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**APPLICATION OF VOLTERRA SERIES IN NONLINEAR
MECHANICAL SYSTEM IDENTIFICATION AND IN STRUCTURAL
HEALTH MONITORING PROBLEMS**

Ilha Solteira
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SIDNEY BRUCE SHIKI

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MECHANICAL SYSTEM IDENTIFICATION AND IN STRUCTURAL
HEALTH MONITORING PROBLEMS

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Knowledge area: Solid Mechanics.

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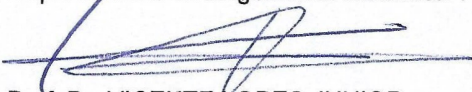
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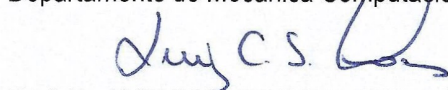
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To my family.

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But see that the imagination of nature is far, far greater than the imagination of man. No one who did not have some inkling of this through observations could ever have imagined such a marvel as nature is.

The meaning of it all - Richard P. Feynman

Abstract

Nonlinear structures are frequent in structural dynamics, specially considering screwed components, with joints, clearance or flexible components presenting large displacements. In this sense the monitoring of systems based on classical linear methods, as the ones based on modal parameters, can drastically fail to characterize nonlinear effects. This thesis proposed the use of Volterra series for nonlinear system identification aiming applications in damage detection and parameter quantification. The property of this model of representing the linear and nonlinear components of the response of a system was used to formulate damage features to make clear the need of nonlinear modeling. Also metrics based on the linear and nonlinear residues of the terms of the Volterra model were employed to identify parametric models of the structure. The proposed methodologies are illustrated in experimental setups to show the relevance of nonlinear phenomena in the structural health monitoring.

Keywords: Nonlinear structures. Structural health monitoring. Volterra models. Kautz filters. Nonlinear model updating.

Resumo

Estruturas com comportamento não-linear são frequentes em dinâmica estrutural, principalmente considerando componentes parafusados, com juntas, folgas ou estruturas flexíveis sujeitas à grandes deslocamentos. Desse modo, o monitoramento de estruturas com métodos lineares clássicos, como os baseados em parâmetros modais, podem falhar drasticamente em caracterizar efeitos não-lineares. Neste trabalho foi proposta a utilização de séries de Volterra para identificação de sistemas mecânicos não-lineares em aplicações de detecção de danos e quantificação de parâmetros. A propriedade deste modelo de representar separadamente os componentes de resposta linear e não-linear do sistema foi aplicada para se construir índices de dano que evidenciam a necessidade de modelagem não-linear. Além disso métricas de resíduo linear e não-linear dos termos do modelo de Volterra são empregadas para identificar modelos paramétricos da estrutura. As metodologias propostas são ilustradas em bancadas experimentais de modo a evidenciar a importância de fenômenos não-lineares para o monitoramento de estruturas.

Palavras-chave: Estruturas não lineares. Monitoramento de integridade estrutural. Modelos de Volterra. Filtros de Kautz. Ajuste de modelos não lineares.

List of Symbols

a_1	-	Parameter of the β function
a_2	-	Parameter of the β function
b_η	-	Parameter of the Kautz orthonormal function
c	-	Damping coefficient
d_η	-	Parameter of the Kautz orthonormal function
e_η	-	Prediction error of order η
$e_{\eta,ref}$	-	Prediction error of order η in the reference condition
$e_{\eta,unk}$	-	Prediction error of order η in the unknown condition
f	-	Input force
H_0	-	Null hypothesis
H_1	-	Alternative hypothesis
h_1	-	First-order Volterra kernel in the continuous-time formulation
h_2	-	Second-order Volterra kernel in the continuous-time formulation
h_3	-	Third-order Volterra kernel in the continuous-time formulation
h_η	-	η -order Volterra kernel in the continuous-time formulation
J_1	-	Number of orthonormal functions to represent the first Volterra kernel
J_2	-	Number of orthonormal functions to represent the second Volterra kernel
J_3	-	Number of orthonormal functions to represent the third Volterra kernel
J_η	-	Number of orthonormal functions to represent the η -th Volterra kernel
k	-	Sample number for the discrete-time formulation in the Volterra series
k_1	-	Linear stiffness
k_2	-	Quadratic stiffness
k_3	-	Cubic stiffness
l_{η,i_j}	-	Input signal filtered by the orthonormal function
m	-	Mass coefficient
N_1	-	Memory length of the first Volterra kernel in the discrete-time formulation
N_2	-	Memory length of the second Volterra kernel in the discrete-time formulation
N_3	-	Memory length of the third Volterra kernel in the discrete-time formulation

N_η	-	Memory length of the η -th Volterra kernel in the discrete-time formulation
n_1	-	First-order sample delay in the discrete-time formulation of Volterra series
n_2	-	Second-order sample delay in the discrete-time formulation of Volterra series
n_3	-	Third-order sample delay in the discrete-time formulation of Volterra series
n_η	-	η -order sample delay in the discrete-time formulation of Volterra series
P	-	Probability density function of the λ_η damage index
$\mathbf{p}$	-	Vector with the parameters to be found
p	-	Probability value of the distribution (p -value)
T	-	transpose of a matrix
t	-	Continuous time variable
u	-	Input signal applied in the Volterra series
V	-	maximum number of used orthonormal filters
v_1	-	Number of degrees of freedom for the prediction error in the unknown condition
v_2	-	Number of degrees of freedom for the prediction error in the reference condition
x	-	Displacement of the Duffing oscillator
$\dot{x}$	-	Velocity of the Duffing oscillator
$\ddot{x}$	-	Acceleration of the Duffing oscillator
$\mathbf{y}$	-	Vector with the total output of the Volterra series
$\mathbf{y}_1$	-	Vector with the first-order output of the Volterra series
$\mathbf{y}_2$	-	Vector with the second-order output of the Volterra series
$\mathbf{y}_3$	-	Vector with the third-order output of the Volterra series
y	-	Total output of the Volterra series
y_{exp}	-	Experimentally measured response
y_1	-	First-order output of the Volterra series
y_2	-	Second-order output of the Volterra series
y_3	-	Third-order output of the Volterra series
y_η	-	η -order output of the Volterra series
z	-	complex variable in the discrete frequency domain

$\mathcal{B}_1$	-	First-order Volterra kernel in the orthonormal basis
$\mathcal{B}_{1,ref}$	-	Reference first-order Volterra kernel in the orthonormal basis
$\mathcal{B}_2$	-	Second-order Volterra kernel in the orthonormal basis
$\mathcal{B}_{2,ref}$	-	Reference second-order Volterra kernel in the orthonormal basis
$\mathcal{B}_3$	-	Third-order Volterra kernel in the orthonormal basis
$\mathcal{B}_{3,ref}$	-	Reference third-order Volterra kernel in the orthonormal basis
$\mathcal{B}_\eta$	-	η -order Volterra kernel in the orthonormal basis
$\mathcal{C}_{lin}$	-	Linear objective function for parameter identification
$\mathcal{C}_{nlin}$	-	Nonlinear objective function for parameter identification
$\mathcal{H}_1$	-	First-order Volterra kernel in the discrete-time formulation
$\mathcal{H}_2$	-	Second-order Volterra kernel in the discrete-time formulation
$\mathcal{H}_3$	-	Third-order Volterra kernel in the discrete-time formulation
$\mathcal{H}_\eta$	-	η -order Volterra kernel in the discrete-time formulation
$\mathcal{S}_\eta$	-	Pole representing the Kautz functions in the continuous frequency domain
$\mathcal{Z}_\eta$	-	Pole representing the Kautz functions in the discrete frequency domain
$\bar{\mathcal{Z}}_\eta$	-	Conjugate of the pole representing the Kautz functions in the discrete frequency domain

Greek letters

α	-	Significance value of the hypothesis test
$\mathbf{\Gamma}$	-	Matrix with combinations of the input signal
$\mathbf{\Gamma}_1$	-	Matrix with combinations of the input signal for the first-order output
$\mathbf{\Gamma}_2$	-	Matrix with combinations of the input signal for the second-order output
$\mathbf{\Gamma}_3$	-	Matrix with combinations of the input signal for the third-order output
ζ_1	-	Damping ratio representing the first Volterra kernel in the orthonormal basis
ζ_2	-	Damping ratio representing the second Volterra kernel in the orthonormal basis
ζ_3	-	Damping ratio representing the third Volterra kernel in the orthonormal basis

η	-	Order of the Volterra series expansion
λ_1	-	Damage index of first-order
λ_2	-	Damage index of second-order
λ_3	-	Damage index of third-order
λ_η	-	Damage index of order η
ξ	-	Damping ratio of the Duffing oscillator
σ	-	Standard deviation operator
τ_1	-	Time delay for the first Volterra kernel in the continuous-time formulation
τ_2	-	Time delay for the second Volterra kernel in the continuous-time formulation
τ_3	-	Time delay for the third Volterra kernel in the continuous-time formulation
τ_η	-	Time delay for the η -th Volterra kernel in the continuous-time formulation
Φ	-	Vector with the considered Volterra kernels
Φ_1	-	Vector with the first Volterra kernel
Φ_2	-	Vector with the second Volterra kernel
Φ_3	-	Vector with the third Volterra kernel
ψ	-	Orthonormal function in the discrete-time domain
Ψ	-	Orthonormal function in the discrete frequency domain
ω_1	-	Frequency representing the first Volterra kernel in the orthonormal basis
ω_2	-	Frequency representing the second Volterra kernel in the orthonormal basis
ω_3	-	Frequency representing the third Volterra kernel in the orthonormal basis

List of abbreviations

CDF	-	Cumulative distribution function
FRF	-	Frequency response function
GVT	-	Ground vibration testing
HBM	-	Harmonic balance method
HOFRF	-	Higher-order frequency response function
LANL	-	Los Alamos National Laboratory
MIMO	-	Multiple inputs/multiple outputs
NOFRF	-	Nonlinear output frequency response function
PDF	-	Probability density function
PSD	-	Power spectral density
RMS	-	Root mean square
ROC	-	Receiver operating characteristic
SHM	-	Structural health monitoring
SIMO	-	Single input/multiple outputs
SISO	-	Single input/single output
SQP	-	Sequential quadratic programming

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1 Introduction

This chapter presents a brief introduction to the subject of the thesis. A motivation of the research is presented in the section 1.1. The main contributions of this thesis and the objectives of the research are presented in the sections 1.2 and 1.3, respectively. Finally, section 1.4 shows the outline of the chapters and the respective content.

1.1 Motivation

The relevance of considering nonlinear phenomena in the dynamics of structures and machines increased due to the need of better performance in terms of comfort, weight reduction, noise among other requirements (KERSCHEN et al., 2006). Nonlinear behavior can bring complex effects as jumps, limit cycle oscillations, harmonic distortion, chaos, among others even in simple academic examples as the ones described in the book of Virgin (2000). These effects can be a problem for the use of traditional techniques fundamentally based on the linear superposition principle (WORDEN; TOMLINSON, 2001) or can cause harm to the systems due to the sudden appearance of unexpected change in the structure vibration as is the case of the jump phenomenon (BRENNAN et al., 2008). On the other hand it is also possible to make use of nonlinear properties in the improvement of mechanical design. A few examples are in the use of nonlinearities for vibration isolators (KOVACIC et al., 2008) and energy harvesters (ERTURK; INMAN, 2011) aiming to improve the performance of these devices. Those cases shows both examples where one would prefer to avoid nonlinear effects and situations where these responses can improve the development of new devices.

Considering the importance of nonlinearities in the dynamical behavior of mechanical systems this topic is highly relevant for the area of structural health monitoring (SHM). This area is concerned with the process of implementing a damage identification strategy for aerospace, civil and mechanical engineering infrastructure (FARRAR; WORDEN, 2007). In the classification proposed by Rytter (1993) the damage state of a system can

be fully described by the following actions:

1. Identification of the presence of damage in the structure;
2. Determination of the geometric location of the damage;
3. To establish the kind of damage present;
4. Quantification of the severity of the damage;
5. Prediction of the remaining useful life of the system.

The nonlinear behavior of structures can be important in all the five steps for the damage identification process since it can change significantly the expected results in these systems. For this reason a big concern in the SHM literature is to take in account nonlinear effects in order to properly identify the damage state of a structure. In this way the nonlinear behavior in mechanical systems is treated by two main approaches (NICHOLS; TODD, 2009) where the analyst must handle with this behavior (or else avoid it):

- First approach: nonlinearity is considered to be caused by the damage in the system;
- Second approach: the systems is inherently nonlinear even before the occurrence of the damage.

In the first approach the damage is directly related to the nonlinear behavior and the structure is considered to be linear before its occurrence. This is an appealing strategy because typical fault modes like cracks, impacts, delamination and rubbing of rotating machinery induce nonlinearity in the response of the system (NICHOLS; TODD, 2009). This situation allows to detect structural changes by identifying and characterizing the presence of nonlinear behavior in the response of the system. For this case many well-established nonlinear tools as coherence functions, Hilbert transform, higher-order spectra and phase-space methods can be employed (CAMBRAIA, 1990; OVERBEY; TODD, 2007; WORDEN et al., 2008). Many recent papers in the SHM literature still use this kind of approach with good results considering damage features based on modal properties (BANDARA et al., 2014), wave propagation (YELVE et al., 2014) and other signal processing tools (GHAZI; BÜYÜKÖZTÜRK, 2015).

In the second approach, the system has to be considered nonlinear before the occurrence of the damage because the nonlinearity is intrinsic to the regime of operation of the

system (BORNN et al., 2010). This case demands to properly model the baseline behavior of the structure in a way that the inherent nonlinear effects of the system are not mistakenly viewed as due to a damage. In this situation nonlinear system identification and signal processing techniques can be used to take in account nonlinear effects before the structural change. Techniques as restoring force surface method, nonlinear autoregressive models, Hilbert transform, principal component analysis, time-frequency analysis among others can be applied to identify such nonlinearity (KERSCHEN et al., 2006; WORDEN et al., 2008; BORNN et al., 2010; CHANPHENG et al., 2012; REYNDERS et al., 2013). However, there is still no general model to deal with nonlinear dynamics in a comprehensive way, and the application of mathematical tools in nonlinear problems tends to be case-specific (KERSCHEN et al., 2006).

In this thesis the focus will be given to problems where the system is nonlinear before the occurrence of the damage. The investigation will aim to present possible ways of applications of the Volterra series expansion in the identification of the presence of the damage and quantification of parameters. This model will be used in the single input/single output (SISO) formulation to represent nonlinear vibrating structures. The Volterra model is identified by minimizing the output error of the series expansion with relationship to the experimental response measured in the system. The properties of this model will be used to show the needs of considering nonlinear models when monitoring structures under the nonlinear regime of motion.

1.2 Main contributions of the thesis

The main contributions of the research presented in this thesis are:

- Proposition of a methodology for the identification of discrete Volterra kernels in vibratory systems using Kautz functions;
- Utilization of the Volterra model for the detection of structural changes and in the quantification of parameters;
- Comparison between the consequences of assuming or not a nonlinear damage index under the nonlinear regime of motion in the structure;
- Presentation of experimental examples trying to make clear the needs of taking in account nonlinear phenomena in SHM problems.

The results of this thesis also allowed a few publications in conference and journal papers. The appendix B summarize the publications submitted during the doctorate.

1.3 Objective

To develop methodologies using Volterra series for the damage detection and parameter quantification aiming applications in SHM of initially nonlinear systems.

1.4 Outline of the thesis

This thesis is organized in 6 chapters and two appendices with the following content:

- **Chapter 1 - Introduction:** shows the motivation and main objectives of the research presented in this thesis;
- **Chapter 2 - Identification of Volterra models:** describe the Volterra series expansion and the formulation of this model using orthonormal functions to represent the Volterra kernels. After that the procedure to make a least-squares estimation of the kernels is summarized and a simulated example of the identification of kernels is performed in a classical asymmetric Duffing oscillator;
- **Chapter 3 - Methodologies for the application in SHM:** introduces the reader to the proposed methodologies of application of the Volterra model as a tool for damage detection and parameter identification problems.
- **Chapter 4 - Identification of Volterra kernels in experimental examples:** the calculation of the Volterra kernels representing two experimental setup is presented in this chapter. The first example is illustrated by a magneto-elastic system with hardening behavior. The second example is a F-16 aircraft tested during a ground vibration testing which showed softening effects in the frequency domain. With the Volterra models representing these systems it is then possible to analyze the components of the Volterra model to show the linear and nonlinear contributions.
- **Chapter 5 - Experimental application of Volterra series for SHM:** in this chapter the use of the discrete-time Volterra model is exemplified in problems of damage detection and parameter identification. The need of considering nonlinear phenomena in SHM applications is emphasized in this part of the thesis.

- **Chapter 6 - Final remarks:** this chapter presents the main final conclusions about the present thesis with proposals for further development in this research area.
- **Appendix A - Details on the identification of Volterra kernels:** the appendix is a complement of the chapter 2 in order to have a better understanding of the Volterra series and the practical identification of the kernels.
- **Appendix B - Publications of the author during the doctorate studies:** this appendix shows the publications of the authors in the subject of this PhD thesis.

