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Three-point functions in $\mathcal{N} = 4$ SYM

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Dedico esta tese à minha mãe, Virgínia.

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"Quem não vacila mesmo derrotado
Quem já perdido nunca desespera,
E envolto em tempestade, decepado
Entre os dentes segura a primavera"

Secos e Molhados, *Primavera entre os Dentes*

Resumo

Na teoria de Yang-Mills planar maximamente supersimétrica, podemos calcular funções de correlação com acoplamento de t'Hooft pequeno usando a hexagonalização. Os principais objetos deste método são chamados de hexágonos, que devem obedecer a um conjunto de axiomas. Revisando as propriedades analíticas desses blocos de construção, aprendemos como fixar completamente o hexágono na primeira ordem de teoria da perturbação. Nossa abordagem oferece lições importantes para a generalização dos resultados para todas as ordens.

Entre outras coisas, somos capazes de calcular a função de três pontos envolvendo três operadores com spin e com polarizações genéricas em N = 4 SYM em primeira ordem de uma forma muito simples. Tais constantes de estrutura são essenciais para o estudo da física de spin grande e fazem parte de uma enorme teia de dualidades na teoria.

Palavras Chaves: Campos de calibre (Física); Física matemática; Supersimetria

Áreas do conhecimento: Teoria Quântica de Campos; Teoria de cordas; Física matemática

Abstract

In planar maximally supersymmetric Yang-Mills, we can compute three-point functions at weak coupling using the so-called hexagonalization formalism. The main objects in this framework are called hexagons, which must obey a set of axioms. By reviewing the analytic properties of these building blocks, we learn how to completely bootstrap the hexagon at tree-level. Our approach offers important lessons for the generalization of the results to all-loops.

Among other things, we are able to compute the three-point function involving three spinning operators with generic polarizations in $\mathcal{N} = 4$ SYM at tree-level in a very simple way. Such correlators are essential to the study of the large spin physics and is part of a huge web of dualities in the theory.

Keywords: Gauge theories; AdS-CFT correspondence; Integrability; Conformal Field Theory.

Knowledge areas: Quantum Fields and Strings; Mathematical Physics

Contents

Chapter 1

Introduction

Classical and quantum integrability are paramount to understanding the AdS/CFT correspondence. Over the last two decades, many integrability techniques have been developed to study a plethora of observables in $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) and its gravity dual at weak, strong, and even at finite coupling [1,2]. Of outstanding success are the computation of the spectrum and structure constants of single-trace operators to all-loops in perturbation theory, using the quantum spectral curve [3] and the hexagonalization procedure [4], respectively. In this thesis, we focus on the latter.

In the hexagonalization framework, we start by taking inspiration from the string side of the AdS/CFT correspondence and regard the correlation function of three single-trace operators as a pair of pants where the excitations of each operator live on the boundary of the corresponding hole (or closed string). The idea behind the method is to cut the pair of pants along the seams to get two - simpler - objects, which we call hexagons. If we conjecture the integrability of the theory at all-loops, these building blocks must satisfy some axioms, known as 'decoupling axioms'. Such axioms are used to fix the hexagons, which can be glued again into a correlation function.

With this very simple idea (conceptually), we are able to compute structure constants in the weak coupling limit up to five-loops (e.g. [5], ..) (!) and possibly more. Even more impressive, it also matches the strong coupling limit of the theory. It's therefore thought to be valid at all-loops; once we accept this, the method becomes also useful to generate perturbative data which can be used to test other ideas in both the gauge theory and the string theory sides.

Although being extremely powerful, allowing for non-perturbative predictions in a four-dimensional non-Abelian gauge theory, this framework can also be obscure, in some sense. Notably, we have very little analytical understanding about the hexagons and structure constants with three generic excited states, even at tree-level! This is way more than a simple technical detail; it's vital if we want to understand the web of dualities surrounding $\mathcal{N} = 4$ SYM, in particular the relation between three-point functions and Wilson loops [6–8].

1.1 Results

In this work, we discover the origin of the hexagonalization from the spin-chain picture [9–12]. This new understanding allow us to understand how to generate solutions to the 'decoupling axioms' for spinning hexagons (derived in [7]) and find - for the first time - closed expressions for the tree-level hexagons and structure constants for three excited operators with arbitrary polarizations. Furthermore, we present evidence that the very same method can, perhaps, be used to also provide concise expressions for all-loop hexagons.

Inverting the logic, we can use these hexagons to present concise formulas for the three-point of spinning operators! These take a remarkably similar form to those with simpler configurations, as the 'type I-I-I' correlator in the SU(2) sector [11, 12]. This fact indicates that many other constructions done for the simpler configurations might have analogous for the generic spinning operators. As an example, we point out the coherent state construction which maps the classical limit of the spin-chain to a classically integrable Landau-Lifschitz model [13].

From studying the analytical structure of the hexagons, we also solve the open problem of deriving the classical limit of 'type I-I-I' from their microscopic description. Our results match the predictions made in [13] using powerful tools of classical integrability. Furthermore, we use the closed formulas we found for spinning correlators to start exploring their large spin limit. We hope these explorations are relevant for understanding the relation between large spin physics and Wilson loops and the interplay between hexagons and pentagons [14].

1.2 Outline

This work is based on [15], in collaboration with Matheus Fabri. In Chapter 2 we review the map between single-trace operators in $\mathcal{N} = 4$ and spin-chains, and compute its spectrum. In Chapter 3, we review the 'tailoring' procedure in spin-chains and use it to re-derive hexagonalization in the weak coupling limit. Then, we use the new insights to compute the hexagon for three excited operators in the SU(2) and SL(2) sectors, our main result. In Chapter 4, we discuss the classical limit of SU(2) correlators and solve the open problem of computing it from the microscopic expressions. We also compute the large spin limit for some very simple spinning correlators. In Chapter 5 conclude the discussion and summarize the results and future prospects.

Chapter 2

$\mathcal{N} = 4$ and integrability: spectrum

$\mathcal{N} = 4$ SYM is a non-abelian gauge theory which enjoys many extra symmetries (beyond Poincaré symmetry), as conformal symmetry and supersymmetry. Being a Conformal Field Theory, one can (in principle) recover all the correlators in the theory only from the dimensions of the operators $\{\Delta_i\}$ and their structure constants $\{C_{ijk}\}$.

This chapter explains how to use integrability to compute the spectrum of $\mathcal{N} = 4$ SYM [1, 16]. We describe the map between the so-called single trace operators and closed spin chains. Then, the problem of computing the anomalous dimensions of operators in the gauge theory is mapped into the very well known problem of computing the energies in an integrable spin-chain.

Then, in the next chapter we show that we can go on with the spin-chain picture and show that Wick contractions can be realized as combinations of overlaps between spin-chain states and use the structure of quantum integrability to compute the structure constants between three excited states.

2.1 $\mathcal{N} = 4$ SYM and spin-chains

The goal of this section is to relate one loop correlation functions in $\mathcal{N} = 4$ SYM and closed integrable spin-chains. $\mathcal{N} = 4$ SYM is a four-dimensional superconformal gauge theory with

- six scalar fields Φ_i, $(i = 1, \ldots, 6)$
- one four-dimensional vector field A_μ, $(\mu = 1, \ldots, 4)$
- one ten-dimensional spinor Ψ_A, $(A = 1, \ldots, 16)$

All fields are in the adjoint representation of the gauge group U(N), which means we can write $\Phi_i = \Phi_i^a T_a$, $A_\mu = A_\mu^a T_a$, $\Psi_A = \Psi_A^a T_a$, where T_a are the generators of U(N) with the normalization $\text{tr}(T_a T_b) = \frac{\delta_{ab}}{2}$. The action of the theory is the following

$$S = \frac{1}{2g_{YM}^2} \int d^4x \ \text{tr} \left[-\frac{1}{2}(F_{\mu\nu})^2 + (D_\mu \Phi)^2 - \frac{1}{2}[\Phi_i, \Phi_j]^2 + i\bar{\Psi} \left(\Gamma^\mu D_\mu \Psi + i\Gamma^i \left[\Phi_i, \Psi\right] \right) \right] \tag{2.1}$$

Here, g_{YM} is the coupling constant of the theory, and $\Gamma = (\Gamma^\mu, \Gamma^i)$ are ten sixteen-dimensional Dirac matrices with normalization $\text{tr}\left(\Gamma^A, \Gamma^B\right) = 16\ \delta^{AB}$. Moreover, F and D are defined as usual.

In the planar limit ($N \to \infty$, $\lambda \equiv g_{YM}^2 N$ finite), we can study the theory as a function of the 't Hooft coupling λ. In particular, we can use perturbation theory techniques to understand the weak coupling regime ($\lambda \ll 1$). In this work we'll work with a class of gauge-invariant operators constructed by taking the trace of a product of scalar fields

$$\mathcal{O}^{\text{bare}}(x) = \text{tr}\left(\Phi_{i_1}^{\text{bare}} \dots \Phi_{i_\ell}^{\text{bare}}\right)(x) \tag{2.2}$$

In particular, we want to compute two- and three-point functions of these *single-trace operators*. These observables are almost completely completely fixed by the conformal symmetry of $\mathcal{N} = 4$ SYM. The two-point function is characterized by the *conformal dimensions* Δ_i of each operator. Concretely, we can always choose a basis of renormalized operators such that

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(y) \rangle = \mathcal{N}_i \frac{\delta_{ij}}{|x-y|^{2\Delta_i}} \tag{2.3}$$

and the three-point function is characterized by the *structure constants* C_{ijk}

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(y) \mathcal{O}_k(z) \rangle = \frac{\sqrt{\mathcal{N}_i \mathcal{N}_j \mathcal{N}_k}}{N} \frac{C_{ijk}}{|x-y|^{\Delta_{ij}} |y-z|^{\Delta_{jk}} |z-x|^{\Delta_{ki}}} \tag{2.4}$$

where $\Delta_{ij} = \frac{\Delta_i + \Delta_j - \Delta_k}{2}$. In the zero-th order of large-N perturbation theory, the dimension of the (bare) single-trace operators is given simply by the sum of the dimensions of the fields inside the trace. When we try to compute the one-loop corrections to the two-point function, the operators both mix and need to be renormalized. Then, we can write

$$\mathcal{O}_i(x) = \left(e^{\hat{\Gamma}\log\Lambda}\right)_i^{\ j}\mathcal{O}_j^{\text{bare}}(x) \tag{2.5}$$

such that the two-point function have the form (2.3). Note that if we take $\mathcal{O}^{\text{bare}}(x)$ to be eigenstate of $\hat{\Gamma}$, $\mathcal{O}$ transforms as follows:

$$\mathcal{O}_i \to \Lambda^{-(\Delta_0+\gamma)}\mathcal{O}_i \tag{2.6}$$

under $x \to \Lambda x$, where γ is an eigenvalue of $\hat{\Gamma}$. Then, to find the one-loop conformal dimension of single trace operators, we need to find the eigenvalues of $\hat{\Gamma}$. In what follows we will find $\hat{\Gamma}$ and in the next section we'll learn the tools to diagonalize it.

To find $\hat{\Gamma}$, we will use the fact that the correlation functions involving renormalized operators are finite. In particular, $\langle \mathcal{O}_{i_1\ldots i_\ell}\Phi^{j_1}\ldots\Phi^{j_\ell}\rangle$ needs to be finite if $\Phi = \sqrt{Z_\Phi}\Phi^{\text{bare}}$ is renormalized, i.e., if Z_Φ is such that $Z_\Phi\langle\Phi^{\text{bare}}\Phi^{\text{bare}}\rangle$ is finite. By imposing this condition, we get the following expression for $\hat{\Gamma}$:[1]

$$\hat{\Gamma} \equiv \frac{\lambda}{8\pi^2}\hat{H}^{(1)} = 2g^2\sum_{n=1}^{\ell}\left(\mathbf{1}_{n,n+1} - \mathbf{P}_{n,n+1} - \frac{1}{2}\mathbf{K}_{n,n+1}\right) \tag{2.7}$$

Here, $\hat{H}$ is the same as the Hamiltonian of a SO(6) Heisenberg spin-chain if we make the identification

$$\mathcal{O}_{i_1\ldots i_\ell} \to |\uparrow_{i_1}\ldots\uparrow_{i_\ell}\rangle \tag{2.8}$$

and

$$\mathbf{1}|\ldots\uparrow_i\uparrow_j\ldots\rangle = |\ldots\uparrow_i\uparrow_j\ldots\rangle \tag{2.9}$$

$$\mathbf{P}|\ldots\uparrow_i\uparrow_j\ldots\rangle = |\ldots\uparrow_j\uparrow_i\ldots\rangle \tag{2.10}$$

$$\mathbf{K}|\ldots\uparrow_i\uparrow_j\ldots\rangle = \delta_{ij}\sum_{k=1}^{6}|\ldots\uparrow_k\uparrow_k\ldots\rangle \tag{2.11}$$

The Hamiltonian for this SO(6) spin-chain is integrable, which means we have an infinite set of conserved quantities and we have a very nice method to compute its spectrum, which we'll develop in the next section.

To simplify our life, we'll start by working with a SU(2) sub-sector of the

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[1] Here we define $g^2 = \frac{\lambda}{16\pi^2}$

scalar sector. Namely, we will consider only operators built from the two complex scalars,

$$Z = \Phi_1 + i\Phi_2 \quad \text{and} \quad X = \Phi_3 + i\Phi_4 \tag{2.12}$$

and make the identification

$$\mathrm{tr}(\dots Z \dots) \to |\dots \uparrow \dots\rangle \quad \text{and} \quad \mathrm{tr}(\dots X \dots) \to |\dots \downarrow \dots\rangle \tag{2.13}$$

Note that **K** vanishes in this sector; for example,

$$\mathbf{K}|\dots \uparrow\uparrow \dots\rangle = \mathbf{K}|\dots \uparrow_1\uparrow_1 \dots\rangle - \mathbf{K}|\dots \uparrow_2\uparrow_2 \dots\rangle = 0 \tag{2.14}$$

Therefore, we get the following Hamiltonian.

$$\hat{H}^{(1)} = \ell - \sum_{n=1}^{\ell} \mathbf{P}_{n,n+1} = \sum_{n=1}^{\ell} \left(\mathbf{1}_{n,n+1} - \mathbf{P}_{n,n+1}\right) \tag{2.15}$$

2.2 The ansatz of Bethe

As we saw in the last section, one can interpret the one-loop dilatation operator of $\mathcal{N} = 4$ SYM as the Hamiltonian of a XXX spin chain, which is *integrable*. In this section, we will try to diagonalize the spin $1/2$ XXX chain using the *algebraic Bethe ansatz*.

$$\hat{H}^{(1)} = \ell - \sum_{n=1}^{\ell} \mathbf{P}_{n,n+1} = \frac{1}{2}\left[\ell - 4\sum_{n=1}^{\ell} \vec{S}_n \cdot \vec{S}_{n+1}\right] \tag{2.16}$$

While we will work out only this specific example, the construction is very general and applies to all integrable spin-chains. The basic ingredient of the method is the *R-matrix*, which acts on $h_i \otimes h_j$, where $h_{i>0}$ is the Hilbert space of the i-th site of the spin chain and h_0 is an auxiliary Hilbert space. The R-matrix of an integrable theory must satisfy the *Yang-Baxter equations*

$$R_{ij}(u)R_{ik}(u+v)R_{jk}(v) = R_{jk}(u)R_{ik}(u+v)R_{ij}(v) \tag{2.17}$$

The complex numbers u and v are called *spectral parameters*. Using the symmetry of the spin-chain and the Yang-Baxter equations, we can fix the R-matrix up to an overall factor by writing down all the invariant tensors. For the SU(N) case, we have

$$R_{ij}(u) = u\mathbf{1}_{ij} + i\mathbf{P}_{ij} \tag{2.18}$$

where $\mathbf{1}_{ij}$ and $\mathbf{P}_{ij}$ are the identity and permutation operators acting in $h_i \otimes h_j$. Now, we construct the *Lax operator*

$$L_n^{(a)}(u) \equiv R_{0_a i}\left(u - \frac{i}{2}\right) = \left(u - \frac{i}{2}\right)\mathbf{1}_{0_a n} + i\mathbf{P}_{0_a n} \tag{2.19}$$

For the SU(2) spin-chain, we can also write

$$L_n^{(a)}(u) = u\mathbf{1} \otimes \mathbf{1} + i\vec{\sigma} \otimes \vec{S}_n = \begin{pmatrix} u\mathbf{1} + iS_n^z & iS_n^- \\ iS_n^+ & u\mathbf{1} - iS_n^z \end{pmatrix} \tag{2.20}$$

Note that $L_n^{(a)}$ acts on $h_{0_a} \otimes h_n$. Since the Lax operators are constructed from R-matrices, we can write the following "almost commutation relations" (see 2.2)

$$R_{0_1 0_2}(u - v)L_n^{(1)}(u)L_n^{(2)}(v) = L_n^{(2)}(v)L_n^{(1)}(u)R_{0_1 0_2}(u - v) \tag{2.21}$$

