\beta \wedge \boldsymbol{\Omega}^\mu{}_\nu + \boldsymbol{e}^\alpha \wedge \boldsymbol{\Omega}^\nu{}_\alpha \wedge \boldsymbol{\Omega}^\mu{}_\nu - \boldsymbol{e}^\nu \wedge d\boldsymbol{\Omega}^\mu{}_\nu \\
&= \frac{1}{2} e_\nu \tau^\mu{}_{\alpha\beta} \, \boldsymbol{e}^\nu \wedge \boldsymbol{e}^\alpha \wedge \boldsymbol{e}^\beta + \frac{1}{2} \tau^\mu{}_{\alpha\beta} \tau^\alpha{}_{\nu\lambda} \, \boldsymbol{e}^\nu \wedge \boldsymbol{e}^\lambda \wedge \boldsymbol{e}^\beta + \frac{1}{2} \Gamma^\alpha{}_{\lambda\nu} \tau^\mu{}_{\alpha\beta} \, \boldsymbol{e}^\lambda \wedge \boldsymbol{e}^\nu \wedge \boldsymbol{e}^\beta \\
&\quad - \frac{1}{2} \Gamma^\beta{}_{\lambda\nu} \tau^\mu{}_{\alpha\beta} \boldsymbol{e}^\alpha \wedge \boldsymbol{e}^\lambda \wedge \boldsymbol{e}^\nu + \frac{1}{2} \Gamma^\mu{}_{\nu\lambda} \tau^\nu{}_{\alpha\beta} \, \boldsymbol{e}^\alpha \wedge \boldsymbol{e}^\beta \wedge \boldsymbol{e}^\lambda + \boldsymbol{e}^\alpha \wedge \boldsymbol{\Omega}^\nu{}_\alpha \wedge \boldsymbol{\Omega}^\mu{}_\nu - \boldsymbol{e}^\nu \wedge d\boldsymbol{\Omega}^\mu{}_\nu \\
&= \frac{1}{2} \left(e_\nu \tau^\mu{}_{\alpha\beta} + \Gamma^\mu{}_{\lambda\nu} \tau^\lambda{}_{\alpha\beta} - \Gamma^\lambda{}_{\alpha\nu} \tau^\mu{}_{\lambda\beta} - \Gamma^\lambda{}_{\beta\nu} \tau^\mu{}_{\alpha\lambda} + \tau^\mu{}_{\lambda\nu} \tau^\lambda{}_{\alpha\beta} \right) \, \boldsymbol{e}^\nu \wedge \boldsymbol{e}^\alpha \wedge \boldsymbol{e}^\beta \\
&\quad + \boldsymbol{e}^\nu \wedge \boldsymbol{\Omega}^\lambda{}_\nu \wedge \boldsymbol{\Omega}^\mu{}_\lambda - \boldsymbol{e}^\nu \wedge d\boldsymbol{\Omega}^\mu{}_\nu \\
&= \frac{1}{2} \left(\nabla_\nu \tau^\mu{}_{\alpha\beta} + \tau^\mu{}_{\lambda\nu} \tau^\lambda{}_{\alpha\beta} \right) \, \boldsymbol{e}^\nu \wedge \boldsymbol{e}^\alpha \wedge \boldsymbol{e}^\beta + \boldsymbol{e}^\nu \wedge \boldsymbol{\Omega}^\lambda{}_\nu \wedge \boldsymbol{\Omega}^\mu{}_\lambda - \boldsymbol{e}^\nu \wedge d\boldsymbol{\Omega}^\mu{}_\nu \\
&= \frac{1}{2} R^\mu{}_{\nu\alpha\beta} \, \boldsymbol{e}^\nu \wedge \boldsymbol{e}^\alpha \wedge \boldsymbol{e}^\beta + \boldsymbol{e}^\nu \wedge \boldsymbol{\Omega}^\lambda{}_\nu \wedge \boldsymbol{\Omega}^\mu{}_\lambda - \boldsymbol{e}^\nu \wedge d\boldsymbol{\Omega}^\mu{}_\nu.
\end{aligned} \tag{2.74}$$

In the last step, the first Bianchi identity (2.60) was employed. By defining the *curvature two-form*

$$\boldsymbol{R}^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\alpha\beta} \, \boldsymbol{e}^\alpha \wedge \boldsymbol{e}^\beta, \tag{2.75}$$

and because of the linear independence of the dual basis $\{\boldsymbol{e}^\nu\}$, the expression (2.74) yields the *second Cartan structure equation*:

$$\boxed{\boldsymbol{R}^\mu{}_\nu = d\boldsymbol{\Omega}^\mu{}_\nu + \boldsymbol{\Omega}^\mu{}_\lambda \wedge \boldsymbol{\Omega}^\lambda{}_\nu.} \tag{2.76}$$

2.4.5 Divergence Theorem

In Euclidean spaces, the flux of a vector field across the boundary of a volume is closely related to its divergence inside through the so called divergence theorem, which is expected to have an analogous version for Lyra manifolds. Consider a hypervolume $V \subset \mathcal{M}$ with boundary ∂V, a vector field $\boldsymbol{A} \in T\mathcal{M}$, and the following integral:

$$\int_V d^n x\, \phi^n \sqrt{|\det(g)|}\nabla_\nu A^\nu = \int_V d^n x\, \phi^n \sqrt{|\det(g)|}\left(\phi^{-1}\partial_\nu A^\nu + \Gamma^\nu{}_{\mu\nu} A^\mu\right), \tag{2.77}$$

where, according to Eq. (2.69):

$$\begin{aligned}\Gamma^\nu{}_{\mu\nu} &= \phi^{-1}\left\{ {}^{\,\nu}_{\mu\nu} \right\} - \gamma^\nu{}_{\mu\nu} + \frac{1}{2} M^\nu{}_{\mu\nu} - \tau^\nu{}_{\mu\nu} \\ &= \frac{1}{2}\phi^{-1} g^{\nu\lambda}\partial_\mu g_{\nu\lambda} + (n-1)\phi^{-2}\partial_\mu\phi - \frac{1}{2} g^{\nu\lambda}\nabla_\mu g_{\nu\lambda} - \tau^\nu{}_{\mu\nu}.\end{aligned} \tag{2.78}$$

By using the property

$$\partial_\mu |\det(g)| = |\det(g)| g^{\nu\lambda}\partial_\mu g_{\nu\lambda}, \tag{2.79}$$

one obtains the following relation:

$$\frac{1}{2} g^{\nu\lambda}\partial_\mu g_{\nu\lambda} = \frac{1}{\sqrt{|\det(g)|}}\partial_\mu \sqrt{|\det(g)|}. \tag{2.80}$$

Now, by substituting Eq. (2.80) into Eq. (2.78), and substituting the result into Eq. (2.77), the first integral can be written as the sum of two others:

$$\begin{aligned} I_1 &= \int_V d^n x \left(\phi^{n-1}\sqrt{|\det(g)|}\partial_\mu A^\mu + \phi^{n-1}\partial_\mu \sqrt{|\det(g)|} A^\mu + (n-1)\phi^{n-2}\partial_\mu \phi \sqrt{|\det(g)|} A^\mu\right) \\ &= \int_V d^n x\, \partial_\mu \left(\phi^{n-1}\sqrt{|\det(g)|} A^\mu\right) = \int_{\partial V} d^{n-1} x\, \phi^{n-1}\sqrt{|\det(g)|} n_\mu A^\mu. \end{aligned}$$

where $\boldsymbol{n} = n^\mu \boldsymbol{e}_\mu$ is the unit vector normal to the boundary ∂V; and

$$I_1 = \int_V d^n x\, \phi^n \sqrt{|\det(g)|}\frac{1}{2} g^{\nu\lambda}(\nabla_\mu g_{\nu\lambda} + 2\tau_{\lambda\mu\nu})A^\lambda.$$

Therefore, there is a generalized divergence theorem for Lyra manifolds:

$$\boxed{\int_V d^n x\, \phi^n \sqrt{|\det(g)|}\nabla_\nu A^\nu = \int_{\partial V} d^{n-1} x\, \phi^{n-1}\sqrt{|\det(g)|} n_\mu A^\mu,} \tag{2.81}$$

which holds only if the following relation is satisfied:

$$\nabla_\mu g_{\alpha\beta} = 2\tau_{(\alpha\beta)\mu}. \tag{2.82}$$

This condition is necessary for the divergence theorem to be a direct extension of the Riemannian version, and it is important for the discussion at the beginning of the next chapter.

2.5 Symmetries in Lyra geometry

Sometimes, certain objects (scalar, vector, tensor, etc.) defined on Lyra manifold may be invariant under the action of some transformation group:

$$\bar{T}(\bar{x}) = T(x). \tag{2.83}$$

When this happens, it is said that there is *a symmetry for the object under such a group*. In particular, if the group is acting only upon Lyra Reference Systems, there is a *Lyra symmetry* for the object.

Consider an infinitesimal LRS transformation. This is parameterized according to

$$\begin{cases} x^\mu & \to \bar{x}^\mu = x^\mu + \epsilon\xi^\mu(x) \\ \phi(x) & \to \bar{\phi}(\bar{x}) = \phi(x) + \epsilon\Pi(x) \end{cases} \tag{2.84}$$

where ϵ is an arbitrary infinitesimal parameter, and ξ^m and Π are Killing fields containing information about the transformation. For the purpose of this work it will only be necessary to consider spacetime transformations—that is $\Pi = 0$—acting upon scalar fields and covariant tensor fields of rank 2:

- Let Ψ be a scalar field defined on $\mathcal{M}$; then, under (2.84), it transforms to first order in ϵ as

  $$\bar{\Psi}(\bar{x}) = \bar{\Psi}(x + \epsilon\xi) = \bar{\Psi}(x) + \epsilon\xi^\mu\partial_\mu\Psi(x). \tag{2.85}$$

  Since Ψ is a scalar field, it follows that $\bar{\Psi}(\bar{x}) = \Psi(x)$. Then,

  $$\Psi(x) = \bar{\Psi}(x) + \epsilon\xi^\mu\partial_\mu\Psi(x).$$

  Now, applying Eq. (2.83) to Ψ implies:

  $$\xi^\mu\partial_\mu\Psi(x) = 0. \tag{2.86}$$

  This is the necessary condition in order to have a Lyra symmetry for the scalar field Ψ.

- Let $\boldsymbol{T}$ be a covariant tensor field of rank 2. It transforms according to

  $$\bar{T}_{\alpha\beta}(\bar{x}) = \left(\frac{\phi(x)}{\bar{\phi}(\bar{x})}\right)^2 \frac{\partial x^\mu}{\partial\bar{x}^\alpha}\frac{\partial x^\nu}{\partial\bar{x}^\beta} T_{\mu\nu}(x). \tag{2.87}$$

  From Eq. (2.84), one obtains

  $$\frac{\partial x^\mu}{\partial\bar{x}^\alpha} = \delta^\mu_\alpha - \epsilon\partial_\alpha\xi^\mu, \tag{2.88}$$

  and recall that $\Pi = 0$, as it was established before. Then, Eq. (2.87) can be rewritten as

  $$\bar{T}_{\alpha\beta}(\bar{x}) = T_{\alpha\beta}(x) - \epsilon\partial_\alpha\xi^\mu T_{\mu\beta} - \epsilon\partial_\beta\xi^\mu T_{\alpha\mu}.$$

  By Taylor expanding the left hand side, the above relation becomes

  $$\bar{T}_{\alpha\beta}(x) = T_{\alpha\beta}(x) - \epsilon\partial_\alpha\xi^\mu T_{\mu\beta} - \epsilon\partial_\beta\xi^\mu T_{\alpha\mu} - \epsilon\xi^\mu\partial_\mu T_{\alpha\beta}(x).$$

Finally, from Eq. (2.83), one obtains the necessary condition for symmetry:

$$\partial_\alpha \xi^\mu T_{\mu\beta} + \partial_\beta \xi^\mu T_{\alpha\mu} + \xi^\mu \partial_\mu T_{\alpha\beta}(x) = 0. \tag{2.89}$$

In particular, when both the Lyra scale function and the metric tensor are symmetric under the action of spacetime transformations, in the sense of Eq. (2.83), it is said that there is a *Lyra isometry*.

Therefore, $g_{\alpha\beta}$ satisfies Eq. (2.89), which can be rewritten as

$$\begin{aligned}\partial_\alpha \xi^\mu g_{\mu\beta} + \partial_\beta \xi^\mu g_{\mu\alpha} + \xi^\mu \partial_\mu g_{\alpha\beta} &= \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha - \xi^\mu (\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta}) \\ &= \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha - 2\xi_\mu \left\{ {}^{\mu}_{\alpha\beta} \right\} \\ &= \phi \nabla_\alpha \xi_\beta + \phi \nabla_\beta \xi_\alpha + 2\phi \Gamma^\mu{}_{(\alpha\beta)} \xi_\mu - 2\xi_\mu \left\{ {}^{\mu}_{\alpha\beta} \right\}.\end{aligned}$$

Finally, by substituting Eq. (2.69), one obtains

$$\begin{aligned}\partial_\alpha \xi^\mu g_{\mu\beta} + \partial_\beta \xi^\mu g_{\mu\alpha} + \xi^\mu \partial_\mu g_{\alpha\beta} &= \phi \nabla_\alpha \xi_\beta + \phi \nabla_\beta \xi_\alpha - 2\phi \xi^\mu \gamma_{(\alpha\beta)\mu} - 2\phi \xi^\mu \tau_{(\alpha\beta)\mu} + \phi \xi^\mu M_{\mu\alpha\beta} \\ &= \nabla_\alpha (\phi \xi_\beta) + \nabla_\beta (\phi \xi_\alpha) - 2 g_{\alpha\beta} \xi^\mu \nabla_\mu \phi - 2\phi \xi^\mu \tau_{(\alpha\beta)\mu} + \phi \xi^\mu M_{\mu\alpha\beta},\end{aligned}$$

and, since ϕ satisfies Eq. (2.86), it follows that

$$\nabla_\alpha (\phi \xi_\beta) + \nabla_\beta (\phi \xi_\alpha) = \xi^\mu \left(2\phi \tau_{(\alpha\beta)\mu} - \phi M_{\mu\alpha\beta} \right), \tag{2.90}$$

This equation provides the necessary condition for Lyra isometry under spacetime transformations, namely, the invariance of the metric tensor under Lyra transformations with $\Pi = 0$. This symmetry plays an important role in the analysis of energy–momentum conservation, as will be discussed in Chapter 3, and it will also prove useful in Chapter 4 for simplifying the field equations.

Chapter 3

Lyra Scalar-Tensor Gravity

Having established the mathematical structure of Lyra manifolds in Chapter 1, this chapter marks the point at which physical considerations enter the construction of the gravitational theory. The transition from pure geometry to physics requires identifying which geometric structures can meaningfully describe gravitational phenomena and how they give rise to dynamical laws. The aim of this chapter is therefore to ground the geometry in physical principles and to show how those principles lead naturally to a determinate spacetime connection, to well-defined dynamical equations, and to a description of matter and its interactions with the gravitational field. Only once these physical inputs are introduced does the general geometric framework acquire predictive power, and it is in this sense that this chapter initiates the actual formulation of the gravitational theory.

3.1 Preliminary Considerations

A gravitational theory must provide an unambiguous description of how freely falling matter moves. The trajectories of test particles are not an auxiliary detail but the very foundation of the theory: they determine how spacetime geometry is detected, how gravitational fields are measured, and how the theory connects with observation. For this reason, the geometric structure of spacetime cannot be chosen arbitrarily; it must be compatible with a physically meaningful notion of free fall.

In the general affine framework of Chapter 1, geodesics (Eq. 2.35) defined by extremizing the length do not necessarily coincide with auto-parallels (Eq. 2.53) defined by the affine connection. This ambiguity raises the question of which curves should represent the motion of test particles and, consequently, which geometric quantities carry physical significance. In this sense, the theory offers two distinct candidates for the motion of test particles, with no physical reason to prefer one over the other. To remove this ambiguity, the two notions must coincide:

$$
\begin{aligned}
&\frac{d^2x^\mu}{dt^2} + \left(\phi \Gamma^\mu{}_{\alpha\beta} + \delta^\mu_\alpha \nabla_\beta \phi\right) \frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = \\
&\frac{d^2x^\mu}{dt^2} + \left\{ {\mu \atop \alpha\beta} \right\} \frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} + \left(\delta^\mu_\alpha \nabla_\beta \phi + \delta^\mu_\beta \nabla_\alpha \phi - g_{\alpha\beta} g^{\mu\nu} \nabla_\nu \phi\right) \frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}.
\end{aligned}
$$

Therefore, the symmetric part of the connection coefficients are

$$
\Gamma^\mu{}_{(\alpha\beta)} = \phi^{-1} \left\{ {\mu \atop \alpha\beta} \right\} + \phi^{-1} \left(\delta^\mu_{(\alpha} \nabla_{\beta)} \phi - g_{\alpha\beta} g^{\mu\nu} \nabla_\nu \phi\right). \tag{3.1}
$$

On the other hand, symmetrization of Eq. (2.69) yields

$$\Gamma^{\mu}{}_{(\alpha\beta)} = \phi^{-1}\left\{{}^{\ \mu}_{\alpha\beta}\right\} + \frac{1}{2}M^{\mu}{}_{\alpha\beta} + \phi^{-1}\left(\delta^{\mu}_{(\alpha}\nabla_{\beta)}\phi - g_{\alpha\beta}g^{\mu\nu}\nabla_{\nu}\phi\right) - g^{\mu\nu}\tau_{(\alpha\beta)\nu}. \tag{3.2}$$

By comparing Eqs. (3.1) and (3.2), it is clear that

$$M_{\mu\alpha\beta} = 2\tau_{(\alpha\beta)\mu}, \tag{3.3}$$

and, along with Eq. (2.82), one obtains the stronger condition:

$$\nabla_{\mu}g_{\alpha\beta} = 0, \quad \text{or} \quad \boldsymbol{\nabla}\boldsymbol{g} = \mathbf{0}. \tag{3.4}$$

When condition (3.4) is satisfied, $\boldsymbol{\nabla}$ is called a *metric-compatible connection*. This type of connection is essential, as it preserves the scalar product of parallel transported vector fields. Indeed, let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{X}$ be vector fields; then, in general:

$$\begin{aligned}
\boldsymbol{X}\big(\boldsymbol{g}(\boldsymbol{u},\boldsymbol{v})\big) &= X^{\alpha}\boldsymbol{e}_{\alpha}(g_{\mu\nu}u^{\mu}v^{\nu}) \\
&= X^{\alpha}\left[\boldsymbol{e}_{\alpha}(g_{\mu\nu})u^{\mu}v^{\nu} + g_{\mu\nu}\boldsymbol{e}_{\alpha}(u^{\mu})v^{\nu} + g_{\mu\nu}u^{\mu}\boldsymbol{e}_{\alpha}(v^{\nu})\right] \\
&= X^{\alpha}\Big[\big(\nabla_{\alpha}g_{\mu\nu} + \Gamma^{\lambda}{}_{\mu\alpha}g_{\lambda\nu} + \Gamma^{\lambda}{}_{\nu\alpha}g_{\mu\lambda}\big)u^{\mu}v^{\nu} \\
&\quad + g_{\mu\nu}\big(\nabla_{\alpha}u^{\mu} - \Gamma^{\mu}{}_{\lambda\alpha}u^{\lambda}\big)v^{\nu} + g_{\mu\nu}u^{\mu}\big(\nabla_{\alpha}v^{\nu} - \Gamma^{\nu}{}_{\lambda\alpha}v^{\lambda}\big)\Big]
\end{aligned}$$

or equivalently,

$$\boldsymbol{X}\big(\boldsymbol{g}(\boldsymbol{u},\boldsymbol{v})\big) = \big(\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{g}\big)(\boldsymbol{u},\boldsymbol{v}) + \boldsymbol{g}\big(\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{u},\boldsymbol{v}\big) + \boldsymbol{g}\big(\boldsymbol{u},\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{v}\big), \tag{3.5}$$

which vanishes if $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel transported along $\boldsymbol{X}$ and $\boldsymbol{\nabla}$ is metric-compatible.