2 Identification of Volterra models

This chapter brings the essential information to understand the methodology for the identification of a Volterra model. Additional explanation of this formulation is given in the appendix A. The section 2.1 starts with an overview of the literature available in the subject of nonlinear system identification, specially in the applications with Volterra series in mechanical systems. The Volterra series formulation is presented in the section 2.2. Sections 2.3 and 2.4 shows the method for the calculation of the kernels of the Volterra model and the orthonormal representation of the kernels respectively. In the section 2.5 the methodology is illustrated in a simple Duffing oscillator. Finally the conclusions about the content of the chapter are presented in section 2.6.

2.1 Overview of the literature in nonlinear system identification with Volterra series

The development of models to represent nonlinear systems is still a major topic in the literature. In the area of structural dynamics techniques as the restoring force surface method (WORDEN; TOMLINSON, 2001), nonlinear autoregressive models (FASSOIS; SAKELLARIOU, 2007), nonlinear normal modes (PEETERS et al., 2011), subspace methods (NOËL; KERSCHEN, 2013), and others are often applied with promising results. Despite all the efforts on this subject there is still no general tool to deal with nonlinearities and more development is still needed in this area of research.

Among the existing techniques, Volterra series is an interesting approach because it is a clear generalization of the linear convolution model (SCHETZEN, 1980). Many researchers already applied this kind of model to represent nonlinear behavior in practical applications. Silva (2005) studied the identification of nonlinear aeroelastic systems by identifying Volterra models using input and output signals of the aileron and the vibration signal measured in an active aeroelastic wing under a wind tunnel test. The author states that there is a great applicability of the Volterra series as a reduced order model for

aeroelastic systems instead of relying on computational fluid dynamics models.

Cafferty and Tomlinson (1997) identified analytic expressions of the Volterra kernels in automotive dampers with the probing method obtaining the higher-order frequency response functions (HOFRF) which are the frequency-domain versions of the kernels. The main diagonals of the HOFRFs were used to estimate the nonlinear parameters of the system. The Volterra model was also quite popular in applications for the modeling of biological systems (ZHANG et al., 1998; MARMARELIS, 2004) and nonlinear electrical devices (BJÖRSELL et al., 2010).

Many authors in the signal processing and system identification literature already mentioned the Volterra series expansion with strong skepticism. One example of this can be found in the book of Bendat and Piersol (2011) where the authors discuss about the utility of higher-order spectra and Volterra kernels: “[...] the desired optimum frequency response functions can be obtained by computing appropriate first-order spectral density functions, second-order bispectral density functions, and third-order trispectral density functions. The second-order and third-order functions are complicated to compute and difficult to interpret [...]”. The authors also made some interesting remarks about the usefulness of the Volterra model: “These Volterra models and techniques should not be used except as a last resort because they can often be replaced by the simpler practical SI/SO nonlinear models and techniques recommended here [...]”. Despite criticisms about the model, it is necessary to emphasize that more practical ways to calculate the Volterra kernels (as the methodology presented in this thesis) are available. Also, some efforts are being made in order to allow a generalization of concepts valid for linear systems to the nonlinear domain. Examples of this can be found in the papers of Tawfiq and Vinh (2003) and Tawfiq and Vinh (2004) which tried to extend the properties in the classical linear modal analysis to the case of nonlinear vibrating systems. In these papers, properties of the linear model are compared to features of the higher-order kernels trying to illustrate similarities, since the nonlinear terms are generalizations of the convolution integral.

Despite the big number of applications of the Volterra model, limitations related to the large number of parameters to represent the model and convergence issues are a frequent problem of this technique. Due to this, many authors tried to observe the limitations of the technique and in some cases proposing methods to minimize known problems. Peng et al. (2008) compared the performance of the harmonic balance method (HBM) and the method of nonlinear output frequency response function (NOFRF) based on Volterra series. The authors studied the capacity of prediction of both methods in a system with polynomial stiffness (Duffing oscillator) and other with polynomial damping. The harmonics of the

responses of the systems to sinusoidal excitation are separated and analyzed according to the variation of the frequency of the excitation signal. In the simulations with the Duffing oscillator, a better performance was found by using the HBM, but this method may have high computational cost and possible failures when higher harmonics are considered. The NOFRF, however, had a poor performance in frequencies next to the resonance of the system, and in this case it was not possible to reproduce the jump phenomenon.

Da Silva et al. (2010) applied the Volterra model in the identification of a cantilever beam with a local stiffness nonlinearity in the free end. The model was used in the form expanded in orthonormal Kautz functions in a way to substantially reduce the number of terms necessary to represent the kernels. The authors stated about the advantages of the Volterra model in separating linear and nonlinear components of the response of the system, and the possibility of the application of this technique in problems of nonlinear model updating.

Rosa et al. (2007) developed a sub-optimal method to find the poles of the orthonormal Kautz basis used to expand the Volterra kernels. The method is based in the minimization of the superior limit of the truncation error of the kernels. The results showed good approximations in a case study using a second order kernel with a oscillatory and exponential decaying behavior.

These papers showed that there is still a lot of efforts on studies aiming to find more robust models that can represent the complex nonlinear dynamics. The Volterra series has an interesting structure of the model but a few limitations of this tool are still a problem. In the next sections the formulation of discrete-time Volterra series is presented and one way to minimize problems of the model is illustrated using orthonormal functions to represent the Volterra kernels.

2.2 Volterra series

The Volterra series was originated in the investigation of the Italian mathematician Vito Volterra between the end of the XIX century and the beginning of the XX century. His research provided a functional expansion that represents the input/output relationship in a continuous and stable nonlinear system with finite memory (DOYLE et al., 2012). Later in the 1940s the American mathematician Norbert Wiener used the ideas conceived by Volterra in modeling and nonlinear system identification problems (MARMARELIS, 2004). The Volterra model is an attractive technique since it is a generalization of the

well-known linear model based on the impulse response function (SCHETZEN, 1980).

In the continuous time (t) formulation of the Volterra series, the response $y(t)$ of a nonlinear system can be described by an infinite sum of η -order responses written as $y_\eta(t)$ (SCHETZEN, 1980):

$$y(t) = \sum_{\eta=1}^{+\infty} y_\eta(t) = y_1(t) + y_2(t) + y_3(t) + \dots \quad (1)$$

where $y_1(t)$, $y_2(t)$ and $y_3(t)$ are the first, second and third order components of the response respectively. The main interesting property of this model is that $y_1(t)$ represents the linear portion of the output and the sum of higher-order terms $y_2(t) + y_3(t) + \dots$ represents the nonlinear response (RUGH, 1991). In this way the linear component is written as a classical convolution integral:

$$y_1(t) = \int_0^\infty h_1(\tau_1)u(t - \tau_1)d\tau_1 \quad (2)$$

where $h_1(\tau_1)$ is the linear impulse response function and $u(t - \tau_1)$ is the input signal with a time delay τ_1 . In this simple representation the response of the system is written as a function of the applied input signal $u(t)$ and the information regarding the system is in the term $h_1(\tau_1)$. In a similar way, the second order response $y_2(t)$ is written as a bidimensional convolution between the second-order impulse response function $h_2(\tau_1, \tau_2)$ and the input signal delayed in τ_1 and τ_2 :

$$y_2(t) = \int_0^\infty \int_0^\infty h_2(\tau_1, \tau_2)u(t - \tau_1)u(t - \tau_2)d\tau_1d\tau_2 \quad (3)$$

It is usual to name the η -order impulse response functions ($h_\eta(\tau_1, \dots, \tau_\eta)$) as the Volterra kernels of the system. In this representation the η -th response $y_\eta(t)$ is represented as multidimensional convolution integrals:

$$y_\eta(t) = \int_0^\infty \dots \int_0^\infty h_\eta(\tau_1, \dots, \tau_\eta)u(t - \tau_1) \dots u(t - \tau_\eta)d\tau_1 \dots d\tau_\eta \quad (4)$$

where $u(t - \tau_\eta)$ is the excitation applied in the system delayed in τ_η . It is clear that the equation 4 is a generalization of the linear convolution showed in equation 2 describing higher-order responses.

Three main considerations should be done about the model to apply it in nonlinear system modelling (EYKHOFF, 1974):

- $h_\eta(\tau_1, \dots, \tau_\eta)$ can be constructed in a symmetric way

- $h_\eta(\tau_1, \dots, \tau_\eta) = 0 \quad \forall \tau_\eta < 0$
- $\lim_{\tau_1, \dots, \tau_\eta \rightarrow \infty} h_\eta(\tau_1, \dots, \tau_\eta) = 0$

The first consideration allows one to consider equal the terms of $h_\eta(\tau_1, \dots, \tau_\eta)$ for all the permutations of $\tau_1, \dots, \tau_\eta$ (e.g. $h_2(\tau_1, \tau_2) = h_2(\tau_2, \tau_1)$). With this it is possible to calculate only the unique terms of the kernels instead of all terms, this will be better explored in the discrete-time representation of the model that will be later explained.

The second statement is to restrict the representation for causal systems with null initial conditions. The causality means in practice that the response of the system does not depend on future input samples (GEROMEL; PALHARES, 2005).

Finally the last statement is done in order to ensure the stability of the system described by the Volterra series. It is necessary that the model have finite length memory which means that the terms inside the Volterra kernels $h_\eta(\tau_1, \dots, \tau_\eta)$ need to reach zero (MARMARELIS, 2004). This is specially important in the discrete-time version of the model to allow one to truncate the model in some finite time.

A more appealing way to use the Volterra series is using the discrete-time representation of this model. This allows one to directly deal with experimental input and output signals measured on a nonlinear system. For this formulation the k -th sample of the discrete output $y(k)$ is the sum of infinite η -order responses $y_\eta(k)$ (RUGH, 1991):

$$y(k) = \sum_{\eta=1}^{+\infty} y_\eta(k) = y_1(k) + y_2(k) + y_3(k) + \dots \quad (5)$$

where $y_1(k)$, $y_2(k)$ and $y_3(k)$ are the first, second and third-order discrete-time components of the response of the Volterra model. Although this representation is an infinite series expansion it is usually necessary to truncate the model in some order η . In the discrete-time version of the model the linear component of the response $y_1(k)$ is a simple convolution sum:

$$y_1(k) = \sum_{n_1=0}^{N_1-1} \mathcal{H}_1(n_1) u(k - n_1) \quad (6)$$

where $\mathcal{H}_1(n_1)$ is the discrete first-order Volterra kernel, $u(k - n_1)$ is the input signal delayed in n_1 samples and N_1 is the memory length of the first kernel (i.e. the number of samples needed to represent the kernel). In a similar way to the expressions in the continuous time, the terms $y_2(k)$ and $y_3(k)$ are described by multidimensional convolution

sums (SCHETZEN, 1980):

$$y_2(k) = \sum_{n_1=0}^{N_2-1} \sum_{n_2=0}^{N_2-1} \mathcal{H}_2(n_1, n_2) u(k - n_1) u(k - n_2) \quad (7)$$

$$y_3(k) = \sum_{n_1=0}^{N_3-1} \sum_{n_2=0}^{N_3-1} \sum_{n_3=0}^{N_3-1} \mathcal{H}_3(n_1, n_2, n_3) u(k - n_1) u(k - n_2) u(k - n_3) \quad (8)$$

where $\mathcal{H}_2(n_1, n_2)$ and $\mathcal{H}_3(n_1, n_2, n_3)$ are the second and third-order kernels respectively, N_2 and N_3 are the memory lengths for $\mathcal{H}_2(n_1, n_2)$ and $\mathcal{H}_3(n_1, n_2, n_3)$ respectively, and n_2 and n_3 are the sample delays for the input signals. Intrinsic in this formulation is the fact that it is considered that the η -th kernel has the same size in all directions of the array (N_η).

Generalizing equations 6, 7 and 8 it is possible to obtain the η -th response $y_\eta(k)$:

$$y_\eta(k) = \sum_{n_1=0}^{N_\eta-1} \cdots \sum_{n_\eta=0}^{N_\eta-1} \mathcal{H}_\eta(n_1, \dots, n_\eta) \prod_{i=1}^{\eta} u(k - n_i) \quad (9)$$

where $\mathcal{H}_\eta(n_1, \dots, n_\eta)$ is the η -th Volterra kernel, N_η is the memory length of the η -th kernel and n_i is the i -th delay in the input signal. Since the Volterra model is linear with respect to the parameters (AGUIRRE, 2007), it is possible to write the vector of the output signal $\mathbf{y}$ as a simple matrix equation:

$$\mathbf{y} = \mathbf{\Gamma} \mathbf{\Phi} \quad (10)$$

where $\mathbf{\Gamma}$ is the matrix with the combinations of the input signal $u(k)$ and $\mathbf{\Phi}$ is the vector with the Volterra kernels considered in the truncation of the model. More details about the construction of the $\mathbf{\Gamma}$ and $\mathbf{\Phi}$ matrices are presented in the appendix A.

2.3 Identification of Volterra kernels

A common way to calculate the Volterra kernels of a certain nonlinear system is the harmonic probing method (WORDEN et al., 1997). In this technique, the kernels in the frequency domain, called higher-order frequency response functions (HOFRF), are calculated using the nonlinear differential equation representing the behavior of the system (WORDEN; TOMLINSON, 2001). The main idea of this method is to probe the system with sinusoidal input signals considering η harmonics in order to calculate the η -th

HOFRF. The main drawback of this technique is that the motion equation or other types of models of the system should be known before to obtain the Volterra kernels. In experimental tests this idea can be applied by mapping the η -th HOFRF using a sum of η sine signals with different frequencies (MARMARELIS, 2004), the main problem with this identification method is in the ammount of time to calculate all the terms in the HOFRFs, specially for higher-order terms. Despite of these limitation, many papers in the already applied the probing method for system identification (CAFFERTY; TOMLINSON, 1997) and damage detection (CHATTERJEE, 2010).

A more practical approach, using the discrete-time version of the model, is to use the matrix form of the Volterra series presented in equation 10 to estimate the vector with the considered Volterra kernels Φ . Since the input signal matrix Γ is not necessarily a square matrix it is possible to apply a least-squares estimation (AGUIRRE, 2007):

$$\Phi = (\Gamma^T \Gamma)^{-1} \Gamma^T \mathbf{y} \quad (11)$$

where ^{T} stands for the transpose of a matrix and $^{-1}$ is the inverse operation. Since the equation 11 has to deal with the inversion of the matrix $\Gamma^T \Gamma$, numerical problems can appear due to small terms in the input signal. In order to minimize this problem a normalization procedure is used in both the input and output signals:

$$\mathbf{u}_n = \mathbf{u} / c_u \quad (12)$$

$$\mathbf{y}_n = \mathbf{y} / c_y \quad (13)$$

where $\mathbf{u}$ is the vector with the collection of the input signal samples, $\mathbf{u}_n$ and $\mathbf{y}_n$ are the normalized input and output vectors respectively, and c_u and c_y are the constant factors used to normalize $\mathbf{u}$ and $\mathbf{y}$ respectively. In this case the matrix equation of the multidimensional convolution becomes:

$$\mathbf{y}_n = \frac{\Gamma}{c_u} \left(\frac{c_u}{c_y} \Phi \right) \quad (14)$$

In this way, using the normalized data, the kernels found by the least-squares approximation of equation 11 are multiplied by $\frac{c_u}{c_y}$ and to be corrected after the estimation the obtained vector has to be multiplied by the inverse term $\frac{c_y}{c_u}$. In this thesis the constants c_u and c_y were assumed to be the maximum values of the vector $\mathbf{u}$ and $\mathbf{y}$ respectively.

Although this seems an appealing approach where the vector with the Volterra kernels Φ can be determined by having a collection of input and output signals to apply equation 11, there are many terms in the vector Φ to be calculated in this way. This is specially relevant when considering higher-order kernels which have additional dimensions in the representation leading to a problem of overparameterization (TAWFIQ; VINH, 2003). One way to minimize this issue is presented in the next section by using orthonormal functions to represent the Volterra kernels.

2.4 Identification of the kernels using orthonormal functions

Problems of overparameterization can appear due to the high number of samples to represent the discrete Volterra kernels with large memory size. One way to minimize this problem is to assume an orthonormal representation instead of directly using the physical form of the kernel $\mathcal{H}_\eta(n_1, \dots, n_\eta)$. In this formulation, the η -order Volterra kernel is written as a combination of orthonormal functions $\psi_{\eta, i_j}(n_j)$ (MARMARELIS, 2004):

$$\mathcal{H}_\eta(n_1, \dots, n_\eta) \approx \sum_{i_1=1}^{J_1} \dots \sum_{i_\eta=1}^{J_\eta} \mathcal{B}_\eta(i_1, \dots, i_\eta) \prod_{j=1}^{\eta} \psi_{\eta, i_j}(n_j) \quad (15)$$

where $\mathcal{B}_\eta(i_1, \dots, i_\eta)$ is an array with η dimensions that can be seen as the orthonormal projection of the physical kernel, and $J_1, \dots, J_\eta$ are the number of orthonormal functions chosen to represent the kernel. In order to have a shorter representation of the output of the nonlinear system the input signal filtered by the orthonormal functions (l_{η, i_j}) is defined by:

$$l_{\eta, i_j}(k) = \sum_{n_i}^{V-1} \psi_{\eta, i_j}(n_i) u(k - n_i) \quad (16)$$

where $V = \max(J_1, \dots, J_\eta)$. Using the filtered input signal $l_{\eta, i_j}(k)$ the η -th output of the Volterra model can be calculated by:

$$y_\eta(k) \approx \sum_{i_1=0}^{J_\eta-1} \dots \sum_{i_\eta=0}^{J_\eta-1} \mathcal{B}_\eta(i_1, \dots, i_\eta) \prod_{j=1}^{\eta} l_{\eta, i_j}(k) \quad (17)$$

With this expression, one can better observe the similarities between the orthonormal version of the Volterra expansion (equation 17) and the original formulation in the physical basis (equation 9). An important question that arise here is about the proper function

$\psi_{\eta,i_j}(n_j)$ that should be applied in order to represent the response of a nonlinear system. The orthonormal representation of the kernels only works in an effective way when the function $\psi_{\eta,i_j}(n_j)$ represents the type of dynamical behavior in the nonlinear system (ROSA et al., 2007).