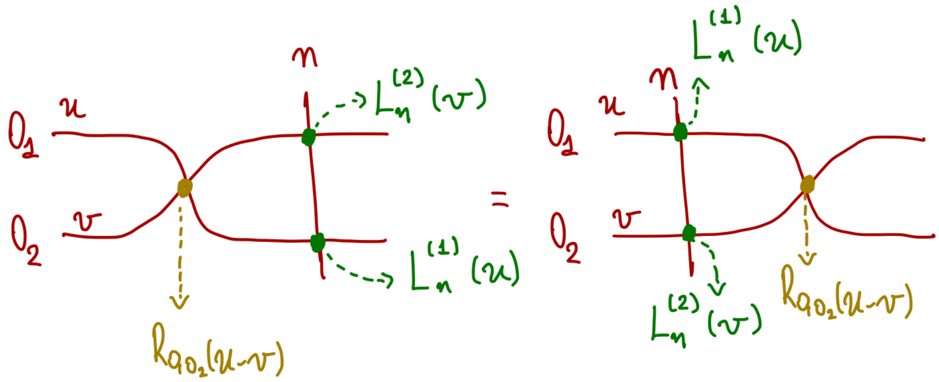

Figure 2.1: We can move lines

Using these operators, we can construct the *monodromy matrix*

$$\Omega^{(a)}(u) = L_1^{(a)}(u) \dots L_\ell^{(a)}(u) = \begin{pmatrix} \hat{A}(u) & \hat{B}(u) \\ \hat{C}(u) & \hat{D}(u) \end{pmatrix} \tag{2.22}$$

which acts on $h_{0_a} \otimes h_1 \otimes \cdots \otimes h_\ell$. This operator inherits the "almost commutation relations" of the Lax operator, meaning

$$R_{0_1 0_2}(u-v)\Omega^{(1)}(u)\Omega^{(2)}(v) = \Omega^{(2)}(v)\Omega^{(1)}(u)R_{0_1 0_2}(u-v) \tag{2.23}$$

By tracing out the auxiliary space of the monodromy matrix, we get the *transfer matrix*, which acts on the full Hilbert space of the spin chain.

$$\hat{T}(u) = \mathrm{Tr}_{0_a}\Omega(u) = \hat{A}(u) + \hat{D}(u) \tag{2.24}$$

The transfer matrix does not depend on the auxiliary space we choose. By multiplying the "almost commutation relations" of Ω by $R_{12}^{-1}(u-v)$ from the right and taking the trace over $h_{0_1} \otimes h_{0_2}$, we show the identity $[\hat{T}(u), \hat{T}(v)] = 0$ holds for any u and v.

2.2.1 Bethe states

Now, if we are lucky enough so that our Hamiltonian can be constructed from T's, we will have a good time when diagonalizing it :D. In fact, the SU(2) spin chain Hamiltonian can be written as

$$\hat{H}^{(1)} = \ell - i\frac{d\log\hat{T}}{du}\bigg|_{u=\frac{i}{2}} \tag{2.25}$$

Therefore, if we want to find the eigenvalues of $\hat{H}$ we only need to diagonalize $\hat{T}$. To do that, first let's see how the monodromy matrix acts on the ferromagnetic vacuum of a spin-chain with ℓ sites $|\uparrow^{\ell}\rangle \equiv \otimes_{i=1}^{\ell}|\uparrow\rangle_{\ell+1-i}$. By looking at one site,

$$L_n(u)|\uparrow\rangle = \begin{pmatrix} \left(u+\frac{i}{2}\right)|\uparrow\rangle & |\text{complicated}\rangle \\ 0 & \left(u-\frac{i}{2}\right)|\uparrow\rangle \end{pmatrix} \tag{2.26}$$

then,

$$\hat{A}(u)|\uparrow^{\ell}\rangle = \left(u+\frac{i}{2}\right)^{\ell}|\uparrow^{\ell}\rangle \quad \text{and} \quad \hat{D}(u) = \left(u-\frac{i}{2}\right)^{\ell}|\uparrow^{\ell}\rangle \tag{2.27}$$

Also, inspired by the fact that $\hat{B}(u)$ should behave like S^{-}, we guess that it should be some kind of creation operator. If that's the case, we can write the following ansatz for an eigenstate of $\hat{T}$:

$$|\mathbf{u}\rangle = \hat{B}(u_1)\dots\hat{B}(u_M)|\uparrow^\ell\rangle \tag{2.28}$$

By applying $\hat{T}$ to this state and using the algebra derived from (2.23), we conclude that $|\mathbf{u}\rangle$ is an eigenstate if $\mathbf{u}$ satisfies

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^\ell = \prod_{k\neq j} \frac{u_j - u_k + i}{u_j - u_k - i} \tag{2.29}$$

Spectral parameters obeying this condition are called *Bethe roots* and states built from it are said to be *on-shell*. These eigenstates have eigenvalue

$$T(u) = \left(u + \frac{i}{2}\right)^\ell \prod_{k=1}^{M} \frac{u - u_k - i}{u - u_k} + \left(u - \frac{i}{2}\right)^\ell \prod_{k=1}^{M} \frac{u - u_k + i}{u - u_k} \tag{2.30}$$

From $T(u)$ we can now compute the energy of the spin-chain:

$$E = \ell - i\frac{d}{du}\log T\left(u + i/2\right)\Big|_{u=0} \tag{2.31}$$

which gives us simply

$$E = \sum_{k=1}^{M} \frac{1}{u_k^2 + \frac{1}{4}} \tag{2.32}$$

Furthermore, we can identify the Bethe roots to the momenta of the magnons by $e^{ip} = \frac{u+\frac{i}{2}}{u-\frac{i}{2}}$. By doing that, we have the following dispersion relation

$$E = \sum_{k=1}^{M} \epsilon(p_k) \qquad \text{where} \qquad \epsilon(p) = 4\sin^2\frac{p}{2} \tag{2.33}$$

Finally, since we have a closed spin-chain, we need to have translation invariance. That requires $e^{iP} = 1$, where P is the total momentum of the magnons. Then, the Bethe roots must satisfy

$$\prod_{k=1}^{M} \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} = 1 \tag{2.34}$$

We concluded the task of diagonalizing the SU(2) Heisenberg spin-chain. We can use a similar procedure to compute the eigensystem of the PSU(2,2 | 4) spin chain, which is equivalent to computing the spectrum of single-trace operators in $\mathcal{N} = 4$ SYM. An important diference between these two groups is that the latter is non-compact. In the following section we discuss the simplest non-compact subgroup of PSU(2,2 | 4).

2.2.2 Non-compact spin chains

Consider operators constructed from a complex scalar (say Z) and covariant derivatives along some light-like direction D. Schematically, they have the form

$$\mathcal{O} = \mathrm{tr}(D^{k_1} Z\, D^{k_2} Z \ldots D^{k_\ell} Z) \to |k_1\, k_2 \ldots k_\ell\rangle \tag{2.35}$$

and correspond to states in the $\mathrm{XXX}_{-\frac{1}{2}}$ spin-chain, labeled by positive integers $\{k_i\}$. Therefore, this sector, known as SL(2), is non-compact. This fact leads to important differences in the construction of the R-matrix.

In the SU(2) case, the R-matrix acts on the tensor product of two sites which are spin-$\frac{1}{2}$ representations of SU(2), which is the direct sum of a spin-0 and a spin-1 representation:

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1 \tag{2.36}$$

Then, we can construct the R-matrix and Hamiltonian from combinations of the projectors in these spaces

$$(\mathcal{P}_{\text{singlet}})_{ij} = \mathbf{1}_{ij} - \mathbf{P}_{ij} \tag{2.37}$$

$$(\mathcal{P}_{\text{triplet}})_{ij} = \mathbf{1}_{ij} + \mathbf{P}_{ij} \tag{2.38}$$

However, the SL(2) algebra is infinite-dimensional: each one-site state is labeled by a positive integer. As consequence, the two-site states are in an infinite direct sum of representations and the R-matrix and Hamiltonian should be constructed from the projectors in each of these spaces, $(\mathcal{P}_{n\geq 0})_{ij}$. Explicitly,

$$\hat{H}^{(1)} = \sum_{n=0}^{\infty} \sum_{j=1}^{\ell} h(n)\; (\mathcal{P}_n)_{jj+1} \tag{2.39}$$

where $h(n)$ is the n-th harmonic number. Also, up to an overall factor,

$$R_{ij}(u) = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(-n-iu)}{\Gamma(-n+iu)} (\mathcal{P}_n)_{ij} \tag{2.40}$$

Despite these complications, we can perform the Bethe ansatz method in the same way and construct states $|\mathbf{u}\rangle$ from $\hat{B}$ operators acting on the vacuum state corresponding to the $\frac{1}{2}$-BPS operator trZ^L. As it turns out, this leads to almost the same Bethe equations of the SU(2) case (pay attention to the flipped signs):

$$\left(\frac{u_j - \frac{i}{2}}{u_j + \frac{i}{2}}\right)^{\ell} = \prod_{k \neq j}^{S} \frac{u_j - u_k + i}{u_j - u_k - i} \tag{2.41}$$

where the spin S is the total number of derivatives, $S = \sum_{j=1}^{j=\ell} k_j$. The energy of such state is given by

$$E = \sum_{k=1}^{S} \frac{1}{u_k^2 + \frac{1}{4}} \tag{2.42}$$

and the state must satisfy the same periodicity condition as before.

But don't be mistaken: the seemly innocent sign flip dramatically changes the nature of the Bethe roots $\{u_j\}$. For example, the roots come in complex conjugate pairs for SU(2) and are all real for SL(2). Another subtle detail is that the equations above don't make any reference to the numbers $\{k_j\}$, but instead contain all possible states with S derivatives; the information about how the derivatives are distributed can be labeled by 'mode numbers'. We will elaborate on these two points when discussing semi-classical limits.

2.2.3 All-loop Bethe ansatz

The interpretation of the anomalous dimensions as energies in integrable spin-chains is not restricted to tree-level and can be extended to some orders in perturbation theory! By computing more Feynman diagrams at two-loops, it's possible to extend (2.7) [17], restricted to the SU(2) sector, to

$$\hat{\Gamma} = 2g^2 \underbrace{\sum_{n=1}^{\ell} (\mathbf{1}_{n,n+1} - \mathbf{P}_{n,n+1})}_{\hat{H}^{(1)}} + 2g^4 \underbrace{\left[\sum_{n=1}^{\ell} (\mathbf{1}_{n,n+2} - \mathbf{P}_{n,n+2}) - 4\sum_{n=1}^{\ell} (\mathbf{1}_{n,n+1} - \mathbf{P}_{n,n+1})\right]}_{\hat{H}^{(2)}} \tag{2.43}$$

Although this full Hamiltonian is not integrable in that it's charges do not commute in general, it's 'perturbatively' integrable, in the sense that its conserved charges commute up to $O(g^8)$ corrections.

We do not expect the integrable spin-chain picture remain for much higher loops. However, one can formulate an all-loop Bethe ansatz valid for asymptotically large operators. For example, in the SU(2) sector [18]:

$$\left(\frac{x_j^+}{x_j^-}\right)^{\ell} = \prod_{k \neq j}^{M} \left(\frac{x_j^+ - x_k^-}{x_j^- - x_k^+}\right) \left(\frac{1 - \frac{1}{x_j^+ x_k^-}}{1 - \frac{1}{x_k^+ x_j^-}}\right) \sigma(x_j, x_k)^2 \tag{2.44}$$

using the notation $x_j \equiv x(u_j)$, $x_j^{\pm} \equiv x(u_j \pm \frac{i}{2})$, with $x(u)$ defined through

$$x(u) + \frac{1}{x(u)} = \frac{u}{g} \tag{2.45}$$

These equations do not come from any spin-chain and can, in fact, can be related to the string sigma-model.

Chapter 3

$\mathcal{N} = 4$ and integrability: correlation functions

Given the success of the map between single-trace operators in $\mathcal{N} = 4$ SYM and spin-chains for the spectrum, one may ask whether is possible to extract other observables in a similar fashion. In this chapter we will explore some ways to compute the three-point functions of single-trace operators both at weak- and finite-coupling.

3.1 Tailoring spin-chains

The first strategy is to simply compute the tree-level three-point functions by Feynman diagrams. The combinatorics of Wick contractions can be reproduced by considering all the ways we can cut the spin-chains corresponding to the operators in sub-chains and overlapping adjacent sub-chains [9–12].

Moreover, in the spirit of mapping the problem to spin-chains, we consider the overlap between individual excitations of the spin-chain to correspond to the propagator between the constituting fields. Schematically,

$$\wick{\mathrm{tr}(\c1 X \ldots \c2 Z \c3 \bar{Y})\,\mathrm{tr}(\c3 Y \c2 \bar{Z} \ldots \c1 \bar{X})} \;\Rightarrow\; \langle X|_{\mathrm{r}}|\bar{X}\rangle \ldots \langle Z|_{\mathrm{r}}|\bar{Z}\rangle \langle \bar{Y}|_{\mathrm{r}}|Y\rangle \tag{3.1}$$

The subscript ‘r’ signalizes that we need to perform some kind of transformation in the bra’s so that the overlap reproduces the propagators in the field theory.

3.1.1 Double spin-chain

The last approach is, in principle, valid to all sectors of $\mathcal{N} = 4$ SYM at tree-level. However, in practice it's very hard to study simple operators (in the SU(2) and SL(2) sectors) embedded in larger spaces. In what follows we will study the SO(4) sub-sector of the theory formed only by Z, $\bar{Z}$, Y, and $\bar{Y}$. In this case, we can write our operators as SU(2) excitations over a double spin-chain $\text{SU}(2)_L \times \text{SU}(2)_R \approx$ SO(4).

In this picture, the fields we are going to use are written as

$$Z \Rightarrow |\uparrow\rangle_L \otimes |\uparrow\rangle_R, \qquad \bar{Z} \Rightarrow |\downarrow\rangle_L \otimes |\downarrow\rangle_R \tag{3.2}$$

$$Y \Rightarrow |\uparrow\rangle_L \otimes |\downarrow\rangle_R, \qquad -\bar{Y} \Rightarrow |\downarrow\rangle_L \otimes |\uparrow\rangle_R \tag{3.3}$$

As we did before, we are going to take Z to be the vacuum and we will put excitations on top of it. In order to remain in the SO(4) sub-sector, we can only put excitation in either the left or the right spin-chain:

$$|\mathbf{u}, \uparrow^{\ell}\rangle_L \equiv B(u_1)\dots B(u_M)|\uparrow^{\ell}\rangle_L \tag{3.4}$$

$$|\tilde{\mathbf{u}}, \uparrow^{\ell}\rangle_R \equiv B(\tilde{u}_1)\dots B(\tilde{u}_M)|\uparrow^{\ell}\rangle_R \tag{3.5}$$

After putting the excitations on the canonical ground state, we rotate the state to the desired direction of SO(4). To perform the Wick contractions between two fields, we introduce the *spin-vertex* operator:

$$\overbracket{F_1 F_2} = \langle\mathbf{1}|(|\mathbf{n}_1\rangle_L \otimes |\mathbf{n}_2\rangle_L)\langle\mathbf{1}|(|\tilde{\mathbf{n}}_1\rangle_R \otimes |\tilde{\mathbf{n}}_2\rangle_R) \tag{3.6}$$

where $\langle\mathbf{1}|$ projects out the singlet:

$$\langle\mathbf{1}| = \epsilon_{ab}\langle a| \otimes \langle b| = \langle\uparrow| \otimes \langle\downarrow| - \langle\downarrow| \otimes \langle\uparrow| \tag{3.7}$$

For example,

$$\overbracket{Z Z} = 0 \quad \text{and} \quad \overbracket{Z \bar{Z}} = 1 \tag{3.8}$$

since

$$\langle\mathbf{1}|(|\uparrow\rangle \otimes |\uparrow\rangle) = 0 \quad \text{and} \quad \langle\mathbf{1}|(|\uparrow\rangle \otimes |\downarrow\rangle) = 1 \tag{3.9}$$

Finally, we define

$$\wick{\c{\mathcal{O}}_1 \c{\mathcal{O}}_2} = \langle |\mathcal{O}_1\rangle_L, |\mathcal{O}_2\rangle_L\rangle \langle |\tilde{\mathcal{O}}_1\rangle_R, |\tilde{\mathcal{O}}_2\rangle_R\rangle \tag{3.10}$$

where the bilinear $\langle\, \cdot\, ,\, \cdot\, \rangle$ is defined for two spin-chains A_1 and A_2 of equal length ℓ as

$$\langle |A_1\rangle, |A_2\rangle\rangle = \left(\prod_{j=1}^{\ell} \langle \mathbf{1}_{j,\ell-j+1}|\right) |A_1\rangle \otimes |A_2\rangle \tag{3.11}$$

We know now how to contract two double spin-chains. To compute the three-point function, we break each operator in two subchains (labeled by l and r) and contract the adjacent parts, schematically[1]:

$$C_{123} \sim \sum_{a,b,c} \langle |\mathcal{O}_{1_a}\rangle^r, |\mathcal{O}_{2_b}\rangle^l\rangle \langle |\mathcal{O}_{2_b}\rangle^r, |\mathcal{O}_{3_c}\rangle^l\rangle \langle |\mathcal{O}_{3_c}\rangle^r, |\mathcal{O}_{1_a}\rangle^l\rangle \tag{3.12}$$

This procedure is done in detail in [11]. A very interesting remark is the similarity between the spin-vertex and the string-vertex from String Field Theory [11,12,19]. In what follows we are going explore a bit the structure of these correlators.