Now, torsion is generally associated with the presence of fermionic matter fields, as it couples naturally to their intrinsic spin. In the absence of such spin sources, torsion does not arise dynamically and has no observable effect on the motion of spinless matter. In classical gravitational theories, where the majority of matter fields are effectively treated as spinless—such as fluids, scalar fields, or electromagnetic fields—the contributions of torsion can therefore be safely neglected:

$$\tau^{\mu}{}_{\alpha\beta} = 0, \quad \text{or} \quad \boldsymbol{\tau} = \mathbf{0}. \tag{3.6}$$

This assumption allows the connection to be uniquely determined, simplifying the structure of the theory while remaining fully consistent with observational evidence.

By replacing conditions (3.4) and (3.6) into Eq. (2.69), and denoting $\nabla^{\mu} := g^{\mu\nu}\nabla_{\nu}$, one obtains the components of the torsion-free, metric-compatible Lyra connection:

$$\boxed{\Gamma^{\mu}{}_{\alpha\beta} = \phi^{-1}\left\{{}^{\ \mu}_{\alpha\beta}\right\} + \phi^{-1}\left(\delta^{\mu}_{\beta}\nabla_{\alpha}\phi - g_{\alpha\beta}\nabla^{\mu}\phi\right).} \tag{3.7}$$

From here on, the connection will be assumed to have this form, and it will be referred to as the *LyST connection*.

3.1.1 Curvature in the LyST Connection

Once the LyST connection has been fully specified, all curvature–related quantities acquire a definite structure. Substituting Eq. (2.26) into Eq. (2.58) yields the expression for the curvature tensor in the form

$$R^{\alpha}{}_{\beta\mu\nu} = \phi^{-2}\partial_{\mu}(\phi\Gamma^{\alpha}{}_{\beta\nu}) - \phi^{-2}\partial_{\nu}(\phi\Gamma^{\alpha}{}_{\beta\mu}) + \Gamma^{\alpha}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\beta\mu}. \tag{3.8}$$

Then, replacing the connection coefficients by their LyST form using Eq. (3.7) and performing the appropriate rearrangements leads to

$$\begin{aligned}
\frac{1}{2}\phi^{2}R^{\alpha}{}_{\beta\mu\nu} &= \partial_{[\mu}\left(\left\{{\alpha \atop \beta\nu]}\right\} + \delta^{\alpha}_{\nu]}\nabla_{\beta}\phi - g_{\beta\nu]}\nabla^{\alpha}\phi\right) \\
&\quad + \left(\left\{{\alpha \atop \lambda[\mu}\right\} + \delta^{\alpha}_{[\mu}\nabla_{\lambda}\phi - g_{\lambda[\mu}\nabla^{\alpha}\phi\right)\left(\left\{{\lambda \atop \beta\nu]}\right\} + \delta^{\lambda}_{\nu]}\nabla_{\beta}\phi - g_{\beta\nu]}\nabla^{\lambda}\phi\right) \\
&= \partial_{[\mu}\left\{{\alpha \atop \beta\nu]}\right\} + \delta^{\alpha}_{[\nu}\partial_{\mu]}\nabla_{\beta}\phi - {\color{red}\partial_{[\mu}g_{\beta\nu]}\nabla^{\alpha}\phi} - g_{\beta[\nu}\partial_{\mu]}\nabla^{\alpha}\phi \\
&\quad + \left\{{\alpha \atop \lambda[\mu}\right\}\left\{{\lambda \atop \beta\nu]}\right\} + \cancel{\left\{{\alpha \atop [\nu\mu]}\right\}}\nabla_{\beta}\phi - g_{\beta[\nu}\left\{{\alpha \atop \lambda\mu]}\right\}\nabla^{\lambda}\phi + \delta^{\alpha}_{[\mu}\phi\Gamma^{\lambda}{}_{\beta\nu]}\nabla_{\lambda}\phi \\
&\quad - {\color{red}\{[\mu\beta\nu]\}\nabla^{\alpha}\phi} - \cancel{g_{[\nu\mu]}}\nabla^{\alpha}\phi\nabla_{\beta}\phi + g_{\beta[\nu}g_{\lambda\mu]}\nabla^{\alpha}\phi\nabla^{\lambda}\phi \\
&= \left(\partial_{[\mu}\left\{{\alpha \atop \beta\nu]}\right\} + \left\{{\alpha \atop \lambda[\mu}\right\}\left\{{\lambda \atop \beta\nu]}\right\}\right) + \phi\delta^{\alpha}_{[\nu}\left(\partial_{\mu]}\nabla_{\beta}\phi - \Gamma^{\lambda}{}_{\beta\mu]}\nabla_{\lambda}\phi\right) \\
&\quad - g_{\beta[\nu}\partial_{\mu]}\nabla^{\alpha}\phi - g_{\beta[\nu}\left(\left\{{\alpha \atop \lambda\mu]}\right\} - g_{\lambda\mu]}\nabla^{\alpha}\phi\right)\nabla^{\lambda}\phi \\
&= \frac{1}{2}\mathcal{R}^{\alpha}{}_{\beta\mu\nu} + \phi\delta^{\alpha}_{[\nu}\nabla_{\mu]}\nabla_{\beta}\phi - \phi g_{\beta[\nu}\nabla_{\mu]}\nabla^{\alpha}\phi + \delta^{\alpha}_{[\mu}g_{\beta\nu]}\nabla^{\lambda}\phi\nabla_{\lambda}\phi,
\end{aligned} \tag{3.9}$$

In this computation, anti-symmetrization over the indices μ and ν is denoted by square brackets $[\ ]$, and the highlighted terms of the same color cancel each other. The quantity $\mathcal{R}^{\alpha}{}_{\beta\mu\nu}$ appearing in the final step corresponds to the usual Riemannian curvature tensor:

$$\mathcal{R}^{\alpha}{}_{\beta\mu\nu} = \partial_{\mu}\left\{{\alpha \atop \beta\nu}\right\} - \partial_{\nu}\left\{{\alpha \atop \beta\mu}\right\} + \left\{{\alpha \atop \lambda\mu}\right\}\left\{{\lambda \atop \beta\nu}\right\} - \left\{{\alpha \atop \lambda\nu}\right\}\left\{{\lambda \atop \beta\mu}\right\}. \tag{3.10}$$

As expected, the curvature tensor acquires the Riemannian form (3.10) when the limit $\phi \to 1$ is taken. The Riemannian Ricci tensor and scalar will be denoted by $\mathcal{R}_{\beta\nu}$ and $\mathcal{R}$.

From Eq. (3.9), the explicit form of the LyST curvature tensor follows:

$$\begin{aligned}
R^{\alpha}{}_{\beta\mu\nu} &= \phi^{-2}\mathcal{R}^{\alpha}{}_{\beta\mu\nu} + \phi^{-2}\left(\delta^{\alpha}_{\mu}g_{\beta\nu} - \delta^{\alpha}_{\nu}g_{\beta\mu}\right)\nabla^{\lambda}\phi\nabla_{\lambda}\phi \\
&\quad + \phi^{-1}\left(\delta^{\alpha}_{\nu}\nabla_{\mu}\nabla_{\beta}\phi - \delta^{\alpha}_{\mu}\nabla_{\nu}\nabla_{\beta}\phi + g_{\beta\mu}\nabla_{\nu}\nabla^{\alpha}\phi - g_{\beta\nu}\nabla_{\mu}\nabla^{\alpha}\phi\right).
\end{aligned} \tag{3.11}$$

Contracting the appropriate indices yields the Ricci tensor,

$$R_{\beta\nu} = \phi^{-2}\mathcal{R}_{\beta\nu} + (n-1)\phi^{-2}g_{\beta\nu}\nabla^{\lambda}\phi\nabla_{\lambda}\phi - (n-2)\phi^{-1}\nabla_{\nu}\nabla_{\beta}\phi - \phi^{-1}g_{\beta\nu}\nabla^{\lambda}\nabla_{\lambda}\phi, \tag{3.12}$$

and its trace gives the Ricci scalar,

$$R = \phi^{-2}\mathcal{R} + n(n-1)\phi^{-2}\nabla^{\lambda}\phi\nabla_{\lambda}\phi - 2(n-1)\phi^{-1}\nabla^{\lambda}\nabla_{\lambda}\phi. \tag{3.13}$$

In addition, Eq. (3.11) makes it evident that the LyST curvature tensor satisfies the following symmetry properties:

$$\boxed{R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu},} \tag{3.14}$$

$$\boxed{R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu},} \tag{3.15}$$

$$\boxed{R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}.} \tag{3.16}$$

The first symmetry follows directly from Eq. (2.58), while the remaining two hold specifically for the LyST connection. These properties, along with the first Bianchi identity (2.60), reduce the number of independent components to $n^2(n^2-1)/12$.

3.1.2 Minimal Coupling Prescription

The gravitational theory to be developed must preserve the invariance of physical laws between LRS, that is, under both coordinate and scale transformations. In many physical systems where gravitational effects can be neglected, the laws of physics are invariant under the Poincaré group and can therefore be formulated in Minkowski spacetime. This motivates the need to extend these laws to the framework of a Lyra spacetime. Such an extension is achieved through the *Minimal Coupling Prescription*.

Given a matter field ψ_a in a Minkowski background with metric $\eta_{\mu\nu}$, the corresponding lagrangian density is extended to a Lyra spacetime through the substitution

$$\mathcal{L}_M(\eta_{\mu\nu}, \psi_a, \partial_\alpha \psi_a) \quad \mapsto \quad \mathcal{L}_M(g_{\mu\nu}, \psi_a, \nabla_\alpha \psi_a). \tag{3.17}$$

Here, $g_{\mu\nu}$ and $\boldsymbol{\nabla}$ denote the metric tensor and connection of the Lyra manifold. In this setup, the scale function ϕ is naturally incorporated through the connection, ensuring that the resulting Lagrangian density remains invariant under general Lyra transformations.

This generalization, together with the volume element of Eq. (2.42), yields the appropriate framework for constructing Lyra-invariant actions for matter fields:

$$S_M = \int_V d^n x\, \phi^n \sqrt{|\det(g)|}\mathcal{L}_M(g_{\mu\nu}, \psi_a, \nabla_\alpha \psi_a). \tag{3.18}$$

Given a variation of the matter field ψ_a, the corresponding variation of the action is

$$\delta S_M = \int_V d^n x\, \phi^n \sqrt{|\det(g)|}\left(\frac{\partial \mathcal{L}_M}{\partial \psi_a}\delta\psi_a + \frac{\partial \mathcal{L}_M}{\partial \nabla_\mu \psi_a}\delta \nabla_\mu \psi_a\right).$$

Since the covariant derivative is additive, it commutes with the variation operator, $\delta\nabla_\mu\psi_a = \nabla_\mu\delta\psi_a$. Applying the Leibniz rule and the divergence theorem (2.81) yields

$$\begin{aligned}\delta S_M = &\int_V d^n x\, \phi^n \sqrt{|\det(g)|}\left(\frac{\partial \mathcal{L}_M}{\partial \psi_a} - \nabla_\mu \frac{\partial \mathcal{L}_M}{\partial \nabla_\mu \psi_a}\right)\delta\psi_a \\ &+ \int_{\partial V} d^{n-1}x\, \phi^{n-1}\sqrt{|\det(g)|}\frac{\partial \mathcal{L}_M}{\partial \nabla_\mu \psi_a} n_\mu \delta\psi_a.\end{aligned}$$

The stationary-action principle $\delta S_M = 0$, with the boundary condition $\delta\psi_a = 0$ on δV, leads to the

Euler-Lagrange field equations for the matter field:

$$\boxed{\frac{\partial \mathcal{L}_M}{\partial \psi_a} - \nabla_\mu \frac{\partial \mathcal{L}_M}{\partial \nabla_\mu \psi_a} = 0.} \tag{3.19}$$

This equation governs the dynamics of the matter field. Its structure guarantees consistency with the underlying background, incorporating the effects of both the metric and the scale function through the associated connection.

3.2 Field Equations

This section derives the gravitational field equations. As a first step, an appropriate gravitational action must be defined. Since any admissible Lagrangian density must be Lyra-invariant, only Lyra scalars are to be considered. In Chapter 1, two such quantities were identified: the Ricci scalar R and the Kretschmann scalar K. For a second-order theory, the Ricci scalar is the natural choice; in contrast, the Kretschmann scalar contains higher derivatives of the metric and is therefore better suited to higher-order gravitational models, which are not considered here. Accordingly, the gravitational action in a four-dimensional Lyra spacetime is taken to be

$$S_G = \frac{1}{2\kappa} \int_V d^4x \, \phi^4 \sqrt{|\det(g)|} R, \tag{3.20}$$

where κ is a constant to be determined later, and the LyST connection is assumed.

Including the matter sector, the total action of the system becomes

$$S = \frac{1}{2\kappa} \int_V d^4x \, \phi^4 \sqrt{|\det(g)|} R + \int_V d^4x \, \phi^4 \sqrt{|\det(g)|} \mathcal{L}_M \tag{3.21}$$

The gravitational field equations follow from the stationary-action principle $\delta S = 0$ under variations of the metric $g_{\mu\nu}$ and the scale function ϕ, with $\delta g_{\mu\nu}$, $\delta\phi$ and $\delta \nabla_\mu \phi$ vanishing on the boundary ∂V.

The variation of the total action takes the form

$$\delta S = \int_V d^4x \left(\frac{\delta S}{\delta g_{\mu\nu}} \delta g_{\mu\nu} + \frac{\delta S}{\delta \phi} \delta\phi \right), \tag{3.22}$$

and, since the metric tensor and the scale function vary independently, the variational principle yields

$$\frac{\delta S}{\delta g_{\mu\nu}} \delta g_{\mu\nu} = 0 \quad \text{and} \quad \frac{\delta S}{\delta \phi} \delta\phi = 0. \tag{3.23}$$

The next two subsections compute these variations separately and provide the corresponding field equations for each sector. Treating both sectors independently allows one to clearly identify the contributions of geometry and scalar dynamics to the full theory, as well as their mutual consistency within the variational framework.