In order to represent nonlinear systems with oscillatory behavior, Kautz functions are often employed (KAUTZ, 1954). These functions were already applied for the representation of nonlinear circuits and magnetic systems (ROSA, 2009). In the frequency domain the Kautz functions can be represented as a pair of discrete functions $\Psi_{\eta,2g}(z)$ and $\Psi_{\eta,2g-1}(z)$:

$$\Psi_{\eta,2g}(z) = \frac{\sqrt{(1-d_\eta^2)(1-b_\eta^2)}z}{z^2 + b_\eta(d_\eta - 1)z - d_\eta} \left[\frac{-d_\eta z^2 + b_\eta(d_\eta - 1)z + 1}{z^2 + b_\eta(d_\eta - 1)z - d_\eta} \right]^{g-1} \quad (18)$$

$$\Psi_{\eta,2g-1}(z) = \frac{z - b_\eta}{\sqrt{1 - b_\eta^2}} \Psi_{\eta,2g}(z) \quad (19)$$

where z is the complex variable in the discrete domain, g is an index where $g \in [1, 2, \dots, J_\eta/2]$ and b_η and d_η can be obtained by:

$$b_\eta = \frac{(\mathcal{Z}_\eta + \bar{\mathcal{Z}}_\eta)}{(1 + \mathcal{Z}_\eta \bar{\mathcal{Z}}_\eta)} \quad (20)$$

$$d_\eta = -\mathcal{Z}_\eta \bar{\mathcal{Z}}_\eta \quad (21)$$

where $\mathcal{Z}_\eta$ and $\bar{\mathcal{Z}}_\eta$ are the pair of conjugate poles that are the parameters of the Kautz functions. Note that the mentioned time-domain orthonormal functions $\psi_{\eta,i_j}(n_j)$ are the impulse response functions of the frequency-domain Kautz filters $\Psi_{\eta,2g}(z)$ and $\Psi_{\eta,2g-1}(z)$. The poles of these functions should be determined somehow in order to identify the Volterra kernels using the least-squares estimation depicted in equation 11. Note that the poles $\mathcal{Z}_\eta$ and $\bar{\mathcal{Z}}_\eta$ are both in the discrete z domain which are related to the continuous version of the poles in the s domain through the equation:

$$\mathcal{Z}_\eta = \exp(\mathcal{S}_\eta/F_s) \quad (22)$$

where $\mathcal{S}_\eta$ is the continuous pole of the Kautz functions and F_s is the sampling frequency used in the discretization of the time-domain signal. In practical applications, F_s will be

the sampling frequency configured in the acquisition system used to capture the input and output signals from the system. The continuous poles $\mathcal{S}_\eta$ can be parameterized as a function of the η -th natural frequency ω_η and damping ratio ζ_η through the equation:

$$\mathcal{S}_\eta = -\zeta_\eta \omega_\eta + j\omega_\eta \sqrt{1 - \zeta_\eta^2} \quad (23)$$

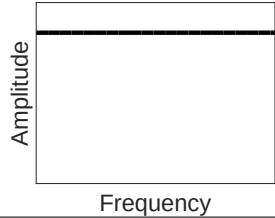
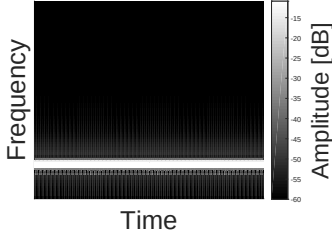
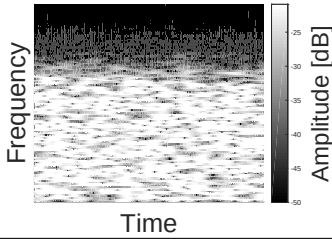
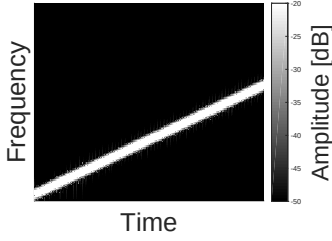
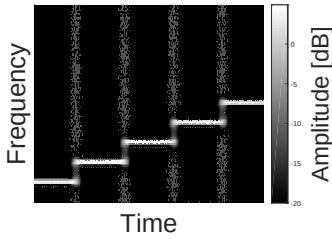
In this case, the pair of conjugate poles of the Kautz functions can be represented in terms of ω_η and ζ_η and these values are usually close to the parameters describing the linear dynamics of the system (i.e the natural frequency ω_n and damping ratio ξ of the mechanical system) (DA SILVA et al., 2010). The complete process of the identification of a Volterra model is separated here in five main steps which are summarized in figure 1:

Step 1 - Data acquisition and pre-processing: data acquisition from the dynamical system in the nonlinear regime and normalization of the data to avoid numerical issues. This task can also include a preliminary analysis of the system using for example classical spectral analysis in order to estimate the FRF of the system (STOICA; MOSES, 2005). A procedure of modal parameter extraction can be performed in the experimental FRF of the system, mainly in the linear regime (FU; HE, 2001). This procedure can extract the natural frequency ω_n and damping ratio ξ of the vibration mode to be studied in order to obtain a first guess for the parameters of the Kautz functions.

One important question in this step is about the selection of the input signal to be used in the calculation of the kernels. While sinusoidal and impact signals are not able to excite all the terms in the Volterra kernels (SCHETZEN, 1980), random excitation tends to linearize the system response (WORDEN; TOMLINSON, 2001). Stepped sine excitations, composed by sequential sines in different frequencies, can be used to show a sudden increase or decrease in vibration response called jump phenomenon (BRENNAN et al., 2008), but with the cost of a long test time. A simpler test can be done by using a chirp signal in which the frequency range to be studied can be adjusted as well as the input level and sweep rate in order to expose the nonlinear behavior. Table 1 summarizes the main features of input signals and the advantages and drawbacks of each type of test (WORDEN; TOMLINSON, 2001).

Step 2 - Convergence analysis and order selection: this step is composed by the definition of the number of terms η considered in the Volterra series expansion and the analysis of the number of orthonormal functions used to represent the Volterra kernels. Although a trial-and-error procedure can be used to set the value η , the harmonic content

Table 1 – Comparison between different input signals in the identification of Volterra kernels.

Input	Time-frequency	Advantages	Drawbacks
Impact 		• Practical and fast testing; • Broadband excitation.	• Excite only the main diagonal of Volterra kernels; • Difficulty to excite nonlinearities.
Sine 		• Energy concentrated in a single frequency; • Can show harmonics in the response.	• Excite only a few terms of Volterra kernels.
Random 		• Usual input in linear modal testing; • Broadband excitation.	• Energy is spread in the frequency range; • Difficulty to excite nonlinearities.
Chirp 		• Sweep range can be adjusted to excite nonlinearities; • Can select a frequency range to analyze; • Can illustrate harmonic distortion in the response.	• Does not let the system to reach a steady state response; • Response is difficult to characterize.
Stepped sine 		• Excite each single frequency during certain time; • Shows harmonics in the response for multiple frequencies; • Able to illustrate jump-up and jump-down.	• Very long test.

Source: Prepared by the author.

of the response of the system can be analyzed since the η -th Volterra kernel can obtain a response representing an η -order harmonic.

For an initial guess in this step the values of the natural frequency and damping ratio extracted in an experimental modal analysis can be used for the parameters ω_η and ζ_η describing the Kautz parameters $\mathcal{Z}_\eta$ and $\bar{\mathcal{Z}}_\eta$. The output error of the model can be analyzed as a function of the number of orthonormal filters ($J_1, J_2, \dots, J_\eta$) in order to decide the amount of functions to represent the kernels. In this thesis, the error was calculated based on the normalized root mean square error (NRMSE) between the experimental response (y_{exp}) and the model response calculated by equation 10:

$$\text{NRMSE} = \frac{\|y_{exp} - \mathbf{\Gamma}\Phi\|}{\|y_{exp} - \text{mean}(y_{exp})\|} \quad (24)$$

The value of NRMSE can be used as a measure of quality of the orders $J_1, J_2, \dots, J_\eta$ that define the number of orthonormal functions representing the kernels. While the NRMSE of the model does not converge, the orders are updated until the convergence criterion is reached.

Step 3 - Calculation of the parameters of the orthonormal functions: in the case of Kautz filters, these parameters are represented as complex conjugate poles that have to be selected before the identification of the model. In the case of oscillating systems it is possible to relate these poles to the natural frequencies and damping ratios of the system through the equation 23. In the procedure adopted in this thesis the NRMSE is calculated for each set of parameters of Kautz functions representing the first η Volterra kernels considered in the expansion $(\omega_1, \zeta_1, \omega_2, \zeta_2, \dots, \omega_\eta, \zeta_\eta)$. If the NRMSE has a convergence to a low value, the Kautz parameters defined by the frequencies and damping ratios are assumed to represent the orthonormal version of the Volterra kernels. Otherwise, these parameters are updated and the step 3 is repeated until the convergence of the NRMSE. For each iteration in this step the orthonormal kernels are calculated using the least-squares procedure depicted in equation 11. Although other methods were already proposed in the literature for the determination of the Kautz orthonormal functions as in the thesis of Rosa (2009), these procedures have limitations related to the need of knowing the original Volterra kernels. A more general solution to this selection may be to use an optimization procedure in order to minimize the NRMSE of the model and it was adopted in this work. After the procedures in the steps 2 and 3 the Volterra kernels in the vector Φ were already calculated by the least-squares procedure using the orthonormal functions defined in the step 3.

Step 4 - Evaluation of the performance of the model: in this last step the kernels were already defined in the previous steps. With these kernels it is possible to evaluate the η -order response $y_\eta(k)$ to an input signal $u(k)$. In order to verify the model it is possible to use a signal $u(k)$ different from the one used in the calculation of the Volterra kernels. The harmonic content of the response can be analyzed to verify the separation of linear and nonlinear components of the output.

2.5 Identification of the Volterra kernels for a Duffing oscillator

The steps of the identification of Volterra kernels are illustrated in this section with a classical asymmetric Duffing oscillator (BRENNAN et al., 2008). In this kind of system the motion equation can be written as:

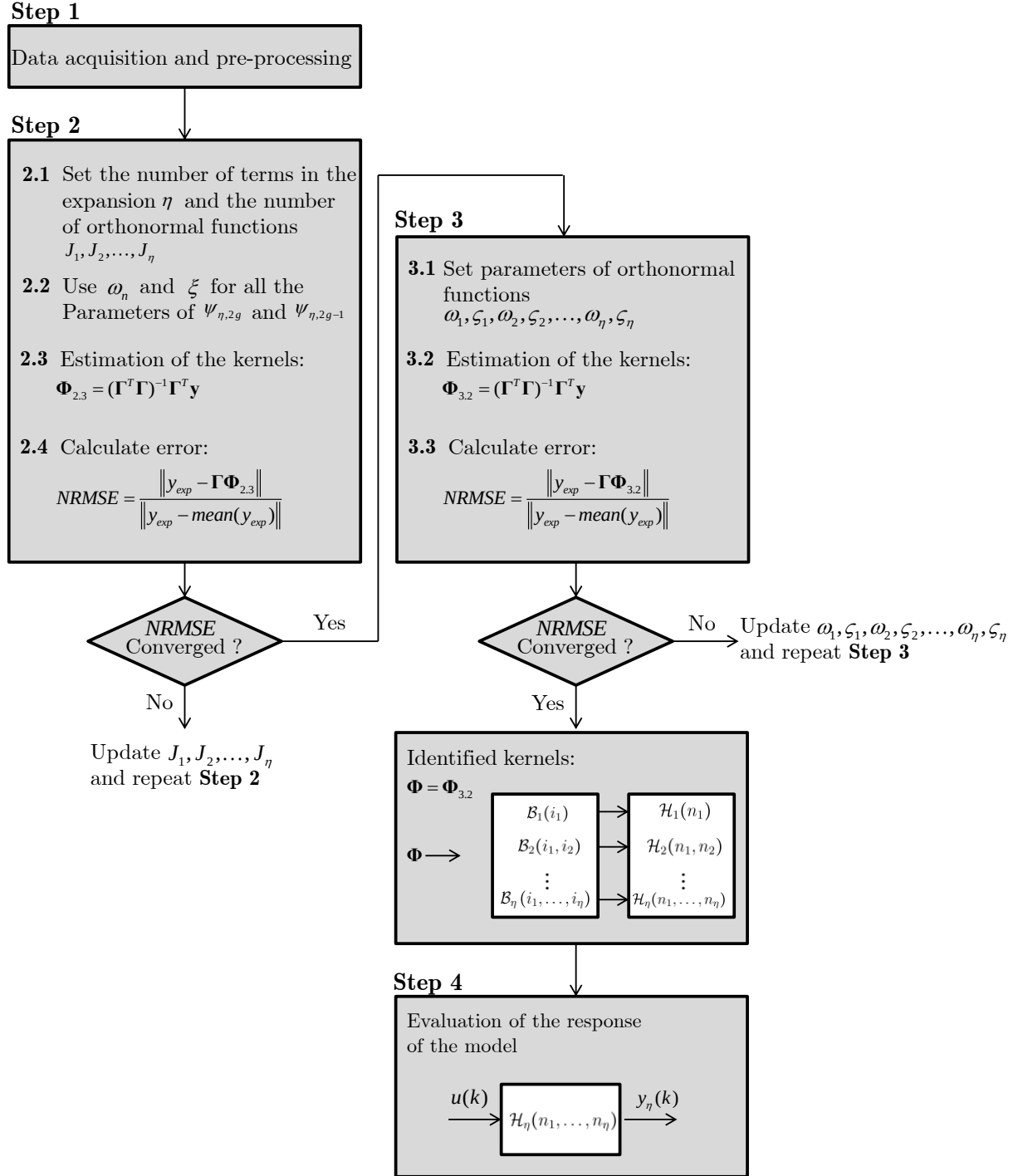
$$m\ddot{x} + c\dot{x} + k_1x + k_2x^2 + k_3x^3 = f(t) \quad (25)$$

where m is the mass coefficient [kg], c is the viscous damping [N.s/m], k_1 , k_2 and k_3 are the linear [N/m], quadratic [N/m²] and cubic stiffness [N/m³] respectively, $\ddot{x}$, $\dot{x}$ and x are the acceleration [m/s²], velocity [m/s] and displacement [m] respectively, and $f(t)$ is the input force in the system [N]. This kind of oscillator can represent many types of physical systems (KOVACIC; BRENNAN, 2011).

In this thesis the parameters of a clamped-clamped beam with a preload in the center of the beam are used as described in Shiki et al. (2014). For this structure it is possible to obtain the values $m = 0.078$ kg, $c = 0.50$ Ns/m, $k_1 = 2200$ N/m, $k_2 = 4.9 \times 10^5$ N/m² and $k_3 = 8.1 \times 10^7$ N/m³.

Figure 2(a) shows the time-frequency representation of the response of the oscillator to a chirp input ranging from 1 to 200 Hz considering 8192 samples with 0.5 N of force amplitude. This chirp signal was configured to sweep the mentioned frequency range with a rate of nearly 24.3 Hz/s. Figure 2(b) shows the frequency response curve to a stepped sine input with amplitudes of 0.1, 0.3 and 0.5 N. This signal is composed by mono-frequency sinusoidal inputs from 25 to 30 Hz with steps of 0.5 Hz and duration of 8192 samples for each single frequency. For both input signals the motion equation was integrated by the Newmark average acceleration method adapted for nonlinear polynomial systems and solved with a Newton-Raphson procedure (NEWMARK, 1959; JACOB; EBECKEN, 1994) using a sampling rate of 1000 Hz. The time-frequency distribution of the response

Figure 1 – Flowchart showing the methodology for the identification of Volterra models.

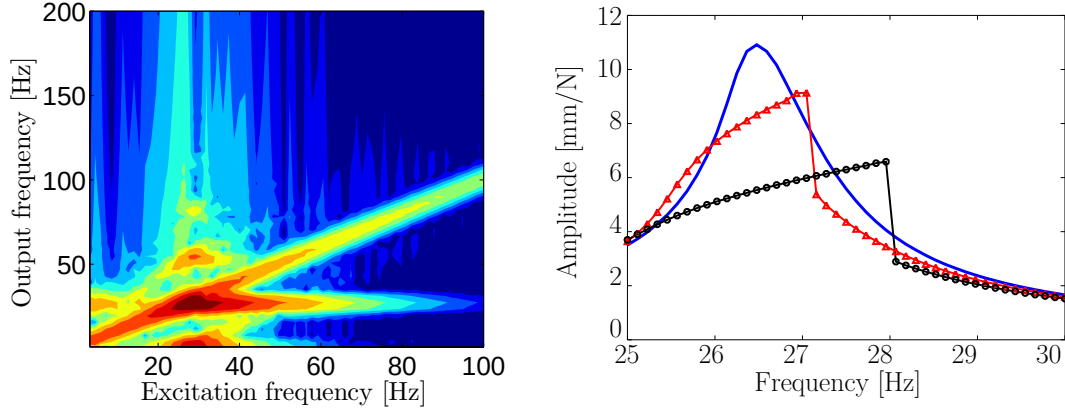


Source: Prepared by the author.

of the system to a chirp excitation clearly shows the presence of even and odd harmonics in the response due to the quadratic and cubic stiffness. The frequency response curve for a stepped sine input shows the hardening behavior of the system as well as the jump (BRENNAN et al., 2008) in the responses of the system subjected to higher levels of force.