3.1.2 Simplest example: mixed correlators

We start by analyzing a simple example, in which there's essentially only one way of cutting the chains and two of the overlaps are trivial. Take, for example,

$$\mathcal{O}_1 = \mathrm{tr}(ZZYYY) \tag{3.13}$$
$$\mathcal{O}_2 = \mathrm{tr}(\bar{Y}\bar{Z}\bar{Z}\bar{Z}\bar{Z}) \tag{3.14}$$
$$\mathcal{O}_3 = \mathrm{tr}(ZZ\bar{Y}\bar{Y}) \tag{3.15}$$

with lengths L_1, L_2, L_3. Schematically, the only type of Wick contraction in this configuration is the following:

Then, there's only one non-trivial overlap - namely the one between $\mathcal{O}_1$ and $\mathcal{O}_2$. The scalar product between these two sub-chains involves all the excitations of the two operators and it's equal to $\mathcal{A}_{\ell_{12}}(\mathbf{u}_1 \cup \mathbf{u}_2)$, with

$$\mathcal{A}_\ell(\mathbf{u}) = \sum_{\alpha\cup\bar{\alpha}=\mathbf{u}} (-1)^{|\alpha|}\, a_\ell(\alpha)\, f(\alpha, \bar{\alpha}) \tag{3.16}$$

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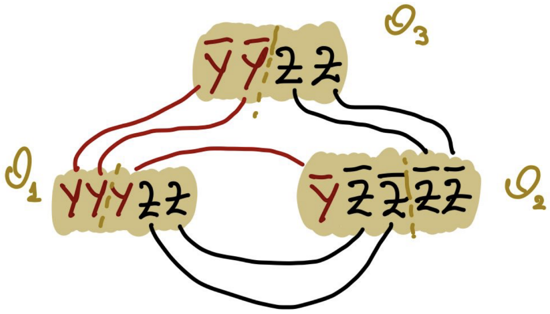
[1] For each sector



where $\ell_{ij} = \frac{L_i + L_j - L_k}{2}$ and

$$a_\ell(\alpha) = \prod_{x \in \alpha} \left(\frac{x - \frac{i}{2}}{x + \frac{i}{2}} \right)^\ell, \qquad f(\alpha, \beta) = \prod_{x \in \alpha} \prod_{y \in \beta} \left(\frac{x - y + i}{x - y} \right) \tag{3.17}$$

This beautiful and simple quantity has many representations and, in particular, can be re-casted as a determinant; it appears in the integrability literature as either partial Domain Wall Partition Function (pDWPF) or simply 'mathcal A'. As an example of its nice structure, we write the following identity [10]:

$$\mathcal{A}_\ell(\mathbf{u}) = \frac{Q_{\mathbf{u}}(i/2)^\ell}{\Delta(\mathbf{u})} \prod_{j=1}^{M} \left(1 - \mathbb{D}_{u_j}^{-1} \right) \frac{\Delta(\mathbf{u})}{Q_{\mathbf{u}}(i/2)^\ell} \tag{3.18}$$

where $\Delta(\mathbf{u}) = \prod_{j<k}^{M} (u_j - u_k)$, $Q_{\mathbf{u}}(v) = \prod_{j=1}^{M} (v - u_j)$ and $\mathbb{D}_x^{-1}$ is the operators that shifts x by $-i$. From this identity, it's straightforward to show that $\mathcal{A}$ is a determinant.

In this example, the both $\mathcal{O}_1$ and (rotated) $\mathcal{O}_2$ have Y excitations over a Z vacuum, while $\mathcal{O}_3$ has $\bar{Y}$ excitations over a Z vacuum. This fact implies that operator $\mathcal{O}_3$ cannot interact with $\mathcal{O}_1$ and $\mathcal{O}_2$. For this reason, we call these *mixed* correlators or *type I-I-II*. This is reminescent of the fact that the global symmetry factorizes, i.e. SO(4) = SU(2)$_L$ × SU(2)$_R$. In this language, the excitations of $\mathcal{O}_1$ and $\mathcal{O}_2$ belong to the left sector and the ones from $\mathcal{O}_3$ belong to the right sector.

3.1.3 General example: pure correlators

We can also have the three operators having the same kind of excitations when rotated to the Z vacuum, we call this correlators *type I-I-I* or *pure*. Consider, for example, the three-point function with the operators

$$\mathcal{O}_1 = \mathrm{tr}(ZZYYY) \tag{3.19}$$

$$\mathcal{O}_2 = \mathrm{tr}(\bar{Z}\bar{Z}\bar{Z}\bar{Z}\bar{Y}) \tag{3.20}$$

$$\mathcal{O}_3 = \mathrm{tr}(\tilde{Z}\tilde{Z}\tilde{Y}\tilde{Y}) \tag{3.21}$$

where

$$\tilde{Z} = Z + \bar{Z} + Y - \bar{Y} \tag{3.22}$$

$$\tilde{Y} = Y + \bar{Z} \tag{3.23}$$

Now, there are many different structures for Wick contractions of the fields. In this case, we have to rotate the vacua of each operator to the same one (which gives rise to polarization factors ${z_{ij}}$[2]); separate the excitations of each operator in two (weighted by a splitting factor); and compute the overlaps between adjacent spin chains (using (3.16)).

The general formula for the properly normalized correlator is

$$\begin{aligned}\mathcal{G}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \sum_{\alpha_i \cup \bar{\alpha}_i = \mathbf{u}_i} & z_{12}^{M_{12}-|\alpha_1|-|\bar{\alpha}_2|} z_{23}^{M_{23}-|\alpha_2|-|\bar{\alpha}_3|} z_{31}^{M_{31}-|\alpha_3|-|\bar{\alpha}_1|} \\ & \times H_{\ell_{31}}(\alpha_1, \bar{\alpha}_1)\, H_{\ell_{12}}(\alpha_2, \bar{\alpha}_2)\, H_{\ell_{23}}(\alpha_3, \bar{\alpha}_3) \\ & \times \mathcal{A}_{\ell_{12}}(\bar{\alpha}_1 \cup \alpha_2)\, \mathcal{A}_{\ell_{23}}(\bar{\alpha}_2 \cup \alpha_3) \mathcal{A}_{\ell_{31}}(\bar{\alpha}_3 \cup \alpha_1)\end{aligned} \tag{3.24}$$

with the splitting factors given by

$$H_\ell(\alpha, \bar{\alpha}) = (-1)^{|\alpha|} \frac{f(\bar{\alpha}, \alpha)}{a_\ell(\alpha)} \tag{3.25}$$

One important remark is that although the three-point function seems to strongly depend on the polarizations z_{ij}, this is only an illusion due to the fact that this answer is an off-shell statement. In our case, the three polarizations are parallel and, therefore, the *on-shell* correlator should not depend on any free parameter. In fact, one can check this is the case when we use sets of Bethe roots $\mathbf{u}_i$ which satisfy the Bethe equations.

The simplicity of the derivation in this section relies on the fact that we were studying configurations with fields that can be embedded in SO(4) = SU(2)$_L\times$

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[2] These numbers satisfy $z_{12} + z_{23} + z_{31} = 0$

SU(2)$_R$, which is very convenient for dealing with SU(2) excitations. This story would not be so simple if we had fields with SU(2) excitations with polarizations exploring the entire SO(6).

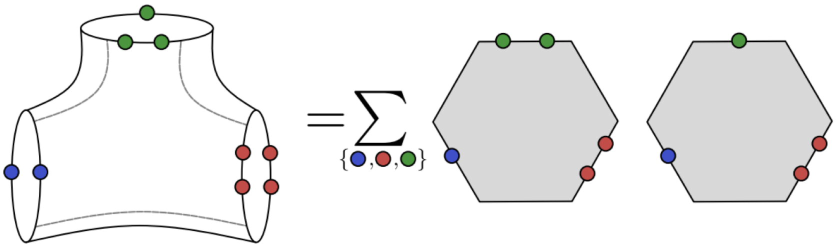
In what follows we will switch gears and briefly introduce the powerful, non-perturbative, framework of hexagonalization. Within this framework, we are going to show, in the next chapter, how to find an expression similar to 3.24 in more general settings. In particular, we are going to show the explicit result correlation functions of spinning operators, i.e. operators with SL(2) excitations, with polarizations exploring the SO(2,4) sector of $\mathcal{N} = 4$ SYM.

3.2 Hexagonalization

The mapping between the correlators in the theory and combinations of spin-chain overlaps is a particular feature of the weak coupling limit. At finite coupling we must have a more generic framework which also accounts for the strong-coupling behaviour of the theory, dual to classical string theory in AdS$_5 \times$ S^5. This is obtained by looking to the correlation functions at finite coupling as a pair of pants, i.e. puncture sphere, in which every hole (puncture) corresponds to a single-trace operator/closed string.

3.2.1 Cutting pairs of pants

While in this work we are not going to focus on the details of the construction, we are going to describe the general idea of the prescription and highlight key properties which will be important for our goals. Starting with the pair of pants, we break them into two simpler building blocks called *hexagons*. Then, we sum over all possible ways of splitting the excitations between the front and back hexagons, as in the following picture:



In addition to that, we glue two hexagons by inserting a complete basis of states along the seams and integrating over all possible flavor/momentum of each intermediate state. We call those states *mirror particles*, which can 'wrap' around the operators. In most of this work, we are going to ignore such effects, since they are damped (roughly) by a factor $e^{-E\ell} \sim g^{2\ell}$, where ℓ is the length of the bridge in which the magnon propagates, and E is its energy. We call the correlators computed without including these corrections 'asymptotic three-point functions'.

Schematically[3],

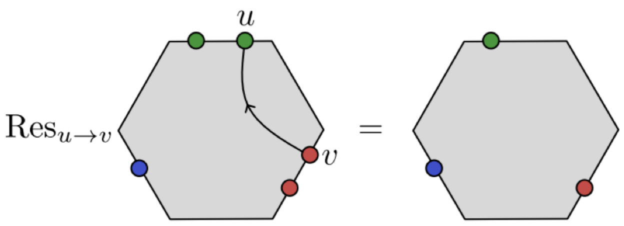
$$\begin{aligned}\mathcal{G}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3) = \sum_{\delta_i\cup\bar{\delta}_i=\mathbf{u}_i} (-1)^{|\delta_1|+|\delta_2|+|\delta_3|}\,\omega(\bar{\delta}_1,\delta_1)\,\omega(\bar{\delta}_2,\delta_2)\,\omega(\bar{\delta}_3,\delta_3)\\ \times\ \mathcal{H}(\delta_1,\delta_2,\delta_3)\,\mathcal{H}(\bar{\delta}_1,\bar{\delta}_3,\bar{\delta}_2)\quad (3.26)\end{aligned}$$

where ω is some simple splitting factor and $\mathcal{H}$ is the hexagon.

Then, the idea of the approach is to find an expression for $\mathcal{H}$. This is obtained by using the fact that, in this picture, the excitations of the operators are particles in an two dimensional integrable field theory, of which our hexagons are form-factors. This observation makes it possible to derive axioms for these form-factors in the spirit of the integrable bootstrap.

3.2.2 Decoupling axioms

To our purposes, we are going to use the axioms in their tree-level version and for simple sectors as, for example, SU(2) and SL(2).



The first condition the form-factors must satisfy is that if we send an rapidity to infinity, it should simply decouple and the hexagon should remain the same

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[3] Note that the second hexagon has the opposite orientation.

(with proper normalization). That is,

$$\lim_{x\to\infty} \mathcal{H}(\delta_1 \cup \{x\}, \delta_2, \delta_3) = \mathcal{H}(\delta_1, \delta_2, \delta_3) \tag{3.27}$$

$$\lim_{x\to\infty} \mathcal{H}(\delta_1, \delta_2 \cup \{x\}, \delta_3) = \mathcal{H}(\delta_1, \delta_2, \delta_3) \tag{3.28}$$

$$\lim_{x\to\infty} \mathcal{H}(\delta_1, \delta_2, \delta_3 \cup \{x\}) = \mathcal{H}(\delta_1, \delta_2, \delta_3) \tag{3.29}$$

The second condition concerns what happens when we take one rapidity from two different operators to be the same. In this framework, we can 'cross' excitations from one operator to other one. When we do that, the particle on the first operator becomes its anti-particle and annihilates the particle in the second operator. In the tree-level case, this reduces to

$$\text{Res}_{y\to x}\mathcal{H}(\delta_1 \cup \{x\}, \delta_2 \cup \{y\}, \delta_3) = \pm\, i\, t_{12}\, f(x,\delta_1)\, f(\delta_2,x)\, \mathcal{H}(\delta_1,\delta_2,\delta_3) \tag{3.30}$$

$$\text{Res}_{y\to x}\mathcal{H}(\delta_1, \delta_2 \cup \{x\}, \delta_3 \cup \{y\}) = \pm\, i\, t_{23}\, f(x,\delta_2)\, f(\delta_3,x)\, \mathcal{H}(\delta_1,\delta_2,\delta_3) \tag{3.31}$$

$$\text{Res}_{y\to x}\mathcal{H}(\delta_1 \cup \{y\}, \delta_2, \delta_3 \cup \{x\}) = \pm\, i\, t_{31}\, f(x,\delta_3)\, f(\delta_1,x)\, \mathcal{H}(\delta_1,\delta_2,\delta_3) \tag{3.32}$$

where for the SU(2) (SL(2)) case we use the + (-) sign. The t_{ij} factors refers to the scalar product between the normalized polarizations of the i-th and j-th operators. At tree-level, these two conditions combined allow us to completely determine $\mathcal{H}$. To discover how to do it, we are going to derive now $\mathcal{H}$ from (3.24) to get inspiration. For the I-I-I correlator, $t_{ij} = 1$ and we pick the $+$ sign.