3.2.1 Field Equations from Metric Variation

The variation of the gravitational action with respect to the metric is

$$\delta S_G = \frac{1}{2\kappa}\int_V d^4x\,\phi^4\sqrt{|\det(g)|}\left(R\frac{\delta\sqrt{|\det(g)|}}{\sqrt{|\det(g)|}}+\delta R\right). \tag{3.24}$$

Using the property

$$\delta|\det(g)| = |\det(g)|g^{\mu\nu}\delta g_{\mu\nu}, \tag{3.25}$$

one finds the relation

$$\frac{\delta\sqrt{|\det(g)|}}{\sqrt{|\det(g)|}} = \frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}. \tag{3.26}$$

Moreover,

$$\delta R = \delta(R_{\mu\nu}g^{\mu\nu}) = g^{\mu\nu}\delta R_{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu}. \tag{3.27}$$

In order to compute the variation of the Ricci tensor, it is convenient to start from the variation of the curvature tensor:

$$\begin{aligned}\delta R^{\alpha}{}_{\mu\beta\nu} = {} & \phi^{-1}\partial_\beta\delta\Gamma^{\alpha}{}_{\mu\nu} - \phi^{-1}\partial_\nu\delta\Gamma^{\alpha}{}_{\mu\beta} + \delta\Gamma^{\alpha}{}_{\lambda\beta}\Gamma^{\lambda}{}_{\mu\nu} + \Gamma^{\alpha}{}_{\lambda\beta}\delta\Gamma^{\lambda}{}_{\mu\nu} \\ & - \delta\Gamma^{\alpha}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\mu\beta} - \Gamma^{\alpha}{}_{\lambda\nu}\delta\Gamma^{\lambda}{}_{\mu\beta} - \Gamma^{\alpha}{}_{\mu\lambda}\gamma^{\lambda}{}_{\beta\nu}.\end{aligned} \tag{3.28}$$

The transformation rule for the connection, Eq. (2.70), shows that the last two terms do not depend on the metric. Consequently, the variation of the connection with respect to the metric transforms according to

$$\delta\Gamma'^{\alpha}{}_{\mu\nu} = \frac{\phi}{\phi'}\frac{\partial x'^{\alpha}}{\partial x^{\lambda}}\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\delta\Gamma^{\lambda}{}_{\rho\sigma}. \tag{3.29}$$

Thus, $\delta\Gamma^{\alpha}{}_{\mu\nu}$ behaves as a tensor, and its covariant derivative is

$$\nabla_\beta\delta\Gamma^{\alpha}{}_{\mu\nu} = \phi^{-1}\partial_\beta\delta\Gamma^{\alpha}{}_{\mu\nu} + \Gamma^{\alpha}{}_{\lambda\beta}\delta\Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\mu\beta}\delta\Gamma^{\alpha}{}_{\lambda\nu} - \Gamma^{\lambda}{}_{\nu\beta}\delta\Gamma^{\alpha}{}_{\mu\lambda}$$

Comparison with Eq. (3.28) yields

$$\delta R^{\alpha}{}_{\mu\beta\nu} = \nabla_\beta\delta\Gamma^{\alpha}{}_{\mu\nu} - \nabla_\nu\delta\Gamma^{\alpha}{}_{\mu\beta}, \tag{3.30}$$

and hence

$$\delta R_{\mu\nu} = \nabla_\alpha\left(\delta\Gamma^{\alpha}{}_{\mu\nu} - \delta^{\alpha}_{\nu}\delta\Gamma^{\lambda}{}_{\mu\lambda}\right). \tag{3.31}$$

Using this result in Eq. (3.27), one finds

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\alpha\left(g^{\mu\nu}\delta\Gamma^{\alpha}{}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^{\lambda}{}_{\mu\lambda}\right). \tag{3.32}$$

Now, substituting Eqs. (3.26) and (3.32) into Eq. (3.24) yields

$$\begin{aligned}\delta S_G = {} & \frac{1}{2\kappa}\int_V d^4x\,\phi^4\sqrt{|\det(g)|}\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu} \\ & + \frac{1}{2\kappa}\int_{\partial V} d^3x\,\phi^3\sqrt{|\det(g)|}n_\alpha\left(g^{\mu\nu}\delta\Gamma^{\alpha}{}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^{\lambda}{}_{\mu\lambda}\right).\end{aligned}$$

Here, the second integral vanishes, as $\delta g_{\mu\nu} = 0$ on ∂V. Thus

$$\delta S_G = \frac{1}{2\kappa} \int_V d^4x\, \phi^4 \sqrt{|\det(g)|} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta g^{\mu\nu}. \tag{3.33}$$

On the other hand, the variation of the matter field action with respect to the metric is easily calculated:

$$\delta S_M = -\frac{1}{2} \int_V d^4x\, \phi^4 \sqrt{|\det(g)|} \left(-2 \frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}_M \right) \delta g^{\mu\nu}. \tag{3.34}$$

The expression in parentheses defines the *energy-momentum tensor*:

$$\boxed{T_{\mu\nu} := -2 \frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}_M.} \tag{3.35}$$

Combining Eqs. (3.33) and (3.34), the variation of the total action with respect to the metric takes the form

$$\delta S = \int_V d^4x\, \phi^4 \sqrt{|\det(g)|} \left[\frac{1}{2\kappa} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) - \frac{1}{2} T_{\mu\nu} \right] \delta g^{\mu\nu},$$

which yields the field equations

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}.} \tag{3.36}$$

Using Eqs. (3.12) and (3.13), these are written explicitly:

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} \mathcal{R} g_{\mu\nu} - 3 g_{\mu\nu} \nabla^\lambda \phi \nabla_\lambda \phi + 2\phi g_{\mu\nu} \nabla^\lambda \nabla_\lambda \phi - 2\phi \nabla_\mu \nabla_\nu \phi = \kappa \phi^2 T_{\mu\nu}. \tag{3.37}$$

As expected, these equations reduce to those of General Relativity in the limit $\phi \to 1$.

3.2.2 Field Equation from Scale Variation

The analysis now turns to variations with respect to the scale function. The corresponding variation of the gravitational action is

$$\delta S_G = \frac{1}{2\kappa} \int_V d^4x\, \phi^4 \sqrt{|\det(g)|} \left(4R \frac{\delta\phi}{\phi} + \delta R \right). \tag{3.38}$$

Using Eq. (3.13), the variation of the Ricci scalar is written as

$$\delta R = -\frac{2}{\phi^3} \delta\phi\, \mathcal{R} - \frac{24}{\phi^3} \delta\phi\, \nabla^\lambda \phi \nabla_\lambda \phi + \frac{24}{\phi^2} \nabla^\lambda \phi\, \delta \nabla_\lambda \phi + \frac{6}{\phi^2} \delta\phi\, \nabla^\lambda \nabla_\lambda \phi - \frac{6}{\phi} \delta \nabla^\lambda \nabla_\lambda \phi. \tag{3.39}$$

To evaluate this expression, the variations $\delta \nabla_\lambda \phi$ and $\delta \nabla^\lambda \nabla_\lambda \phi$ must be computed. The first variation reads

$$\delta \nabla_\nu \phi = \delta \left(\frac{1}{\phi} \partial_\nu \phi \right) = -\frac{1}{\phi^2} \delta\phi\, \partial_\nu \phi + \frac{1}{\phi} \partial_\nu \delta\phi = -\frac{1}{\phi} \delta\phi\, \nabla_\nu \phi + \nabla_\nu \delta\phi. \tag{3.40}$$

Next, consider

$$\begin{aligned}
\delta\nabla_\mu\nabla_\nu\phi &= \delta\left[\frac{1}{\phi}\left(\partial_\mu\nabla_\nu\phi - \phi\Gamma^\lambda{}_{\nu\mu}\nabla_\lambda\phi\right)\right]\\
&= -\frac{1}{\phi^2}\delta\phi\left(\partial_\mu\nabla_\nu\phi - \phi\Gamma^\lambda{}_{\nu\mu}\nabla_\lambda\phi\right) + \frac{1}{\phi}\left(\partial_\mu\delta\nabla_\nu\phi - \phi\Gamma^\lambda{}_{\nu\mu}\delta\nabla_\lambda\phi - \delta(\phi\Gamma^\lambda{}_{\nu\mu})\nabla_\lambda\phi\right)\\
&= -\frac{1}{\phi}\delta\phi\,\nabla_\mu\nabla_\nu\phi + \nabla_\mu\delta\nabla_\nu\phi - \frac{1}{\phi}\delta\left(\left\{{}^{\lambda}_{\nu\mu}\right\} + \delta^\lambda_\mu\nabla_\nu\phi - g_{\nu\mu}\nabla^\lambda\phi\right)\nabla_\lambda\phi\\
&= -\frac{1}{\phi}\delta\phi\,\nabla_\mu\nabla_\nu\phi + \nabla_\mu\delta\nabla_\nu\phi - \frac{1}{\phi}\nabla_\mu\phi\,\delta\nabla_\nu\phi + \frac{1}{\phi}g_{\mu\nu}\nabla^\lambda\phi\,\delta\nabla_\lambda\phi,
\end{aligned}$$

which, after replacement of Eq. (3.40), becomes

$$\begin{aligned}
\delta\nabla_\mu\nabla_\nu\phi &= -\frac{1}{\phi}\delta\phi\,\nabla_\mu\nabla_\nu\phi + \left(\frac{1}{\phi^2}\delta\phi\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{\phi}\nabla_\mu\delta\phi\,\nabla_\nu\phi - \frac{1}{\phi}\delta\phi\nabla_\mu\nabla_\nu\phi + \nabla_\mu\nabla_\nu\delta\phi\right)\\
&\quad - \frac{1}{\phi}\nabla_\mu\phi\left(-\frac{1}{\phi}\delta\phi\,\nabla_\nu\phi + \nabla_\nu\delta\phi\right) + \frac{1}{\phi}g_{\mu\nu}\nabla^\lambda\phi\left(-\frac{1}{\phi}\delta\phi\,\nabla_\lambda\phi + \nabla_\lambda\delta\phi\right)\\
&= -\frac{2}{\phi}\delta\phi\nabla_\mu\nabla_\nu\phi + \frac{2}{\phi^2}\delta\phi\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{\phi}\nabla_\mu\delta\phi\,\nabla_\nu\phi - \frac{1}{\phi}\nabla_\mu\phi\nabla_\nu\delta\phi\\
&\quad + \nabla_\mu\nabla_\nu\delta\phi - \frac{1}{\phi^2}g_{\mu\nu}\delta\phi\nabla^\lambda\phi\nabla_\lambda\phi + \frac{1}{\phi}g_{\mu\nu}\nabla^\lambda\phi\nabla_\lambda\delta\phi. \qquad (3.41)
\end{aligned}$$

The quantity actually required is the trace of this expression:

$$\delta\nabla^\lambda\nabla_\lambda\phi = -\frac{2}{\phi}\delta\phi\nabla^\lambda\nabla_\lambda\phi - \frac{2}{\phi^2}\delta\phi\nabla^\lambda\phi\nabla_\lambda\phi + \frac{2}{\phi}\nabla^\lambda\phi\nabla_\lambda\delta\phi + \nabla^\lambda\nabla_\lambda\delta\phi. \qquad (3.42)$$

Insetion of Eqs. (3.40) and (3.42) in Eq. (3.39) leads to

$$\begin{aligned}
\delta R &= -\frac{2}{\phi^3}\delta\phi\,\mathcal{R} - \frac{36}{\phi^3}\delta\phi\nabla^\lambda\phi\nabla_\lambda\phi + \frac{18}{\phi^2}\delta\phi\,\nabla^\lambda\nabla_\lambda\phi + \frac{12}{\phi^2}\nabla^\lambda\phi\nabla_\lambda\delta\phi - \frac{6}{\phi}\nabla^\lambda\nabla_\lambda\delta\phi\\
&= -2R\frac{\delta\phi}{\phi} - \frac{12}{\phi^3}\delta\phi\nabla^\lambda\phi\nabla_\lambda\phi + \frac{6}{\phi^2}\delta\phi\nabla^\lambda\nabla_\lambda\phi + \frac{12}{\phi^2}\nabla^\lambda\phi\nabla_\lambda\delta\phi - \frac{6}{\phi}\nabla^\lambda\nabla_\lambda\delta\phi\\
&= -2R\frac{\delta\phi}{\phi} - \nabla_\lambda\nabla^\lambda\left(\frac{6}{\phi}\delta\phi\right), \qquad (3.43)
\end{aligned}$$

which, when placed into Eq. (3.38), yields

$$\delta S_G = \int_V d^4x\,\phi^4\sqrt{|\det(g)|}\left(\frac{2}{\phi}R\right)\delta\phi - \int_{\partial V} d^3x\,\phi^3\sqrt{|\det(g)|}n_\lambda\nabla^\lambda\left(\frac{6}{\phi}\delta\phi\right).$$

In this case, the second integral vanishes because $\delta\phi = 0$ and $\nabla_\lambda\phi = 0$ on ∂V. Thus

$$\delta S_G = \frac{1}{2\kappa}\int_V d^4x\,\phi^4\sqrt{|\det(g)|}\left(\frac{2}{\phi}R\right)\delta\phi. \qquad (3.44)$$

On the other hand, the variation of the matter field action with respect to the scale is

$$\delta S_M = \int_V d^4x\,\phi^4\sqrt{|\det(g)|}\left(\frac{4}{\phi}\delta\phi\,\mathcal{L}_M + \delta\mathcal{L}_M\right), \qquad (3.45)$$

with

$$\delta\mathcal{L}_M = \frac{\partial\mathcal{L}_M}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}_M}{\partial\nabla_\mu\phi}\left(-\frac{1}{\phi}\delta\phi\nabla_\mu\phi + \nabla_\mu\delta\phi\right)$$
$$= \left(\frac{\partial\mathcal{L}_M}{\partial\phi}\delta\phi - \nabla_\mu\frac{\partial\mathcal{L}_M}{\partial\nabla_\mu\phi} - \frac{1}{\phi}\frac{\partial\mathcal{L}_M}{\partial\nabla_\mu\phi}\nabla_\mu\phi\right)\delta\phi + \nabla_\mu\left(\frac{\partial\mathcal{L}_M}{\partial\nabla_\mu\phi}\delta\phi\right).$$

The total divergence term again yields a vanishing boundary integral. Consequently,

$$\delta S_M = -\int_V d^4x\,\phi^4\sqrt{|\det(g)|}\left(\frac{1}{\phi}M\right)\delta\phi, \tag{3.46}$$

where

$$M = -4\mathcal{L}_M + \frac{\partial\mathcal{L}_M}{\partial\nabla_\mu\phi}\nabla_\mu\phi - \phi\left(\frac{\partial\mathcal{L}_M}{\partial\phi} - \nabla_\mu\frac{\partial\mathcal{L}_M}{\partial\nabla_\mu\phi}\right). \tag{3.47}$$

By combining Eqs. (3.44) and (3.46), the variation of the total action with respect to the scale function is obtained:

$$\delta S = \int_V d^4x\,\phi^4\sqrt{|\det(g)|}\left(\frac{1}{\kappa}R - M\right)\frac{\delta\phi}{\phi},$$

which leads to the corresponding field equation,

$$\boxed{R = \kappa M.} \tag{3.48}$$

Finally, comparing Eq. (3.48) with the trace of Eq. (3.36) yields the relation

$$M = -T. \tag{3.49}$$

Since there is no reason for this relation to hold identically for arbitrary matter fields, the field equations (3.36) and (3.48) must be regarded, in principle, as independent. Nonetheless, in vacuum—meaning either the total absence of matter fields or regions where existing fields vanish—the condition is satisfied automatically.

3.2.3 Energy-Momentum Conservation

Once the field equations have been established, it becomes essential to understand how the gravitational dynamics constrain the behavior of matter fields. Any consistent theory of gravitation must specify under what conditions energy and momentum are preserved, and how these quantities flow through spacetime.

Consider the second Bianchi identity (2.61) and write it explicitly:

$$\nabla_\lambda R_{\alpha\mu\beta\nu} + \nabla_\beta R_{\alpha\mu\nu\lambda} + \nabla_\nu R_{\alpha\mu\lambda\beta} = 0. \tag{3.50}$$

By contracting with $g^{\alpha\beta}g^{\lambda\mu}$, and using the properties (3.14) and (3.15), one finds

$$\nabla_\mu\left(R^\mu{}_\nu - \frac{1}{2}R\delta^\mu_\nu\right) = 0, \tag{3.51}$$

which, because of the field equations (3.36), implies the *local conservation* of energy-momentum:

$$\boxed{\nabla_\mu T^\mu{}_\nu = 0.} \tag{3.52}$$

Unlike in flat spacetime, Eq. (3.52) does not, by itself, ensure the existence of globally conserved quantities. In curved geometry the very notions of total energy and total momentum depend on the global structure of spacetime, and in the absence of additional symmetries or asymptotic conditions, no invariant global charges can be defined. In particular, global conservation requires a mechanism to convert the local conservation law into a flux integral that is independent of the choice of spatial hypersurface.

A natural way to achieve this is through spacetime symmetries. Suppose the spacetime admits a timelike Killing vector field ξ^μ, satisfying Eq. (2.90):

$$\nabla_\mu(\phi\xi_\nu) + \nabla_\nu(\phi\xi_\mu) = 0.$$

From this symmetry one may construct the current $J^\mu := T^{\mu\nu}\phi\xi_\nu$. Using the vanishing divergence of the energy–momentum tensor, and the Killing equation, one obtains

$$\nabla_\mu J^\mu = \nabla_\mu T^{\mu\nu}\phi\xi_\nu + T^{\mu\nu}\nabla_\mu(\phi\xi_\nu) = \frac{1}{2}T^{\mu\nu}\left[\nabla_\mu(\phi\xi_\nu) + \nabla_\nu(\phi\xi_\mu)\right] = 0.$$

Integrating this relation over a spacetime region V and applying the divergence theorem yields

$$\int_{\partial V} d^3x\, \phi^3\sqrt{|\det(g)|}n_\mu J^\mu = 0.$$

Let V be the region bounded by two spacelike hypersurfaces Σ_1 and Σ_2, together with the boundary at spatial infinity W,

$$\partial V = \Sigma_1 \cup \Sigma_2 \cup W.$$

If the current J^μ decays sufficiently fast so that its flux through W vanisheso, then the above equation implies

$$\int_{\Sigma_1} d\Sigma_1\, \phi^3\sqrt{|\det(g)|}n^{(1)}_\mu J^\mu = \int_{\Sigma_2} d\Sigma_2\, \phi^3\sqrt{|\det(g)|}n^{(2)}_\mu J^\mu.$$

Therefore, the quantity

$$Q = \int_\Sigma d\Sigma\, \phi^3\sqrt{|\det(g)|}n_\mu J^\mu \tag{3.53}$$

is conserved, in the sense that its value is independent of the particular spacelike hypersurface Σ. This hypersurface independence identifies Q as a genuine global conserved charge. In the present context, where the Killing vector is timelike, Q represents the conserved energy associated with the underlying symmetry. Moreover, if the spacetime admits additional spatial or rotational Killing vectors, the same construction yields conserved momentum or angular-momentum charges, illustrating how each continuous symmetry gives rise to a corresponding globally conserved quantity.

3.3 Newtonian Limit

As it is well known, Newtonian gravity accurately describes a broad range of gravitational phenomena on Solar-System scales, provided certain conditions are satisfied. Any consistent gravitational theory formulated in Lyra geometry must therefore reproduce these Newtonian results in the appropriate regime. Accordingly, the field equations and equations of motion should reduce to their Newtonian counterparts when those limiting conditions are imposed.

The first condition requires that massive particles move with non-relativistic velocities in spacetime:

$$\left|\frac{dx^i}{dt}\right| \ll 1 \quad \Longrightarrow \quad \frac{dt}{d\tau} \approx 1, \tag{3.54}$$

where τ is the proper time affine parameter. Thus, geodesic equation (2.35) reduces to

$$\frac{d^2x^\mu}{dt^2} + \left\{ {\mu \atop 00} \right\} + 2\delta^\mu_0 \nabla_0 \phi - g_{00} g^{\mu\nu} \nabla_\nu \phi \approx 0. \tag{3.55}$$

The second condition is that the gravitational field is static, meaning both the metric and the scale factor are independent of time:

$$\partial_0 g_{\mu\nu} = 0 \quad \text{and} \quad \partial_0 \phi = 0. \tag{3.56}$$

Consequently, the Christoffel symbols simplify to

$$\left\{ {\mu \atop 00} \right\} = \frac{1}{2} g^{\mu\alpha} (2\partial_0 g_{\alpha 0} - \partial_\alpha g_{00}) = -\frac{1}{2} g^{\mu\alpha} \partial_\alpha g_{00},$$

so that (3.55) becomes

$$\frac{d^2x^\mu}{dt^2} \approx \frac{1}{2} g^{\mu\alpha} \partial_\alpha g_{00} + \frac{1}{\phi} g_{00} g^{\mu\nu} \partial_\nu \phi \tag{3.57}$$

The third condition is the weak-field limit, in which the gravitational field acts as a small perturbation of flat Minkowski spacetime:

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1; \tag{3.58}$$

$$\phi \approx 1 + \delta\phi, \quad |\delta\phi| \ll 1; \tag{3.59}$$

where Cartesian coordinates are used so that $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. This allows one to linearize equations, keeping only first-order terms in the perturbations.