In the **Step 1** of the proposed procedure the chirp input and the corresponding displacement (x) to low (0.005 N) and high levels (0.5 N) were used, as the input and output signals respectively, to estimate the Volterra kernels. The amplitudes and sweep rate mentioned previously were chosen by a trial-and-error process aiming to obtain the minimum number of samples in the input signal that still shows some nonlinear behavior as illustrated in figure 2. For the identification of the linear kernels a low input level dataset was used while a high input level dataset was applied in order to calculate the higher-order kernels. The informations regarding the input signals are summarized in table 2.

Figure 2 – Detection of nonlinearity in the Duffing oscillator. The continuous line — is the low input (0.1 N), -△- is the medium input (0.3 N) and -○- is the high input (0.5 N).



(a) Time-frequency distribution of the response with a chirp with input amplitude of 0.5 N. (b) Frequency response curve to a stepped sine input.

Source: Prepared by the author.

Table 2 – Input signal used for the identification of the Volterra kernels representing the Duffing oscillator.

Parameter	Value
Input signal type	Chirp
Frequency range [Hz]	1 – 200
Number of samples	8192
Sweep rate [Hz/s]	24.3
Input levels [N]	0.005 N (low) and 0.5 N (high)

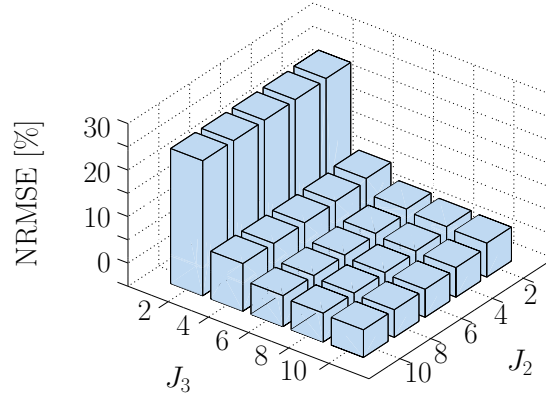
Source: Prepared by the author.

Since the oscillator responded with second and third-order harmonics the first three Volterra kernels were considered to represent the nonlinear system. This choice is justified by the fact that the second and third kernels are responsible to represent the second and third order harmonics respectively (SCHETZEN, 1980). In order to choose the number

of orthonormal functions of the model, a convergence analysis was performed by varying the number of Kautz filters used to represent the kernels $\mathcal{H}_2(n_1, n_2)$ and $\mathcal{H}_3(n_1, n_2, n_3)$ while the first kernel was chosen to be represented by 2 orthonormal functions due to the second order nature of linear oscillatory systems.

Figure 3 shows the NRMSE of the model as function of the number of orthonormal bases in $\mathcal{H}_2(n_1, n_2)$ and $\mathcal{H}_3(n_1, n_2, n_3)$. Analyzing figure 3, it is possible to observe that the second kernel does not have a great effect in the response of the system while the third kernel has the convergence of the error close to 6 functions. Considering this information the order of the model was considered to be $J_1 = 2$, $J_2 = 2$ and $J_3 = 6$ representing the first, second and third order kernels respectively (**Step 2**).

Figure 3 – Convergence analysis of the Volterra model representing the Duffing oscillator.



Source: Prepared by the author.

After deciding the number of orthonormal functions to be used in the Volterra kernels, the Kautz parameters should be defined (**Step 3**). An optimization procedure based on sequential quadratic programming (SQP) was used in order to define these parameters (LUENBERGER; YE, 2008). Since the frequencies and damping ratios of the Kautz parameters are generally close to the values describing the linear dynamics of the system, the optimization can search values close to the ones observed in a frequency-domain analysis of the identification data (SHIKI et al., 2014). In this study the Kautz parameters describing the first Volterra kernel (ω_1 and ζ_1) were fixed in the values of the known natural frequency and damping ratio of the Duffing oscillator. In this way the parameters $\omega_2, \zeta_2, \omega_3$ and ζ_3 were optimized to minimize the NRMSE of the model. The parameters of the SQP optimization are summarized in table 3.

The calculated Kautz parameters are illustrated in table 4 by the natural frequencies (ω_1, ω_2 and ω_3) and damping ratios (ζ_1, ζ_2 and ζ_3) of each Volterra kernel.

Table 3 – Optimization parameters for the calculation of the Kautz parameters in the Duffing oscillator.

Parameter of the optimization	Value
Algorithm	SQP
Objective function	NRMSE
Number of parameters	4 ($[\omega_2, \zeta_2, \omega_3, \zeta_3]$)
Maximum number of iterations	1000
Termination tolerance on the variable	10^{-10}
Termination tolerance on the objective function	10^{-10}
Initial guess	[26.7 Hz, 1.9 %, 26.7 Hz, 1.9 %]
Lower search bound	[21.3 Hz, 0.1 %, 21.3 Hz, 0.1 %]
Upper search bound	[32.1 Hz, 5.0 %, 32.1 Hz, 5.0 %]

Source: Prepared by the author.

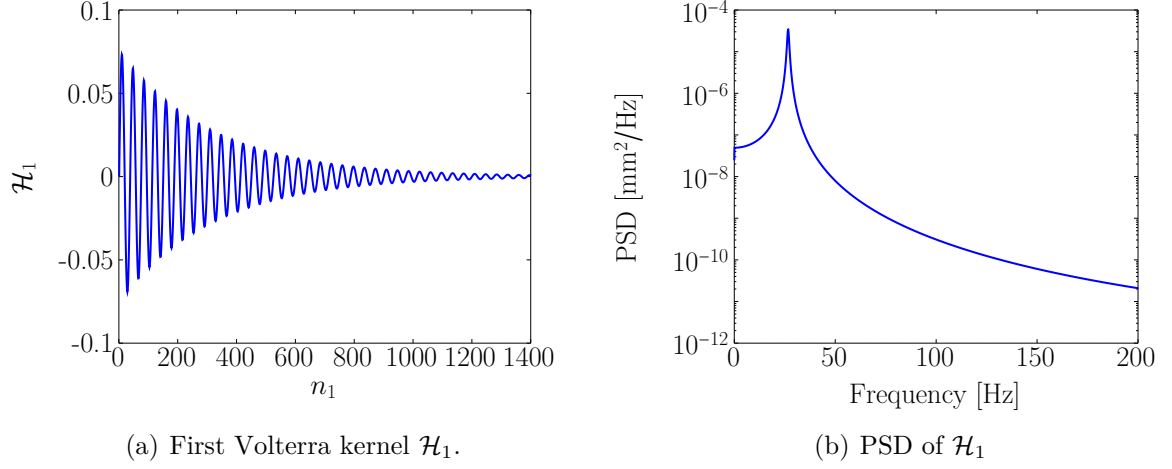
Table 4 – Information for the identification of the Kautz functions of the Duffing oscillator.

ω_1 [Hz]	ζ_1 [%]	ω_2 [Hz]	ζ_2 [%]	ω_3 [Hz]	ζ_3 [%]
26.7	1.9	26.9	2.2	26.1	1.3

Source: Prepared by the author.

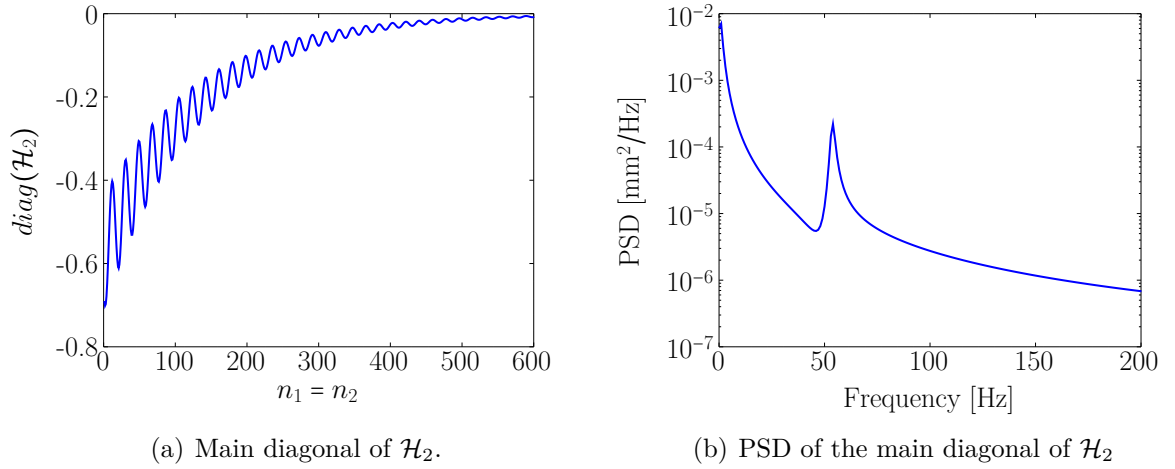
With the definition of the Kautz orthonormal functions, it is then possible to estimate the kernels with a least-squares approximation. The first, second and third Volterra kernels estimated for the Duffing oscillator are illustrated through the main diagonals in figures 4, 5 and 6 respectively. The power spectral density (PSD) of the main diagonal of these kernels are also illustrated showing the presence of harmonics of the natural frequency in the nonlinear kernels. The estimated model can be used to predict the response of the system and to analyze the linear and nonlinear components of the response (**Step 4**).

Figure 4 – Illustration of the first Volterra kernel in the physical basis representing the Duffing oscillator.



Source: Prepared by the author.

Figure 5 – Illustration of the second Volterra kernel in the physical basis representing the Duffing oscillator.

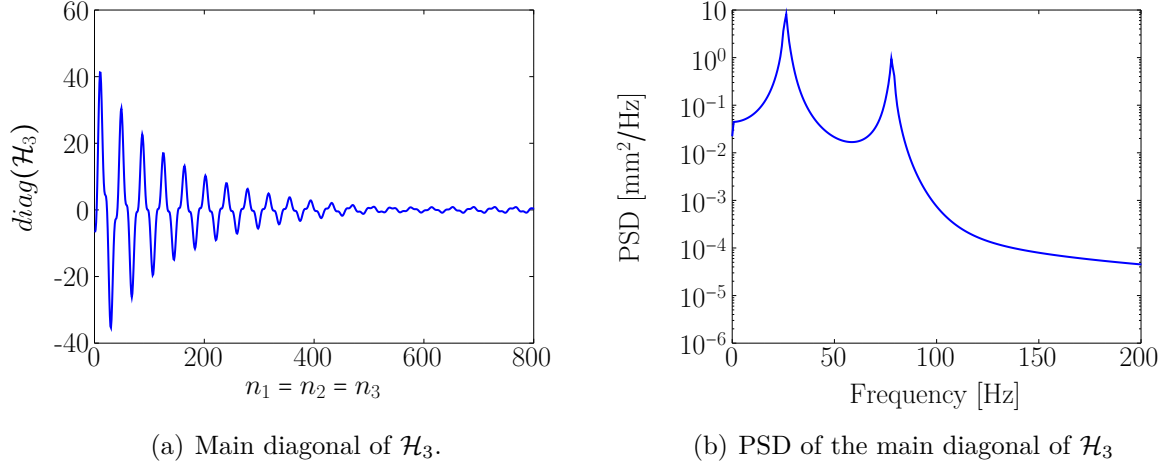


Source: Prepared by the author.

The direct comparison between the response obtained from the numerical integration of the Duffing oscillator and through the Volterra model is illustrated in figure 7 showing a good agreement for a low and high level of displacement. The components of the response of the Volterra model to a chirp excitation in the lowest level of amplitude (0.1 N) and in the highest one (0.5 N) can be observed in figure 8. The increase in y_2 and y_3 with respect to y_1 is evident in this figure. Also it is interesting to see that y_2 represents the asymmetry of the displacement in the oscillator due to the quadratic stiffness k_2 .

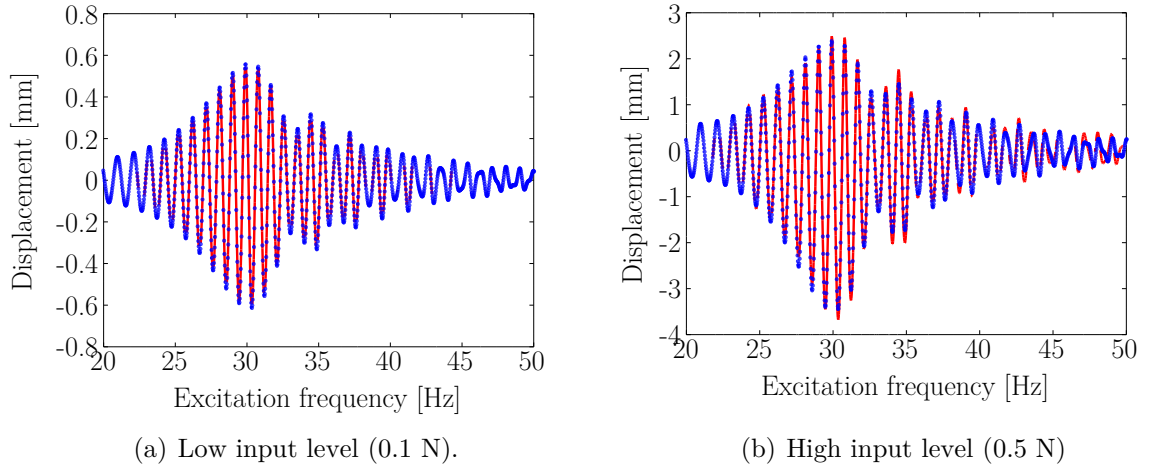
The system was also excited by a sinusoidal input with 0.1 N in the natural frequency

Figure 6 – Illustration of the third Volterra kernel in the physical basis representing the Duffing oscillator.



Source: Prepared by the author.

Figure 7 – Output of the identified Volterra model. The continuous line --- represents the response of the model and, $\circ$ is the response of the Duffing oscillator.

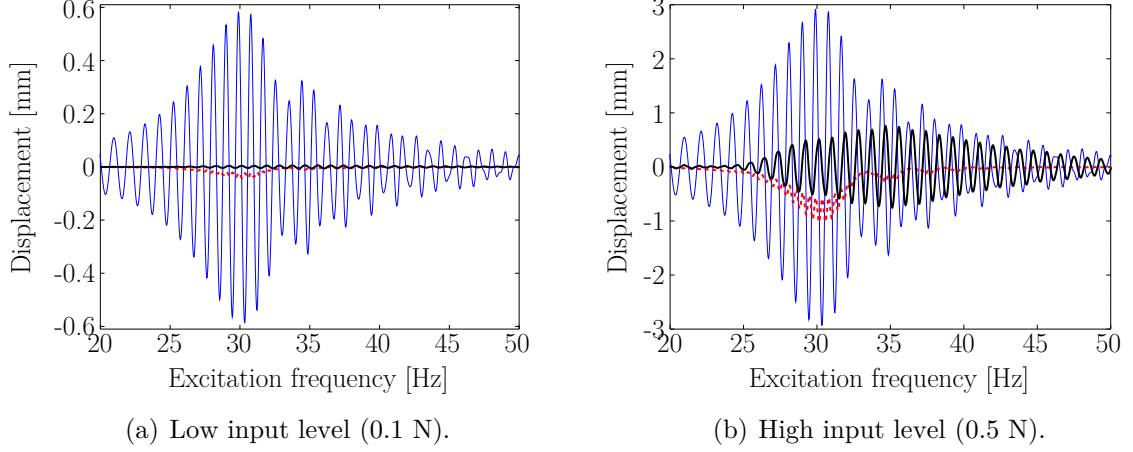


Source: Prepared by the author.

of the system (26.7 Hz). The Fourier series coefficients of the response of the Volterra model as well as its components are depicted in the figures 9 and 10 respectively.

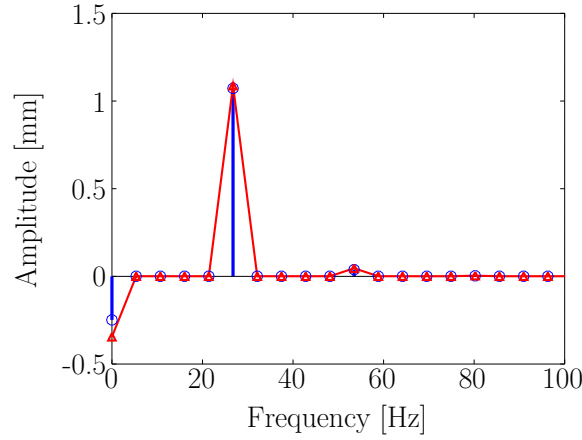
This representation makes possible the observation of harmonic components of the response of the system to a sine input. Figure 9 shows a very small contribution of harmonics in the response, while in figure 10 it is interesting to note that each of the η -th kernels is responsible for the representation of the respective η -order harmonic. This clearly illustrates the separation of the linear and nonlinear components of the response by the Volterra theory.