3.2.3 From tailoring to hexagonalization

In what follows, we will reshuffle the sum and show that, upon using Bethe equations, our expression for the correlator can be recast in a hexagon-like formula. By inserting the explicit expression for $\mathcal{A}$, (3.16), in equation (3.24):

$$\begin{aligned}\mathcal{G} = \sum_{\substack{\alpha_i \cup \bar{\alpha}_i = \mathbf{u}_i \\ \beta_i \cup \bar{\beta}_i = \bar{\alpha}_i \cup \alpha_{i+1}}} & z_{12}^{M_{12}-|\alpha_1|-|\bar{\alpha}_2|} z_{23}^{M_{23}-|\alpha_2|-|\bar{\alpha}_3|} z_{31}^{M_{31}-|\alpha_3|-|\bar{\alpha}_1|} \, (-1)^{|\bar{\alpha}_1|+|\bar{\alpha}_2|+|\bar{\alpha}_3|+|\beta_1|+|\beta_2|+|\beta_3|} \\ & \times f(\bar{\alpha}_1,\alpha_1) f(\bar{\alpha}_2,\alpha_2) f(\bar{\alpha}_2,\alpha_2) \times f(\beta_1,\bar{\beta}_1) f(\beta_2,\bar{\beta}_2) f(\beta_3,\bar{\beta}_3) \\ & \times \frac{a_{\ell_{31}}(\alpha_1) a_{\ell_{12}}(\alpha_2) a_{\ell_{23}}(\alpha_3)}{a_{\ell_{12}}(\beta_1) a_{\ell_{23}}(\beta_2) a_{\ell_{31}}(\beta_3)}\end{aligned} \tag{3.33}$$

Then, separating the roots in the intersections $\alpha_i \cap \beta_{i-1}$, etc:

$$\frac{a_{\ell_{31}}(\alpha_1)}{a_{\ell_{31}}(\beta_3)} = \frac{a_{\ell_{31}}(\alpha_1 \cap \beta_3) a_{\ell_{31}}(\alpha_1 \cap \bar{\beta}_3)}{a_{\ell_{31}}(\alpha_1 \cap \beta_3) a_{\ell_{31}}(\bar{\alpha}_3 \cap \beta_3)} = \frac{a_{\ell_{31}}(\alpha_1 \cap \bar{\beta}_3)}{a_{\ell_{31}}(\bar{\alpha}_3 \cap \beta_3)} \tag{3.34}$$

Note that all the roots in the argument of $a_{\ell_{31}}$ in the denominator belongs to the third operator. Then, we can write the following version of the Bethe equations:

$$a_{\ell_{31}}(x) a_{\ell_{23}}(x) = -\frac{f(\mathbf{u}_3, x)}{f(x, \mathbf{u}_3)} \tag{3.35}$$

Using this,

$$\frac{a_{\ell_{31}}(\alpha_1)}{a_{\ell_{31}}(\beta_3)} = (-1)^{|\bar{\alpha}_3 \cap \beta_3|} \times a_{\ell_{31}}(\alpha_1 \cap \bar{\beta}_3) a_{\ell_{23}}(\bar{\alpha}_3 \cap \beta_3) \times \frac{f(\bar{\alpha}_3 \cap \beta_3, \mathbf{u}_3)}{f(\mathbf{u}_3, \bar{\alpha}_3 \cap \beta_3)} \tag{3.36}$$

And, finally,

$$\begin{aligned} \mathcal{G} = \sum_{\substack{\alpha_i \cup \bar{\alpha}_i = \mathbf{u}_i \\ \beta_i \cup \bar{\beta}_i = \bar{\alpha}_i \cup \alpha_{i+1}}} & z_{12}^{M_{12} - |\alpha_1| - |\bar{\alpha}_2|} z_{23}^{M_{23} - |\alpha_2| - |\bar{\alpha}_3|} z_{31}^{M_{31} - |\alpha_3| - |\bar{\alpha}_1|} (-1)^{|\bar{\alpha}_1| + |\bar{\alpha}_2| + |\bar{\alpha}_3| + |\beta_1| + |\beta_2| + |\beta_3|} \\ & \times f(\bar{\alpha}_1, \alpha_1) f(\bar{\alpha}_2, \alpha_2) f(\bar{\alpha}_2, \alpha_2) \times f(\beta_1, \bar{\beta}_1) f(\beta_2, \bar{\beta}_2) f(\beta_3, \bar{\beta}_3) \\ & \times a_{\ell_{31}}(\alpha_1 \cap \bar{\beta}_3) a_{\ell_{23}}(\bar{\alpha}_3 \cap \beta_3) a_{\ell_{12}}(\alpha_2 \cap \bar{\beta}_1) a_{\ell_{31}}(\bar{\alpha}_1 \cap \beta_1) a_{\ell_{23}}(\alpha_3 \cap \bar{\beta}_2) a_{\ell_{12}}(\bar{\alpha}_2 \cap \beta_2) \\ & \times (-1)^{|\bar{\alpha}_1 \cap \beta_1| + |\bar{\alpha}_2 \cap \beta_2| + |\bar{\alpha}_3 \cap \beta_3|} \times \frac{f(\bar{\alpha}_1 \cap \beta_1, \mathbf{u}_1)}{f(\mathbf{u}_1, \bar{\alpha}_1 \cap \beta_1)} \frac{f(\bar{\alpha}_2 \cap \beta_2, \mathbf{u}_2)}{f(\mathbf{u}_2, \bar{\alpha}_2 \cap \beta_2)} \frac{f(\bar{\alpha}_3 \cap \beta_3, \mathbf{u}_3)}{f(\mathbf{u}_3, \bar{\alpha}_3 \cap \beta_3)} \end{aligned} \tag{3.37}$$

This is seemingly a very messy expression, but we'll see it contains some nice features! First, one can check that using this expression, $\mathcal{G}$ is independent of the polarizations for any choice of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$. Therefore, this is a good off-shell representation for the three-point function as it makes the global SO(4) symmetry of our setting explicit.

To discover the hidden structure of such a complicated expression, let's try to write the terms of it which don't contain any momentum factor $a_{\ell_{ij}}$. This term is given by the partitions such that

$$\bar{\beta}_1 \cap \alpha_2 = \bar{\beta}_2 \cap \alpha_3 = \bar{\beta}_3 \cap \alpha_1 = \beta_1 \cap \bar{\alpha}_1 = \beta_2 \cap \bar{\alpha}_2 = \beta_3 \cap \bar{\alpha}_3 = \emptyset. \tag{3.38}$$

These conditions fix the subsets β_j and $\bar{\beta}_j$ to be

$$\beta_1 = \alpha_2,\ \beta_2 = \alpha_3,\ \beta_3 = \alpha_1,\ \bar{\beta}_1 = \bar{\alpha}_1,\ \bar{\beta}_2 = \bar{\alpha}_2,\ \bar{\beta}_3 = \bar{\alpha}_3. \tag{3.39}$$

Therefore, plugging these back in (3.37) we get

$$\mathcal{G}_0 = (-1)^{M_1+M_2+M_3} \sum_{\substack{\alpha_1\cup\bar{\alpha}_1=\mathbf{u}_1\\ \alpha_2\cup\bar{\alpha}_2=\mathbf{u}_2\\ \alpha_3\cup\bar{\alpha}_3=\mathbf{u}_3}} z_{12}^{M_2-M_3-|\alpha_2|+|\alpha_1|} z_{23}^{M_3-M_1-|\alpha_3|+|\alpha_2|} \times z_{31}^{M_1-M_2-|\alpha_1|+|\alpha_3|} \\ \times f(\bar{\alpha}_1,\alpha_1) f(\bar{\alpha}_2,\alpha_2) f(\bar{\alpha}_3,\alpha_3) f(\alpha_2,\bar{\alpha}_1) f(\alpha_3,\bar{\alpha}_2) f(\alpha_1,\bar{\alpha}_3). \quad (3.40)$$

The term above has a very nice structure: it only contains interactions between "neighbor" partitions $f(\alpha_i,\bar{\alpha}_j)$ and $f(\bar{\alpha}_i,\alpha_i)$. More than that, this quantity exactly corresponds to the hexagon for the SU(2) correlator! To show that, we generalize what we just did to define $\mathcal{G}_0$: we analyze the terms in $\mathcal{G}$ proportional to

$$a_{\ell_{31}}(\delta_1) a_{\ell_{12}}(\delta_2) a_{\ell_{23}}(\delta_3), \quad (3.41)$$

where $\delta_i \cup \bar{\delta}_i = \mathbf{u}_i$. The partitions β_i need to satisfy $\bar{\beta}_i \cap \alpha_j = \bar{\gamma}_j$ and $\beta_i \cap \bar{\alpha}_i = \gamma_i$ with $\gamma_i \cup \bar{\gamma}_i = \delta_i$.

Then these are fixed to

$$\beta_i = \alpha_j/\bar{\gamma}_j \cup \gamma_i \text{ and } \bar{\beta}_i = \bar{\alpha}_i/\gamma_i \cup \bar{\gamma}_j. \quad (3.42)$$

Plugging this back in (3.37) yields us our first result[4]

$$\mathcal{G}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3) = \sum_{\substack{\delta_1\cup\bar{\delta}_1=\mathbf{u}_1\\ \delta_2\cup\bar{\delta}_2=\mathbf{u}_2\\ \delta_3\cup\bar{\delta}_3=\mathbf{u}_3}} (-1)^{|\bar{\delta}_1|+|\bar{\delta}_2|+|\bar{\delta}_3|} f(\delta_1,\bar{\delta}_1) f(\delta_2,\bar{\delta}_2) f(\delta_3,\bar{\delta}_3) \\ \times a_{\ell_{31}}(\bar{\delta}_1) a_{\ell_{12}}(\bar{\delta}_2) a_{\ell_{23}}(\bar{\delta}_3) \mathcal{G}_0(\delta_1,\delta_2,\delta_3) \mathcal{G}_0(\bar{\delta}_1,\bar{\delta}_3,\bar{\delta}_2). \quad (3.43)$$

Therefore, we gave an alternative derivation for the hexagon form-factor at tree-level for three non-BPS operators in the 'SO(4) sector'! In doing so, we established that the hexagon itself can also be written as a sum over partitions (3.40). On the other hand, we know the hexagons must satisfy the decoupling axioms mentioned previously. It's very instructive to analyze how the sum over partitions ensure the axioms are satisfied.

For the first axiom, note that $\lim_{x\to\infty} f(x,\text{anything}) = 1$. Moreover, if we look at $\mathcal{G}_0(\delta_1 \cup \{x\},\delta_2,\delta_3)$, we can put x in either α_1 or $\bar{\alpha}_1$. If it's in α_1, we get a contribution $-z_{12}\, z_{23}^{-1}\, z_{31}^{0}\, \mathcal{G}_0(\delta_1,\delta_2,\delta_3)$. Similarly, if it's in $\bar{\alpha}_1$, we get a contribution

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[4] The details of this derivation are left to the Appendix XXX.

$-z_{12}^{0}\, z_{23}^{-1}\, z_{31}\, \mathcal{G}_0$. Summing the two, we obtain:

$$\lim_{x\to\infty} \mathcal{G}_0(\delta_1 \cup \{x\}, \delta_2, \delta_3) = -\left(\frac{z_{12}}{z_{23}} + \frac{z_{31}}{z_{23}}\right) \mathcal{G}_0(\delta_1, \delta_2, \delta_3) = \mathcal{G}_0(\delta_1, \delta_2, \delta_3) \tag{3.44}$$

Using the same kind of reasoning we can verify the second axiom. This time, we note the only contribution comes from $x \in \bar{\alpha}_1$ and $y \in \alpha_2$, and $\text{Res}_{y\to x} f(x,y) = i$. Therefore,

$$\begin{aligned} &\text{Res}_{y\to x}\mathcal{G}_0(\delta_1 \cup \{x\}, \delta_2 \cup \{y\}, \delta_3) \\ &\quad = (-1)^{M_1+M_2+M_3} \sum_{\alpha_i \cup \bar{\alpha}_i = \mathbf{u}_i} (\dots) \times \times f(x,\alpha_1) f(\bar{\alpha}_2, x)\; i\; f(x,\bar{\alpha}_1) f(\alpha_2, x) \end{aligned} \tag{3.45}$$

That is,

$$\text{Res}_{y\to x}\mathcal{G}_0(\delta_1 \cup \{x\}, \delta_2 \cup \{y\}, \delta_3) = i\, f(x,\delta_1) f(\delta_2, x)\; \mathcal{G}_0(\delta_1, \delta_2, \delta_3) \tag{3.46}$$

3.2.4 Spinning hexagons

Now we extend the same analysis for the non-compact SL(2) sector with generic polarizations, i.e., with polarizations pointing to any direction of SO(2,4). In this sector instead of polarized scalars like before we consider now operators of the form

$$\mathcal{O}_S(x, L, R) = L_{\alpha_1} R_{\dot{\alpha}_1} \cdots L_{\alpha_S} R_{\dot{\alpha}_S}\; \mathcal{O}_S^{\alpha_1\dot{\alpha}_1\cdots\alpha_S\dot{\alpha}_S}(x), \tag{3.47}$$

where $\mathcal{O}_S^{\alpha_1\dot{\alpha}_1\cdots\alpha_S\dot{\alpha}_S}(x)$ is a BPS vacuum with S covariant derivatives $D^{\alpha\dot{\alpha}}$ applied on it. Also L_α and $R_{\dot{\alpha}}$ are the left and right polarization spinors.

The prescription for computing the hexagons starts by putting the the excitations on the same edge. Each excitation belongs to a bi-fundamental representation of PSU(2 | 2)^2, $\chi^A \dot{\chi}^{\dot{A}}$:

$$\chi^A = (\varphi^1, \varphi^2, \psi^1, \psi^2) \qquad \dot{\chi}^{\dot{A}} = (\dot{\varphi}^1, \dot{\varphi}^2, \dot{\psi}^1, \dot{\psi}^2) \tag{3.48}$$

For example,

$$X = \varphi^1\dot{\varphi}^1, \quad -\bar{X} = \varphi^2\dot{\varphi}^2 \tag{3.49}$$

$$Y = \varphi^1\dot{\varphi}^2, \qquad \bar{Y} = \varphi^2\dot{\varphi}^1 \tag{3.50}$$

$$D^{\alpha\dot{\alpha}}Z = \psi^\alpha\dot{\psi}^{\dot{\alpha}} \tag{3.51}$$

Then, we should put all dotted excitations on one side and all undotted ones on the other. In doing so, they scatter with the Beisert S-matrix and finally contracted with the polarizations L and R. The resulting quantity can be interpreted as a partition function in a Kagome lattice.

Although there is this honest way to compute the hexagons, it's very messy and hard to see how it really works. For example, even for the tree-level computation of the SU(2) hexagon with only one excitation in each edge, we already have to sum over all PSU(2 | 2) flavours. For more details (and images!), see [7].

Instead, it's possible to find recursion relations for the hexagon with operators in this configuration.

$$\text{Res}_{y\to x}\, \mathcal{G}_0(\mathbf{u}_1 \cup \{x\}, \mathbf{u}_2 \cup \{y\}, \mathbf{u}_3) = -it_{12}\, f(x, \mathbf{u}_1) f(\mathbf{u}_2, x) \mathcal{G}_0(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3), \quad (3.52)$$

$$\text{Res}_{y\to x}\, \mathcal{G}_0(\mathbf{u}_1, \mathbf{u}_2 \cup \{x\}, \mathbf{u}_3 \cup \{y\}) = -it_{23}\, f(x, \mathbf{u}_2) f(\mathbf{u}_3, x) \mathcal{G}_0(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3), \quad (3.53)$$

$$\text{Res}_{y\to x}\, \mathcal{G}_0(\mathbf{u}_1 \cup \{y\}, \mathbf{u}_2, \mathbf{u}_3 \cup \{x\}) = -it_{31}\, f(x, \mathbf{u}_3) f(\mathbf{u}_1, x) \mathcal{G}_0(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3). \quad (3.54)$$

where t_{ij} and $f(\mathbf{x}, \mathbf{y})$ are:

$$t_{ij} = \frac{\langle L_i, R_j\rangle \langle L_j, R_i\rangle}{\langle L_i, R_i\rangle \langle L_j, R_j\rangle} \quad \text{with} \quad \langle L, R\rangle = \det(L, R), \quad (3.55)$$

$$f(\mathbf{x}, \mathbf{y}) = \prod_{u\in\mathbf{x}} \prod_{v\in\mathbf{y}} \left(\frac{u - v - i}{u - v}\right) \quad (3.56)$$

These relations are very similar to the $SU(2)$ ones, note however the presence of the polarizations t_{ij} and the sign change in $f(\mathbf{x}, \mathbf{y})$. The latter is related to the fact that the S-matrix for SU(2) and SL(2) are inverse of one another.

Now we need remember the way the sum over partitions (3.40) solved the recursion relations for the SU(2) case. With that inspiration, we derive the following formula as our second result:

$$\mathcal{G}_0(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \sum_{\substack{\alpha_1\cup\bar{\alpha}_1=\mathbf{u}_1\\ \alpha_2\cup\bar{\alpha}_2=\mathbf{u}_2\\ \alpha_3\cup\bar{\alpha}_3=\mathbf{u}_3}} h_1^{|\alpha_1|}(1-h_1)^{|\bar{\alpha}_1|} h_2^{|\alpha_2|}(1-h_2)^{|\bar{\alpha}_2|} h_3^{|\alpha_3|}(1-h_3)^{|\bar{\alpha}_3|} f(\bar{\alpha}_1, \alpha_1)$$
$$\times f(\bar{\alpha}_2, \alpha_2) f(\bar{\alpha}_3, \alpha_3) f(\alpha_2, \bar{\alpha}_1) f(\alpha_3, \bar{\alpha}_2) f(\alpha_1, \bar{\alpha}_3), \quad (3.57)$$

Note that in order to satisfy the first axiom, we need to have the weights of α_i and $\bar{\alpha}_i$ summed to one, hence we choose them to be h_i and $1 - h_i$. Moreover, when we take the residue of a root from $\mathbf{u}_i$ going to a root of $\mathbf{u}_j$, we will get the product of the weights of $\bar{\alpha}_i$ and α_j. Therefore, choosing

$$h_2(1-h_1) = t_{12}, \quad h_3(1-h_2) = t_{23}, \quad \text{and} \quad h_1(1-h_3) = t_{31}. \tag{3.58}$$

we manage to construct a solution to the recursion relations.

Therefore, we have found a closed expression for the hexagon! We remember that in [7] the hexagon partition function was a complicated sum over intermediate states in a Kagome lattice and remarkably (3.57) reproduces it exactly. It would be interesting understand how to derive this from the partition function approach.