To first order in $h_{\mu\nu}$ and $\delta\varphi$, Eq. (3.57) reduces to

$$\frac{d^2x^\mu}{dt^2} \approx \eta^{\mu\nu} \partial_\nu \left(\frac{1}{2} h_{00} + \delta\phi \right). \tag{3.60}$$

Because of the two first conditions, the temporal component

$$\frac{d^2x^0}{dt^2} \approx \partial_0 \left(\frac{1}{2} h_{00} + \delta\phi \right),$$

is identically satisfied, while the spatial components reduce to

$$\frac{d^2\vec{x}}{dt^2} \approx -\nabla U, \tag{3.61}$$

which is the Newton law of motion, with *effective Newtonian potential* defined as

$$U := \frac{1}{2}h_{00} + \delta\phi. \tag{3.62}$$

This shows explicitly that, in LyST gravity, the Newtonian potential receives contributions both from the perturbation of the metric and from the perturbation of the scale factor, reflecting the additional geometric structure introduced by the theory.

Now, consider the field equations (3.37), which can be rewritten as

$$\mathcal{R}_{\mu\nu} + 3g_{\mu\nu}\nabla^\lambda\phi\nabla_\lambda\phi - \phi g_{\mu\nu}\nabla^\lambda\nabla_\lambda\phi - 2\phi\nabla_\mu\nabla_\nu\phi = \kappa\phi^2\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right). \tag{3.63}$$

To first order in $h_{\mu\nu}$ and $\delta\phi$, the 00 component in the static limit becomes

$$\mathcal{R}_{00} - \partial^i\partial_i\delta\phi \approx \kappa\left(T_{00} - \frac{1}{2}T\right), \tag{3.64}$$

where

$$\mathcal{R}_{00} \approx \partial_\mu\left\{{\mu \atop 00}\right\} \approx \frac{1}{2}\eta^{\mu\lambda}\partial_\mu\left(2\partial_0 g_{0\lambda} - \partial_\lambda g_{00}\right) \approx -\frac{1}{2}\partial^i\partial_i h_{00}. \tag{3.65}$$

Most classical matter sources can be modeled as perfect fluids with negligible pressure, $p \ll \rho$. The Lagrangian density for a perfect fluid is

$$\mathcal{L}_M = \varepsilon(n,\sigma), \tag{3.66}$$

where ε is the rest-frame energy density, n the corresponding particle number density, and σ the specific entropy. Here, the specific entropy is regarded as independent of the metric. Then, using Eq. (3.66) in Eq. (3.35) gives

$$T_{\mu\nu} = -2\frac{\partial\varepsilon}{\partial n}\frac{\partial n}{\partial g^{\mu\nu}} + \varepsilon g_{\mu\nu}. \tag{3.67}$$

Let u^μ be the four-velocity vector field of the matter distribution. The particle number current is defined as

$$j^\mu = \phi^3\sqrt{|\det(g)|}\,n u^\mu, \tag{3.68}$$

from which

$$n = \frac{\sqrt{g_{\mu\nu}j^\mu j^\nu}}{\phi^3\sqrt{|\det(g)|}}. \tag{3.69}$$

Varying this expression with respect to the metric yields

$$\delta n = -\frac{\delta|\det(g)|}{2\phi^3\sqrt{|\det(g)|}^3}\sqrt{g_{\mu\nu}j^\mu j^\nu} + \frac{1}{\phi^3\sqrt{|\det(g)|}}\frac{j^\mu j^\nu\delta g_{\mu\nu}}{2\sqrt{g_{\alpha\beta}j^\alpha j^\beta}} = \frac{n}{2}\left(g_{\mu\nu} - u_\mu u_\nu\right)\delta g^{\mu\nu}. \tag{3.70}$$

For fixed specific entropy, the local version of the Gibbs-Duhem relation is

$$\frac{\partial \varepsilon}{\partial n} = \frac{\varepsilon + p}{n}. \tag{3.71}$$

Using Eqs. (3.70) and (3.71) in Eq. (3.67) leads to the familiar form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}, \tag{3.72}$$

where $\rho = \varepsilon$ is the mass density in natural units. For negligible pressure,

$$T_{\mu\nu} \approx \rho u_\mu u_\nu, \tag{3.73}$$

Substituting Eqs. (3.65) and (3.73) into Eq. (3.64) yields

$$\nabla^2 U \approx \frac{\kappa}{2}\rho, \tag{3.74}$$

so that consistency with Newtonian gravity requires

$$\kappa = 8\pi. \tag{3.75}$$

Thus, the Newtonian limit of LyST gravity is correctly recovered: the theory reproduces classical gravitational dynamics under non-relativistic, static, and weak-field conditions. Notably, the effective Newtonian potential depends on both h_{00} and the Lyra scalar φ, showing that the Lyra scale factor contributes genuinely to the gravitational field. This distinguishes Lyra gravity from General Relativity while preserving full agreement with Newtonian physics in the appropriate regime.

Chapter 4

Spherically Symmetric Solution

Spherical symmetry occupies a fundamental place in gravitational theory because many physically relevant configurations—such as the exterior fields of compact sources or the idealized environments employed in scattering and lensing analyses—can be effectively approached within this framework. For this reason, the study of spherically symmetric solutions often serves as a crucial step in evaluating how a modified theory compares with, or departs from, the predictions of General Relativity.

In this chapter, attention is directed to static, spherically symmetric solutions in Lyra gravity. The goal is to understand how the displacement field characteristic of Lyra's geometry shapes these configurations and to determine the extent to which it alters their geometric and physical properties. By examining this symmetry sector, the chapter provides a clear and tractable setting in which the distinctive features of Lyra gravity become evident and can be assessed in a physically meaningful way.

4.1 Spherical Symmetry

The analysis of spherical symmetry begins with the systematic identification of the Killing vectors that characterize this class of spacetimes. Determining these vectors is essential, as they encode the underlying geometric constraints that any admissible line element must satisfy within LyST geometry. By establishing the corresponding Killing algebra, this section lays the foundation for selecting the appropriate metric structure to be employed throughout the chapter.

Spherical symmetry is associated with invariance under spatial rotation transformations, which form a three-parameter group. In local Cartesian coordinates, an infinitesimal rotation is written as

$$x'^{\mu} = x^{\mu} + \epsilon^{i} \left[L_{(i)}\right]^{\mu}{}_{\nu} x^{\nu}, \tag{4.1}$$

where $L_{(i)}$ is generator around the i-th Cartesian axis and ϵ^{i} are infinitesimal parameters. In matrix form, these generators are expressed as

$$L_{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad L_{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad L_{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.2}$$

or more compactly,

$$\left[L_{(i)}\right]^{\mu}{}_{\nu} = -\delta^{\mu j} \delta^{k}_{\nu} \epsilon_{ijk}. \tag{4.3}$$

Thus, the associated Killing vectors are

$$\xi^{\mu}_{(i)} = \left[L_{(i)}\right]^{\mu}{}_{\nu} x^{\nu} = -\delta^{\mu j}\epsilon_{ijk}x^k. \tag{4.4}$$

It is convenient to perform a transformation from Cartesian to spherical coordinates. The transformation is

$$\begin{aligned} t &= t, & \qquad\qquad t &= t, \\ x &= r\sin\theta\cos\varphi, & r &= \sqrt{x^2+y^2+z^2}, \\ y &= r\sin\theta\sin\varphi, \quad \Longleftrightarrow & \theta &= \arctan\frac{\sqrt{x^2+y^2}}{z}, \\ z &= r\cos\theta, & \varphi &= \arctan\frac{y}{x}, \end{aligned} \tag{4.5}$$

with Jacobian matrix

$$J = \frac{\partial(t,r,\theta,\varphi)}{\partial(t,x,y,z)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ 0 & \frac{1}{r}\cos\theta\cos\varphi & \frac{1}{r}\cos\theta\sin\varphi & -\frac{1}{r}\sin\theta \\ 0 & -\frac{1}{r}\csc\theta\sin\varphi & \frac{1}{r}\csc\theta\cos\varphi & 0 \end{pmatrix}. \tag{4.6}$$

Therefore, the spherical components of the Killing vectors are

$$\bar{\xi}^{\mu}_{(i)} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\nu}}\xi^{\nu}_{(i)} = -J^{\mu}{}_{\nu}\delta^{\nu j}\epsilon_{ijk}x^k,$$

or, specifically,

$$\begin{aligned} \bar{\xi}^{\mu}_{(1)} &= (0,0,-\sin\varphi,-\cot\theta\cos\varphi), \\ \bar{\xi}^{\mu}_{(2)} &= (0,0,\cos\varphi,-\cot\theta\sin\varphi), \\ \bar{\xi}^{\mu}_{(3)} &= (0,0,0,1). \end{aligned} \tag{4.7}$$

Now, spherical symmetry or rotational invariance physically means that the structure of spacetime—defined by the metric tensor and the scale function—remains unchanged under the transformations generated by the Killing vectors in Eq. (4.7).

First, the scale function ϕ must satisfy Eq. (2.86) for each of the Killing vectors. Then, one obtains the following system of equations:

$$\xi^{\mu}_{(i)}\partial_{\mu}\phi = 0 \qquad \Longrightarrow \qquad \begin{cases} \tan\theta\tan\varphi\,\partial_{\theta}\phi + \partial_{\varphi}\phi = 0, \\ \tan\theta\cot\varphi\,\partial_{\theta}\phi - \partial_{\varphi}\phi = 0, \\ \partial_{\varphi}\phi = 0, \end{cases}$$

whose solution is given by

$$\phi = \phi(t,r). \tag{4.8}$$

On the other hand, the metric tensor $g_{\mu\nu}$ must satisfy Eq. (2.89). Note that spatial rotations do not involve a change of scale function, which means $\Phi = 0$. Therefore,

$$\xi^{\alpha}_{(i)}\partial_{\alpha}g_{\mu\nu} + \partial_{\mu}\xi^{\alpha}_{(i)}g_{\alpha\nu} + \partial_{\nu}\xi^{\alpha}_{(i)}g_{\mu\alpha} = 0. \tag{4.9}$$

Immediately, for $i = 3$, one obtains

$$\partial_\varphi g_{\mu\nu} = 0. \tag{4.10}$$

There remains a system of two equations for each pair of indices.

- For $\mu = 0$ and $\nu = 0$:

$$\xi^\alpha_{(i)} \partial_\alpha g_{00} = 0 \quad \Longrightarrow \quad \begin{cases} \sin\varphi\, \partial_\theta g_{00} = 0, \\ \cos\varphi\, \partial_\theta g_{00} = 0, \end{cases}$$

  from which

$$\partial_\theta g_{00} = 0. \tag{4.11}$$

- For $\mu = 0$ and $\nu = 1$:

$$\xi^\alpha_{(i)} \partial_\alpha g_{01} = 0 \quad \Longrightarrow \quad \begin{cases} \sin\varphi\, \partial_\theta g_{01} = 0, \\ \cos\varphi\, \partial_\theta g_{01} = 0, \end{cases}$$

  from which

$$\partial_\theta g_{01} = 0. \tag{4.12}$$

- For $\mu = 0$ and $\nu = 2$:

$$\xi^\alpha_{(i)} \partial_\alpha g_{02} + \partial_2 \xi^\alpha_{(i)} g_{0\alpha} = 0 \quad \Longrightarrow \quad \begin{cases} \sin\varphi\, \partial_\theta g_{02} - \csc^2\theta \cos\varphi\, g_{03} = 0, \\ \cos\varphi\, \partial_\theta g_{02} + \csc^2\theta \sin\varphi\, g_{03} = 0, \end{cases}$$

  from which

$$\partial_\theta g_{02} = 0 \quad \text{and} \quad g_{03} = 0. \tag{4.13}$$

- For $\mu = 0$ and $\nu = 3$:

$$\partial_3 \xi^\alpha_{(i)} g_{0\alpha} = 0 \quad \Longrightarrow \quad \begin{cases} \cos\varphi\, g_{02} = 0, \\ \sin\varphi\, g_{02} = 0, \end{cases}$$

  from which

$$g_{02} = 0. \tag{4.14}$$

- For $\mu = 1$ and $\nu = 1$:

$$\xi^\alpha_{(i)} \partial_\alpha g_{11} = 0 \quad \Longrightarrow \quad \begin{cases} \sin\varphi\, \partial_\theta g_{11} = 0, \\ \cos\varphi\, \partial_\theta g_{11} = 0, \end{cases}$$

  from which

$$\partial_\theta g_{11} = 0. \tag{4.15}$$

- For $\mu = 1$ and $\nu = 2$:

$$\xi^\alpha_{(i)} \partial_\alpha g_{12} + \partial_2 \xi^\alpha_{(i)} g_{1\alpha} = 0 \quad \Longrightarrow \quad \begin{cases} \sin\varphi\, \partial_\theta g_{12} - \csc^2\theta \cos\varphi\, g_{13} = 0, \\ \cos\varphi\, \partial_\theta g_{12} + \csc^2\theta \sin\varphi\, g_{13} = 0, \end{cases}$$

from which

$$\partial_\theta g_{12} = 0 \quad \text{and} \quad g_{13} = 0. \tag{4.16}$$

- For $\mu = 1$ and $\nu = 3$:

$$\partial_3 \xi^\alpha_{(i)} g_{1\alpha} = 0 \quad \Longrightarrow \quad \begin{cases} \cos\varphi\, g_{12} = 0, \\ \sin\varphi\, g_{12} = 0, \end{cases}$$

from which

$$g_{12} = 0. \tag{4.17}$$

- For $\mu = 2$ and $\nu = 2$:

$$\xi^\alpha_{(i)} \partial_\alpha g_{22} + 2\partial_2 \xi^\alpha_{(i)} g_{2\alpha} = 0 \quad \Longrightarrow \quad \begin{cases} \sin\varphi\, \partial_\theta g_{22} - 2\csc^2\theta \cos\varphi\, g_{23} = 0, \\ \cos\varphi\, \partial_\theta g_{22} + 2\csc^2\theta \sin\varphi\, g_{23} = 0, \end{cases}$$

from which

$$\partial_\theta g_{22} = 0 \quad \text{and} \quad g_{23} = 0. \tag{4.18}$$

- For $\mu = 2$ and $\nu = 3$:

$$\partial_2 \xi^\alpha_{(i)} g_{\alpha 3} + \partial_3 \xi^\alpha_{(i)} g_{2\alpha} = 0 \quad \Longrightarrow \quad \begin{cases} \csc^2\theta \cos\varphi\, g_{33} - \cos\varphi\, g_{22} = 0, \\ \csc^2\theta \sin\varphi\, g_{33} - \sin\varphi\, g_{22} = 0, \end{cases}$$

from which

$$g_{33} = g_{22} \sin^2\theta. \tag{4.19}$$

- For $\mu = 3$ and $\nu = 3$:

$$\xi^\alpha_{(i)} \partial_\alpha g_{33} + 2\partial_3 \xi^\alpha_{(i)} g_{\alpha 3} = 0 \quad \Longrightarrow \quad \begin{cases} \sin\varphi\, \partial_\theta g_{33} - 2\cot\theta \sin\varphi\, g_{33} = 0, \\ \cos\varphi\, \partial_\theta g_{33} - 2\cot\theta \cos\varphi\, g_{33} = 0, \end{cases}$$

which comes from the derivative of the previous system of equations, thus giving again Eq. (4.19).

Therefore, using Eq. (4.8) along with Eqs. (4.10)-(4.19), the line element reduces to

$$ds^2 = \phi^2(t,r)\left[g_{00}(t,r)\, dt^2 + 2g_{01}(t,r)\, dtdr + g_{11}(t,r)\, dr^2 + g_{22}(t,r)\, d\Omega^2\right], \tag{4.20}$$

where $d\Omega^2$ denotes the standard line element on the 2-sphere,

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2. \tag{4.21}$$

Expression (4.20) can be simplified by suitable coordinate transformations. For convenience, introduce the functions

$$\begin{aligned} A(t,r) &= \phi^2(t,r) g_{00}(t,r), & B(t,r) &= -\phi^2(t,r) g_{11}(t,r), \\ C(t,r) &= -\phi^2(t,r) g_{01}(t,r), & D^2(t,r) &= -\phi^2(t,r) g_{22}(t,r). \end{aligned}$$

Now define a new radial coordinate r' by requiring

$$r'\phi'(t,r') = D(t,r), \tag{4.22}$$

where ϕ' is a new arbitrary scale function. From this relation, one obtains

$$dr = \frac{\partial_{r'}(r'\phi')}{\partial_t D}dr' - \frac{\partial_t D}{\partial_r D}dt, \tag{4.23}$$

which preserves the general structure of the line element. Extracting an overall factor $\phi'^2(r')$ then yields

$$ds^2 = \phi'^2(r')\left[\bar{A}(t,r')\,dt^2 - \bar{B}(t,r')\,dr'^2 - 2\bar{C}(t,r')\,dtdr' - r'^2\,d\Omega^2\right],$$

where the expressions for $\bar{A}$, $\bar{B}$ and $\bar{C}$ can be computed but are not necessary here.

A further redefinition of the temporal coordinate allows the metric to be expressed in a diagonal form. Locally, this is achieved through

$$dt' = \frac{1}{E(t,r')}\left[\bar{A}(t,r')\,dt - \bar{C}(t,r')\,dr'\right], \tag{4.24}$$

where E is an arbitrary function that is independent of $\bar{A}$, $\bar{B}$ and $\bar{C}$.