Figure 8 – Components of the response to a chirp input for the Duffing oscillator. The continuous line y_1 is the linear component (y_1), y_2 is the second order nonlinear response (y_2) and y_3 is the third order nonlinear response (y_3).



Source: Prepared by the author.

Figure 9 – Comparison between the Fourier series coefficients of the responses to a sine excitation of the oscillator and the identified model. y_1 is the reference signal and y_2 is the response of the Volterra model.

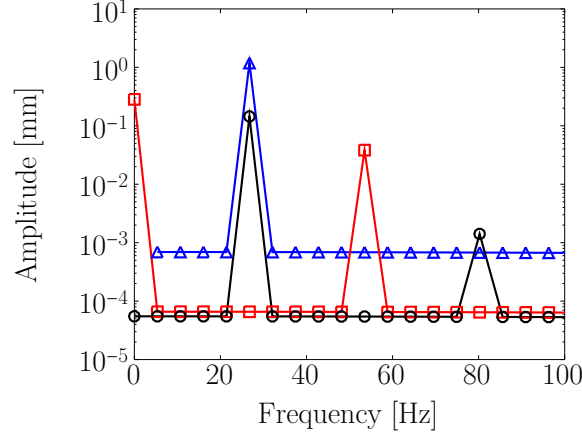


Source: Prepared by the author.

These separated components can be used to calculate a metric of the degree of non-linearity in the system response modeled by the Volterra model. Figure 11 shows a ratio between the root mean square (RMS) level of the nonlinear components and the root mean square of the total response ($RMS(y_2 + y_3)/RMS(y)$) as a function of the input level for the case of chirp excitation. Due to the polynomial nature of the stiffness, an increase in the level of nonlinear response can be observed as the displacement of the system is increased.

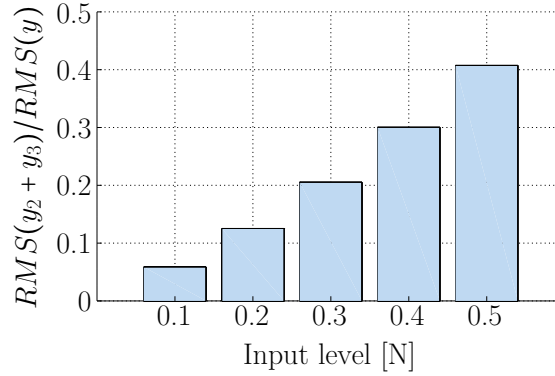
This methodology for the identification of a Volterra model can be applied in experimental data under a nonlinear regime of motion. In the next sections similar procedures

Figure 10 – Comparison between the Fourier series coefficients of the response components to a sine excitation. $\triangle$ is the linear component, $\square$ is the second order component and $\circ$ is the third order component.



Source: Prepared by the author.

Figure 11 – Nonlinearity index for the Duffing oscillator.



Source: Prepared by the author.

will be used to analyze other nonlinear systems.

2.6 Conclusions

This chapter presented a brief description of the methodology for the identification of a Volterra model using input and output time series. Orthonormal Kautz functions were applied to represent the Volterra kernels and minimize problems of overparameterization and convergence of the model. The steps for the identification of the Volterra series were illustrated in a simple Duffing oscillator. The Volterra series shows to be an interesting tool to represent the response of a nonlinear system since it is possible to analyze the

linear and nonlinear responses in a separated way. Although only a single input/single output (SISO) formulation of the model was presented, single input/multiple outputs (SIMO) and multiple inputs/multiple outputs (MIMO) are also perfectly possible but are not explored in this thesis.

In the next sections the Volterra model will be used as a reference of the structure in the reference condition in order to detect structural variations. Also the model will be applied to quantify parameters of an assumed grey-box model.

3 Methodologies for the application in SHM

This chapter presents strategies for the application of Volterra series for structural health monitoring. Section 3.1 shows a brief review of the literature concerning the aspects of SHM involving nonlinearities. Section 3.2 shows a possible option of application for an identified Volterra model as representing the reference state of a structure in a way that the prediction error of the model can be monitored to detect structural variations. Section 3.3 shows the methodology for the parameter quantification using the reference Volterra kernels to estimate physical information of the system. Finally in the section 3.4 some partial conclusions about the methodology are presented.

3.1 Overview of the literature for nonlinear techniques in SHM

As previously mentioned in chapter 1, a frequently used approach in the damage identification literature is to consider the nonlinear effects caused by the occurrence of damage. Many researchers already used this appealing idea to detect nonlinear damages like cracks, impacts, delamination or rotor-stator rub (NICHOLS; TODD, 2009). Haroon and Adams (2007) shows the characterization of nonlinear behavior in experimental tests on a vehicular suspension system under different simulated damage conditions. Mechanical faults in the suspensions are done by the loosening of bolts and cuts in the structure. These damages are identified in the experimental data using nonlinear restoring force surface method and autoregressive models with exogenous input.

Chanpheng et al. (2012) applied a damage index based on the degree of nonlinearity of the structure. This index is based on the Hilbert transform of the frequency response function between the ground motion and the structural vibration during earthquakes. In simulations with simple nonlinear oscillators, the metric showed to be representative of

the nonlinearity of the system. However, using experimental data acquired during five different earthquakes in a bridge, the data showed that the system behaved in a linear way and the method was not able to accurately detect structural variations in this situation.

Reynders et al. (2013) deals with the monitoring of the Z24 bridge in Switzerland using the kernel principal component analysis method. The authors state that the method can take in account the nonlinear relation between the environmental factors and the natural frequencies of the system that significantly varies with these factors. The nonlinear damage index showed to be sensitive to structural variations. Also, the authors applied an updating in the methodology considering new data acquired through the time to improve the reference model.

However, the consideration of inherent nonlinearities in the SHM literature is much less frequent. In this case, nonlinear system identification techniques are usually employed to have a characterization of the structure in the reference condition (KERSCHEN et al., 2006). Fassois and Sakellariou (2007) present an overview of time-series methods for damage detection using vibration data. One of the examples presented in this paper is a simple simulated structure with two degrees of freedom and a grounded cubic spring that gives the nonlinearity to this system. Linear structural variations are introduced to simulate damage and are detected in the system using a nonlinear autoregressive model with exogenous input.

Bornn et al. (2010) deals with the detection of damage in initially nonlinear systems. The authors states that most of the techniques that assume linearity are not able to reliably identify damage in nonlinear systems. The paper applies autoregressive support vector machines in a benchmark nonlinear structure with 4 degrees of freedom from the Los Alamos National Laboratory (LANL). The study illustrate the possibility of finding false-negatives when monitoring nonlinear structures with linear tools.

As a system identification technique, Volterra series was also been present in the literature for applications in damage detection and parameter identification. The frequency-domain version of this model is often employed where the system is represented by the higher-order frequency response functions (HOFRF) that can be calculated by using the harmonic probing method. Cafferty and Tomlinson (1997) identifies analytical expression of the HOFRFs representing an automotive damper using the harmonic probing method. The main diagonals of the HOFRFs are used in the identification of the nonlinear parameters of the damper. In a similar research Khan and Vyas (2001) estimated a cubic nonlinear stiffness in flexible rotor bearing with Volterra series. Numerical simulations

were carried to identify HOFRFs of first and third order using white noise signal as the input and the vibration of the system as response. Nonlinear parameters of the system were estimated using the third-order HOFRF.

Tang et al. (2010) applied the Volterra model in the fault diagnosis of a rotor-bearing system. The first three Volterra kernels are identified using acceleration data from the system. Unbalance and rubbing conditions are compared with the baseline using a metric based on the Volterra kernels. When considering the first three kernels, the metric allowed a good separation between different states of the system.

Surace et al. (2011) showed the effects of cracks in the HOFRFs which are represented as a bilinear nonlinearity. The authors calculated the HOFRFs with the probing method showing that the nonlinear kernels are more sensitive to the crack parameters. In a similar research line Chatterjee (2010) uses the HOFRF to calculate the components of the response of a beam with breathing crack.

Da Silva (2011) applied the Volterra model to calculate a metric to solve a nonlinear mode updating problem of a frame structure with local nonlinearity. The Volterra kernels were represented using Kautz orthonormal functions and the results showed applications in simulated polynomial and bilinear nonlinearities.

Cheng et al. (2014) applies a damage index based on the Volterra kernels to detect stiffness changes in a piecewise linear system representing a beam with a breathing crack and a beam under a nonlinear elastic foundation with a crack damage. A damage indicator is used based on an assurance criterion of the higher-order impulse response functions. Both simulations shows that the nonlinear kernels are more sensitive to grow of the cracks in the system.

Rébillat et al. (2014) applies Hammerstein models using damage features based on nonlinear components of the response in order to detect damage. The method is applied in linear systems subjected to a bilinear nonlinearity that represents a crack in the system. The technique is also applied in a composite plate do detect impact damage.

The literature review on the monitoring of structures with nonlinearities before and/or after the damage shows that this is still an important topic. Since there is no general technique to deal with nonlinear systems a lot of effort is needed in order to obtain better tools for initially nonlinear structures. More details on the techniques for the detection of nonlinear damage modes are available in the the paper of Worden et al. (2008) and the book chapter of Nichols and Todd (2009). Nonlinear system identification techniques can also be used for this purpose and can be seen in the book of Worden and Tomlinson (2001)

and the classical survey paper of Kerschen et al. (2006). A technical report from Farrar et al. (2007) made in the LANL also presents many system identification techniques dedicated for the case where the damage is a source of nonlinear behavior to be identified. A few examples of experimental setups with nonlinear damage modes are also presented in this report.

The next sections shows the proposed methodologies for the use of Volterra series in problems of damage detection and parameter quantification.

3.2 Damage detection using Volterra series

By using the methodology explained in the chapter 2 it, is possible to identify a model of a nonlinear system by processing discrete-time input and output data to estimate the Volterra kernels. Assuming that the identified kernels represent the response of system in the reference structural condition, it is possible to use this model to detect discrepancies in the measured response of a nonlinear structure which might be caused by structural variations. This approach follows the idea of unsupervised learning (i.e novelty detection) from the area of machine learning (FARRAR; WORDEN, 2012b). In this case a model representing the system in the reference condition is built. Subsequent measured responses from the system can then be tested to observe changes in the structure.

A straightforward way to perform this task is by analyzing the prediction error of the reference model. Since the Volterra series is a series expansion, it is possible to obtain prediction errors considering different numbers of terms in the expansion:

$$e_\eta = y_{exp} - \sum_{m=1}^{\eta} y_m \quad (26)$$

where e_η is the η -order prediction error, y_{exp} is the measured response and the sum represents the response of the model truncated in an order η . This was formulated in this manner in order to allow the comparison between considering a purely linear error ($\eta = 1$) and a higher-order nonlinear model ($\eta > 1$). This allows to observe the drawbacks of considering a linear model when monitoring a structure under the nonlinear regime of response.

In order to have a lower dimension feature to analyze, an index can be defined based on the ratio between the standard deviation of the prediction errors in the unknown

structural condition and in the reference state:

$$\lambda_\eta = \frac{\sigma(e_{\eta,unk})}{\sigma(e_{\eta,ref})} \quad (27)$$

where λ_η is the damage index of order η , $e_{\eta,unk}$ and $e_{\eta,ref}$ are the η -order prediction errors in the unknown and reference states respectively, and σ is the standard deviation operator. This kind of index was already applied in the work of Sohn and Farrar (2001) in order to detect damage in an experimental system with 8 degrees of freedom.

For the classification of the calculated index λ_η , a statistical model of this variable can be determined by assuming a Gaussian distribution in the prediction errors of the Volterra model. Although this seems to be a strong consideration, it is possible to correct the distribution when there is some deviation on the assumed distribution (SOHN; FARRAR, 2001). With the Gaussian assumption, it is possible to relate the index in equation 27 to a F distribution which describes the ratio between the variances of two variables (BENDAT; PIERSOL, 2011). This distribution can be modified by a simple change of variables which gives the distribution of the damage index:

$$P(\lambda) = \frac{2\left[\frac{v_1}{v_2}\right]^{\frac{v_1}{2}} \lambda^{v_1-1}}{\beta\left(\frac{v_1}{2}, \frac{v_2}{2}\right) \left[1 + \frac{v_1}{v_2} \lambda^2\right]^{\left(\frac{v_1+v_2}{2}\right)}} \quad (28)$$

where $P(\lambda)$ is the probability density function (PDF) of the index, v_1 and v_2 are the number of degrees of freedom for the prediction errors in the unknown and reference conditions respectively and β is the function:

$$\beta(a_1, a_2) = \frac{(a_1 - 1)!(a_2 - 1)!}{(a_1 + a_2 - 1)!} \quad (29)$$

where a_1 and a_2 are arguments of the β function. Knowing the distribution of the damage index it is possible to use a statistical hypothesis test to detect a significant increase in the prediction error under a significance level α :

$$\begin{aligned} H_0 : \sigma(e_{\eta,unk}) &= \sigma(e_{\eta,ref}) \\ H_1 : \sigma(e_{\eta,unk}) &> \sigma(e_{\eta,ref}) \end{aligned} \quad (30)$$

where H_0 is the null hypothesis and H_1 is the alternative hypothesis. In this formulation H_0 is the condition where is likely that the system is still in the reference condition. Meanwhile H_1 represents the case where a statistical variable of the prediction error, in this case the standard deviation, presented a significant increase due to some modification

in the system. One should note here that the type of change in the system does not need to be a damage since any kind of effect that changes the response of the system will be detected as a violation of the null hypothesis. In practice this means that in this formulation it is considered that no other effects are changing the response of the system (e.g. environmental variations).

In order to classify a certain system between damaged or undamaged, the experimental input and output u_{unk} and y_{unk} are measured and the reference Volterra model is used to calculate the expected response of the system in the reference state. With the prediction errors in the unknown state $e_{\eta,unk}$ and in the reference state $e_{\eta,ref}$, the λ_η index is calculated as demonstrated in the equation 27. With the value of λ_η it is possible to calculate the probability of the observed λ_η to happen in the case where H_0 is true named p -value (HOGG; LEDOLTER, 1987). This probability (p) can be calculated assuming a right-tailed test as:

$$p = 1 - \int_0^{\lambda_\eta} P(\lambda) d\lambda \quad (31)$$

where the integral is the cumulative distribution function (CDF). The p -value should be compared with the significance level α of the statistical test and two possible options can happen in the case of a right-tailed test:

- $p > \alpha$: it means that one is likely to see the calculated sample λ_η when there is no structural variation. In other words, the null hypothesis H_0 can not be rejected (no damage);
- $p < \alpha$: in this case one is not likely to see the sample λ_η when there is no structural variation. This means that H_0 should be rejected (damage).

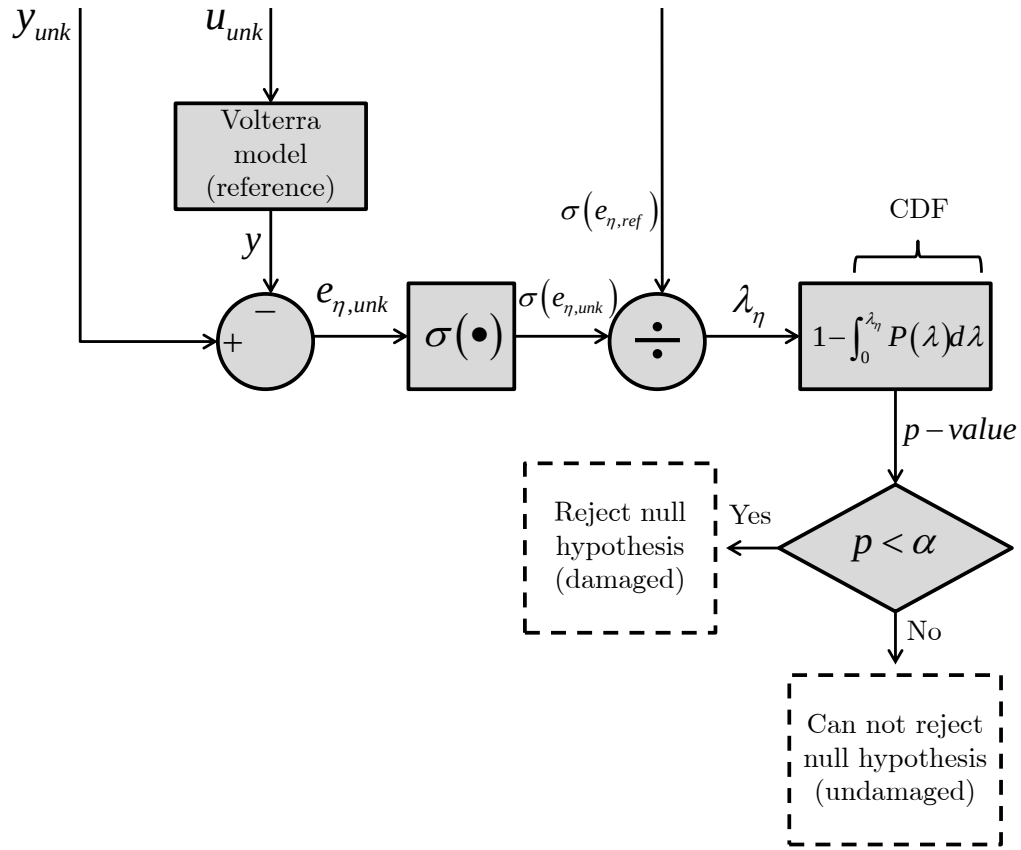
The entire process of damage detection based on a reference Volterra model can be divided in 4 main steps:

- **Step 1 - Calculation of the reference Volterra model:** this step comprises the measurement of the input and output signals from the nonlinear system and the identification of the Volterra kernels as it was explained in the chapter 2;
- **Step 2 - Measurement of the input and output signals of the system in an unknown structural condition:** after having the reference model the input and output data of the system can be measured to detect structural variations;

- **Step 3 - Calculation of the prediction errors with the reference model:** the residue between the reference model response and the measured data can be calculated to represent the structural state of the system;
- **Step 4 - Hypothesis test using the prediction error:** the proposed distribution can be used to detect significant changes in the prediction error statistical properties.