Although it's not manifest, all hexagons with excitations in only one operator is equal to one at tree-level, as one can see by noticing the only potential poles are at the locations $u_i = u_j$ and seeing that they all cancel. Explicitly,

$$\text{Res}_{u_1 \to u_2} \sum_{\alpha_1 \cup \bar{\alpha}_1 = \mathbf{u}_1} h_1^{|\alpha_1|}(1-h_1)^{|\bar{\alpha}_1|} \prod_{\substack{u \in \alpha_1 \\ v \in \bar{\alpha}_1}} \left(\frac{u - v + i}{u - v} \right) \tag{3.59}$$

has two contributions: $u_1 \in \alpha_1$, $u_2 \in \bar{\alpha}_1$ and vice-versa. The residue in these two situations only differ by a sign and, therefore, cancel. Since there are no poles and the sum must be a rational function whose numerator's degree is at most the degree of the denominator, it's a constant. Setting all the roots to infinity, one by one, one see that the sum must be equal to

$$\mathcal{H} = (h_1 + (1-h_1))^{M_1} = 1 \tag{3.60}$$

This seemingly innocent fact is indeed the main problem in the generalization of our method to all-loops as we'll see. The reason why it's so important is that this is what ensures that our decoupling axioms will always result in hexagons which are polynomials in the (physical) polarizations t_{ij} and don't depend on other combinations of h_i's, since the recursion relations can only generate t_{ij}. At the end of this chapter we will discuss the importance of these observations for the all-loop hexagons.

As anticipated before, the SL(2) hexagon has the same structure as the one for SU(2) operators (3.40). However, this result did not come from spin chains as before; it is therefore an interesting question to find some spin chain origin of this sum. Indeed, to begin the explorations in this direction, we can go backwards and state the following formula correlators with general polarizations:

$$\begin{aligned}\mathcal{G}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3) = \sum_{\alpha_i\cup\bar{\alpha}_i=\mathbf{u}_i} & h_1^{|\alpha_1|}(1-h_1)^{|\bar{\alpha}_1|}h_2^{|\alpha_2|}(1-h_2)^{|\bar{\alpha}_2|}h_3^{|\alpha_3|}(1-h_3)^{|\bar{\alpha}_3|} \\ & \times H_{\ell_{31}}(\alpha_1,\bar{\alpha}_1)\, H_{\ell_{12}}(\alpha_2,\bar{\alpha}_2)\, H_{\ell_{23}}(\alpha_3,\bar{\alpha}_3) \\ & \times \mathcal{A}_{\ell_{12}}(\bar{\alpha}_1\cup\alpha_2)\, \mathcal{A}_{\ell_{23}}(\bar{\alpha}_2\cup\alpha_3)\mathcal{A}_{\ell_{31}}(\bar{\alpha}_3\cup\alpha_1)\end{aligned} \tag{3.61}$$

It would be a good idea to investigate if there is some modification in the tailoring procedure or in the global rotations to adapt the full construction to the SL(2) sector. That would allow to recast the problem in terms of a classically integrable sigma-model. One could speculate that in this case we would arrive in some kind of higher-rank Hitchin system, as in the Thermodynamical Bubble Ansatz [20]. If this is indeed true, it could also clarify how to compute these more complicated correlation functions at strong coupling - currently only the SO(4) and SO(2,2) case are done, which correspond to strings rotating in S^3 and in AdS_3, respectively.

3.2.5 Special cases

For completeness, now that we found the solution we can explore it a bit. First, there is the very simple choice $(t_{12}, t_{23}, t_{31}) = (1, 0, 0)$, analogous to the I-I-II correlators in the SU(2) sector. For this choice of polarizations the sum over partitions (3.57) truncate and we find

$$\mathcal{H}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3) = f(\mathbf{u}_2,\mathbf{u}_1), \tag{3.62}$$

That configuration, also called *Abelian*, can be evaluated at finite coupling and, furthermore, its three-point function has a Pfaffian representation. Using (3.61),

$$\mathcal{G}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3) \sim \mathcal{A}_{\ell_{12}}(\mathbf{u}_1\cup\mathbf{u}_2)\, \mathcal{A}_{\ell_{23}}(\mathbf{u}_3) \sim \mathcal{A}_{\ell_{12}}(\mathbf{u}_1\cup\mathbf{u}_2)\, \mathcal{A}_{\ell_{31}}(\mathbf{u}_3) \tag{3.63}$$

as expected. Note that we have two equivalent choices in this case, $(h_1, h_2, h_3) = (0, 1, 0)$ or $(h_1, h_2, h_3) = (0, 1, 1)$, hence the two possible representations. We ommited some momentum factors to make more explicit the symmetry of the correlators. For the yet simpler $(t_{12}, t_{23}, t_{31}) = (0, 0, 0)$ case, $\mathcal{H} = 1$ and we can even find a closed form for the structure constant, given by

$$\mathcal{G} = \mathcal{A}_{\ell_{31}}(\mathbf{u}_1)\mathcal{A}_{\ell_{12}}(\mathbf{u}_2)\mathcal{A}_{\ell_{23}}(\mathbf{u}_3) \tag{3.64}$$

Note that it is completely factorized since the interactions between roots of distinct operators are given by the hexagons, which are trivial here. Up to now, the previous discussed cases had simple analogs in SU(2). A slightly more general case we can consider is $(t_{12}, t_{23}, t_{31}) = (t, 0, 0)$, where $0 < t < 1$. This choice translates to $(h_1, h_2, h_3) = (0, t, 0)$ and then we have

$$\mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \sum_{\alpha_2 \cup \bar{\alpha}_2 = \mathbf{u}_2} t^{|\alpha_2|}(1-t)^{|\bar{\alpha}_2|} f(\bar{\alpha}_2, \alpha_2) f(\alpha_2, \mathbf{u}_1). \tag{3.65}$$

Note that $\mathbf{u}_3$ decouples entirely and the $t = 1$ and $t = 0$ cases easily reduce to what we found before. We also note this expression has the exact form of the $\mathcal{A}$ functional of $\mathbf{u}_2$ with a modified $a_\ell(\alpha_2)$ given now by $f(\alpha_2, \mathbf{u}_1)$ and can, therefore, be expressed by a determinant. More explicitly

$$\mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = (1-t)^{|\mathbf{u}_2|} \mathcal{A}^{+}_{\mathbf{u}_2}\left[\frac{t}{t-1} f(z, \mathbf{u}_1)\right]. \tag{3.66}$$

Plugging this choice of polarization in 3.61, we get

$$\mathcal{G} \;=\; \mathcal{A}_{\ell_{31}}(\mathbf{u}_3) \sum_{\alpha_i \cup \bar{\alpha}_i = \mathbf{u}_i} t^{|\alpha_2|}(1-t)^{|\bar{\alpha}_2|}\; H_{\ell_{12}}(\alpha_2, \bar{\alpha}_2)\; \mathcal{A}_{\ell_{12}}(\mathbf{u}_1 \cup \alpha_2)\; \mathcal{A}_{\ell_{23}}(\bar{\alpha}_2) \tag{3.67}$$

Which reduces in the $t \to 0$ limit to the previous factorized expression. Note that although we could not work out the sum over partitions in this case, the dependence on the polarizations is explicit unlike in the hexagon partition function.

3.2.6 All-loop explorations

As derived in the [7,21], the abelian hexagon is given at all loops by

$$\mathcal{H}_{\text{abelian}}(\delta_1, \delta_2, \delta_3) = h(\delta_1^{4\gamma}, \delta_2) h(\delta_1^{4\gamma}, \delta_3^{2\gamma}) h(\delta_3^{2\gamma}, \delta_2) = f(\delta_2, \delta_1)\frac{b(\delta_3, \delta_2)}{b(\delta_3, \delta_1)} \tag{3.68}$$

where

$$h(u^{4\gamma}, v) = \frac{1}{h(v, u)} = f(v, u) \tag{3.69}$$

$$h(u^{2\gamma}, v) = b(u, v) = \frac{1}{b(v, u)} \tag{3.70}$$

Note that b is anti-symmetric[5]. At tree-level, f is the same as in the section before and $b = 1$. In general, denoting the dressing phase[6] as $\sigma(u, v)$, we have

$$h(u,v) = \frac{x^- - y^-}{x^- - y^+} \frac{1 - \frac{1}{x^- y^+}}{1 - \frac{1}{x^+ y^+}} \frac{1}{\sigma(u,v)} \tag{3.71}$$

$$b(u,v) = \frac{1 - \frac{1}{x^+ y^-}}{1 - \frac{1}{x^- y^+}} \sigma(u,v) \tag{3.72}$$

If we are to find a representation for the all-loop non-abelian hexagons as sums over partitions, we need to make sure to match the 3 abelian cases. From the last section, we see that a sensible guess might be, in terms of f and b (in a super compact representation)

$$\mathcal{H}(\delta_1, \delta_2, \delta_3) = \sum_{\alpha_i \cup \bar{\alpha}_i = \delta_i} h_i^{|\alpha_i|} (1 - h_i)^{|\bar{\alpha}_i|} f(\bar{\alpha}_i, \alpha_i) f(\alpha_j, \bar{\alpha}_i) \frac{b(\bar{\alpha}_j, \alpha_i)}{b(\bar{\alpha}_i, \bar{\alpha}_j)} g_1(\alpha_i, \alpha_j) g_2(\bar{\alpha}_i, \alpha_i) \tag{3.73}$$

where we cannot determine g_1 and g_2 from the matching with the abelian cases alone, since they are always 1 for the only contributing term. We require now some kind of decoupling property, meaning, for example

$$\text{Res}_{y \to x} \mathcal{H}(\delta_1 \cup x, \delta_2 \cup y, \delta_3) = -\frac{i t_{12}}{\mu(x)} f(x, \delta_1) f(\delta_2, x) \mathcal{H}(\delta_1, \delta_2, \delta_3) \tag{3.74}$$

with and we conclude the following sum is our perfect candidate:

$$\mathcal{H}(\delta_1, \delta_2, \delta_3) = b(\delta_j, \delta_i) \times \sum_{\alpha_i \cup \bar{\alpha}_i = \delta_i} h_i^{|\alpha_i|} (1 - h_i)^{|\bar{\alpha}_i|} \frac{f(\bar{\alpha}_i, \alpha_i)}{b(\bar{\alpha}_i, \alpha_i)} \frac{f(\alpha_j, \bar{\alpha}_i)}{b(\alpha_j, \bar{\alpha}_i)} \tag{3.75}$$

To see this, note that the only pole as $y \to x$ in $\mathcal{H}(\delta_1 \cup x, \delta_2 \cup y, \delta_3)$ comes from $x \in \bar{\alpha}_1$ and $y \in \alpha_2$ due to $f(\alpha_2, \bar{\alpha}_1)$. Then, we can pull out all the dependence of x and y from sum to get the decoupling above[7]. By a similar reasoning, it also

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[5] In the sense $b(u, v) b(v, u) = 1$

[6] Which is anti-symmetric, i.e. $\sigma(u, v)\sigma(v, u) = 1$

[7] To get the exact decoupling, note that we also need to strip out the x and y dependence from the b's in the pre-factor.

satisfies the decoupling of a root going to infinity[8]:

$$\lim_{x\to\infty} \mathcal{H}(\delta_1 \cup x, \delta_2, \delta_3) = \mathcal{H}(\delta_1, \delta_2, \delta_3) \tag{3.76}$$

One very nice observation here is that the combination $\frac{f}{b}$ can be written as

$$\frac{f(x,y)}{b(x,y)} = f_t(u,v)s(u,v) \tag{3.77}$$

where $f_t(u,v)$ is the f we used at tree-level, and $s(u,v) = 1 + O(g^2)$ is a symmetric function **at all loops** and does not depend on the infamous dressing phase. At one-loop, it's given by:

$$s(u,v) = 1 + \frac{2g^2}{\left(u^2+\frac{1}{4}\right)\left(v^2+\frac{1}{4}\right)} \tag{3.78}$$

The expression we found for the hexagon satisfies the decoupling conditions and matches the Abelian cases. But what about more complex configurations? In fact, we can show that for **any** choice of polarizations, our hexagons matches the complicated computation with the partition function of a Hubbard model in a Kagome lattice when there is at most 1 root in each operator, including the case $\mathcal{H}(u,v,w)$. For one-loop, this is:

$$\begin{aligned}\mathcal{H}(u,v,w) = 1 + \frac{i\,t_{12}}{u-v} + \frac{i\,t_{31}}{w-u} + \frac{i\,t_{23}}{v-w} + \\ g^2\left(\frac{t_{12}(u-v+i)}{\left(u^2+\frac{1}{4}\right)\left(v^2+\frac{1}{4}\right)(u-v)} + \text{perms}\right)\end{aligned} \tag{3.79}$$

However, this construction fails when you have more than one root in each operator. To see why this is true, let's look at $\mathcal{H}(\mathbf{u}_1, \{\}, \{\})$ for a generic $\mathbf{u}_1$. One can see

$$\mathcal{H} = \sum_{\alpha_1 \cup \bar{\alpha}_1 = \mathbf{u}_1} h_1^{|\alpha_1|}(1-h_1)^{|\bar{\alpha}_1|} \frac{f(\bar{\alpha}_1, \alpha_1)}{b(\bar{\alpha}_1, \alpha_1)} \tag{3.80}$$

This time, we cannot carry out the tree-level argument since nothing prevents

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[8] Since $\lim_{x\to\infty} f(x,y) = \lim_{x\to\infty} b(x,y) = 1$

this sum of having poles at, for example, $\pm\frac{i}{2}$. In fact, for $\mathbf{u}_1 = \{u_1, u_2\}$:

$$\mathcal{H} = 1 + 2g^2 \frac{h_1(1-h_1)}{\left(u_1^2 + \frac{1}{4}\right)\left(u_2^2 + \frac{1}{4}\right)} = 1 + 2h_1(1-h_1)(s(u_1, u_2) - 1) \qquad (3.81)$$

As a consequence of this fact, the recursion relations keep propagating these factors with h_i and the full hexagon becomes cursed with this problem. It's an open problem how to overcome this obstacle and generalize our construction to all loops.

Chapter 4

Clasical limits

In this chapter, we describe the classical limit three-point functions in both the SU(2) and SL(2) sector: we still take the weak coupling limit but now we take the operators to have very large charges, which imply a very large number of Bethe roots. Furthermore, in the appropriate regime, these roots condense into 'cuts' described by some density $\rho(u)$. Remarkably, one can often find very compact representations for correlators in this limit as functions of the densities.

What's more, these quantities are very similar in form to those obtained for the strong coupling limit of the theory, described by the integrability of classical strings. In fact, these similarities can be made explicit by considering the classical limit of coherent states made of spin-chain Bethe states [XXX], described by the integrable Landau-Lifschitz model. With such construction, it was possible to compute the I-I-I correlators of heavy states. One of the outcomes of the present work is showing how to derive this result directly from the microscopic picture.

4.1 Setup and Bethe equations

The classical limit in the SU(2) sector is defined as

$$\ell \sim M \to \infty. \tag{4.1}$$

In this regime, the roots of each operator grows with ℓ, ($u_i \sim \ell \to \infty$), and have a separation of order $\mathcal{O}(1)$. Thus, after a $1/\ell$ re-scaling, the roots' separation are of order $\mathcal{O}(1/\ell)$ and they condense into a continuous contour $\mathcal{C}_i$ [22], as shown in figure 4.1. Each set of roots is characterized by a density $\rho_i(u)$ in the complex plane defined at the contour $\mathcal{C}_i$.

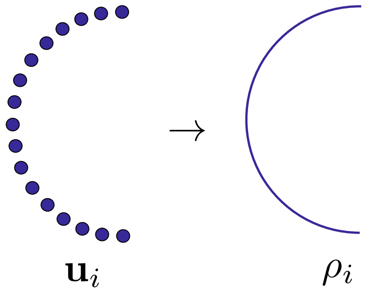

Figure 4.1: Condensation of roots. In the classical limit, the roots condense into cuts and we describe the set of roots $\mathbf{u}_i$ by a root density ρ_i in the complex plane.

As shown before, our basic ingredients are f and a_ℓ. Upon this limit, they become

$$f(z,\mathbf{u}) \to e^{iG(z)} \quad \text{and} \quad a_\ell(z) \to e^{-i\frac{\ell}{z}}, \tag{4.2}$$

where $G(z)$ is the resolvent

$$G(z) = \int_{\mathcal{C}} dx\, \frac{\rho(x)}{z-x}. \tag{4.3}$$

Unlike the density, it is clear that this function is defined at the entire complex plane modulo $\{\mathcal{C}_j\}$. with discontinuity given by

$$G(z+i0) - G(z-i0) = 2\pi i \rho(z) \tag{4.4}$$

We can go further and extend $G(z)$ to the full plane with the definition.

$$\not{G}(z) \equiv \frac{G(z+i0)+G(z-i0)}{2} \tag{4.5}$$

With these notations, the SU(2) Bethe equations

$$\left(\frac{u_j+\frac{i}{2}}{u_j-\frac{i}{2}}\right)^\ell = \prod_{k\neq j} \frac{u_j-u_k-i}{u_j-u_k+i} \tag{4.6}$$

become simply, upon taking the logarithm and replacing the sums with integrals against the density $\rho(x)$,

$$\not{G}(z) = \frac{\ell}{2z} + \pi n \tag{4.7}$$

where n is the mode number, which is associated to which branch of the logarithm we choose.