After performing this transformation and suppressing the prime symbols for simplicity, the line element can be written as

$$\boxed{ds^2 = \phi^2(t,r)\left[A(t,r)\,dt^2 - B(t,r)\,dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,\varphi^2\right)\right],} \tag{4.25}$$

which is the most general expression for a spherically symmetric line element in diagonal form. The functions ϕ, A and B are different from the former ones and will be determined by the field equations in Section 4.3.

4.2 Spherically Symmetric Field Equations

In order to explicitly compute the field equations for the spherically symmetric spacetime described by (4.25), the Cartan formalism is adopted. This approach is particularly convenient, as it simplifies the derivation of curvature components and reduces algebraic complexity. As a first step, the undetermined functions are written in exponential form:

$$\phi(t,r) = e^{\gamma(t,r)}, \quad A(t,r) = e^{\alpha(t,r)}, \quad B(t,r) = e^{\beta(t,r)}, \tag{4.26}$$

and define the following orthonormal 1-form tetrad basis:

$$h^t := e^{\gamma+\alpha}\,dt, \quad h^r := e^{\gamma+\beta}\,dr, \quad h^\theta := e^{\gamma}r\,d\theta, \quad h^\varphi := e^{\gamma}r\sin\theta\,d\varphi. \tag{4.27}$$

The corresponding exterior derivatives for these quantities are

$$\begin{aligned}
dh^t &= \phi^{-1}(\alpha' + \gamma')e^{-\beta} h^r \wedge h^t \,, \\
dh^r &= \phi^{-1}(\dot\beta + \dot\gamma)e^{-\alpha} h^t \wedge h^r \,, \\
dh^\theta &= \phi^{-1}\dot\gamma e^{-\alpha} h^t \wedge h^\theta + \phi^{-1}\left(\gamma' + \frac{1}{r}\right) e^{-\beta} h^r \wedge h^\theta \,, \\
dh^\varphi &= \phi^{-1}\dot\gamma e^{-\alpha} h^t \wedge h^\varphi + \phi^{-1}\left(\gamma' + \frac{1}{r}\right) e^{-\beta} h^r \wedge h^\varphi + \phi^{-1}\left(\frac{1}{r}\cot\theta\right) h^\theta \wedge h^\varphi \,.
\end{aligned} \tag{4.28}$$

By comparing with the first Cartan structure equation, (2.73), one obtains the non-zero components of the connection one-form:

$$\begin{aligned}
&\Omega^t{}_r = \phi^{-1}(\alpha' + \gamma')e^{-\beta} h^t + \phi^{-1}(\dot\beta + \dot\gamma)e^{-\alpha} h^r, &&\Omega^t{}_\theta = \phi^{-1}\dot\gamma e^{-\alpha} h^\theta, \\
&\Omega^t{}_\varphi = \phi^{-1}\dot\gamma e^{-\alpha} h^\varphi, &&\Omega^r{}_\theta = -\phi^{-1}\left(\gamma' + \frac{1}{r}\right) e^{-\beta} h^\theta, \\
&\Omega^r{}_\varphi = -\phi^{-1}\left(\gamma' + \frac{1}{r}\right) e^{-\beta} h^\varphi, &&\Omega^\theta{}_\varphi = -\phi^{-1}\left(\frac{1}{r}\cot\theta\right) h^\varphi.
\end{aligned} \tag{4.29}$$

The second Cartan structure equation, (2.76), yields the curvature two-form:

$$\begin{aligned}
\mathbf{R}^t{}_r &= \phi^{-2}\Big[(\ddot\beta + \ddot\gamma + \dot\beta^2 + \dot\gamma\dot\beta - \dot\beta\dot\alpha - \dot\gamma\dot\alpha)e^{-2\alpha} \\
&\qquad - (\alpha'' + \gamma'' + \alpha'^2 + \gamma'\alpha' - \alpha'\beta' - \gamma'\beta')e^{-2\beta}\Big] h^t \wedge h^r, \\
\mathbf{R}^t{}_\theta &= \phi^{-2}\left[\left((\ddot\gamma - \dot\gamma\dot\alpha)e^{-2\alpha} - \left(\gamma'^2 + \gamma'\alpha' + \frac{\gamma'}{r} + \frac{\alpha'}{r}\right) e^{-2\beta}\right) h^t \wedge h^\theta \right. \\
&\qquad \left. + \left(\dot\gamma' - \dot\gamma\alpha' - \gamma'\dot\beta - \gamma'\dot\gamma - \frac{\dot\beta}{r}\right) e^{-\alpha-\beta} h^r \wedge h^\theta \right], \\
\mathbf{R}^t{}_\varphi &= \phi^{-2}\left[\left((\ddot\gamma - \dot\gamma\dot\alpha)e^{-2\alpha} - \left(\gamma'^2 + \gamma'\alpha' + \frac{\gamma'}{r} + \frac{\alpha'}{r}\right) e^{-2\beta}\right) h^t \wedge h^\varphi \right. \\
&\qquad \left. + \left(\dot\gamma' - \dot\gamma\alpha' - \gamma'\dot\beta - \gamma'\dot\gamma - \frac{\dot\beta}{r}\right) e^{-\alpha-\beta} h^r \wedge h^\varphi \right], \\
\mathbf{R}^r{}_\theta &= \phi^{-2}\left[- \left(\dot\gamma' - \dot\gamma\alpha' - \gamma'\dot\beta - \gamma'\dot\gamma - \frac{\dot\beta}{r}\right) e^{-\alpha-\beta} h^t \wedge h^\theta \right. \\
&\qquad \left. + \left((\dot\gamma^2 + \dot\gamma\dot\beta)e^{-2\alpha} - \left(\gamma'' - \gamma'\beta' + \frac{\gamma'}{r} - \frac{\beta'}{r}\right) e^{-2\beta}\right) h^r \wedge h^\theta \right], \\
\mathbf{R}^r{}_\varphi &= \phi^{-2}\left[- \left(\dot\gamma' - \dot\gamma\alpha' - \gamma'\dot\beta - \gamma'\dot\gamma - \frac{\dot\beta}{r}\right) e^{-\alpha-\beta} t^\varphi \right. \\
&\qquad \left. + \left((\dot\gamma^2 + \dot\gamma\dot\beta)e^{-2\alpha} - \left(\gamma'' - \gamma'\beta' + \frac{\gamma'}{r} - \frac{\beta'}{r}\right) e^{-2\beta}\right) h^r \wedge h^\varphi \right], \\
\mathbf{R}^\theta{}_\varphi &= \phi^{-2}\left[\dot\gamma^2 e^{-2\alpha} - \left(\gamma'^2 + \frac{2\gamma'}{r} + \frac{1}{r^2}\right) e^{-2\beta} + \frac{1}{r^2}\right] h^\theta \wedge h^\varphi.
\end{aligned} \tag{4.30}$$

From here, the components of the curvature tensor are obtained:

$$
\begin{aligned}
R^{t}{}_{rtr} &= \phi^{-2}\Big[(\ddot{\beta}+\ddot{\gamma}+\dot{\beta}^2+\dot{\gamma}\dot{\beta}-\dot{\beta}\dot{\alpha}-\dot{\gamma}\dot{\alpha})e^{-2\alpha} \\
&\qquad -(\alpha''+\gamma''+\alpha'^2+\gamma'\alpha'-\alpha'\beta'-\gamma'\beta')e^{-2\beta}\Big], \\
R^{t}{}_{\theta t\theta} = R^{t}{}_{\varphi t\varphi} &= \phi^{-2}\left[(\ddot{\gamma}-\dot{\gamma}\dot{\alpha})e^{-2\alpha}-\left(\gamma'^2+\gamma'\alpha'+\frac{\gamma'}{r}+\frac{\alpha'}{r}\right)e^{-2\beta}\right], \\
R^{t}{}_{\theta r\theta} = R^{t}{}_{\varphi r\varphi} &= \phi^{-2}\left[\dot{\gamma}'-\dot{\gamma}\alpha'-\gamma'\dot{\beta}\gamma'\dot{\gamma}-\frac{\dot{\beta}}{r}\right]e^{-\alpha-\beta}, \\
R^{r}{}_{\theta r\theta} = R^{r}{}_{\varphi r\varphi} &= \phi^{-2}\left[(\dot{\gamma}^2+\dot{\gamma}\dot{\beta})e^{-2\alpha}-\left(\gamma''-\gamma'\beta'+\frac{\gamma'}{r}-\frac{\beta'}{r}\right)e^{-2\beta}\right], \\
R^{\theta}{}_{\varphi\theta\varphi} &= \phi^{-2}\left[\dot{\gamma}^2e^{-2\alpha}-\left(\gamma'^2+\frac{2\gamma'}{r}+\frac{1}{r^2}\right)e^{-2\beta}+\frac{1}{r^2}\right].
\end{aligned}
\tag{4.31}
$$

Therefore, the components of the Ricci tensor are

$$
\begin{aligned}
R^{t}{}_{t} &= -\phi^{-2}\Bigg[\left(\ddot{\beta}+3\ddot{\gamma}+\dot{\beta}^2+\dot{\gamma}\dot{\beta}-\dot{\beta}\dot{\alpha}-3\dot{\gamma}\dot{\alpha}\right)e^{-2\alpha} \\
&\qquad -\left(\alpha''+\gamma''+\alpha'^2+2\gamma'^2+3\gamma'\alpha'-\alpha'\beta'-\gamma'\beta'+\frac{2\gamma'}{r}+\frac{2\alpha'}{r}\right)e^{-2\beta}\Bigg], \\
R^{t}{}_{r} &= -\phi^{-2}\left[2\left(\dot{\gamma}'-\dot{\gamma}\alpha'-\gamma'\dot{\beta}-\gamma'\dot{\gamma}-\frac{\dot{\beta}}{r}\right)e^{-\alpha-\beta}\right], \\
R^{r}{}_{r} &= -\phi^{-2}\Bigg[\left(\ddot{\beta}+\ddot{\gamma}+\dot{\beta}^2+2\dot{\gamma}^2+3\dot{\gamma}\dot{\beta}-\dot{\beta}\dot{\alpha}-\dot{\gamma}\dot{\alpha}\right)e^{-2\alpha} \\
&\qquad -\left(\alpha''+3\gamma''+\alpha'^2+\gamma'\alpha'-\alpha'\beta'-3\gamma'\beta'+\frac{2\gamma'}{r}-\frac{2\beta'}{r}\right)e^{-2\beta}\Bigg], \\
R^{\theta}{}_{\theta} = R^{\varphi}{}_{\varphi} &= -\phi^{-2}\Bigg[\frac{1}{r^2}+\left(\ddot{\gamma}+2\dot{\gamma}^2-\dot{\gamma}\dot{\alpha}+\dot{\gamma}\dot{\beta}\right)e^{-2\alpha} \\
&\qquad -\left(\gamma''+2\gamma'^2+\gamma'\alpha'-\gamma'\beta'+\frac{4\gamma'}{r}+\frac{\alpha'}{r}-\frac{\beta'}{r}+\frac{1}{r^2}\right)e^{-2\beta}\Bigg],
\end{aligned}
\tag{4.32}
$$

and the Ricci scalar is

$$
\begin{aligned}
R = -2\phi^{-2}\Bigg[&\frac{1}{r^2}+\left(\ddot{\beta}+3\ddot{\gamma}+\dot{\beta}^2+3\dot{\gamma}^2+3\dot{\gamma}\dot{\beta}-3\dot{\gamma}\dot{\alpha}-\dot{\beta}\dot{\alpha}\right)e^{-2\alpha} \\
&-\left(\alpha''+3\gamma''+\alpha'^2+3\gamma'^2+3\gamma'\alpha'-3\gamma'\beta'-\alpha'\beta'+\frac{6\gamma'}{r}+\frac{2\alpha'}{r}-\frac{2\beta'}{r}+\frac{1}{r^2}\right)e^{-2\beta}\Bigg].
\end{aligned}
\tag{4.33}
$$

Finally, substituting these expressions into (3.36) and multiplying by ϕ^2, field equations in the absence of matter fields take the form

$$\frac{1}{r^2} + \left(3\dot{\gamma}^2 + 2\dot{\gamma}\dot{\beta}\right) e^{-2\alpha} - \left(2\gamma'' + \gamma'^2 - 2\gamma'\beta' + \frac{4\gamma'}{r} - \frac{2\beta'}{r} + \frac{1}{r^2}\right) e^{-2\beta} = 0, \tag{4.34a}$$

$$-2\left(\dot{\gamma}' - \dot{\gamma}\alpha' - \gamma'\dot{\beta} - \gamma'\dot{\gamma} - \frac{\dot{\beta}}{r}\right) e^{-\alpha-\beta} = 0, \tag{4.34b}$$

$$\frac{1}{r^2} + \left(2\ddot{\gamma} + \dot{\gamma}^2 - 2\dot{\gamma}\dot{\alpha}\right) e^{-2\alpha} - \left(3\gamma'^2 + 2\gamma'\alpha' + \frac{4\gamma'}{r} + \frac{2\alpha'}{r} + \frac{1}{r^2}\right) e^{-2\beta} = 0, \tag{4.34c}$$

$$\begin{aligned}&(\ddot{\beta} + 2\ddot{\gamma} + \dot{\beta}^2 + \dot{\gamma}^2 + 2\dot{\gamma}\dot{\beta} - 2\dot{\gamma}\dot{\alpha} - \dot{\alpha}\dot{\beta})e^{-2\alpha} \\ &- \left(\alpha'' + 2\gamma'' + \alpha'^2 + \gamma'^2 + 2\gamma'\alpha' - 2\gamma'\beta' - \alpha'\beta' + \frac{2\gamma'}{r} + \frac{\alpha'}{r} - \frac{\beta'}{r}\right) e^{-2\beta} = 0,\end{aligned} \tag{4.34d}$$

When no matter field is present, the condition (3.49) is automatically satisfied, so the field equation with respect to the scale function, (3.48), is just the trace of Eq. (3.36).

In Eq. (4.22), the new scale function was introduced as arbitrary; it can therefore be chosen as a pure radial function $\phi(r)$. This implies $\dot{\gamma} = 0$, and Eq. (4.34b) reduces to

$$\left(\gamma' + \frac{1}{r}\right)\dot{\beta} = 0. \tag{4.35}$$

The solution given by $\gamma' = -\frac{1}{r}$ is incompatible with Eqs. (4.34a) and (4.34c); thus one obtains $\dot{\beta} = 0$. In this case, Eq. (4.34c) allows isolating α' in terms of γ' and β', showing that $\alpha' = \alpha'(r)$. Hence, $\alpha(t,r) = \alpha_1(t) + \alpha_2(r)$, and the only time-dependent coefficient in Eq. (4.25) will be the function A.

Because the integrating factor introduced in Eq. (4.24) remains free, the temporal contribution α_2 can be absorbed into the time coordinate through the redefinition

$$e^{\alpha_1(t)}dt \to dt.$$

Since α_1 is assumed smooth within the present framework, it is continuous and therefore integrable, ensuring that the above transformation is well defined. Moreover, because γ and β are time-independent, the system (4.34) imposes no further restriction on α_1. One may therefore fix it conveniently; choosing $\alpha_2 = 0$ reproduces the same result as before.

With these considerations, the field equations reduce to the following system:

$$\frac{1}{r^2} - \left(2\gamma'' + \gamma'^2 - 2\gamma'\beta' + \frac{4\gamma'}{r} - \frac{2\beta'}{r} + \frac{1}{r^2}\right) e^{-2\beta} = 0, \tag{4.36a}$$

$$\frac{1}{r^2} - \left(3\gamma'^2 + 2\gamma'\alpha' + \frac{4\gamma'}{r} + \frac{2\alpha'}{r} + \frac{1}{r^2}\right) e^{-2\beta} = 0, \tag{4.36b}$$

$$-\left(2\gamma'' + \alpha'' + \gamma'^2 + \alpha'^2 + 2\gamma'\alpha' - 2\gamma'\beta' - \alpha'\beta' + \frac{2\gamma'}{r} + \frac{\alpha'}{r} - \frac{\beta'}{r}\right) e^{-2\beta} = 0. \tag{4.36c}$$

In conclusion, Birkhoff's theorem holds whenever the scale function is taken to be time-independent.

Furthermore, in the limit $\phi \to 1$ (equivalently, $\gamma \to 0$), the field equations reduce to those of General Relativity, reproducing the Schwarzschild solution and thereby recovering the familiar structure of spherically symmetric vacuum spacetimes.