The process for the classification of the structural state of a system based on the proposed damage feature is also summarized in the flowchart of the figure 12.

Figure 12 – Flowchart of the damage detection methodology using Volterra series.



Source: Prepared by the author.

This strategy will be later exemplified in an experimental system during both the linear and nonlinear regime of motion.

3.3 Parameter identification using the Volterra kernels

The Volterra model is a nonparametric representation of the system since it is represented by the kernels of the series expansion. However, it is possible to quantify equivalent parameters of the structure as mass, damping and stiffness coefficients for example in an indirect way using the kernels representing the structure. Papers as in the work of Cafferty and Tomlinson (1997) performed the parameter identification using analytical versions of the Volterra kernels calculated with the harmonic probing method. This technique has the drawback of needing the motion equation or other model of the system to calculate the kernels.

The metric for the update of the parameters of the structure can be based in the residue between the reference kernels identified with the experimental data, and the kernels of the parametric model. In a similar way to the damage indexes previously presented it is possible to consider a linear ($\mathcal{C}_{lin}$) and nonlinear function ($\mathcal{C}_{nlin}$). Considering the unknown parameters to be grouped in a vector $\mathbf{p}$, a linear objective function can be proposed:

$$\mathcal{C}_{lin}(\mathbf{p}) = \frac{\|\mathcal{B}_{1,ref} - \mathcal{B}_1(\mathbf{p})\|}{\|\mathcal{B}_{1,ref} - \text{mean}(\mathcal{B}_{1,ref})\|} \quad (32)$$

where $\mathcal{B}_{1,ref}$ is the orthonormal projection of the Volterra kernel in the reference condition (in which the parameters need to be determined), and $\mathcal{B}_1(\mathbf{p})$ is the kernel calculated with the parametric model which has the unknown parameters in $\mathbf{p}$.

For the nonlinear objective function the higher-order kernels can be used. By considering a model up to a order $\eta = 3$, the function $\mathcal{C}_{nlin}$ can be given by:

$$\mathcal{C}_{nlin}(\mathbf{p}) = \frac{\|\mathcal{B}_{2,ref} - \mathcal{B}_2(\mathbf{p})\|}{\|\mathcal{B}_{2,ref} - \text{mean}(\mathcal{B}_{2,ref})\|} + \frac{\|\mathcal{B}_{3,ref} - \mathcal{B}_3(\mathbf{p})\|}{\|\mathcal{B}_{3,ref} - \text{mean}(\mathcal{B}_{3,ref})\|} \quad (33)$$

where $\mathcal{B}_{2,ref}$ and $\mathcal{B}_{3,ref}$ are the reference kernels of second and third-order respectively, and $\mathcal{B}_2(\mathbf{p})$ and $\mathcal{B}_3(\mathbf{p})$ are the orthonormal kernels extracted in the model with parameters in the vector $\mathbf{p}$.

The main idea of this methodology, in a similar way to what was done previously by da Silva (2011), is to minimize the residue in the objective functions of the equations 32 and 33 in order to obtain the same Volterra kernels in both the experimental data and in the parametric model. While the parameters of the model are not updated, an increment

in the vector $\mathbf{p}$ is done trying to search for an optimum value that minimizes the objective functions.

The main procedure for the identification of parameters of a model can be summarized in 3 main steps:

- **Step 1 - Calculation of the kernels representing the structure:** this step comprises the identification of the Volterra kernels representing the experimental system as it was explained in chapter 2;
- **Step 2 - Definition of the parametric model to be updated:** the mathematical expression of the model to be fitted and the parameters to be updated have to be selected in this step. Usually, the analysis of the nonlinear dynamics of the structure can help to considerate some simplified model of the system as a finite element model with local nonlinearity or even a simple oscillator with one degree of freedom;
- **Step 3 - Minimization of the residue of the kernels:** in this last step the Volterra kernels of the parametric model have to be extracted, and a residue metric as showed in equations 32 and 33 is then calculated and minimized to find an optimum vector of parameters.

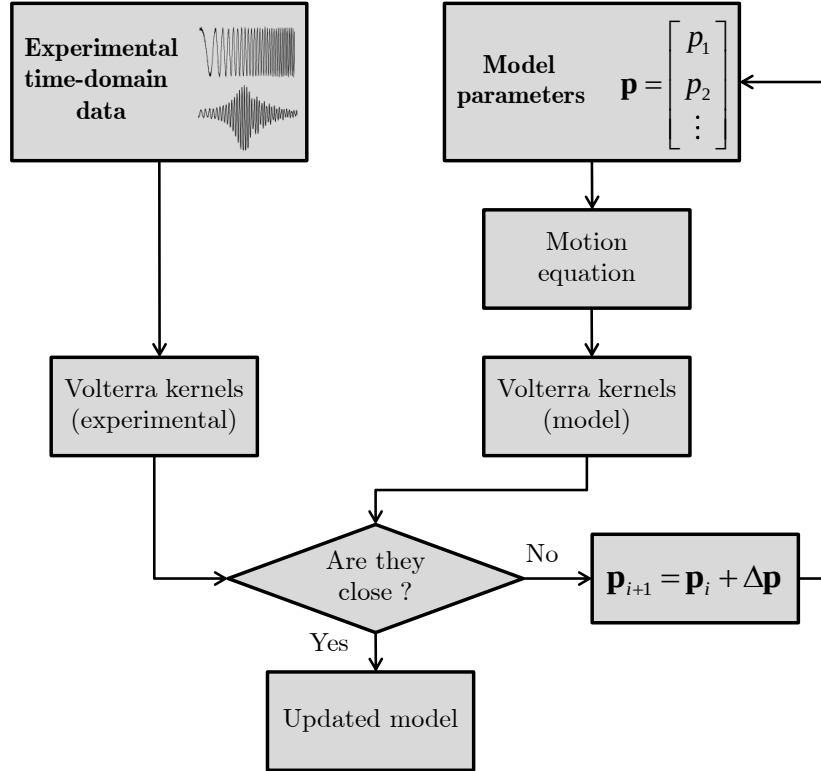
A flowchart showing the methodology for parameter identification is illustrated in figure 13.

3.4 Conclusions

In this chapter the methodology for the application of Volterra series in damage detection was presented. The prediction error of the reference model was used as a damage feature and a low-dimensional index was formulatd. The statistical distribution of the index was determined for application in hypothesis tests in order to classify the system state as damaged or undamaged. The index was formulated in a way that it is possible to compare a linear version of the damage indicator and a nonlinear one. After that, the objective functions for the quantification of parameters in nonlinear models were presented based on the residue between the experimentally obtained kernels and the ones from the parametric model to be found.

The methodologies can be used as tools in the area of structural health monitoring in order to detect damage and quantify parameters in systems with nonlinear behavior.

Figure 13 – Flowchart for the methodology of parameter quantification using the Volterra kernels.



Source: Prepared by the author.

These tools are specially important when considering cases where the nonlinearities are inherent from the structure and are present even before the occurrence of the structural variation. However, one must note that even by considering the Volterra kernels representation of the system it is possible that effects that are not taken in consideration as environmental variations or higher-order nonlinear effects might affect the performance of the presented nonlinear tools. This problem can bring lack of accuracy in the detection of damage and the poor identification of parametric models representing the nonlinear structure. In the field of nonlinear system identification it means that more robust models have to be still investigated and the environmental effects need to be rejected in order to obtain satisfactory representation of the dynamical behavior of structures.

4 Identification of Volterra kernels in experimental examples

This chapter presents the identification of Volterra models in two experimental examples. First in the section 4.1 a magneto-elastic nonlinear system is analyzed with the Volterra series expansion extracted during the modal testing in this simple academic system. After that in the section 4.2 the time series data collected during a ground vibration testing in an F-16 aircraft is used to identify the Volterra kernels representing the vibrating behavior of this system. Finally section 4.3 presents the partial conclusions about the results of the chapter.

4.1 Magneto-elastic system with hardening behavior

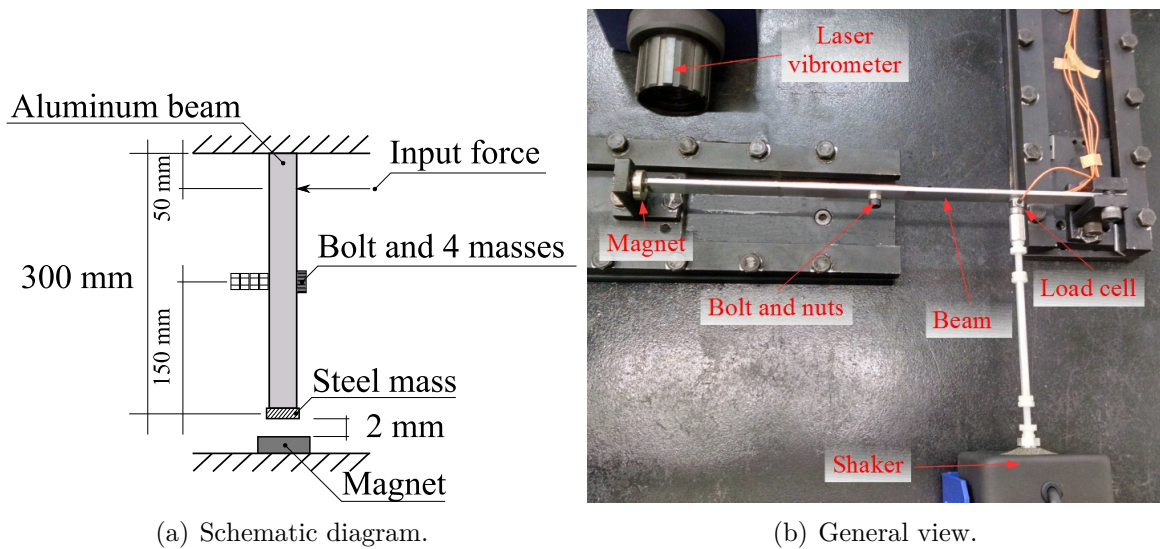
The system used for the calculations is a magneto-elastic system composed by a clamped beam and a permanent magnet. Similar structures were already used in the literature to investigate the interesting nonlinear phenomena in this kind of system. Barton et al. (2010) and Erturk and Inman (2011) used magneto-elastic devices in a bi-stable configuration with multiple magnets in order to design nonlinear oscillators for vibration energy harvesting. The main idea of these studies is to take use of the nonlinear phenomena in order to build energy harvesting devices with a greater useful frequency range. In this thesis a mono-stable configuration is used instead in order to have simpler nonlinear effects in the system.

The experimental setup is composed by a cantilever aluminum beam with dimensions of $300 \times 19 \times 3.2$ mm. This structure has a small steel mass attached to the free end of the beam and a neodymium magnet with a gap of 2 mm from the mass. A shaker is connected 50 mm away from the clamped end instrumented with a load cell. One laser vibrometer is used to measure the velocity on the free end of the beam. The equipments, sensors and actuators used for the experimental tests are:

- Modal Shop Shaker 2004E;
- Polytec OFV-525/-5000-S modular laser vibrometer;
- Dytran load cell model 1022V (IEPE force sensor);
- m+p VibPilot acquisition system.

A schematic diagram and a picture of this structure are illustrated in figure 14.

Figure 14 – Experimental setup of the magneto-elastic system.



Source: Prepared by the author.

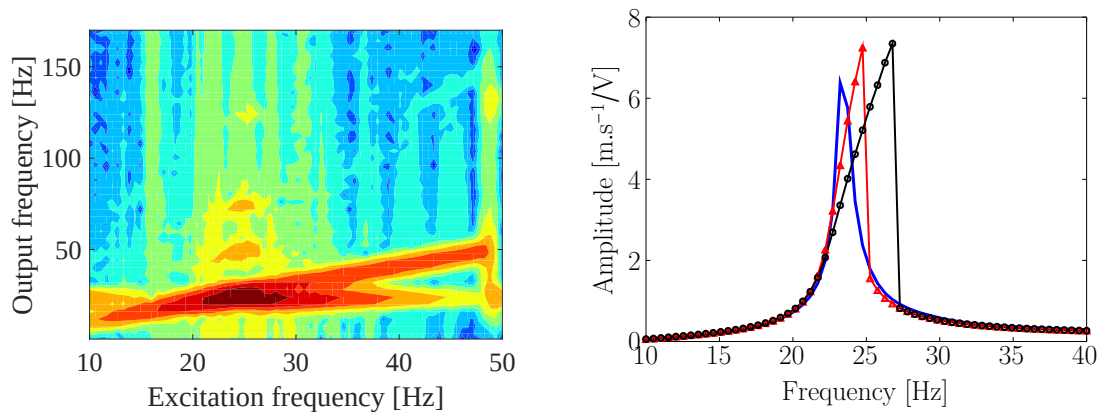
The nonlinearity in this system comes from the magnetic interactions between the permanent magnet and the steel mass placed in the free end of the beam. The magnet is positioned in a way that the beam is attracted in the axial direction and also during the bending movement. In this system, considering the gap of 2 mm, a hardening type of behavior was observed during the experimental tests in this structure. This effect was probably due to the influence of the magnet that was stretching the beam in the axial direction during the movement.

In order to observe the nonlinear behavior in the system, a chirp and a stepped sine input were used to excite the system under different amplitudes. The signals were captured using a 1024 Hz sampling rate. The chirp input was set to continuously sweep the range from 10 to 50 Hz around the first mode of the system with 4096 samples using 3 different voltage levels applied in the shaker: 0.01, 0.10 and 0.15 V. The stepped sine signal was set to sweep the range from 10 to 40 Hz with steps of 0.5 Hz using input levels of 0.01, 0.10 and 0.15 V. As it is possible to observe the input levels are all specified by

the amplitude of the applied voltage signal in the shaker power amplifier. This was done because it is easier to keep the supplied voltage signal with a fixed level while the force measured in the load cell can have significant variation in the amplitude. This happens specially near the resonance region since little force is needed to cause large levels of displacement in the system.

The time-frequency representation of the response to a chirp input and the frequency response curve under the stepped sine excitation can be seen in figure 15. It is possible to observe the second and third order harmonics in the time-frequency plot as a result of the nonlinear interactions in the structure. The hardening behavior which bends the peak of the curve to higher frequency values is visible as it is the jump phenomenon that is represented by the sudden fall in the amplitude of vibration. In this response to the stepped sine input the input/output relationship is specified by the ratio between the response amplitude in m.s^{-1} and the applied voltage in V. Due to the shaker-structure interaction during the test it was easier to supply a constant voltage for this type of experiment. In this sense the figure 15(b) shows what is actually the frequency response curve of the system together with the shaker instead of the isolated structure. A similar procedure was done in the paper of Tang et al. (2015).

Figure 15 – Detection of nonlinearity in the magneto-elastic system. The continuous line — is the low input (0.01 V), $-\triangle-$ is the medium input (0.10 V) and $-\circ-$ is the high input (0.15 V).



(a) Time-frequency distribution of the response with an input amplitude of 0.15 V. (b) Frequency response curve to a stepped sine input.

Source: Prepared by the author.

A Volterra model was used to represent the behavior of the magneto-elastic system considering the first three kernels in order to model the even and odd order harmonics. The chirp data in the low (0.01 V) and high input level (0.15 V) were used to identify the Volterra kernels, the information on the signal used for the calculation of the Volterra

kernels are summarized in table 5. The parameters of the input signal were defined by trial-and-error during the modal tests in order to minimize the length of the signal while maintaining relevant nonlinear effects in the harmonics of the system response in the case of high level input signal.

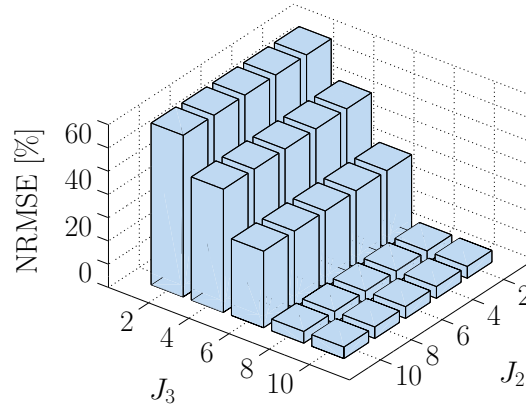
Table 5 – Input signal used for the identification of the Volterra kernels representing the magneto-elastic system.

Parameter	Value
Input signal type	Chirp
Frequency range [Hz]	10 – 50
Number of samples	4096
Sweep rate [Hz/s]	10
Input levels [V]	0.01 V (low) and 0.15 V (high)

Source: Prepared by the author.