As we'll see, the structure constants do not depend on the densities explicitly, but rather on the quasi-momentum,

$$p(z) = G(z) - \frac{\ell}{2z}. \tag{4.8}$$

which is a meromorphic function on the complex plane with a pole at the origin and branch cuts at $\{\mathcal{C}_j\}$. Naturally, this function can be extended to a Riemann surfaces with many sheets. Finally, the Bethe equations become

$$e^{2i\not{p}(x)} = 1 \tag{4.9}$$

The solutions of the SU(2) Bethe equations which satisfy the momentum conservation condition are given by configurations of Bethe roots symmetric with respect to the real axis and have the shape of an "umbrella" (see figure 4.1). Also, in the classical limit, these cuts are separated by a macroscopic distance (order ℓ).

4.2 Path-integral and saddle-point equation

The previous results for the three-point function consisted of taking the classical limit of the $\mathcal{A}$-functional, which we recall here:

$$\mathcal{A}_\ell(\mathbf{u}) = \sum_{\alpha \cup \bar{\alpha} = \mathbf{u}} (-1)^{|\alpha|}\, a_\ell(\alpha)\, f(\alpha, \bar{\alpha}) \tag{4.10}$$

In the proposed limit, besides replacing a_ℓ and f with the classical counterparts, we need to deal with the sum over partitions. Let's go by parts: the summand becomes:

$$a_\ell(\alpha)\, f(\alpha, \bar{\alpha}) \to \exp\left(\int_{\mathcal{C}} du\, \rho_\alpha(u) \left[-i\frac{\ell}{u} + i\pi + i \int_{\mathcal{C}} dv\, \frac{\rho_{\bar{\alpha}}(u)}{u - v} \right] \right) \tag{4.11}$$

where ρ_α represents the density of the roots in α (and the same for $\bar{\alpha}$) and $\rho_\alpha + \rho_{\bar{\alpha}} = \rho$. Then, we replace the sum over partitions by a path integral over ρ_α with an appropriate measure to reflect the fact that many microscopic configurations correspond to the same density:

$$\mathcal{A}_\ell(\rho) = \int [D\rho_\alpha][\text{measure}] \exp\left(\int_{\mathcal{C}} du\, \rho_\alpha(u) \left[-i\frac{\ell}{u} + i\pi + i \fint_{\mathcal{C}} dv\, \frac{\rho_{\bar{\alpha}}(u)}{u-v} \right] \right) \tag{4.12}$$

with

$$\log\,[\text{measure}] = \int_{\mathcal{C}} du\;\; (\rho \log \rho - \rho_\alpha \log \rho_\alpha - \rho_{\bar{\alpha}} \log \rho_{\bar{\alpha}}) \tag{4.13}$$

One can see that in the classical limit the exponent of the integrand is of order ℓ and then we can use saddle point to compute the integral. One must note however that this is not the full story, since we took u and v large in $f(u,v)$ and we need to be careful about the situation $u - v \sim O(1)$. It's possible to show that by doing this one gets the "corrected" measure

$$\log\,[\text{measure}] = \int_{\mathcal{C}} du\;\; (F(\rho) - F(\rho_\alpha) - F(\rho_{\bar{\alpha}})) \tag{4.14}$$

where

$$F'(\rho) = \log \sinh \pi\rho \tag{4.15}$$

Then, the result of the path integral in the classical limit is

$$\log \mathcal{A}_\ell(\rho) = \oint_{\mathcal{C}} \frac{dz}{2\pi} \mathrm{Li}_2\left(e^{iq(u)}\right) \tag{4.16}$$

where

$$q(u) = -\frac{\ell}{z} + G(z) \tag{4.17}$$

Now we present some known results in the literature. The classical limit of I-I-II correlators can be readily computed

$$\begin{aligned} \log\left(\frac{C^{\bullet\bullet\bullet}}{C^{\circ\circ\circ}}\right) = \oint_{\mathcal{C}_1 \cup \mathcal{C}_2} \frac{dz}{2\pi} \mathrm{Li}_2\left(e^{ip_1 + ip_2 + i\ell_3/2z}\right) + \oint_{\mathcal{C}_3} \frac{dz}{2\pi} \mathrm{Li}_2\left(e^{ip_3 + i(\ell_2 - \ell_1)/2z}\right) \\ - \frac{1}{2} \sum_{j=1}^{3} \oint_{\mathcal{C}_j} \frac{dz}{2\pi} \mathrm{Li}_2\left(e^{2ip_j}\right) + \texttt{pols} \end{aligned} \tag{4.18}$$

where $C^{\circ\circ\circ}$ is the structure constant of three BPS operators with the same lengths as the excited operators and `pols` is just the terms containing all the polarizations which is fixed by SU(2) invariance. Such configuration is considerably simpler than I-I-I, since each sector has at most two interacting operators.

To overcome these difficulties, the authors of [13] used the strategy of using the classical limit to map the problem from the starting point to a classically integrable model, known as Landau-Lifschitz. By doing so, they were able to use the powerful machinery of classical integrability previously used to compute the same quantities at strong coupling. In this way, besides finding the result above for the I-I-II case, they also found the following result for the I-I-I correlator:

$$\log\left(\frac{C^{\bullet\bullet\bullet}}{C^{\circ\circ\circ}}\right) = \frac{1}{2}\sum_{\{i,j,k\}=\text{cperm}\{1,2,3\}} \oint_{\mathcal{C}_i\cup\mathcal{C}_j} \frac{dz}{2\pi}\,\text{Li}_2\left(e^{ip_i+ip_j-ip_k}\right) + \\ -\frac{1}{2}\sum_{j=1}^{3}\oint_{\mathcal{C}_j}\frac{dz}{2\pi}\text{Li}_2\left(e^{2ip_j}\right) + \texttt{pols} \qquad (4.19)$$

where $\text{cperm}\{1,2,3\}$ are all the cyclic permutations of $\{1,2,3\}$. Although the structure of pure and mixed correlators are clearly similar, a derivation coming directly from the microscopic expressions was lacking.

In principle, one can proceed in the same way as before and write the saddle-point equations for the more complicated I-I-I correlator. The path integral would be

$$\mathcal{G}_0 = (-1)^{M_1+M_2+M_3} z_{12}^{M_2-M_3} z_{23}^{M_3-M_1} z_{31}^{M_1-M_2} \int [\mathcal{D}\rho_{\alpha_1}]\,[\mathcal{D}\rho_{\alpha_2}]\,[\mathcal{D}\rho_{\alpha_3}] \\ \times \exp\left[\sum_{\text{cperm}}\int_{\mathcal{C}_1}\left(\rho_1\log\rho_1 - \rho_{\alpha_1}\log\rho_{\alpha_1} - (\rho_1-\rho_{\alpha_1})\log(\rho_1-\rho_{\alpha_1}) + \rho_{\alpha_1}\log\frac{z_{12}}{z_{31}}\right)\right] \\ \times \exp\left[\sum_{\text{cperm}}\left(i\int_{\mathcal{C}_1}\int_{\mathcal{C}_1}\frac{\rho_1(u)\rho_{\alpha_1}(v)}{u-v} + i\int_{\mathcal{C}_2}\int_{\mathcal{C}_1}\frac{\rho_{\alpha_2}(u)(\rho_1(v)-\rho_{\alpha_1}(v))}{u-v}\right)\right] \qquad (4.20)$$

It's necessary to introduce 8 more densities which interact among themselves! This fact makes it very hard to solve the equations, which are the following system of coupled integral equations:[1].

$$\log\left(\frac{\rho_1}{\rho_{\alpha_1}}-1\right)+\log\frac{z_{12}}{z_{31}}-iG_1+iG_{\alpha_2}+iG_3-iG_{\alpha_3}=0 \quad \text{for } u\in\mathcal{C}_1 \tag{4.21}$$

$$\log\left(\frac{\rho_2}{\rho_{\alpha_2}}-1\right)+\log\frac{z_{23}}{z_{12}}-iG_2+iG_{\alpha_3}+iG_1-iG_{\alpha_1}=0 \quad \text{for } u\in\mathcal{C}_2 \tag{4.22}$$

$$\log\left(\frac{\rho_3}{\rho_{\alpha_3}}-1\right)+\log\frac{z_{31}}{z_{23}}-iG_3+iG_{\alpha_1}+iG_2-iG_{\alpha_2}=0 \quad \text{for } u\in\mathcal{C}_3 \tag{4.23}$$

We did not fo this way. Instead, we took a shortcut.

4.3 Microscopic derivation

Going back to the hexagon representation of $\mathcal{G}$ in (3.43), we see that the classical limit of all the quantities in the sum over partitions are known with the exception of the hexagon. Consider two excited operators only, then the recursion relations (3.29) and (3.32) are solved by $\mathcal{H}(\mathbf{u}_1,\mathbf{u}_2)=f(\mathbf{u}_2,\mathbf{u}_1)$. Plugging this result in (3.43), we find the left side of I-I-II structure constants. A natural guess to solve the recurrence relations for three excited operators and type I-I-I correlators is

$$\mathcal{H}_{\text{try}}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3)=f(\mathbf{u}_2,\mathbf{u}_1)f(\mathbf{u}_3,\mathbf{u}_2)f(\mathbf{u}_1,\mathbf{u}_3). \tag{4.24}$$

This has the poles in the correct places and possess the appropriate asymptotic behavior. However, it does not satisfy the relations. Indeed for the first one we have:

$$\begin{aligned}&\operatorname{Res}_{y\to x}\mathcal{H}_{\text{try}}(\mathbf{u}_1\cup\{x\},\mathbf{u}_2\cup\{y\},\mathbf{u}_3)\\&\qquad=if(x,\mathbf{u}_1)f(\mathbf{u}_2,x)\underline{f(\mathbf{u}_3,x)f(x,\mathbf{u}_3)}\mathcal{H}_{\text{try}}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3).\end{aligned} \tag{4.25}$$

This is very similar to what we were looking for, apart from the underlined term. In the classical limit, the roots are large and the sets of roots are well separated; then, we can say

$$f(\mathbf{u}_j,z)f(z,\mathbf{u}_j)\approx 1, \tag{4.26}$$

with z being a root in $\mathbf{u}_k$ with $k\neq j$. That is for (4.24) to be a solution of the recursion relations we must have the cuts to be macroscopically distant, e.g. the distance between roots $\mathbf{u}_1$ and $\mathbf{u}_3$ is of order $\mathcal{O}(\ell)$.

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[1] ignoring the correction of the measure (4.14)

Therefore, *in the classical limit regime*, $\mathcal{H}$ is given by (4.24) for three excited operators. This derivation sounds a bit unnatural because we have mixed the microscopic analytic structure (recursion relations) of $\mathcal{H}$ with a macroscopic limit. It also does not seem to carry over more complicated cases, as the SL(2) correlators. We propose it's a good idea to find a coherent-state-like approach for them as well.

With all the ingredients we now move to the computation of the full three-point function. Plugging our classical limit solution for $\mathcal{H}$ in (3.43), we have

$$\frac{\mathcal{G}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3)}{\mathcal{H}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3)} = \prod_{\{i,j,k\}=\text{cperm}\{1,2,3\}} \left(\sum_{\alpha_i \cup \bar{\alpha}_i = \mathbf{u}_i} (-1)^{|\alpha_i|} a_{\ell_{ki}}(\alpha_i) \frac{f(\bar{\alpha}_i,\alpha_i)}{f(\alpha_i,\mathbf{u}_k) f(\mathbf{u}_j,\alpha_i)} \right). \tag{4.27}$$

where cperm$\{1,2,3\}$ is all the cyclic permutations of $\{1,2,3\}$. The partitions being summed are now untangled, and the problem factorizes into three sums. In Appendix E, we transform each of the sums into a path-integral and compute their classical limit using the methods in [9]. In the end we can write the structure constant as:

$$\log \mathcal{G}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3) = \frac{1}{2} \sum_{\{i,j,k\}=\text{cperm}\{1,2,3\}} \oint_{\mathcal{C}_i \cup \mathcal{C}_j} \frac{dz}{2\pi} \, \text{Li}_2\left(e^{ip_i + ip_j - ip_k}\right), \tag{4.28}$$

This result exactly matches the correlator (4.19) found in [13][2].

4.4 Spinning correlators

Let's consider now the SL(2) sector. First, if we take the operators to have very large spin and very large length, $S_i \to \infty$, $\ell \to \infty$, all roots will be large, ($u \sim \ell$) [23]. Although this seems similar to the SU(2) case, there are differences. For example, now all the cuts are located at the real axis. For the configurations we are interested, the cuts are symmetric with respect to the imaginary axis. More precisely, they are of the form $u \in [-b,-a] \cup [a,b]$, with $b > a \gg 1$. Despite those differences, it's still possible to carry out most of the analysis.

Much more interesting is the case of large spin but finite length, $S \gg \ell \sim 1$. In this case, the roots are real and spread in the interval $[-b,b]$, $b \gg 1$. If we choose the mode numbers in the negative axis to be 1 and the mode numbers in

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[2] After we include the contributions from polarizations and Gaudin norms, the latter given in Appendix A.

the positive axis to be -1, we are studying folded strings, i.e. with two cusps. This means all of the spin of a given operator are distributed only between 2 of its sites. In this case, each operator splits in two pieces due to the "centrifugal force". In fact, their correlation function becomes a hexagonal null Wilson loop.

Note that in this regime we cannot:

- Separate the cuts.
- Ignore the contributions near $u \sim 0$.

In the problem of the spectrum, the only consequence of the second observation above is that we cannot, for example, drop the $\frac{1}{4}$ when computing the energy[3],

$$\delta = \frac{E - J - S}{g^2} = 2 \int du \, \frac{\rho(u)}{u^2 + \frac{1}{4}} \tag{4.29}$$

Otherwise, it was completely valid to describe the roots by a density and rescale the Bethe roots [24]. While we have to be careful with the second observation, the first one can be a feature.

4.4.1 Saddle-point equations

In the saddle point equations for the I-I-I three-point function, equation (4.23), we had one equation for each cut. Now, we will have the three equations on the same cut, which seems much better! In this case, we have the following equations:

$$\not{G}_1 - G_3 + G_{\alpha_3} - G_{\alpha_2} = \log \frac{\sinh \pi(\rho_1 - \rho_{\alpha_1})}{\sinh \pi \rho_{\alpha_1}} \frac{h_1}{1 - h_1} \tag{4.30}$$

$$\not{G}_2 - G_1 + G_{\alpha_1} - G_{\alpha_3} = \log \frac{\sinh \pi(\rho_2 - \rho_{\alpha_2})}{\sinh \pi \rho_{\alpha_2}} \frac{h_2}{1 - h_2} \tag{4.31}$$

$$\not{G}_3 - G_2 + G_{\alpha_2} - G_{\alpha_1} = \log \frac{\sinh \pi(\rho_3 - \rho_{\alpha_3})}{\sinh \pi \rho_{\alpha_3}} \frac{h_3}{1 - h_3} \tag{4.32}$$

While we could not solve these equations in general, it might be interesting to investigate the possibility of recasting these equations in a TBA-like fashion. Other possible paths are studying what the solutions of this equation look like for $h_1 = h_2 = h_3 = \frac{1}{2}$; this choice corresponds to the special value in the Wilson loop literature [8], $U_1 = U_2 = U_3 = \frac{1}{4}$. Another possible path might be to take $G \to \zeta G$ and study the limits $\zeta \to 0$ and $\zeta \to \infty$. In all of these cases the equations simplify

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[3] With the appropriate rescaling

a lot and one can get a piece of information. We hope to report progress on this question.

4.4.2 Wilson loop dictionary and collinear limit

In what follows, we will sketch a simple application of our results. Following [25], there is a map between the polarizations t_{ij} and the Wilson loop cross ratios (U_1, U_2, U_3)

$$t_{23} = \frac{\ell_1^2}{(J_3 - \ell_1 - \ell_2)(J_2 - \ell_1 - \ell_3)} \tag{4.33}$$

$$t_{31} = \frac{\ell_2^2}{(J_1 - \ell_2 - \ell_3)(J_3 - \ell_2 - \ell_1)} \tag{4.34}$$

$$t_{12} = \frac{\ell_3^2}{(J_2 - \ell_3 - \ell_1)(J_1 - \ell_3 - \ell_2)} \tag{4.35}$$

where

$$\ell_1 = \frac{J_2 J_3}{J_2 + J_3 + J_1\sqrt{\frac{U_2}{U_1 U_3}}} \tag{4.36}$$

$$\ell_2 = \frac{J_3 J_1}{J_3 + J_1 + J_2\sqrt{\frac{U_1}{U_3 U_2}}} \tag{4.37}$$

$$\ell_3 = \frac{J_1 J_2}{J_1 + J_2 + J_3\sqrt{\frac{U_3}{U_2 U_1}}} \tag{4.38}$$

In the collinear limit [8, 14, 20], one of the U's, say U_2, goes to zero. Then, $\ell_2 = \ell_3 = 0$ and

$$\ell_1 = \frac{J_2 J_3}{J_2 + J_3}$$

Consequently, $t_{12} = t_{31} = 0$, and

$$t_{23} = \left(\frac{J_3}{J_2}\right)^2 \tag{4.39}$$

The corresponding three-point function is given by (3.67). We leave for the future the task of computing its classical limit.