4.3 Vacuum Solution

This section derives the general solution to the static, spherically symmetric vacuum field equations, namely the system (4.36). To begin, subtract Eq. (4.36a) from Eq. (4.36b) and then multiply by $\frac{r}{2}e^{-\alpha+\beta-\gamma}$. This procedure leads to

$$\left[(r\gamma' + 1)e^{-\alpha-\beta-\gamma}\right]' = 0, \tag{4.37}$$

whose integration gives

$$e^{\alpha+\beta+\gamma} = k(r\gamma' + 1), \tag{4.38}$$

where k is a dimensionless integration constant. Moreover, multiplying Eq. (4.38) by e^{γ}, one may rewrite it as

$$e^{\alpha+\beta+2\gamma} = k(re^{\gamma})'. \tag{4.39}$$

Next, add Eqs. (4.36a) and (4.36b) and multiply by $\frac{kr^2}{2}e^{\alpha+\beta+2\gamma}$. The resulting expression is

$$\left[kr(r\gamma' + 1)e^{\alpha-\beta+2\gamma}\right]' = ke^{\alpha+\beta+2\gamma}. \tag{4.40}$$

Using Eq. (4.38) on the left-hand side and Eq. (4.39) on the right-hand side gives

$$\left(re^{2\alpha+3\gamma}\right)' = k^2(re^{\gamma})'. \tag{4.41}$$

Integrating this equation and solving for $e^{2\alpha}$ leads to

$$e^{2\alpha} = k^2e^{-2\gamma} - \frac{k^2 a}{r}e^{-3\gamma} = \frac{k^2}{\phi^2}\left(1 - \frac{a}{r\phi}\right), \tag{4.42}$$

where a is an integration constant. Since k^2 may be absorbed into the time coordinate, one may set $k = 1$ without loss of generality. Therefore,

$$e^{2\alpha} = \frac{1}{\phi^2}\left(1 - \frac{a}{r\phi}\right). \tag{4.43}$$

Similarly, by squaring Eq. (4.39) and substituting Eq. (4.43), one obtains

$$e^{2\beta} = \frac{(r\phi)'^2}{\phi^2}\left(1 - \frac{a}{r\phi}\right)^{-1}. \tag{4.44}$$

Now, Eq. (4.36c) may be used, in principle, to determine the scale function. Multiplying this equation by $e^{\alpha+\beta}$ gives

$$\left[\left(\alpha' + 2\gamma' + \frac{1}{r}\right)e^{\alpha-\beta}\right]' + \left(\gamma' + \frac{1}{r}\right)^2 e^{\alpha-\beta} = 0. \tag{4.45}$$

From Eqs. (4.43) and (4.44), it follows that

$$e^{\alpha-\beta} = \frac{1}{(r\phi)'}\left(1 - \frac{a}{r\phi}\right), \tag{4.46}$$

and Eq. (4.39) implies

$$\alpha' + \beta' + 2\gamma' = \frac{(r\phi)''}{(r\phi)'}. \tag{4.47}$$

Thus,

$$\alpha' + 2\gamma' + \frac{1}{r} = \frac{(r\phi)''}{(r\phi)'} - \frac{(e^{\beta})'}{e^{\beta}} + \frac{1}{r} = \frac{d}{dr}\ln\left[\frac{r(r\phi)'}{e^{\beta}}\right],$$

and, substituting Eq. (4.44),

$$\alpha' + 2\gamma' + \frac{1}{r} = \frac{d}{dr}\ln\left[r\phi\sqrt{1 - \frac{a}{r\phi}}\right] = \frac{(2r\phi - a)(r\phi)'}{2(r\phi)^2\left(1 - \frac{a}{r\phi}\right)}. \tag{4.48}$$

Multiplying Eqs. (4.46) and (4.48) and then differentiating yields

$$\left[\left(\alpha' + 2\gamma' + \frac{1}{r}\right)e^{\alpha-\beta}\right]' = \left[\frac{1}{r\phi} - \frac{a}{2(r\phi)^2}\right]' = -\frac{(r\phi)'}{(r\phi)^2}\left(1 - \frac{a}{r\phi}\right). \tag{4.49}$$

Moreover,

$$\left(\gamma' + \frac{1}{r}\right)^2 e^{\alpha-\beta} = \left(\frac{(r\phi)'}{r\phi}\right)^2 e^{\alpha-\beta} = \frac{(r\phi)'}{(r\phi)^2}\left(1 - \frac{a}{r\phi}\right). \tag{4.50}$$

Eqs. (4.49) and (4.50) show that Eq. (4.45) is automatically satisfied and imposes no condition on ϕ. This reflects a scale gauge freedom, inherited from the invariance of the action under pure scale transformations. Therefore, the vacuum solution is

$$ds^2 = \left(1 - \frac{a}{r\phi}\right)dt^2 - \frac{(r\phi)'^2}{\left(1 - \frac{a}{r\phi}\right)}dr^2 - r^2\phi^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{4.51}$$

Performing the transformation

$$\bar{r} = r\phi(r), \tag{4.52}$$

reduces Eq. (4.51) to the Schwarzschild line element with Schwarzschild radius $\bar{r}_S = a$:

$$ds^2 = \left(1 - \frac{\bar{r}_S}{\bar{r}}\right)dt^2 - \frac{d\bar{r}^2}{\left(1 - \frac{\bar{r}_S}{\bar{r}}\right)} - \bar{r}^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{4.53}$$

This equivalence is expected, since the transformation (4.52) is derived from Eq. (4.22) with $\phi = 1$, which naturally reproduces the results of General Relativity. A time-dependent expression of the line element may be obtained through the transformation

$$\bar{r} = r\phi(t, r),$$

which, after the corresponding calculations, yields

$$ds^2 = \left[\left(1 - \frac{a}{r\phi}\right) - \frac{(r\dot{\phi})^2}{\left(1 - \frac{a}{r\phi}\right)} \right] dt^2 - 2\frac{(r\dot{\phi})\,(r\phi)'}{\left(1 - \frac{a}{r\phi}\right)} dtdr - \frac{(r\phi)'^2}{\left(1 - \frac{a}{r\phi}\right)} dr^2 - r^2\phi^2\, d\Omega^2.$$

However, this expression is not considered further, as it merely reflects a coordinate choice and introduces unnecessary complications.

4.3.1 The Lyra-Schwarzschild solution

In order to obtain explicit results, it is convenient to fix the scale function. In particular, the *Schwarzschild-like gauge fixing* is adopted:

$$A(r) = \frac{1}{B(r)} \qquad \Longrightarrow \qquad e^{2\alpha} = e^{-2\beta}. \tag{4.54}$$

Substituting Eqs. (4.43) and (4.44), one obtains

$$(r\phi)' = \pm\phi^2.$$

Because both r and $\bar{r}$ represent radial coordinates, they must satisfy $\frac{d\bar{r}}{dr} > 0$, which implies $(r\phi)' > 0$. Therefore, integrating the equation above gives

$$\frac{1}{r\phi} = \frac{1}{r} - b,$$

where b is an integration constant. If r_L denotes the value of r when $\bar{r} \to \infty$, then $b = \frac{1}{r_L}$, and the scale function takes the form

$$\phi(r) = \left(1 - \frac{r}{r_L}\right)^{-1}. \tag{4.55}$$

Finally, the resulting expression for the line element in the Schwarzschild gauge is

$$\boxed{ds^2 = \left(1 - \frac{r}{r_L}\right)^{-2} \left[\mu(r)\, dt^2 - \frac{dr^2}{\mu(r)} - r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right],} \tag{4.56}$$

where

$$\mu(r) = \left(1 - \frac{r}{r_L}\right)^2 \left(1 - \frac{a}{r} + \frac{a}{r_L}\right) = \left(1 - \frac{r}{r_L}\right)^2 \frac{1 - \frac{r_S}{r}}{1 - \frac{r_S}{r_L}}. \tag{4.57}$$

The parameter r_L will be referred to as the *Lyra radius*, while r_S is the Schwarzschild radius in the Lyra reference system with non-zero scale function:

$$r_S = \frac{a}{1 + \frac{a}{r_L}}. \tag{4.58}$$

Note that Eq. (4.53) can be obtained not only through the transformation (4.52), but also as the particular case of Eq. (4.56) in the limit $r_L \to \infty$. Therefore, the parameter r_S may be consistently regarded as the natural generalization of the Schwarzschild radius.

If the line element (4.56) is interpreted as an exterior solution, the function $\mu(r)$ must remain positive, which holds for $r_S < r$. Moreover, $\bar{r} > 0$ implies $r < r_L$, hence suggesting a spatially limited spacetime. Cosmological observations indicate the existence of structures on extremely large scales, implying that spacetime is either infinite or bounded by a value of r_L far beyond the currently observed scales. The latter scenario is mathematically plausible, provided that cosmological observations occur in a range $r_S \ll r \ll r_L$—a limit where the coordinates r and $\bar{r}$ are practically indistinguishable. Because of the significant implications, expression (4.56) is worthy of attention, and it will be referred to as the *Lyra-Schwarzschild* line element.

4.3.2 Correction to Newtonian Gravity

Consider the spherically symmetric spacetime described by Eq. (4.56) and Taylor-expand the expressions (4.55) and (4.57) in the regime $r_S \ll r \ll r_L$. To second order, this computation yields

$$\phi(r) \approx 1 + \frac{r}{r_L} + \frac{r^2}{r_L^2},$$

and

$$\mu(r) \approx 1 + \frac{3r_S}{r_L}\left(1 - \frac{r}{r_L} + \frac{r_S}{r_L}\right) - \frac{r_S}{r}\left(1 + \frac{r_S}{r_L}\right) - \frac{2r}{r_L} + \frac{r^2}{r_L^2}.$$

Comparing these results with Eqs. (3.58) and (3.59) gives the corresponding forms of $\delta\phi$ and h_{00}. By substituting those expressions into Eq. (3.62), one obtains the effective Newtonian potential:

$$U = \frac{3r_S}{2r_L}\left(1 - \frac{r}{r_L} + \frac{r_S}{r_L}\right) - \frac{r_S}{2r}\left(1 + \frac{r_S}{r_L}\right) + \frac{3r^2}{2r_L^2}. \tag{4.59}$$

In a spherically symmetric spacetime, the equation of motion in the Newtonian limit, as given in (3.61), reduces to

$$\frac{d^2r}{dt^2} = -\frac{dU}{dr}.$$

Using Eq. (4.59), one obtains

$$\frac{d^2r}{dt^2} = -\frac{r_S}{2r^2}\left(1 + \frac{r_S}{r_L}\right) + \frac{3r_S}{2r_L^2} - \frac{3r}{r_L^2}. \tag{4.60}$$

Now, the *geometric mass* is defined as

$$m_G = \frac{\bar{r}_S}{2} = \frac{r_S}{2\left(1 - \frac{r_S}{r_L}\right)} \approx \frac{r_S}{2}\left(1 + \frac{r_S}{r_L}\right), \tag{4.61}$$

which is just the coefficient of the Newton-type contribution.

Using this definition, Eq. (4.60) becomes

$$\frac{d^2r}{dt^2} = -\frac{m_G}{r^2} + \frac{3m_G}{r_L^2}\left(1 - \frac{r_S}{r_L}\right) - \frac{3r}{r_L^2}. \tag{4.62}$$

This form shows that, in LyST geometry, the geometric mass—acting as the primary effective gravitational source—depends on both r_S and r_L. Moreover, the second-order corrections give rise to a constant repulsive acceleration and a linear term in r, analogous to an anti–de Sitter contribution.

From Eq. (4.61), the Schwarzschild radius may be rewritten in terms of m_G and r_L:

$$r_S = \frac{1}{\dfrac{1}{2m_G} + \dfrac{1}{r_L}}, \tag{4.63}$$

and substituting this expression into Eq. (4.56) leads to

$$\begin{aligned} ds^2 &= \left(1 - \frac{2m_G}{r} + \frac{2m_G}{r_L}\right) dt^2 \\ &\quad - \left(1 - \frac{r}{r_L}\right)^{-4} \left(1 - \frac{2m_G}{r} + \frac{2m_G}{r_L}\right)^{-1} dr^2 - \left(1 - \frac{r}{r_L}\right)^{-2} r^2\, d\Omega^2. \end{aligned} \tag{4.64}$$

Once again, the limit $r_L \to \infty$ recovers the Schwarzschild solution. However, Eq. (4.64) does not admit an asymptotically flat region, since the radial coordinate is restricted by the upper bound r_L.

Finally, the equation of motion in the Newtonian limit for a Schwarzschild spacetime with cosmological constant Λ reads

$$\frac{d^2r}{dt^2} = -\frac{m_G}{r^2} + \frac{\Lambda r}{3}. \tag{4.65}$$

By comparing Eqs. (4.62) and (4.65) when $r_S \ll r$, one finds that the cosmological constant is related to the Lyra radius through

$$\Lambda \approx -\frac{9}{r_L^2}, \tag{4.66}$$

showing that the spacetime behaves as in anti-de Sitter geometry $r_S \ll r \ll r_L$. These results indicate that LyST geometry naturally reproduces the expected Schwarzschild behavior at short distances while introducing a well-defined large-distance modification that becomes relevant only near the geometric boundary set by r_L.

Chapter 5

Free Particle Motion

The analysis of free particle motion provides a direct way to test the dynamical content of Lyra gravity and to assess how its geometric modifications affect observable trajectories. In particular, the Lyra–Schwarzschild spacetime offers a natural setting in which departures from the standard Schwarzschild geometry can be examined in a controlled and highly symmetric background. Within this context, the Hamilton–Jacobi approach plays a central role, as it allows the effects of the Lyra scale function to be incorporated directly into the canonical generators of motion, without relying on the explicit integration of the geodesic equations. By encoding the dynamics in a single generating function, this formalism makes conserved quantities and spacetime symmetries manifest and provides a systematic framework for separating the dynamical variables in spherically symmetric spacetimes. For these reasons, the Hamilton–Jacobi formulation is particularly well suited to the study of both massive and massless particle motion in the Lyra–Schwarzschild geometry, and it forms the basis of the analysis developed in this chapter.

5.1 The Hamilton-Jacobi Approach

It was established in Chapter 3 that free particle motion in Lyra spacetime occurs along geodesic curves. For an affine parametrization λ, the dynamics of the system admits a canonical description that provides the appropriate setting for the Hamilton–Jacobi formulation. In this context, the Lagrangian and Hamiltonian encode the influence of both the spacetime metric and the Lyra scale function on the particle dynamics, while defining the phase-space structure underlying the Hamilton–Jacobi equation. The corresponding expressions are given by:

$$L(x, \dot{x}, \lambda) = \frac{1}{2} m\phi^2 g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \quad \text{and} \quad H(x, p, \lambda) = \frac{1}{2m\phi^2} g^{\mu\nu} p_\mu p_\nu, \tag{5.1}$$

where m is the rest mass, $\dot{x}^\mu$ are the velocities with respect to λ, and $p_\mu = \partial L / \partial \dot{x}^\mu$ are the canonical conjugate momenta. Moreover, the following constraint holds:

$$\phi^2 g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \varepsilon. \tag{5.2}$$

For massive particles, ε is set to 1 and λ is interpreted as the proper time. In contrast, for massless particles, ε is set to 0.

At this point, the Hamilton principal function $S(x, P, \lambda)$ is introduced. This function generates a canonical transformation from (x, p) to (X, P), where the coordinates X^μ and P_μ are constants. The canonical relations are given by

$$p_\mu = \frac{\partial S(x, P, \lambda)}{\partial x^\mu} \quad \text{and} \quad X^\mu = \frac{\partial S(x, P, \lambda)}{\partial P_\mu}, \tag{5.3}$$

and the evolution of the system is governed by the Hamilton-Jacobi equation:

$$H\left(x, \frac{\partial S}{\partial x}, \lambda\right) + \frac{\partial S}{\partial \lambda} = 0. \tag{5.4}$$

From Eq. (4.56), the Hamiltonian in a Lyra-Schwarzschild spacetime is written

$$H = \frac{1}{2m\phi^2}\left(\frac{p_t^2}{\mu} - \mu p_r^2 - \frac{p_\theta^2}{r^2} - \frac{p_\varphi^2}{r^2\sin^2\theta}\right) = \frac{\varepsilon m}{2}, \tag{5.5}$$

where ϕ and μ are given by Eqs. (4.55) and (4.57). Since the Hamiltonian does not explicitly depend on λ, Eq. (5.4) implies

$$S(x, P, \lambda) = -\frac{\varepsilon m \lambda}{2} + W(x, P), \tag{5.6}$$

and the Hamilton-Jacobi equation takes the form

$$H\left(x, \frac{\partial W}{\partial x}\right) = \frac{\varepsilon m}{2}. \tag{5.7}$$

Moreover, the Hamiltonian does not depend on t or φ either, so the corresponding conjugate momenta are constants:

$$cte : E = p_t = \frac{\partial S}{\partial t} = \frac{\partial W}{\partial t}, \qquad cte : -\ell = p_\varphi = \frac{\partial S}{\partial \varphi} = \frac{\partial W}{\partial \varphi},$$

thus leading to the expression

$$W(x, P) = Et - \ell\varphi + W'(r, \theta). \tag{5.8}$$

This result and Eq. (5.5) allow to write the Hamilton-Jacobi equation as follows:

$$\frac{E^2}{\mu} - \mu\left(\frac{\partial W'}{\partial r}\right)^2 - \frac{1}{r^2}\left[\left(\frac{\partial W'}{\partial \theta}\right)^2 + \frac{\ell^2}{\sin^2\theta}\right] = \varepsilon m^2\phi^2, \tag{5.9}$$

which is a separable equation. In fact, by writing

$$W'(r, \theta) = W_1(r) + W_2(\theta), \tag{5.10}$$

Eq. (5.9) yields

$$\frac{E^2 r^2}{\mu} - \varepsilon m^2 r^2 \phi^2 - \mu r^2\left(\frac{dW_1}{dr}\right)^2 = \left(\frac{dW_2}{d\theta}\right)^2 + \frac{\ell^2}{\sin^2\theta}, \tag{5.11}$$

where both members are equal to a constant in order for the equation to make sense. Beginning with the right hand side, and recognizing $p_\theta = dW_2/d\theta$, one obtains

$$p_\theta^2 + \ell^2\csc^2\theta = (p_{\theta,0})^2 + \ell^2\csc^2\theta_0,$$

where $(\theta_0, p_{\theta,0})$ are canonical coordinates for a particular λ_0. Because of the spherical symmetry, the coordinate system (θ, φ) can be oriented such that $\theta_0 = \frac{\pi}{2}$ and $p_{\theta,0} = 0$. Then,

$$p_\theta^2 + \ell^2\cot^2\theta = 0,$$

whose unique solution is given by

$$\theta = \frac{\pi}{2}, \qquad p_\theta = 0.$$

It follows that W_2 is a constant, which can be perfectly included into W_1, thus giving $W_2 = 0$. With this, Eq. (5.11) becomes an ordinary differential equation, with solution

$$W_1(r) = \pm \int dr \sqrt{\frac{E^2}{\mu^2} - \frac{\varepsilon m^2 \phi^2}{\mu} - \frac{\ell^2}{\mu r^2}}. \tag{5.12}$$

Therefore, the Hamilton principal function is written

$$S(t, r, \varphi, P_t, P_r, P_\varphi, \sigma) = -\frac{\varepsilon m \lambda}{2} + Et - \ell\varphi \pm \int dr \sqrt{\frac{E^2}{\mu^2} - \frac{\varepsilon m^2 \phi^2}{\mu} - \frac{\ell^2}{\mu r^2}}, \tag{5.13}$$

where $P_t = E$, $P_r = -\varepsilon m/2$ and $P_\varphi = -\ell$ are chosen as the new momenta.

Finally, the motion of the particle is determined by the second relation in Eq. (5.3):

$$cte : X_t = \frac{\partial S}{\partial P_t} = t \pm k \int \frac{dr}{\mu\sqrt{k^2 - \varepsilon\mu\phi^2 - \frac{h^2\mu}{r^2}}}, \tag{5.14a}$$

$$cte : X_r = \frac{\partial S}{\partial P_r} = \lambda \pm \int \frac{\phi^2\, dr}{\sqrt{k^2 - \varepsilon\mu\phi^2 - \frac{h^2\mu}{r^2}}}, \tag{5.14b}$$

$$cte : X_\varphi = \frac{\partial S}{\partial P_\varphi} = \varphi \pm h \int \frac{dr}{r^2\sqrt{k^2 - \varepsilon\mu\phi^2 - \frac{h^2\mu}{r^2}}}, \tag{5.14c}$$

where $k = E/m$ is the specific energy, and $h = \ell/m$ the specific angular momentum.