In this case, the dataset with a low input level was used to calculate only the first kernel since the behavior in this case was nearly linear. The nonlinear terms were then calculated with the higher input dataset. The convergence of the order of the model in the Kautz representation of the second and third order kernels are illustrated in figure 16 with the variation of NRMSE with the increase in the number of orthonormal functions. The number of terms in the second kernel (J_2) does not modify significantly the performance of the model, while the number of Kautz functions in the third kernel (J_3) converges for the use of 8 functions representing the third kernel. Therefore, a model with $J_1 = 2$, $J_2 = 2$ and $J_3 = 8$ orthonormal functions for each respective kernel was considered.

Figure 16 – Convergence analysis of the Volterra model representing the magneto-elastic system.



Source: Prepared by the author.

In the next step the Kautz parameters were identified through SQP optimization

in order to minimize the prediction error of the model. While the Kautz parameters representing the first-order kernel (ω_1 and ζ_1) were kept fixed with the values describing the FRF of the system under the linear regime, the parameters of the nonlinear kernels (ω_2 , ζ_2 , ω_3 and ζ_3) were optimized using the algorithm described by the parameters in table 6.

Table 6 – Optimization parameters for the calculation of the Kautz parameters in the magneto-elastic system.

Parameter of the optimization	Value
Algorithm	SQP
Objective function	NRMSE
Number of parameters	4 ($[\omega_2, \zeta_2, \omega_3, \zeta_3]$)
Maximum number of iterations	1000
Termination tolerance on the variable	10^{-10}
Termination tolerance on the objective function	10^{-10}
Initial guess	[20.65 Hz, 0.65 %, 20.65 Hz, 0.65 %]
Lower search bound	[16.5 Hz, 0.1 %, 16.5 Hz, 0.1 %]
Upper search bound	[24.7 Hz, 5.0 %, 24.7 Hz, 5.0 %]

Source: Prepared by the author.

The frequencies and damping ratios of the first three Volterra kernels are presented in table 7.

Table 7 – Information for the identification of the Kautz functions of the magneto-elastic system.

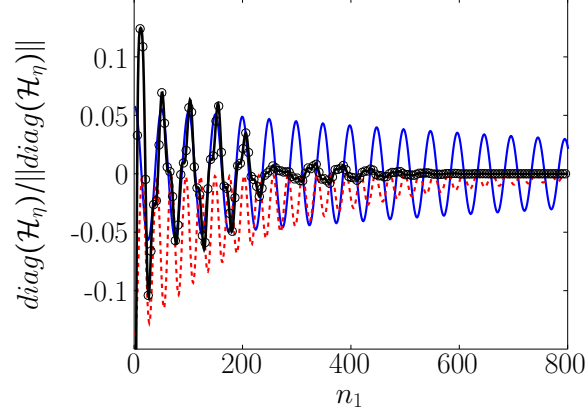
ω_1 [Hz]	ζ_1 [%]	ω_2 [Hz]	ζ_2 [%]	ω_3 [Hz]	ζ_3 [%]
20.65	0.65	19.03	0.27	19.91	1.38

Source: Prepared by the author.

With the defined values of the Kautz parameters, the kernels can be estimated by using the equation 11. The main diagonals of the first three kernels are illustrated in figure 17.

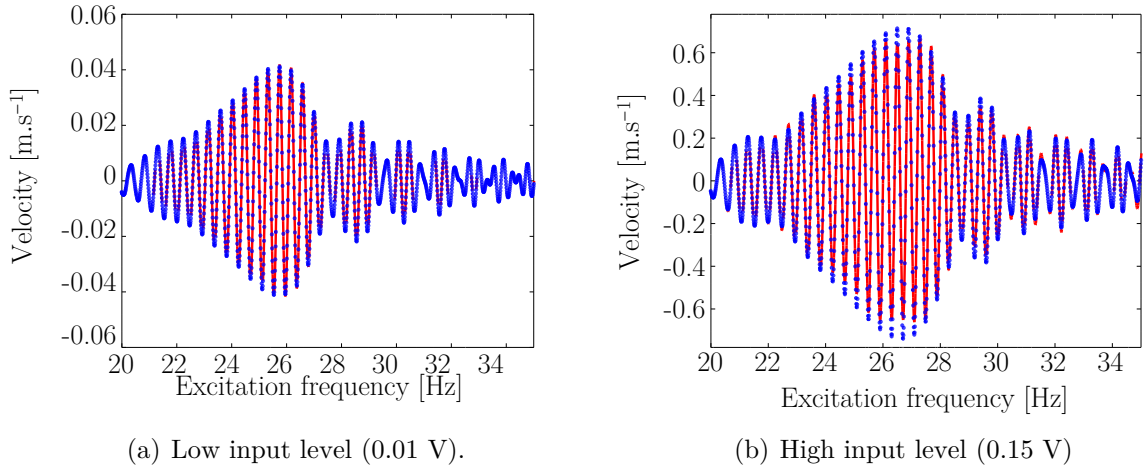
With the identified Volterra model it is possible to retrieve the linear and nonlinear components of the response of the magneto-elastic system. A direct comparison between the experimental response and the one obtained by the Volterra series is illustrated in figure 18 for the low (0.01 V) and high level chirp input (0.15 V). The response components to the low and high level inputs are illustrated in figure 19. In both levels it is observed that the second order component (y_2) has a small contribution. Also the third order component (y_3) shows to increase with the input.

Figure 17 – Identified Volterra kernels representing the magneto-elastic system. The continuous line — is the normalized 1st kernel ($\mathcal{H}_1/\|\mathcal{H}_1\|$), --- is the normalized main diagonal of the 2nd kernel ($diag(\mathcal{H}_2)/\|diag(\mathcal{H}_2)\|$) and -o- is the normalized main diagonal of the 3rd kernel ($diag(\mathcal{H}_3)/\|(diag(\mathcal{H}_3))\|$).



Source: Prepared by the author.

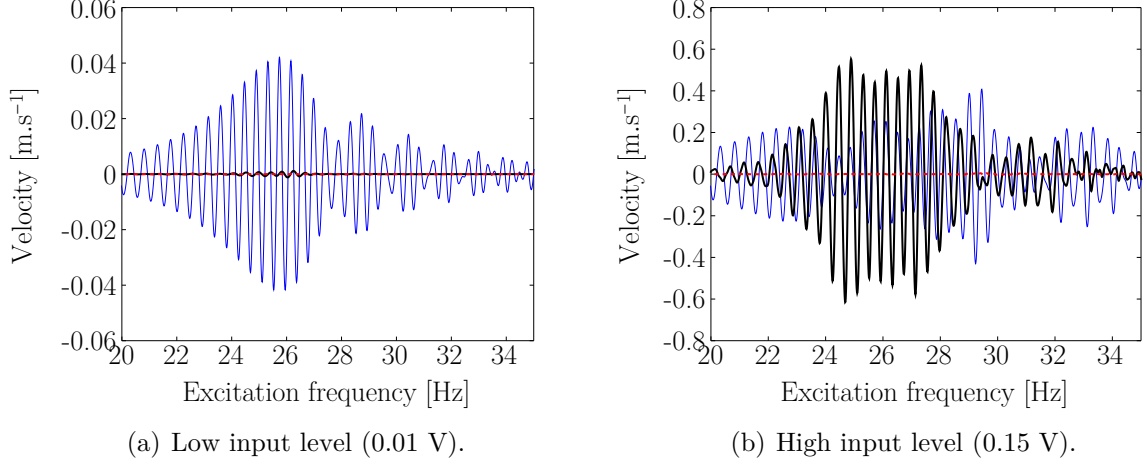
Figure 18 – Output of the identified Volterra model. The continuous line --- represents the response of the model and, $\circ$ is the response of the Duffing oscillator.



Source: Prepared by the author.

The separation of the linear and nonlinear components of the response can be used to calculate an index based on the ratio between the RMS levels of the nonlinear components and the total response. This value represents the degree of the nonlinearity of the response of the system. The metric is illustrated for the 4 amplitudes of chirp input in figure 20. The relative increase in the nonlinear response due to the polynomial nature of the stiffness was already observed in the time-domain components presented in figure 19 and it is summarized in this figure. The identified model will be used in the following chapters to represent the reference state of this system aiming the detection of structural variations

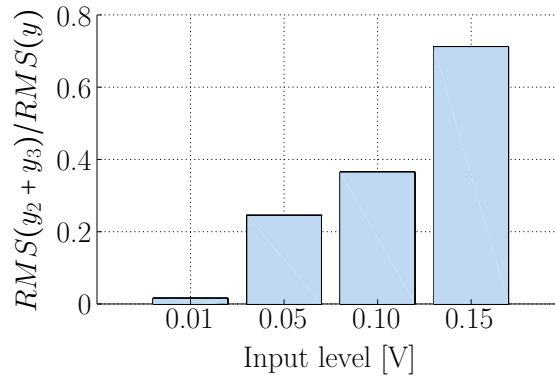
Figure 19 – Components of the response to a chirp input for the magneto-elastic system. The continuous line y_1 is the linear component (y_1), y_2 is the second order nonlinear response (y_2) and y_3 is the third order nonlinear response (y_3).



Source: Prepared by the author.

simulated in the magneto-elastic system.

Figure 20 – Nonlinearity index for the magneto-elastic system.



Source: Prepared by the author.

4.2 Ground vibration testing in an F-16 aircraft

Time-domain data from a ground vibration testing (GVT) was performed in an F-16 aircraft on the occasion of the LMS ground vibration testing master class (LAU et al., 2011). The dataset was provided in a partnership with the Eng. Dr. Fabio L. M. dos Santos and the Eng. Dr. Bart Peeters of Siemens Industry Software in Leuven (Belgium). The experimental setup of the tests is illustrated in figures 21 and 22. The dataset used

for the analysis in this paper consists of a chirp excitation from 6.5 to 7.5 Hz with three different levels of amplitude: 5, 10 and 15 N. This frequency range was set in order to take in account a mode of the aircraft subjected to nonlinear effects. The structure was excited by a shaker in a distance of $2/3$ of the length of one of the wings and the acceleration signal was collected in the driving point. The time-domain signals were acquired with 40910 samples and a sampling rate of 50 Hz. The mode shape of the system subjected to nonlinear effects is illustrated in figure 23. The presence of nonlinearity was also detected by Noël et al. (2013) which attributed this effect to the connection between the wing to the payload attached in the end as illustrated in figure 22(b).

Figure 24 shows the time-frequency distribution of the acceleration signals during the test using 15 N of input level. It is possible to observe mainly the presence of third order harmonics indicating the nonlinear behavior. Nonlinearity can also be detected in the estimated FRFs showed in the figure 25. The shift to the left in the amplitude peak of the FRF which characterizes a softening nonlinearity in this system.

Figure 21 – Photo illustrating the GVT in the F-16 aircraft.



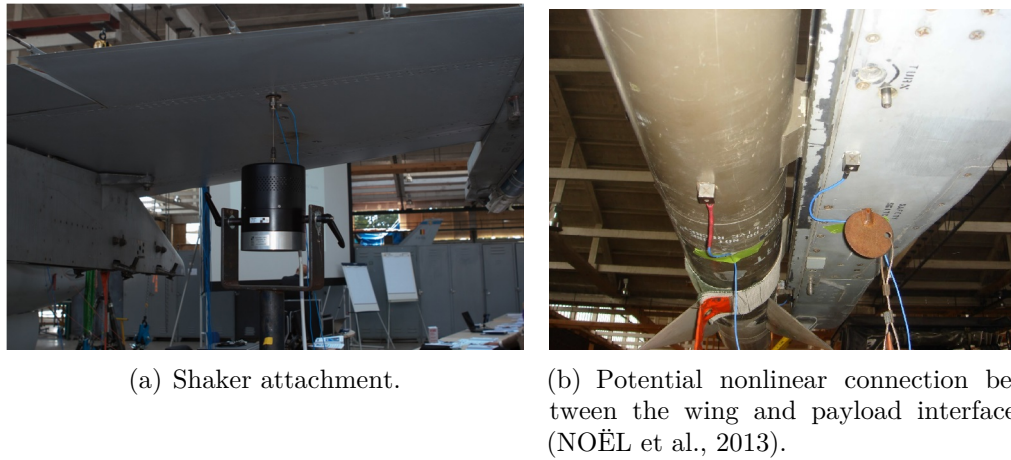
Source: Lau et al. (2011).

The acquired time-domain dataset was used for the identification of the Volterra kernels, the input signals used for the identification are summarized in table 8. Note that in this case the sweep rate is very low since a small frequency range was mapped during a long time.

The model of this system was considered to have the first and third order terms while the second order kernel was neglected for this study. This was done because of the low contributions of the even order harmonics in the response of the system. A convergence analysis was performed in order to define the number of orthonormal functions for the representation of the nonlinear kernel and it is illustrated in figure 26.

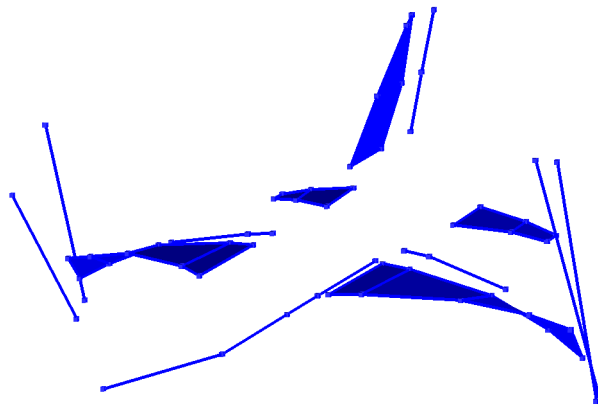
It is possible to observe that the prediction error converged around 8 Kautz functions. In this way a model with $J_1 = 2$ and $J_3 = 8$ was considered for this system. For the

Figure 22 – Detailed photos of the GVT in the F-16 fight aircraft.



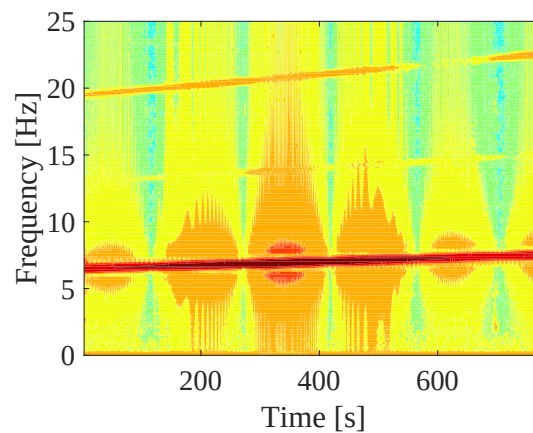
Source: Lau et al. (2011).

Figure 23 – Mode shape of the F-16 aircraft subjected to nonlinear effects.



Source: Prepared by the author.

Figure 24 – Time-frequency distribution of the response of the F-16 aircraft to a 15 N chirp excitation.



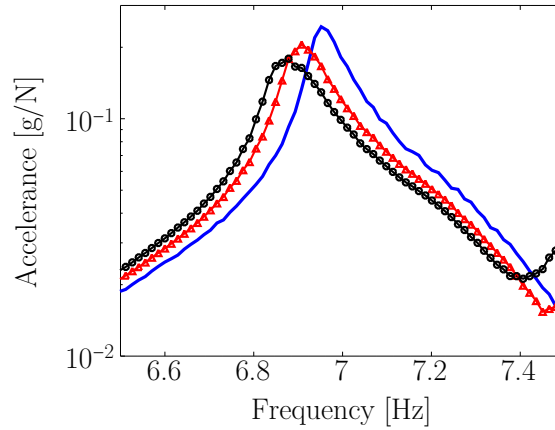
Source: Prepared by the author.

Table 8 – Input signal used for the identification of the Volterra kernels representing the F-16 aircraft.

Parameter	Value
Input signal type	Chirp
Frequency range [Hz]	6.5 – 7.5
Number of samples	40910
Sweep rate [Hz/s]	0.0012
Input levels [N]	5 N (low) and 15 N (high)

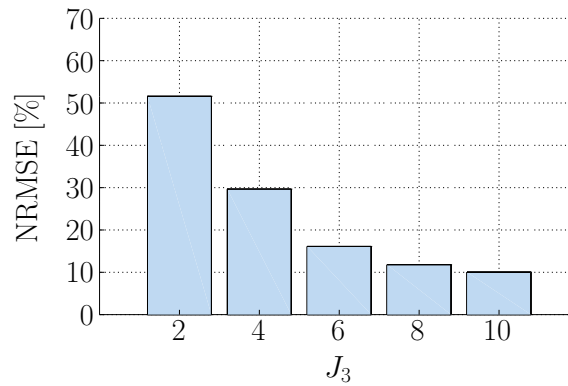
Source: Prepared by the author.

Figure 25 – Experimental FRFs of the F-16 aircraft using a chirp input. The continuous line — is the low input (5 N), -△- is the medium input (10 N) and -○- is the high input (15 N).



Source: Prepared by the author.

Figure 26 – Convergence analysis of the Volterra model representing the F-16 aircraft.



Source: Prepared by the author.

calculation of the Kautz parameters the SQP optimization scheme was again used with the parameters described in the table 9 and the results are illustrated in table 10.

Table 9 – Optimization parameters for the calculation of the Kautz parameters of the F-16 aircraft.

Parameter of the optimization	Value
Algorithm	SQP
Objective function	NRMSE
Number of parameters	2 ($[\omega_3, \zeta_3]$)
Maximum number of iterations	1000
Termination tolerance on the variable	10^{-10}
Termination tolerance on the objective function	10^{-10}
Initial guess	[20.65 Hz, 0.65 %]
Lower search bound	[5.56 Hz, 0.1 %]
Upper search bound	[8.36 Hz, 5.0 %]

Source: Prepared by the author.