Chapter 5

Conclusion

The hexagonalization formalism to compute structure constants in $\mathcal{N} = 4$ SYM was a super powerful achievement, however it's still an important task to understand it better to use all its power. The partition function interpretation for the hexagons introduced in [7] yielded us recursion relations that allowed their computation at tree level ((3.57)). In other words, we solved the hexagon "matrix part", reducing the sum over intermediate states in PSU$(2|2)^2$ in a Kagome lattice to a simple sum over partitions. We tried to do the same for the asymptotic, all-loop Hexagon and, although it didn't work, there is indication that it might be a good way of solving the matrix part in general. We remark that we can work backwards in our construction to also recover the correlation functions from the hexagons. This provides a concrete and useful formula for anyone willing to study the large-spin dynamics of spinning operators in $\mathcal{N} = 4SYM$

We also computed the classical limit of I-I-I correlation functions in the SU(2) sector, which are in agreement with the predictions made in [13] at weak coupling. Furthermore, we discussed some of our attempts to extract information about the classical limit/large spin limit of more complicated families of correlators.

Now we discuss some important topics related to our work.

WL/OPE duality

As suggested in [20], the large spin limit of structure constants involving three spinning operators should be connected with null hexagonal Wilson loops (WL); a precise map for the kinematics of this duality was worked out in [25]. This relation is the WL/OPE duality. It is one piece in a big web of dualities present in $\mathcal{N} = 4$ SYM, which involves polygonal Wilson loops, scattering amplitudes and three-point functions of spinning operators. All these quantities have different integrability descriptions: the polygonal Wilson loops and scattering amplitudes can be computed from the operator product expansion (OPE) of integrable pentagons, while three-point functions (and therefore polygonal Wilson loops) can be computed from the integrable hexagons. These describe the interactions of open and closed strings, respectively. To understand all these dualities from an integrability point of view and relate hexagons to pentagons is a very exciting challenge.

Our analysis may capture some of this large spin dynamics. Concretely, two important families of correlators are given by

$$(U_1, U_2, U_3) = (u, u, u) \Rightarrow (t_{12}, t_{23}, t_{31}) = (t, t, t)$$

and

$$(U_1, U_2, U_3) = (t, 0, 1-t) \Rightarrow (t_{12}, t_{23}, t_{31}) = \left(0, \left(\frac{J_3}{J_2}\right)^2, 0\right)$$

. In our setup, the second limit hugely simplifies and we can almost get a closed formula for it at tree-level... and more! In this limit (collinear limit), we can try to use the one-loop expressions for $\mathcal{A}$ and then we just have to compute **one** path integral, possibly by saddle point.

Classical and quantum integrability

In our paper, we studied the microscopic description of three-point functions at weak coupling in the SL(2) sector. It's given by (3.61). On the other hand, we can study these correlators directly in the macroscopic regime: for SU(2), the problem is mapped to a Landau-Lifschitz sigma-model and can be solved by using classical integrability tools at weak and strong coupling [13]. In this language, the weak coupling correlators have the same structure as the strong coupling ones, which describe the scattering of three classical closed strings rotating in S^3.

In the future, we hope to explore the possibility of mapping the computation of correlators in the full SL(2) sector at weak coupling to a classically integrable sigma model as in the SU(2) case. Furthermore, we expect these objects to be closely related to their strong coupling counterparts, which describe the three-point function of closed strings rotating in AdS_5.

Finally, having in mind the WL/OPE duality, we expect this problem to be related to the computation of Wilson loops living in AdS_5, in which the result can be expressed as the free energy of a thermodynamics Bethe ansatz system [8]. Note that for three-point functions at strong coupling and with polarizations on a plane, this was already computed in [26,27], which would serve as a test for any proposal. A good starting point for achieving this goal may be to find the spin-vertex for this sector to compute the three-point functions and use their monodromy relations to formulate the sigma-model construction [12].

Appendix A

Gaudin norm of Bethe states

The norm of a Bethe state is called the Gaudin norm. It can be computed directly from the algebraic Bethe ansatz [28]. Given a set of roots $\mathbf{u}$ that describes a certain Bethe state, its norm is

$$\mathcal{N}(\mathbf{u}) = \sqrt{\prod_{x\in\mathbf{u}} (x - i/2)^{\ell} \prod_{y\in\mathbf{u}} (y + i/2)^{\ell}}\, \mathcal{B}(\mathbf{u}), \tag{A.1}$$

where ℓ is the length of the spin-chain and

$$\mathcal{B}(\mathbf{u}) = \sqrt{\prod_{n\neq m} \left(\frac{u_n - u_m + i}{u_n - u_m} \right) \det \mathcal{M}}\,, \tag{A.2}$$

$$\mathcal{M}_{nm} = \frac{2}{(u_n - u_m)^2 + 1} + \delta_{nm} \left(\frac{\ell}{u_n^2 + 1/4} - \sum_{i=1}^{M} \frac{2}{(u_n - u_i)^2 + 1} \right). \tag{A.3}$$

The polynomial factors in the norm can be absorbed in the definition of $\mathcal{D}(\alpha_j, \bar{\alpha}_j)$ as we do in Appendix B. So it remains to compute the classical limit of the determinant factor $\mathcal{B}(\mathbf{u})$. It was already found in [10, 29] and it is just

$$\log \mathcal{B}(\mathbf{u}) = \frac{1}{2} \oint_{\mathcal{C}_{\mathbf{u}}} \frac{dz}{2\pi} \mathrm{Li}_2 \left(e^{2ip} \right). \tag{A.4}$$

This explain the presence of the single contour terms in equations (4.18) and (4.19).

Appendix B

Bethe equations and rewriting $\mathcal{D}(\alpha_j, \bar{\alpha}_j)$

In this appendix we rewrite $\mathcal{D}(\alpha_j, \bar{\alpha}_j)$ using Bethe equations such that the final structure constant will then be independent of the polarizations z_{ij} for off-shell roots. First we define the Baxter polynomials as usual and a different normalization for the contraction of Bethe states:

$$Q^{\pm}_{\ell}(\mathbf{x}) = \prod_{u\in\mathbf{x}} (u \pm i/2)^{\ell}, \tag{B.1}$$

$$Z_p(\alpha_i \cup \bar{\alpha}_j|\ell_{ij}) = Q^{+}_{\ell_{ij}}(\bar{\alpha}_j) Q^{-}_{\ell_{ij}}(\alpha_i) \tilde{Z}_p(\alpha_i \cup \bar{\alpha}_j|\ell_{ij}). \tag{B.2}$$

This rewriting of the scalar products suits our purposes since it is $\tilde{Z}_p(\alpha_i \cup \bar{\alpha}_j|\ell_{ij})$ that has a good classical limit and not the former definition. We can write $\mathcal{D}(\alpha_j, \bar{\alpha}_j)$ as

$$\begin{aligned}\mathcal{D}(\alpha_j, \bar{\alpha}_j) = (-1)^{|\alpha_1|+|\alpha_2|+|\alpha_3|} \prod_{i=1}^{3} f(\alpha_i, \bar{\alpha}_i) \prod_{i=1}^{3} Q^{-}_{\ell_i}(\alpha_i) Q^{+}_{\ell_i}(\bar{\alpha}_i) \\ \times \tilde{Z}_p(\alpha_1 \cup \bar{\alpha}_3|\ell_{13}) \tilde{Z}_p(\alpha_2 \cup \bar{\alpha}_1|\ell_{12}) \tilde{Z}_p(\alpha_3 \cup \bar{\alpha}_2|\ell_{23}).\end{aligned} \tag{B.3}$$

Now one uses the $\mathcal{A}$-functional defined in [9,10] by

$$\mathcal{A}^{\pm}_{\mathbf{u}}[g] = \frac{\det_{ab}\left(u_a^{b-1} - g(u_a)(u_a \pm i)^{b-1}\right)}{\det_{ab}(u_a^{b-1})}, \tag{B.4}$$

which has a sum over partitions expression given by

$$\mathcal{A}^{\pm}_{\mathbf{u}}[g] = \sum_{\alpha\cup\bar{\alpha}=\mathbf{u}} (-1)^{|\alpha|} g(\alpha) f(\alpha, \bar{\alpha}). \tag{B.5}$$

Where $+$ is for the SU(2) $f(u,v)$ and $-$ for the SL(2) one. We note that the choice $g(z) = a_\ell(z)$ yields the determinant part of the pDWPF and it is the closed form expression for the scalar product of a Bethe state and a descendant of the vacuum. Let $\mathcal{A}^+_{\mathbf{u}}[a_\ell] = \mathcal{A}_\ell(\mathbf{u})$, then

$$\mathcal{D}(\alpha_j, \bar{\alpha}_j) = (-1)^{|\alpha_1|+|\alpha_2|+|\alpha_3|} \prod_{i=1}^{3} f(\alpha_i, \bar{\alpha}_i) \frac{\prod_{i=1}^{3} Q^-_{\ell_i}(\alpha_i) Q^+_{\ell_i}(\bar{\alpha}_i)}{a_{\ell_{13}}(\bar{\alpha}_3) a_{\ell_{12}}(\bar{\alpha}_1) a_{\ell_{23}}(\bar{\alpha}_2)} \\ \times \mathcal{A}_{\ell_{13}}(\alpha_1 \cup \bar{\alpha}_3) \mathcal{A}_{\ell_{12}}(\alpha_2 \cup \bar{\alpha}_1) \mathcal{A}_{\ell_{23}}(\alpha_3 \cup \bar{\alpha}_2). \quad \text{(B.6)}$$

It remains to divide the structure constant by the norms of the operators. As said in Appendix A the determinant part of the norm has a well defined classical limit, then we define $\tilde{\mathcal{D}}(\alpha_j, \bar{\alpha}_j)$ as the previous summand but divided by the polynomial part of the norms. Therefore

$$\tilde{\mathcal{D}}(\alpha_j, \bar{\alpha}_j) = \frac{(-1)^{|\alpha_1|+|\alpha_2|+|\alpha_3|}}{a_{\ell_{13}}(\bar{\alpha}_3) a_{\ell_{12}}(\bar{\alpha}_1) a_{\ell_{23}}(\bar{\alpha}_2)} \prod_{i=1}^{3} f(\alpha_i, \bar{\alpha}_i) \prod_{i=1}^{3} \sqrt{\frac{a_{\ell_i}(\bar{\alpha}_i)}{a_{\ell_i}(\alpha_i)}} \\ \times \mathcal{A}_{\ell_{13}}(\alpha_1 \cup \bar{\alpha}_3) \mathcal{A}_{\ell_{12}}(\alpha_2 \cup \bar{\alpha}_1) \mathcal{A}_{\ell_{23}}(\alpha_3 \cup \bar{\alpha}_2). \quad \text{(B.7)}$$

Since we are dealing with physical operators, the total momentum of each one vanishes. Then

$$a_{\ell_j}(\mathbf{u}_j) = 1. \quad \text{(B.8)}$$

We can take this identity and redefine the variables in the sum and obtain the new summand

$$\tilde{\mathcal{D}}(\alpha_j, \bar{\alpha}_j) = (-1)^{|\alpha_1|+|\alpha_2|+|\alpha_3|} \prod_{i=1}^{3} f(\alpha_i, \bar{\alpha}_i) a_{\ell_{23}}(\bar{\alpha}_3) a_{\ell_{13}}(\bar{\alpha}_1) a_{\ell_{12}}(\bar{\alpha}_2) \\ \times \mathcal{A}_{\ell_{13}}(\alpha_1 \cup \bar{\alpha}_3) \mathcal{A}_{\ell_{12}}(\alpha_2 \cup \bar{\alpha}_1) \mathcal{A}_{\ell_{23}}(\alpha_3 \cup \bar{\alpha}_2). \quad \text{(B.9)}$$

The full structure constant $\mathcal{G}$ is z_{ij} independent if the roots satisfy the Bethe equations. Using the sum over partitions expression for the $\mathcal{A}$-functional, the full summand is just

$$\tilde{\mathcal{D}} = (-1)^{|\bar{\alpha}_1|+|\bar{\alpha}_2|+|\bar{\alpha}_3|} \sum_{\substack{\beta_1\cup\bar{\beta}_1=\bar{\alpha}_1\cup\alpha_2 \\ \beta_2\cup\bar{\beta}_2=\bar{\alpha}_2\cup\alpha_3 \\ \beta_3\cup\bar{\beta}_3=\bar{\alpha}_3\cup\alpha_1}} (-1)^{|\beta_1|+|\beta_2|+|\beta_3|} \frac{a_{\ell_{12}}(\alpha_2)a_{\ell_{23}}(\alpha_3)a_{\ell_{31}}(\alpha_1)}{a_{\ell_{12}}(\beta_1)a_{\ell_{23}}(\beta_2)a_{\ell_{31}}(\beta_3)}$$
$$\times f(\bar{\alpha}_1,\alpha_1)f(\bar{\alpha}_2,\alpha_2)f(\bar{\alpha}_3,\alpha_3)f(\beta_1,\bar{\beta}_1)f(\beta_2,\bar{\beta}_2)f(\beta_3,\bar{\beta}_3). \quad \text{(B.10)}$$

Note that:

$$\frac{a_{\ell_{12}}(\alpha_2)a_{\ell_{23}}(\alpha_3)a_{\ell_{31}}(\alpha_1)}{a_{\ell_{12}}(\beta_1)a_{\ell_{23}}(\beta_2)a_{\ell_{31}}(\beta_3)} = \frac{a_{\ell_{12}}(\bar{\beta}_1\cap\alpha_2)a_{\ell_{23}}(\bar{\beta}_2\cap\alpha_3)a_{\ell_{31}}(\bar{\beta}_3\cap\alpha_1)}{a_{\ell_{12}}(\beta_1\cap\bar{\alpha}_1)a_{\ell_{23}}(\beta_2\cap\bar{\alpha}_2)a_{\ell_{31}}(\beta_3\cap\bar{\alpha}_3)}. \quad \text{(B.11)}$$

Clearly $\beta_j \cap \bar{\alpha}_j \subset \mathbf{u}_j$ then one can use the Bethe equations to invert $a_\ell(x)$ in the denominator. In our notation the Bethe equations are simply

$$a_{\ell_j}(x) = S(\{x\}, \mathbf{u}_j). \quad \text{(B.12)}$$

In the end we obtain the final form

$$\tilde{\mathcal{D}}(\alpha_j, \bar{\alpha}_j) = (-1)^{|\bar{\alpha}_1|+|\bar{\alpha}_2|+|\bar{\alpha}_3|} \sum_{\substack{\beta_1\cup\bar{\beta}_1=\bar{\alpha}_1\cup\alpha_2 \\ \beta_2\cup\bar{\beta}_2=\bar{\alpha}_2\cup\alpha_3 \\ \beta_3\cup\bar{\beta}_3=\bar{\alpha}_3\cup\alpha_1}} (-1)^{|\beta_1|+|\beta_2|+|\beta_3|}$$
$$\times a_{\ell_{12}}(\bar{\beta}_1\cap\alpha_2)a_{\ell_{23}}(\bar{\beta}_2\cap\alpha_3)a_{\ell_{31}}(\bar{\beta}_3\cap\alpha_1)a_{\ell_{31}}(\beta_1\cap\bar{\alpha}_1)a_{\ell_{12}}(\beta_2\cap\bar{\alpha}_2)a_{\ell_{23}}(\beta_3\cap\bar{\alpha}_3)$$
$$\times S\left(\mathbf{u}_1, \beta_1\cap\bar{\alpha}_1\right) S\left(\mathbf{u}_2, \beta_2\cap\bar{\alpha}_2\right) S\left(\mathbf{u}_3, \beta_3\cap\bar{\alpha}_3\right)$$
$$\times f(\bar{\alpha}_1,\alpha_1)f(\bar{\alpha}_2,\alpha_2)f(\bar{\alpha}_3,\alpha_3)f(\beta_1,\bar{\beta}_1)f(\beta_2,\bar{\beta}_2)f(\beta_3,\bar{\beta}_3). \quad \text{(B.13)}$$

Similar manipulations were made in [4] to check that $\mathcal{G}$ is independent of the polarizations. However they only tested it for one excitation on each operator. Here we work for a general number of them and establish the above relation.