5.2 Massive Particles

As it was mentioned before, for massive particles, $\varepsilon = 1$ with λ as the corresponding proper time. Then, from Eq. (5.14b), one obtains

$$k^2 = 1 + \left(\phi^4 \dot{r}^2 + \frac{h^2}{r^2\phi^2}\right) - \frac{2m_G}{r\phi}\left(1 + \frac{h^2}{r^2\phi^2}\right). \tag{5.15}$$

Here, the unity corresponds to the rest energy; the second term, between parentheses, is the kinetic contribution; and the remaining contribution is related to the gravitational interaction, as it is evident from the presence of the geometric mass m_G. Thus Eq. (5.15) is an extension of the energy-momentum mathematical relation from especial relativity.

Rearranging the terms of Eq. (5.15) gives

$$\mathcal{E} = \frac{1}{2}\phi^4 \dot{r}^2 + V_{\text{eff}}, \tag{5.16}$$

where $\mathcal{E} = (k^2 - 1)/2$ and V_{eff} is the *effective potential*:

$$V_{\text{eff}} = -\frac{m_G}{r}\left(1 - \frac{r}{r_L}\right) + \frac{h^2}{2r^2}\left(1 - \frac{r}{r_L}\right)^2 - \frac{m_G h^2}{r^3}\left(1 - \frac{r}{r_L}\right)^3. \tag{5.17}$$

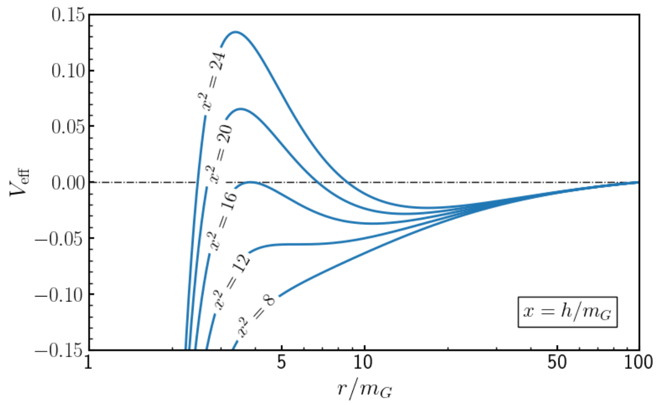

Figure 5.1: Effective potential for different values of h/m_G.

This potential is used in order to determine the types of motion for massive particles in terms of the parameters ℓ and m_G. For this purpose, the two first derivatives of the potential are to be considered:

$$\frac{dV_{\text{eff}}}{dr} = \frac{d\tilde{r}}{dr}\frac{dV_{\text{eff}}}{d\tilde{r}} = \phi^2 \frac{dV_{\text{eff}}}{d\tilde{r}},$$

$$\frac{d^2V_{\text{eff}}}{dr^2} = \frac{d^2\tilde{r}}{dr^2}\frac{dV_{\text{eff}}}{d\tilde{r}} + \left(\frac{d\tilde{r}}{dr}\right)^2 \frac{d^2V_{\text{eff}}}{d\tilde{r}^2} = \frac{2\phi^3}{r_L}\frac{dV_{\text{eff}}}{d\tilde{r}} + \phi^4 \frac{d^2V_{\text{eff}}}{d\tilde{r}^2}.$$

Since $\phi \neq 0$, the scale function does not increase the amount of critical points, thereby giving the same condition:

$$\left(\frac{dV_{\text{eff}}}{dr}\right)_{r_{crit}} = \left(\frac{dV_{\text{eff}}}{d\tilde{r}}\right)_{\tilde{r}_{crit}} = 0. \tag{5.18}$$

Similarly, the stability of the critical points is no affected:

$$\operatorname{sgn}\left(\frac{d^2V_{\text{eff}}}{dr^2}\right)_{r_{crit}} = \operatorname{sgn}\left(\frac{d^2V_{\text{eff}}}{d\tilde{r}^2}\right)_{\tilde{r}_{crit}}. \tag{5.19}$$

Thus, the analysis of critical points can be made in any of these two coordinate systems; for simplicity, the critical points are calculated with respect to $\tilde{r}$ in the first place, and then transformed the result for the coordinate r. From Eq. (5.18), one obtains the critical points:

$$\tilde{r}_{\pm} = \frac{h^2}{2m_G}\left(1 \pm \sqrt{1 - \frac{12m_G^2}{h^2}}\right) = 6m_G\left(1 \mp \sqrt{1 - \frac{12m_G^2}{h^2}}\right)^{-1}, \tag{5.20}$$

or equivalently,

$$r_{\pm} = 6m_G \left[\left(1 + \frac{6m_G}{r_L}\right) \mp \sqrt{1 - \frac{12m_G^2}{h^2}} \right]^{-1}. \tag{5.21}$$

Moreover, the second derivative of the effective potential at these points is given by

$$\left(\frac{d^2 V_{\text{eff}}}{d\tilde{r}^2}\right)_{\tilde{r}_{\pm}} = \frac{h^2}{\tilde{r}_{\pm}^5}(\tilde{r}_{\pm} - 6m_G), \tag{5.22}$$

which yields the following cases:

- $h^2 < 12m_G^2$: No critical points.
- $h^2 = 12m_G^2$: One inflection point with zero slope at $r_{inf} = 6m_G \left(1 + \frac{6m_G}{r_L}\right)^{-1}$.
- $h^2 > 12m_G^2$: Two equilibrium points: r_+ (stable) and r_- (unstable).

On the other hand, the roots of the effective potential are

$$\tilde{R}_{\pm} = \frac{h^2}{4m_G}\left(1 \pm \sqrt{1 - \frac{16m_G^2}{h^2}}\right) = 4m_G \left(1 \mp \sqrt{1 - \frac{16m_G^2}{h^2}}\right)^{-1}, \tag{5.23}$$

or equivalently,

$$R_{\pm} = 4m_G \left[\left(1 + \frac{4m_G}{r_L}\right) \mp \sqrt{1 - \frac{16m_G^2}{h^2}} \right]^{-1}, \tag{5.24}$$

which gives the cases:

- $h^2 < 16m_G^2$: No roots.
- $h^2 = 16m_G^2$: One root given by $r_- = R_0 \equiv 4m_G \left(1 + \frac{4m_G}{r_L}\right)^{-1}$.
- $h^2 > 16m_G^2$: Two roots given by Eq. (5.24), where $R_- < R_0 < R_+$.

The aspect of the effective potential is shown in Figure 5.2, where the following asymptotic behavior is observed:

$$V_{eff}(r \to r_L) \to 0^-, \tag{5.25}$$

$$V_{eff}(r \to 0) \to -\infty. \tag{5.26}$$

On the other hand, the aspect of a specific type of motion is given by Eq. (5.14c). By differentiating this equation with respect to φ, one obtains

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{2r^4}{h^2}(\mathcal{E} - V_{\text{eff}}) \quad \Longrightarrow \quad \left(\frac{du}{d\varphi}\right)^2 = \frac{2}{h^2}(\mathcal{E} - V_{\text{eff}}), \tag{5.27}$$

where $u = 1/r$. Differentiating this expression gives

$$\frac{d^2u}{d\varphi^2} = -\frac{1}{h^2}\frac{dV_{\text{eff}}}{du},$$

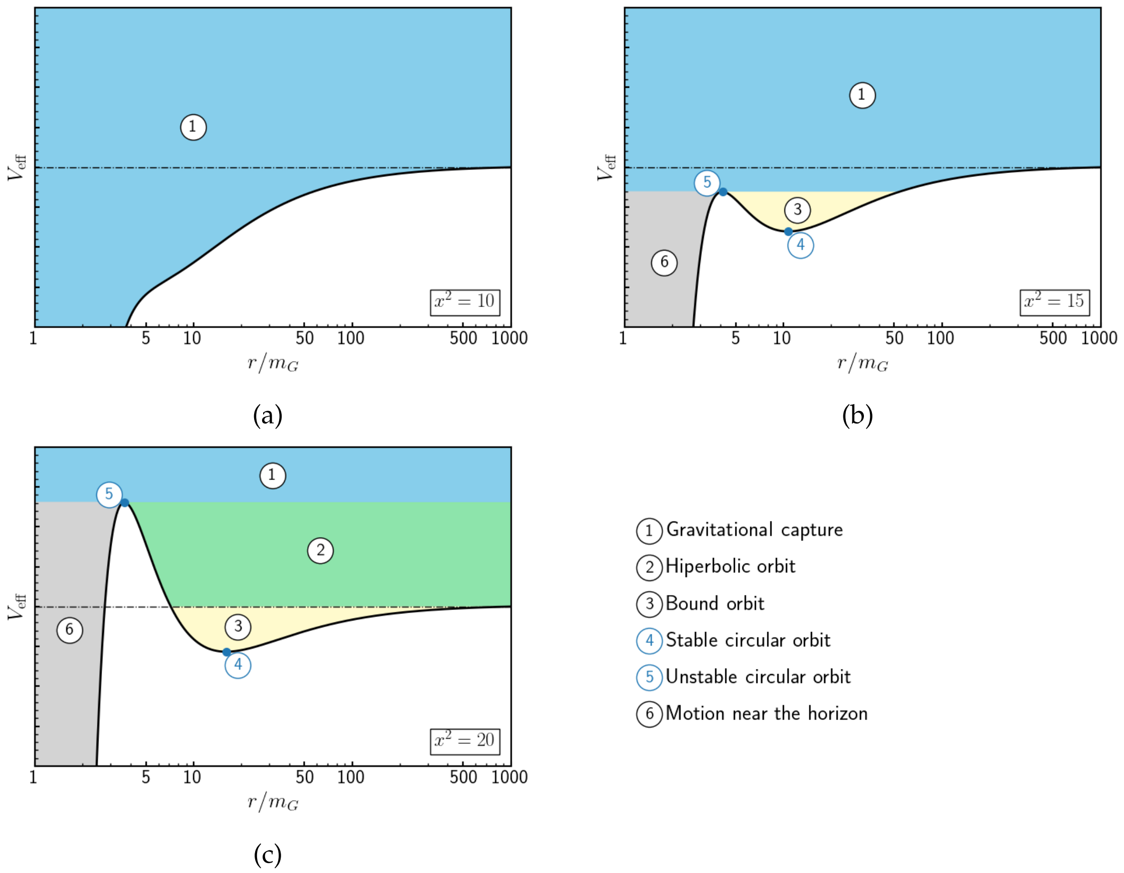

Figure 5.2: Three aspects of the effective potential: (a) $x^2 \leq 12$, (b) $12 < x^2 \leq 16$ and (c) $x^2 > 16$; with $r_L = 1000m_G$. Each region represents a type of motion according to the value of the parameter $\mathcal{E}$.

so the trajectory of the massive particle is given by the following equation:

$$\frac{d^2u}{d\varphi^2} + u = u_L + \frac{m_G}{h^2} + 3m_G(u - u_L)^2. \tag{5.28}$$

The different types of trajectories are described below:

1. **Gravitational capture.** In the general case, this type of trajectory is defined by

$$\mathcal{E} > V_{\text{eff}}(r_-). \tag{5.29}$$

According to Eq. (4.64), if $r \approx r_L$, the line element can be approximated to $ds^2 \approx dt^2 - d\tilde{r}^2 - \tilde{r}^2\, d\Omega^2$. So the particle motion can be locally described as in Minkowski spacetime. Let (x, y, z) be local cartesian coordinates such that, for an instant related to this regime, the position and velocity of the particle are given by

$$(x, y, z) = (X, b, 0) \quad \text{and} \quad (p_x, p_y, p_z) = (-p, 0, 0),$$

where b is the *impact parameter*. Consequently, the specific angular momentum is $h = bp/m$

and, since $E^2 = m^2 + p^2$, one obtains

$$\mathcal{E} = \frac{h^2}{2b^2}.$$

Thus, this approach only works for $\mathcal{E} > 0$, where there are no turning points. In this case, the capture condition (5.29) is satisfied automatically for $h^2 \leq 16m_G^2$, so the impact parameter can take any value. For $h^2 > 16m_G^2$, the capture condition provides a maximum value for the impact parameter in terms of h:

$$b_{cap} = \frac{h^2}{2V_{\text{eff}}(r_-)} = 54m_G^2 \left[1 - \frac{18m_G^2}{h^2} + \sqrt{1 - \frac{12m_G^2}{h^2}}^{\,3}\right]^{-1}. \tag{5.30}$$

Note that $b_{cap} \to \infty$ when $h^2 \to 16m_G^2$, thus recovering the case of unavoidable capture. It should be mentioned that, although expression (5.30) is the same for both Schwarzschild and Lyra-Schwarzschild, the difference lies on the definition of the geometric mass.

2. **Hyperbolic motion.** This type of trajectory is characterized by

$$0 < \mathcal{E} < V_{\text{eff}}(r_-), \tag{5.31}$$

which is valid only for $h^2 > 16m_G^2$ and, consequently, for $b > b_{cap}$. According to Figure 5.2, the equation $V_{\text{eff}}(r) = \mathcal{E}$ has two solutions, the larger of which defines the radius of closest approach r_H.

If a massive particle comes from direction $\varphi = 0$ and gets scattered into direction $\varphi = \varphi_0$, then the scattering angle is $\alpha = \varphi_0 - \pi$. The angle φ_0 can be calculates from Eq. (5.14c):

$$\frac{\varphi_0}{2} = -h \int_{r_{ini}}^{r_H} \frac{dr}{r^2\sqrt{2(\mathcal{E} - V_{\text{eff}}(r))}},$$

or equivalently,

$$\alpha = -\sqrt{2}h \int_{r_{ini}}^{r_H} \frac{dr}{r^2\sqrt{2(\mathcal{E} - V_{\text{eff}}(r))}} - \pi. \tag{5.32}$$

3. **Bound motion.** This type of trajectory is characterized by

$$V_{\text{eff}}(r_+) < \mathcal{E} < \begin{cases} V_{\text{eff}}(r_-), & \text{for } 12m_G^2 < h^2 \leq 16m_G^2; \\ 0, & \text{for } h^2 > 16m_G^2. \end{cases} \tag{5.33}$$

Either way, the turning points are determined as the two largest solutions of the equation $V_{\text{eff}}(r) = \mathcal{E}$.

In this case, it is particularly useful to consider Eq. (5.28) to first order in $m_G/r_L \ll m_G^2/h^2$, which gives

$$\frac{d^2u}{d\varphi^2} + \left(1 + \frac{6m_G}{r_L}\right)u = \frac{m_G}{h^2} + \frac{1}{r_L} + 3m_G u^2. \tag{5.34}$$

Write the solution as $u = u_0 + \Delta u$, where u_0 is the solution for the equation when the quadratic term is ignored:

$$u_0 = \frac{m_G}{h^2\omega^2}\left(1+\frac{h^2}{m_G r_L}\right)[1+e\cos(\omega\varphi)], \quad \text{with } \omega = \sqrt{1+\frac{6m_G}{r_L}}. \tag{5.35}$$

The remaining equation for Δu is given by

$$\frac{d^2\Delta u}{d\varphi^2} + \omega^2\Delta u = 3m_G(u_0+\Delta u)^2, \tag{5.36}$$

whose first order solution is

$$\Delta u = \frac{3m_G^3}{h^4\omega^6}\left(1+\frac{2h^2}{m_G r_L}\right)\left[1+\frac{e^2}{2}+e\omega\varphi\sin(\omega\varphi)-\frac{e^2}{6}\cos(2\omega\varphi)\right]. \tag{5.37}$$

Since the constant $3m_G^3/h^4\omega^6$ is very small, the only term to be retained is that with the factor φ, which is cumulative. Thus, the approximated solution reads

$$\begin{aligned} u &= \frac{m_G}{h^2\omega^2}\left(1+\frac{h^2}{m_G r_L}\right)\left\{1+e\cos\left[\left(1-\frac{3m_G^2}{h^2\omega^4}\left(1+\frac{h^2}{m_G r_L}\right)\right)\omega\varphi\right]\right\} \\ &\approx \frac{m_G}{h^2\omega^2}\left(1+\frac{h^2}{m_G r_L}\right)\left\{1+e\cos\left[\left(1-\frac{3m_G^2}{h^2}\left(1+\frac{h^2}{m_G r_L}\right)\right)\varphi\right]\right\}, \end{aligned} \tag{5.38}$$

so in one revolution, the orbit will rotate about the focus by an amount

$$\Delta\varphi \approx \frac{6\pi m_G^2}{h^2}\left(1+\frac{h^2}{m_G r_L}\right) = \frac{6\pi m_G^2}{h^2} + \frac{6\pi m_G}{r_L}. \tag{5.39}$$

With modern radar measurements and planetary ephemerides, any additional contribution to Mercury's perihelion precession must be smaller than the instrumental uncertainty, corresponding to about 0.005% of the relativistic contribution. This consideration establishes a limit $r_L \geq 1.10 \times 10^{15}$m.

4. **Circular motion.** Stable and unstable circular motion take place, respectively, at the radii r_+ and r_-, given by Eq. (5.21). The *innermost stable circular orbit* (ISCO), as its name suggests, corresponds to the innermost circular orbit of marginal stability, which is obtained through the condition $d^2V_{\text{eff}}/dr^2 = 0$. The corresponding radius is

$$r_{ISCO} = 6m_G\left(1+\frac{6m_G}{r_L}\right)^{-1}, \quad \text{for } h^2 = 12m_G^2. \tag{5.40}$$

Any perturbation around this orbit cause the particle to fall towards the central object. A similar scenario occurs when a perturbation is made to the unstable circular orbit, causing the particle to get into one of the previous types of motion, depending on h^2 and the variation of $\mathcal{E}$ due to the perturbation.

5. **Motion near the horizon.** This type of trajectory is characterized by

$$\mathcal{E} < V_{\text{eff}}(r_-), \tag{5.41}$$

and it has nothing particularly interesting: the particle just falls towards the central object. It is mathematically similar to the hyperbolic orbit but in the opposite direction. If the initial velocity is such that $\dot{r} > 0$, the particle reaches a radius of maximum distance r_D given by the smallest solution of the equation $V_{\rm eff}(r_D) = \mathcal{E}$.

5.3 Photons

In the case of massless particles, $\varepsilon = 0$. Then, from Eq. (5.14b), one obtains

$$\frac{m_G^2}{b^2} = \frac{m_G^2}{h^2}\phi^4 \dot{r}^2 + V_{\rm eff}, \tag{5.42}$$

where $b = h/k$ and $V_{\rm eff}$ is the effective potential:

$$V_{\rm eff} = \frac{m_G^2}{r^2}\left(1 - \frac{r}{r_L}\right)^2 \left(1 - \frac{2m_G}{r} + \frac{2m_G}{r_L}\right). \tag{5.43}$$

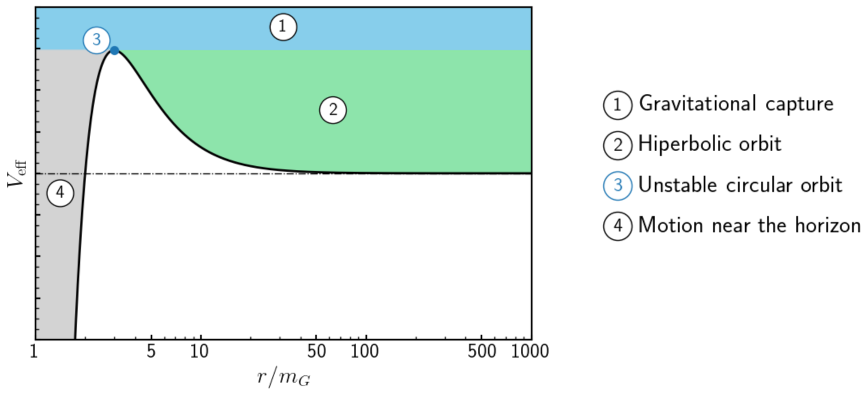

Figure 5.3: Effective potential in the case of photons.

Although different in form, this potential serves the same function as in the case of massive particles and satisfies conditions (5.18) and (5.19). For massless particle motion, there is only one critical point:

$$r_c = 3m_G\left(1 + \frac{3m_G}{r_L}\right)^{-1}, \tag{5.44}$$

which is related to unstable equilibrium, giving the maximum value for the effective potential:

$$V_{\rm eff}(r_c) = \frac{1}{27}. \tag{5.45}$$

There is also only one root of the effective potential, which is the Schwarzschild radius:

$$r_S = 2m_G\left(1 + \frac{2m_G}{r_L}\right)^{-1}, \tag{5.46}$$

and the same asymptotic behavior of Eqs. (5.25) and (5.26) is observed.

On the other hand, the aspect of a specific type of motion is given by Eq. (5.14c). By differentiating this equation with respect to φ, one obtains

$$\frac{d^2u}{d\varphi^2} = -\frac{1}{2m_G^2}\frac{dV_{\text{eff}}}{du},$$

so the trajectory of the massive particle is given by the following equation:

$$\frac{d^2u}{d\varphi^2} + u = u_L + 3m_G(u - u_L)^2. \tag{5.47}$$

The different types of trajectories are described below:

1. **Gravitational capture.** In the general case, this type of trajectory is defined by

$$\frac{m_G^2}{b^2} > V_{\text{eff}}(r_c). \tag{5.48}$$

This provides directly a maximum value for the impact parameter:

$$b_{cap} = \sqrt{27}m_G. \tag{5.49}$$

2. **Hyperbolic motion.** This type of trajectory is characterized by

$$0 < \frac{m_G^2}{b^2} < V_{\text{eff}}(r_c), \tag{5.50}$$

which implies $b > b_{cap}$. According to Figure 5.2, the equation $V_{\text{eff}}(r) = m_G^2/b^2$ has two solutions, the larger of which defines the radius of closest approach.

This case is particularly interesting, as it describes the bending of light. Using Eq. (5.47) to first order in $m_G/r_L \ll m_G/b$, which gives

$$\frac{d^2u}{d\varphi^2} + \left(1 + \frac{6m_G}{r_L}\right)u = \frac{1}{r_L} + 3m_G u^2. \tag{5.51}$$

Write the solution as $u = u_0 + \Delta u$, where u_0 is the solution for the equation when the quadratic term is ignored:

$$u_0 = \frac{1}{b}\left[\frac{b}{\omega^2 r_L} + \sin(\omega\varphi)\right], \quad \text{with } \omega = \sqrt{1 + \frac{6m_G}{r_L}}. \tag{5.52}$$

The remaining equation for Δu is given by

$$\frac{d^2\Delta u}{d\varphi^2} + \omega^2\Delta u = 3m_G(u_0 + \Delta u)^2, \tag{5.53}$$

which first order solution is

$$\Delta u = \frac{3m_G}{b^2\omega^2}\left[\frac{1}{2} - \frac{b}{\omega r_L}\varphi\cos(\omega\varphi) + \frac{1}{6}\cos(2\omega\varphi)\right]. \tag{5.54}$$

Thus, the approximated solution reads

$$u = \frac{1}{b}\left[\frac{b}{\omega^2 r_L} + \frac{3m_G}{2b\omega^2} + \sin(\omega\varphi) - \frac{3m_G}{\omega^3 r_L}\varphi\cos(\omega\varphi) + \frac{m_G}{2b\omega^2}\cos(2\omega\varphi)\right]. \tag{5.55}$$

Consider the limit where the particle is far away from the central object, that is $u \to 1/r_L$. Since $\omega \approx 1$, for a slight deflection, one can take $\sin(\omega\varphi) \approx \omega\varphi$ and $\cos(\omega\varphi) \approx 1$ at such regime. This yields the equation

$$\frac{1}{r_L} = \frac{1}{\omega^2 r_L} + \frac{2m_G}{b^2\omega^2} + \left(1 - \frac{3m_G}{\omega^4 r_L}\right)\frac{\omega\varphi}{b}$$

whose solution is

$$\varphi = -\frac{2m_G}{b} - \frac{6m_G^2}{br_L}. \tag{5.56}$$

Now, the gravitational bending angle is two times the magnitude of this value. Modern measurements of the bending angle for light grazing the Sun surface establish that any additional contribution must be smaller than the instrumental uncertainty, corresponding to about 0.0006% of the relativistic contribution. This consideration yields a limit of $r_L \geq 7.4 \times 10^8$m. However, according to the limit found in the previous section for bound orbits, this means that the Lyra contribution is negligible for bending light in this case, a possible explanation being the great proximity to the central object.

As to the unstable circular orbit and the motion near the horizon, there is nothing particularly interesting beyond the numerical computation of the trajectory, the qualitative description being similar to the case of massive particles.

5.3.1 Gravitational Redshift

Consider an observer and a light source, both fixed in space. The source emits a pulse at event A, with coordinates $(t_E, r_E, \theta_E, \varphi_E)$, which is subsequently received by the observer at event B, with coordinates $(t_R, r_R, \theta_R, \varphi_R)$. Another pulse is emitted by the source at event C, with coordinates $(t_E + \Delta t_E, r_E, \theta_E, \varphi_E)$, and received by the observer at event D, with coordinates $(t_R + \Delta t_R, r_R, \theta_R, \varphi_R)$. As it is well known, the photons will follow null geodesics, where the line element is $ds^2 = 0$ at any point. Therefore:

$$\mu(r)dt^2 = \mu^{-1}(r)\,dr^2 + r^2\,d\Omega^2 = -g_{ij}\,dx^i dx^j.$$

By choosing a parametrization λ, one can write:

$$t_R - t_E = \int_{\lambda_E}^{\lambda_R} d\lambda\,\mu^{-1/2}\left(-g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\right)^{1/2}. \tag{5.57}$$

It is important to note that the integral on the right-hand side of this equation does not strictly depend on the parametrization, but only on the initial and final points. Then:

$$t_R - t_E = (t_R + \Delta t_R) - (t_E + \Delta t_E) \qquad \Rightarrow \qquad \Delta t_E = \Delta t_R.$$

Now, recall that the source and the observer have fixed spatial coordinates. So the curves described by them have a line element of the form $ds^2 = d\tau^2 = \phi^2 g_{00}\, dt^2$, with τ being the corresponding proper time. Consequently, one obtains:

$$\frac{\Delta\tau_E}{\Delta\tau_R} = \frac{\sqrt{\phi^2(r_E)\mu(r_E)}\Delta t_E}{\sqrt{\phi^2(r_R)\mu(r_R)}\Delta t_R} = \sqrt{\frac{1 - \frac{2m_G}{r_E} + \frac{2m_G}{r_L}}{1 - \frac{2m_G}{r_R} + \frac{2m_G}{r_L}}}$$

The gravitational redshift is defined in terms of the emission and reception wavelengths:

$$z = \frac{\lambda_R - \lambda_E}{\lambda_E}. \tag{5.58}$$

It is worth mentioning that here λ represents wavelength instead of the parameter in Eq. (5.57). In the current units, wavelengths and frequencies are related through $\lambda = \nu^{-1} = \Delta\tau$. Thus, the general form of the gravitational redshift in Lyra spacetime is

$$z = \sqrt{\frac{1 - \frac{2m_G}{r_R} + \frac{2m_G}{r_L}}{1 - \frac{2m_G}{r_E} + \frac{2m_G}{r_L}}} - 1. \tag{5.59}$$

In particular, when the light source and the observer are far away from the gravitational source, such that $r_E, r_R \gg r_S$, expression (5.59) simplifies to:

$$z \approx \frac{m_G}{r_E}\left(1 - \frac{r_E}{r_R}\right). \tag{5.60}$$

The conclusion is that redshift occurs only for $r_E < r_R$, while the opposite case causes blueshift. This classification holds even for the general case in Eq. (5.59).

5.3.2 Causal structure

The radial motion ($h^2 = 0$) for photons ($\varepsilon = 0$) can be determined from Eq. (5.14a):

$$t = \pm \int \frac{dr}{\left(1 - \frac{r}{r_L}\right)^2 \left(1 - \frac{2m_G}{r} + \frac{2m_G}{r_L}\right)},$$

whose solution is given by:

$$t_\pm = c \pm \left[\frac{r}{1 - \frac{r}{r_L}} + 2m_G \ln\left(\frac{r}{1 - \frac{r}{r_L}} - 2m_G\right)\right], \tag{5.61}$$

where c is an integration constant. Note that this expression diverges for $r = r_S$ and $r = r_L$. However, the only physical singularity is $r = 0$, as one can verify from the expression of the Kretschmann scalar:

$$K = \frac{12r_S^2}{r^6} \frac{\left(1 - \frac{r}{r_L}\right)^6}{\left(1 - \frac{r_S}{r_L}\right)^2}. \tag{5.62}$$

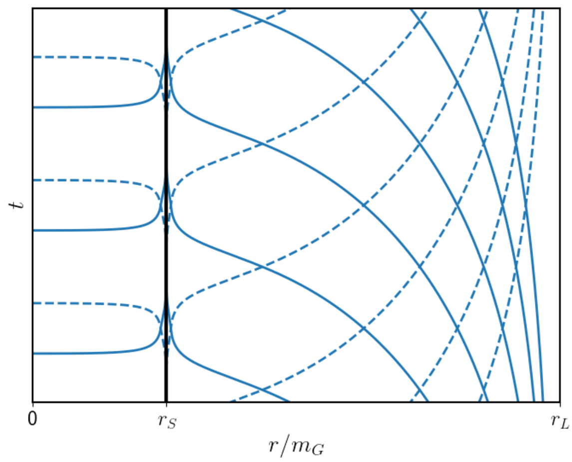

Figure 5.4: Lightcone diagram of the Lyra-Schwarzschild solution.

Therefore, $r = r_S$ and $r = r_L$ are just coordinate singularities, as shown in Fig. 5.4. The singularity $r = r_S$ can be removed in a similar way to the Schwarzschild case. Meanwhile, the singularity $r = r_L$ is a consequence of the transformation $\tilde{r} = r\phi(r)$. This makes sense when considering that $r = r_L$ corresponds to $\tilde{r} = \infty$, which is not a physical singularity, nor a physical place.

Chapter 6

Conclusions and Perspectives

Lyra scalar-tensor theory is formulated on a Lyra manifold with a metric-compatible, torsion-free connection. The fundamental fields of this theory are the metric tensor $g_{\mu\nu}$ and the scale function ϕ. The presence of the scalar field modifies the canonical basis of the tensor bundles, leading to a transformation law for Lyra reference systems that differs from that of General Relativity. The metric-compatible and torsion-free conditions are imposed to make LyST theory as similar to General Relativity as possible. In this sense, LyST theory can be viewed as an extension of General Relativity on a Lyra manifold with a nontrivial scale function.

Field equations in LyST theory are derived from the standard variational principle applied to a Lyra-invariant action. This action generalizes the Einstein-Hilbert action via the minimal coupling principle, incorporating a Lyra-invariant volume element. As shown in Eqs. (3.61) and (3.74), LyST theory is compatible with Newtonian gravity in the linearized and static approximation. The resulting Newtonian gravitational potential is given by $\mathcal{U} = \frac{1}{2}h_{00} + \delta\phi$, where $\delta\phi$ denotes the perturbation of the Lyra scale function. Thus, gravity in LyST theory is also governed by this perturbation, as it represents a quantity independent of the metric.

The most general spherically symmetric solution in LyST theory depends on a single parameter and a freely selectable scale function, as shown in Eq. (4.51). This freedom suggests the existence of an infinite family of solutions satisfying the field equations. However, by defining a new radial coordinate $\tilde{r} = r\phi(r)$, one obtains the exact expression of the standard Schwarzschild solution, where the aforementioned parameter is the Schwarzschild radius $\tilde{r}_S$. Therefore, the Schwarzschild solution is the only physical solution, and the general expression is simply a result of a coordinate transformation involving the scale function. Requiring that the general expression preserve the structure of the Schwarzschild metric leads to a specific choice of the scale function: $\phi(r) = (1 - r/r_L)^{-1}$, where $r_L > 0$ is a new parameter, known as the Lyra radius, with $r < r_L$. The resulting metric is referred to as the Lyra-Schwarzschild solution and has a coordinate singularity at $r_S = (1/\tilde{r}_S + 1/r_L)^{-1}$ and $r = r_L$, yet it is equivalent to the Schwarzschild metric under the transformation defined above. In the limit $r_L \to \infty$, one recovers the Schwarzschild metric, with the radial coordinates becoming identical, $\tilde{r} = r$. This suggests that the Lyra–Schwarzschild metric is merely an alternative representation of the Schwarzschild solution, adapted to the case of a spatially finite universe bounded by r_L.

Could this apparent finiteness be an actual feature of the spatial extent of our universe, or is it merely a consequence of adopting a particular coordinate system? According to the standard ΛCDM model, the universe is generally considered spatially flat, which implies an infinite spatial extension. Thus, the finiteness in the Lyra-Schwarzschild expression seems to be just a matter of the coordinate system. However, the converse is also plausible: the universe might indeed be spatially finite, characterized by an extremely large Lyra radius, well beyond the bounds of any cosmological scale currently accessible to observation. In this case, the Lyra-Schwarzschild description would be nearly indistinguishable from that of the Schwarzschild solution, since the relevant scales satisfy $r \lll r_L$. This raises the possibility that we may be inadvertently confusing coordinate systems, interpreting physical results within an unintended frame, even though the numerical results are practically identical.

In the Newtonian approximation of the spherically symmetric solution, the equation of motion for a massive particle, formulated in the Lyra-Schwarzschild description, is given by Eq. (4.60). The geometric mass m_G, expressed in both coordinate systems, is given by Eq. (4.61). To first order in r_L^{-1} the equation of motion reduces to the standard expression in Newtonian gravity, with the geometric mass still influenced by the Lyra radius. Second order effects introduce a constant repulsive acceleration and an additional term analogous to an anti-de Sitter contribution. It is worth noting that particle motion remains affected by the Lyra radius even in the absence of matter distributions, as the anti-de Sitter term persists when $m_G = 0$. The contribution of this term to the equation of motion increases as r approaches r_L. Therefore, while uncertainties in measurements at solar-system-size scales can be used to set lower bounds to r_L, Eq. (4.60) could be employed to estimate its value by observing particle trajectories at large scales, where $r \sim r_L$.

The Hamilton-Jacobi method offers a formal and straightforward approach to deriving the equations of motion for the spherically symmetric solution. In the Lyra-Schwarzschild description, these equations are given by Eqs. (5.14) and are equivalent under the transformation $\tilde{r} = r\phi(r)$ to those of the standard Schwarzschild case. An analysis of the effective potential in both coordinate systems suggests that r_L appears as a new critical point introduced by the scale function. However, this value lies outside the physical domain, since $r > r_L$ is forbidden. As a result, the overall behavior of the effective potential, whether expressed in terms of r or $\tilde{r}$, remains nearly identical, yielding the same types of motion in both cases. Consequently, the particle trajectories in the Lyra-Schwarzschild description can be directly obtained from the well-known trajectories in the standard Schwarzschild case via the aforementioned coordinate transformation. In fact, the solutions expressed in terms of the inverse radial coordinate differ only by the constant $u_L = 1/r_L$. In the case of photons, the redshift is given by Eq. (5.59), which reduces to the standard Schwarzschild expression when both the source and the observer are located far from the event horizon at r_S, for a fixed $r_L \gg r_S$. Two specific physical phenomena—Mercury's precession and light bending near the Sun—were considered in order to estimate lower bounds for the Lyra radius. The obtained values were of the order of 10^{15}m and 10^{9}m.

Finally, the causal structure presented by the Lyra-Schwarzschild description is very similar to that of the Schwarzschild case in the regime $r \ll r_L$. The only physical singularity remains at the coordinate origin, $r = 0$, as indicated by Eq. (5.62). The main difference in the causal structure is that the lightcones become significantly narrower in the regime $r \gg r_S$, eventually collapsing at r_L. All the results presented here have analogous versions for other type of spherically symmetric solutions, such as the Reissner-Nordstrom metric. This provides encouragement to work on more cases of the LyST gravity for the resolution of field equations with different symmetries, possible applications to galaxy rotation curves, as well as gravitational waves, and even more general constructions like the Einstein-Cartan theory.

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