Appendix C

Simplifying hexagon recursion relations

Recently in [7] some recursion relations were found for spinning hexagons. In this appendix we will rewrite them in a more appropriate form for us. Let $\mathcal{Z}(\mathbf{u},\mathbf{v},\mathbf{w})$ denote the hexagon partition function with three excited operators with J_i excitations each one. The recursion relations found [7] take the following form:

$$\text{Res}_{v_j\to u_i}\mathcal{Z}(\mathbf{u},\mathbf{v},\mathbf{w}) = i\langle L_1,R_2\rangle\langle L_2,R_1\rangle \prod_{i'=1}^{i-1}\frac{f(u_i,u_{i'})}{f(u_{i'},u_i)}\prod_{j'=j+1}^{J_2}\frac{f(v_{j'},u_i)}{f(u_i,v_{j'})}\times \\ \times\,\mathcal{Z}(\mathbf{u}/\{u_i\},\mathbf{v}/\{v_j\},\mathbf{w}), \quad \text{(C.1)}$$

$$\lim_{u_i\to\infty}\mathcal{Z}(\mathbf{u},\mathbf{v},\mathbf{w}) = (-1)^{J_1+J_2+J_3+1}\langle L_1,R_1\rangle\times\mathcal{Z}(\mathbf{u}/\{u_i\},\mathbf{v},\mathbf{w}). \quad \text{(C.2)}$$

It turns out we can greatly simplify these recursion relations by redefining the hexagon partition functions as

$$\mathcal{H}(\mathbf{u},\mathbf{v},\mathbf{w}) = \mathcal{Z}(\mathbf{u},\mathbf{v},\mathbf{w})\times\frac{f_<(\mathbf{u}_1)f_<(\mathbf{u}_2)f_<(\mathbf{u}_3)}{V_1^{J_1}V_2^{J_2}V_3^{J_3}}\times(-1)^{\frac{(J_1+J_2+J_3)(J_1+J_2+J_3-1)}{2}}, \quad \text{(C.3)}$$

where

$$f_<(\mathbf{u}) = \prod_{i<j}\left(\frac{u_i-u_j-i}{u_i-u_j}\right). \quad \text{(C.4)}$$

The first recursion relation for $\mathcal{H}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ then become

$$\frac{1}{f_<(\mathbf{u}_1) f_<(\mathbf{u}_2/\{v_j\}) f_<(\mathbf{u}_3)} \prod_{j'=1}^{j-1} \frac{1}{f(v_{j'}, u_i)} \prod_{j'=j+1}^{J_2} \frac{1}{f(u_i, v_{j'})} \times \mathrm{Res}_{v_j \to u_i} \mathcal{H}(\mathbf{u}, \mathbf{v}, \mathbf{w}) =$$
$$- i t_{12} \prod_{i'=1}^{i-1} \frac{f(u_i, u_{i'})}{f(u_{i'}, u_i)} \prod_{j'=j+1}^{J_2} \frac{f(v_{j'}, u_i)}{f(u_i, v_{j'})} \times \frac{\mathcal{H}(\mathbf{u}/\{u_i\}, \mathbf{v}/\{v_j\}, \mathbf{w})}{f_<(\mathbf{u}_1/\{u_i\}) f_<(\mathbf{u}_2/\{v_j\}) f_<(\mathbf{u}_3)}, \tag{C.5}$$

which, after simplifications, it becomes (3.52):

$$\mathrm{Res}_{v_j \to u_i} \mathcal{H}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -i t_{12} f(u_i, \mathbf{u}_1/\{u_i\}) f(\mathbf{u}_2/\{v_j\}, u_i) \mathcal{H}(\mathbf{u}/\{u_i\}, \mathbf{v}/\{v_j\}, \mathbf{w}). \tag{C.6}$$

And the second recursion relation becomes

$$\lim_{u_i \to \infty} \mathcal{H}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathcal{H}(\mathbf{u}/\{u_i\}, \mathbf{v}, \mathbf{w}), \tag{C.7}$$

Which is exactly (3.29) in Chapter 3. The remaining recursion relations are obtained with similar manipulations.

Appendix D

Redefining the sum over partitions

In this appendix we rewrite the sum over partitions (3.33) to reproduce the hexagon formula (3.43). As done in the Chapter 3 we define the partitions $\{\gamma_j, \bar{\gamma}_j\}$ such that $\bar{\beta}_i \cap \alpha_j = \bar{\gamma}_j$ and $\beta_i \cap \bar{\alpha}_i = \gamma_i$ where $\gamma_i \cup \bar{\gamma}_i = \delta_i$ and we partition each operator's roots as $\delta_i \cup \bar{\delta}_i = \mathbf{u}_i$. This is defined such that the all the terms containing a_ℓ's in (B.13) are joined. Then the partitions $\{\beta_j, \bar{\beta}_j\}$ are given by

$$\beta_i = \alpha_j / \bar{\gamma}_j \cup \gamma_i \text{ and } \bar{\beta}_i = \bar{\alpha}_i / \gamma_i \cup \bar{\gamma}_j. \tag{D.1}$$

Applying this at $\tilde{\mathcal{D}}$ results in

$$\begin{aligned} \tilde{\mathcal{D}} = (-1)^{|\bar{\delta}_1|+|\bar{\delta}_2|+|\bar{\delta}_3|} &\times a_{\ell_{31}}(\delta_1) a_{\ell_{12}}(\delta_2) a_{\ell_{23}}(\delta_3) \\ &\times \frac{f(\bar{\gamma}_1, \gamma_1)}{f(\gamma_1, \bar{\gamma}_1)} \frac{f(\bar{\gamma}_2, \gamma_2)}{f(\gamma_2, \bar{\gamma}_2)} \frac{f(\bar{\gamma}_3, \gamma_3)}{f(\gamma_3, \bar{\gamma}_3)} \frac{f(\bar{\delta}_1, \gamma_1)}{f(\gamma_1, \bar{\delta}_1)} \frac{f(\bar{\delta}_2, \gamma_2)}{f(\gamma_2, \bar{\delta}_2)} \frac{f(\bar{\delta}_3, \gamma_3)}{f(\gamma_3, \bar{\delta}_3)} \\ &\times f(\bar{\alpha}_1/\gamma_1, \alpha_1) f(\gamma_1, \alpha_1) f(\bar{\alpha}_2/\gamma_2, \alpha_2) f(\gamma_2, \alpha_2) f(\bar{\alpha}_3/\gamma_3, \alpha_3) f(\gamma_3, \alpha_3) \\ &\times f(\alpha_2/\bar{\gamma}_2, \bar{\alpha}_1/\gamma_1) f(\gamma_1, \bar{\alpha}_1/\gamma_1) f(\alpha_2/\bar{\gamma}_2, \bar{\gamma}_2) f(\gamma_1, \bar{\gamma}_2) \\ &\times f(\alpha_3/\bar{\gamma}_3, \bar{\alpha}_2/\gamma_2) f(\gamma_2, \bar{\alpha}_2/\gamma_2) f(\alpha_3/\bar{\gamma}_3, \bar{\gamma}_3) f(\gamma_2, \bar{\gamma}_3) \\ &\times f(\alpha_1/\bar{\gamma}_1, \bar{\alpha}_3/\gamma_3) f(\gamma_3, \bar{\alpha}_3/\gamma_3) f(\alpha_1/\bar{\gamma}_1, \bar{\gamma}_1) f(\gamma_3, \bar{\gamma}_1). \end{aligned} \tag{D.2}$$

After some algebraic manipulations one can see that is useful to define partitions for the set $\bar{\delta}_i = \omega_i \cup \bar{\omega}_i$ such that $\alpha_i = \bar{\omega}_i \cup \bar{\gamma}_i$ and $\bar{\alpha}_i = \omega_i \cup \gamma_i$. This yields us the following simple expression

$$\begin{aligned}\tilde{\mathcal{D}} = (-1)^{|\bar\delta_1|+|\bar\delta_2|+|\bar\delta_3|} \times a_{\ell_{31}}(\delta_1)a_{\ell_{12}}(\delta_2)a_{\ell_{23}}(\delta_3) \times f(\bar\delta_1,\delta_1)f(\bar\delta_2,\delta_2)f(\bar\delta_3,\delta_3)\\ \times f(\omega_1,\bar\omega_1)f(\bar\omega_1,\omega_3)f(\omega_3,\bar\omega_3)f(\bar\omega_3,\omega_2)f(\omega_2,\bar\omega_2)f(\bar\omega_2,\omega_1)\\ \times f(\bar\gamma_1,\gamma_1)f(\gamma_1,\bar\gamma_2)f(\bar\gamma_2,\gamma_2)f(\gamma_2,\bar\gamma_3)f(\bar\gamma_3,\gamma_3)f(\gamma_3,\bar\gamma_1). \quad \text{(D.3)}\end{aligned}$$

Which has the same structure as the one in $\mathcal{H}$. Now the only remaining factors in the sum over partitions to deal with are the polarizations. For these we can do the following split:

$$z_{12}^{M_{12}-|\alpha_2|-|\bar\alpha_1|} = z_{12}^{|\bar\delta_1|+|\bar\delta_2|-|\bar\delta_3|-|\bar\omega_2|-|\omega_1|} z_{12}^{|\delta_1|+|\delta_2|-|\delta_3|-|\bar\gamma_2|-|\gamma_1|}, \quad \text{(D.4)}$$

$$z_{23}^{M_{23}-|\alpha_3|-|\bar\alpha_2|} = z_{23}^{|\bar\delta_2|+|\bar\delta_3|-|\bar\delta_1|-|\bar\omega_3|-|\omega_2|} z_{12}^{|\delta_2|+|\delta_3|-|\delta_1|-|\bar\gamma_3|-|\gamma_2|}, \quad \text{(D.5)}$$

$$z_{31}^{M_{31}-|\alpha_1|-|\bar\alpha_3|} = z_{31}^{|\bar\delta_3|+|\bar\delta_1|-|\bar\delta_2|-|\bar\omega_1|-|\omega_3|} z_{31}^{|\delta_3|+|\delta_1|-|\delta_2|-|\bar\gamma_1|-|\gamma_3|}. \quad \text{(D.6)}$$

Thus each one can be absorbed in the sum over partitions $\gamma_i \cup \bar\gamma_i = \delta_i$ and $\omega_i \cup \bar\omega_i = \bar\delta_i$ to obtain exactly:

$$\begin{aligned}\mathcal{G}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3) = \sum_{\substack{\delta_1\cup\bar\delta_1=\mathbf{u}_1\\ \delta_2\cup\bar\delta_2=\mathbf{u}_2\\ \delta_3\cup\bar\delta_3=\mathbf{u}_3}} (-1)^{|\delta_1|+|\delta_2|+|\delta_3|} f(\bar\delta_1,\delta_1)f(\bar\delta_2,\delta_2)f(\bar\delta_3,\delta_3)\\ \times a_{\ell_{31}}(\delta_1)a_{\ell_{12}}(\delta_2)a_{\ell_{23}}(\delta_3)\mathcal{H}(\delta_1,\delta_3,\delta_2)\mathcal{H}(\bar\delta_1,\bar\delta_2,\bar\delta_3). \quad \text{(D.7)}\end{aligned}$$

With $\mathcal{H}$ given by (3.40). This establishes the equality between double spin-chain formalism and hexagonalization.

Appendix E

Contour manipulations in the classical limit

In this appendix we show how to find the final form of the classical limit of type I-I-I correlators (4.28). The first step is compute the classical limit by saddle-point of path integrals as in [29]. We can substitute each sum in (4.27) by path integrals of the form

$$\sum_{\alpha_i \cup \bar{\alpha}_i = \mathbf{u}_i} (-1)^{|\alpha_i|} a_{\ell_{ki}}(\alpha_i) \frac{f(\bar{\alpha}_i, \alpha_i)}{f(\alpha_i, \mathbf{u}_k) f(\mathbf{u}_j, \alpha_i)} \to \int [D\rho_{\alpha_i}] \exp\left(\int_{\mathcal{C}_i} du \; \mathcal{F}[\rho_{\alpha_i}] + i \rho_{\alpha_i} \not{q}_i \right). \tag{E.1}$$

Where $\mathcal{F}[\rho_{\alpha_i}]$ is the stochastic anomaly term, given by

$$\mathcal{F}[\rho_{\alpha_i}] = F(\rho_i) - F(\rho_{\alpha_i}) - F(\rho_{\bar{\alpha}_i}) \quad \text{with} \quad F(\rho) = \int_0^\rho dx \; \log \sinh(\pi x), \tag{E.2}$$

and $i\rho_{\alpha_i} \not{q}_i$ is the classical limit of the summand. Here, q_i is a function of the quasimomenta given by

$$q_i = \pi + p_j - p_k - p_i, \tag{E.3}$$

and $\not{q}_i$ denotes the average of q_i evaluated above and below the cut $\mathcal{C}_i$.

Then one does these path integrals by saddle-point and finds

$$\log \frac{\mathcal{G}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)}{\mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)} = - \sum_{\{i,j,k\} = \text{cperm}\{1,2,3\}} \oint_{\mathcal{C}_i} \frac{du}{2\pi} \, \text{Li}_2\left(e^{-ip_k - ip_i + ip_j} \right). \tag{E.4}$$

To find the same as in [13] we have to join the contours $\mathcal{C}_i$. To do this we use Bethe equations in the classical limit and contour manipulations. The Bethe equations in the classical limit are given by [22]

$$p_j(u+i0)+p_j(u-i0)=2\pi n_j^{(k)} \quad \text{for} \quad u\in\mathcal{C}_j^{(k)}. \tag{E.5}$$

Where $\cup_k \mathcal{C}_j^{(k)}=\mathcal{C}_j$ and $n_j^{(k)}\in\mathbb{Z}$. The Bethe equations tell us that $p(u)$ is defined in a two sheet Riemann surface. Then when one crosses a rapidity cut we simply do the changes $p(u)\to -p(u)$ and $\mathcal{C}_j\to-\mathcal{C}_j$.

Consider the first term in (E.4). We start by splitting it as

$$\begin{aligned}\oint_{\mathcal{C}_1}\frac{du}{2\pi}\,\mathrm{Li}_2\left(e^{-ip_3-ip_1+ip_2}\right)=&-\frac{1}{2}\oint_{\mathcal{C}_1}\frac{du}{2\pi}\,\mathrm{Li}_2\left(e^{ip_1+ip_2-ip_3}\right)\\&+\frac{1}{2}\oint_{\mathcal{C}_1}\frac{du}{2\pi}\,\mathrm{Li}_2\left(e^{-ip_3-ip_1+ip_2}\right),\end{aligned} \tag{E.6}$$

where in the first term we used the Bethe equations to go to the second sheet, thus changing the sign of p_1. We consider now the following dilogarithm identity

$$\mathrm{Li}_2(x)=-\mathrm{Li}_2(-1/x)-\frac{\pi^2}{6}-\frac{\log^2(1/x)}{2}. \tag{E.7}$$

Which can be used to invert p_j in the second term thus yielding

$$\begin{aligned}\oint_{\mathcal{C}_1}\frac{du}{2\pi}\,\mathrm{Li}_2\left(e^{-ip_3-ip_1+ip_2}\right)=&-\frac{1}{4}\oint_{\mathcal{C}_1}\frac{du}{2\pi}\,\log^2\left(-e^{ip_3+ip_1-ip_2}\right)\\&-\frac{1}{2}\oint_{\mathcal{C}_1}\frac{du}{2\pi}\,\mathrm{Li}_2\left(e^{ip_1+ip_2-ip_3}\right)-\frac{1}{2}\oint_{\mathcal{C}_1}\frac{du}{2\pi}\,\mathrm{Li}_2\left(e^{ip_1+ip_3-ip_2}\right)\end{aligned} \tag{E.8}$$

Similar manipulations can be done in the remaining terms in (E.4) to obtain

$$\log\frac{\mathcal{G}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3)}{\mathcal{H}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3)}=\frac{1}{2}\sum_{\{i,j,k\}=\mathrm{cperm}\{1,2,3\}}\left(\oint_{\mathcal{C}_i\cup\mathcal{C}_j}\frac{dz}{2\pi}\,\mathrm{Li}_2\left(e^{ip_i+ip_j-ip_k}\right)\right)+\texttt{rest}. \tag{E.9}$$

Where `rest` is simply given by

$$\texttt{rest}=\sum_{\{i,j,k\}=\mathrm{cperm}\{1,2,3\}}\frac{1}{4}\oint_{\mathcal{C}_i}\frac{du}{2\pi}\,\log^2\left(-e^{ip_i-ip_j+ip_k}\right). \tag{E.10}$$

After some painstaking, but simple, manipulations one sees that $\log\mathcal{H}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3)$ indeed cancels with `rest` up to phases that can be absorbed by redefining the operators. Therefore the classical limit (E.4) is indeed equivalent to the classical limit (4.28) given in the main text.

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