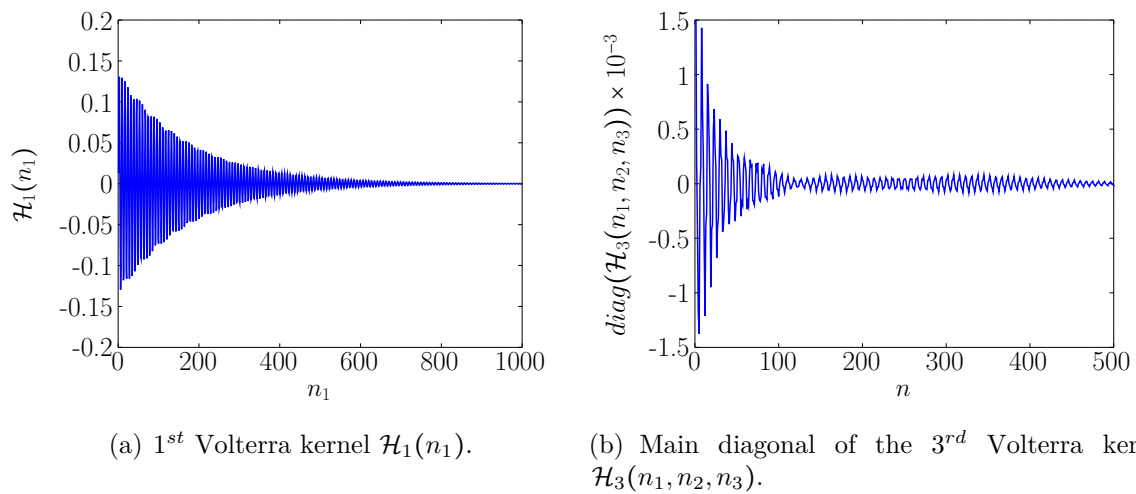
Table 10 – Information for the identification of the Kautz functions of the F-16 aircraft.

ω_1 [Hz]	ζ_1 [%]	ω_3 [Hz]	ζ_3 [%]
6.96	0.66	6.94	1.13

Source: Prepared by the author.

With the orthonormal functions it was then possible to least-squares estimate the Volterra kernels which are illustrated in figure 27. In the case of the third-order kernel only the main diagonal is illustrated since a full representation is not possible in a conventional way due to high number of dimensions in the array representing $\mathcal{H}_3(n_1, n_2, n_3)$.

Figure 27 – Volterra kernels of the identified model representing the F-16 aircraft.

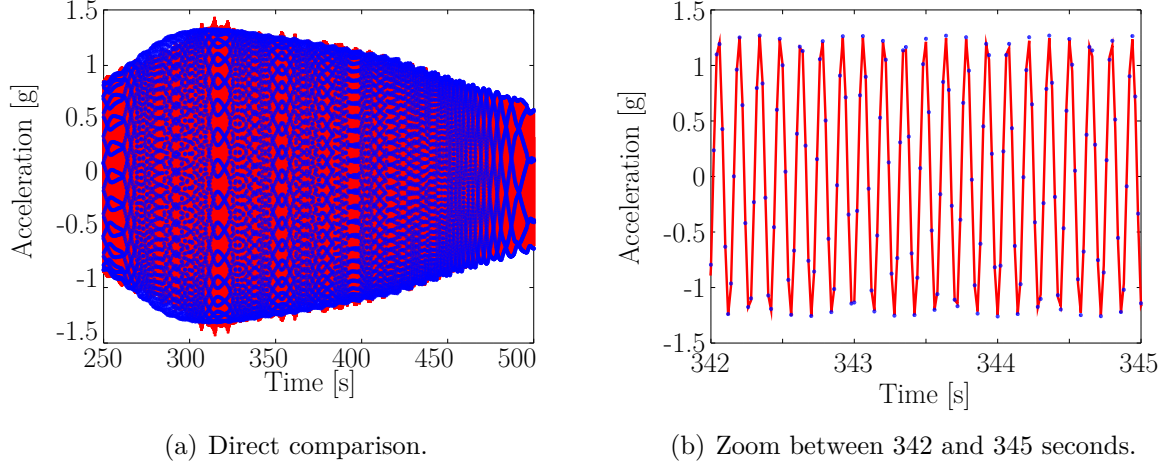


Source: Prepared by the author.

The kernels were then used on the multiple convolutions with the input signal in order to estimate y_1 and y_3 . The direct comparison between the experimental output and the response of the Volterra model are represented in figures 28 and 29 for the input level of

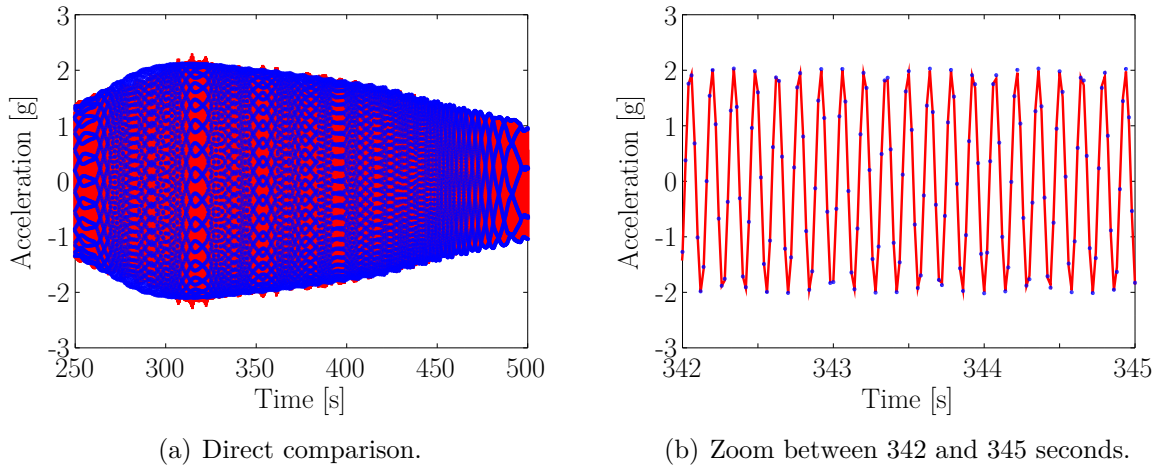
5 and 15 N respectively. The components are depicted in figure 30 with the time-domain components in the lower and higher force levels. Due to the polynomial nature of the nonlinearity, y_3 grows in relation to y_1 with the increase in the input level.

Figure 28 – Output of the identified Volterra model representing the F-16 aircraft for an input level of 5 N. The continuous line --- represents the response of the model and, $\circ$ is the measured response.



Source: Prepared by the author.

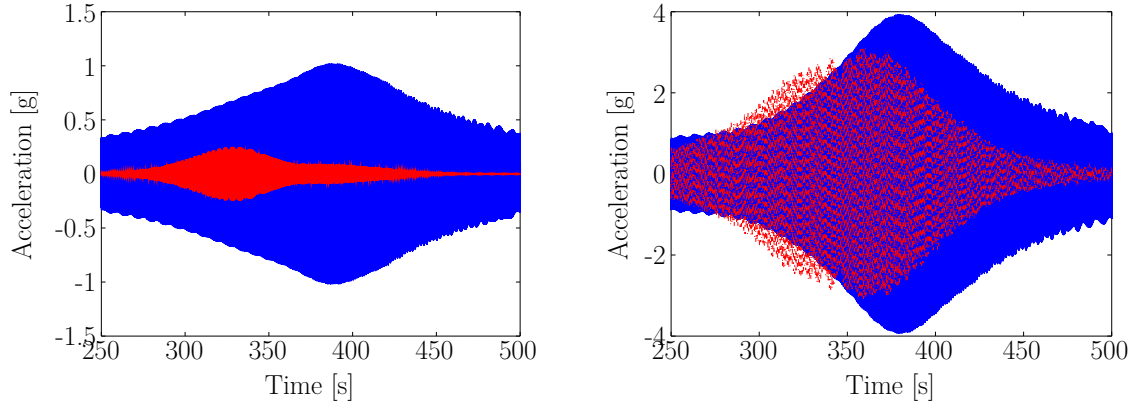
Figure 29 – Output of the identified Volterra model representing the F-16 aircraft for an input level of 15 N. The continuous line --- represents the response of the model and, $\circ$ is the measured response.



Source: Prepared by the author.

The components of the response in the high level input are also illustrated in the time-frequency distribution in figure 31. It is possible to observe the separation of the harmonic components of the response since only the nonlinear output y_3 responds with

Figure 30 – Components of the output of the Volterra model representing the F-16 aircraft. The blue continuous line — is the linear component (y_1) and the red line --- is the nonlinear component (y_3).



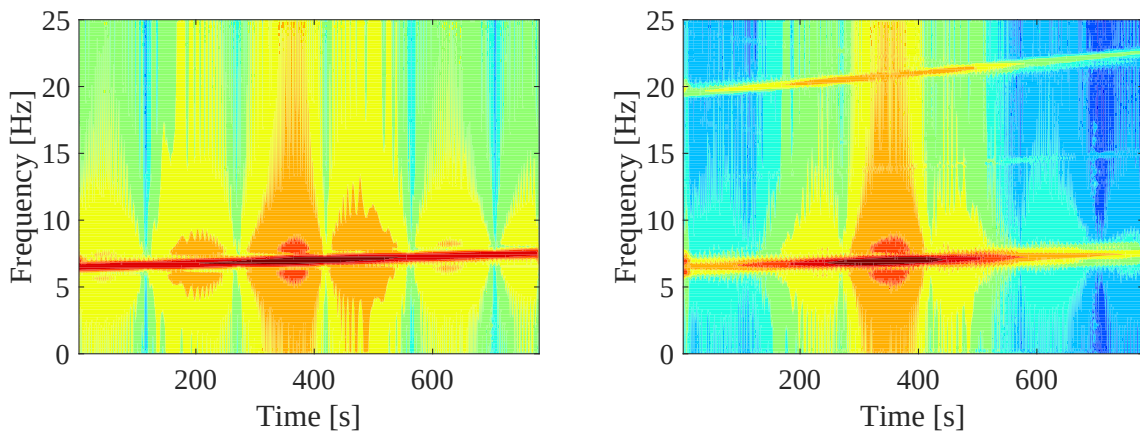
(a) Components of the output to an input of 5 N. (b) Components of the output to an input of 15 N.

Source: Prepared by the author.

harmonics of the excitation frequency. The softening behavior is also reproduced by the model which is illustrated in figure 32 by a shift of the peak frequency to the left.

Figure 33 shows the nonlinear index composed by the ratio between the RMS level of y_3 and the RMS level of the total response. As with the time-domain components shown in figure 30 it is possible to see the relative increase of nonlinearity in the response.

Figure 31 – Time-frequency distribution of the components of the response to a chirp excitation obtained by the Volterra model of the F-16 aircraft.

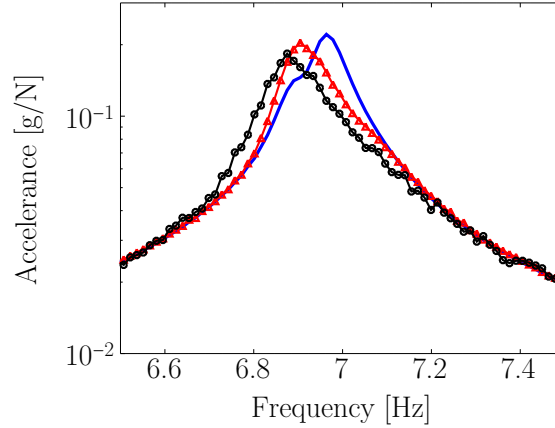


(a) Linear component y_1 .

(b) Nonlinear component y_3 .

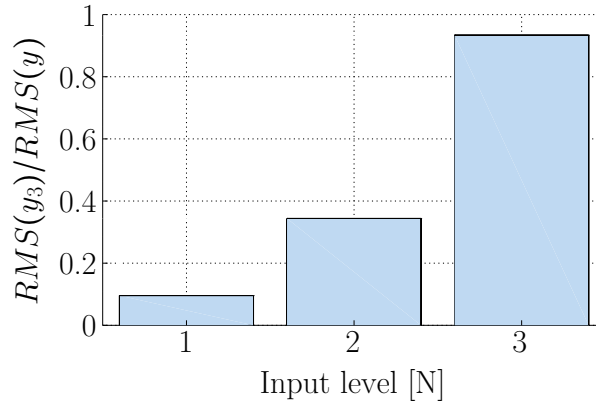
Source: Prepared by the author.

Figure 32 – FRFs of the output of the Volterra kernels using a chirp input. The continuous line — is the low input (5 N), $\text{--}\triangle\text{--}$ is the medium input (10 N) and $\text{--}\circ\text{--}$ is the high input (15 N).



Source: Prepared by the author.

Figure 33 – Nonlinearity index for the F-16 aircraft.



Source: Prepared by the author.

4.3 Conclusions

In this chapter the Volterra model was applied to represent the oscillatory behavior of two different nonlinear systems. The first example consisted in a cantilever beam under magnetic interactions between the free end of the beam and a permanent magnet. This system showed to have a hardening behavior where the apparent natural frequency was increasing with the increase in the displacement level of the beam. The second example was an F-16 aircraft under a ground vibration testing which showed to have softening effects as the input level increased.

With the Volterra kernels representing each of the systems it was possible to calculate

the linear and nonlinear components of the response. The analysis of the components of the output made evident the separation of response specially when analyzing the harmonic content of the output. Due to the structure of the Volterra series the first-order response showed the first-order harmonics as it was expected in a response of a linear system. Meanwhile the η - *th* components showed to respond with η -order harmonics. Also a measure of the degree of nonlinearity of the response of the model was calculated based on the ratio between the RMS amplitude of the nonlinear part of the response and the one from the linear component. This enabled to have a feeling on the degree of nonlinearity in the response of the Volterra model as the input level was increased.

However the Volterra model is highly dependent on the number of terms considered in the expansion meaning that for stronger effects of nonlinearity more terms can be necessary in the expansion. In this case it is possible that even with the orthonormal expansion the higher-order kernels can still suffer the problem of overparameterization making difficult to obtain a model of strong nonlinearities. Also some effects are still not reproducible with the orthonormal Volterra series as is the case of the jump phenomenon and chaotic behavior. It is clear that this can be a problem when trying to use this kind of model for the monitoring of structures since these unaccounted effects are not explained by this model. Solutions to this problem are still to be found in the literature in order to obtain more robust models.

5 Experimental application of Volterra series for SHM

This chapter presents experimental applications of the Volterra model for damage detection and parameter quantification. These tools will be used to show the importance of considering nonlinear behavior when monitoring initially nonlinear structures. In the section 5.1, the results of damage detection in the magneto-elastic system are presented. Section 5.2 shows the identification of parameters based on the Volterra kernels representing the reference system. Finally, in the section 5.3 the conclusions are presented about the results obtained in this chapter.

5.1 Application of Volterra series in the detection of structural variations

A damage detection study was performed in the magneto-elastic system identified in the section 4.1. In order to simulate structural changes, 8 states were defined by removing up to 3 masses and placing them back in the bolted connection located in the center of the beam where each mass was a small nut with nearly 2 grams. This was done to observe if the indexes showed the structure returning to the reference state by simulating the repair of the system. Each state was tested using a chirp input signal sweeping the frequency range starting in 10 to 50 Hz for a low (0.01 V) and a high level excitation (0.15 V) representing the linear and nonlinear regime of the structure respectively. This test was repeated 40 times in each structural condition for the statistical characterization of the proposed damage index. Table 11 summarizes the simulated structural conditions in the magneto-elastic system. Figure 34 shows an illustration with the simulated conditions.

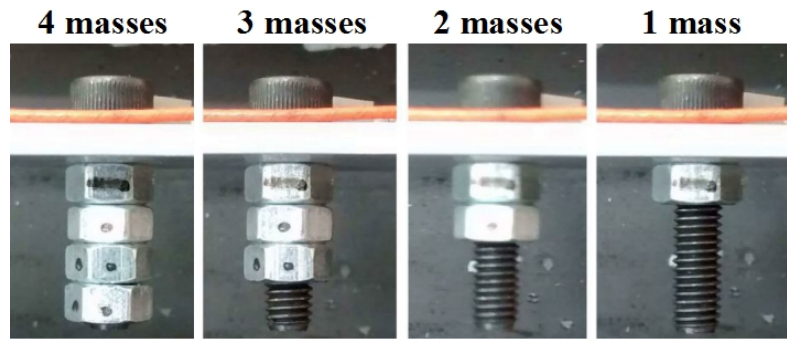
In the 40 repetitions of the test, each first realization was rejected in order to avoid the transient response during the start of the shaker testing. This resulted in a total of 39 blocks to be analyzed. For this magneto-elastic system the most relevant components were the first and third-order responses y_1 and y_3 . With this, only the indexes of the same

Table 11 – Structural states simulated in the nonlinear system.

State	Condition
1	4 nuts (reference)
2	3 nuts (damaged)
3	2 nuts (damaged)
4	1 nut (damaged)
5	1 nut (repair)
6	2 nuts (repair)
7	3 nuts (repair)
8	4 nuts (repair)

Source: Prepared by the author.

Figure 34 – Illustration of the damage simulation

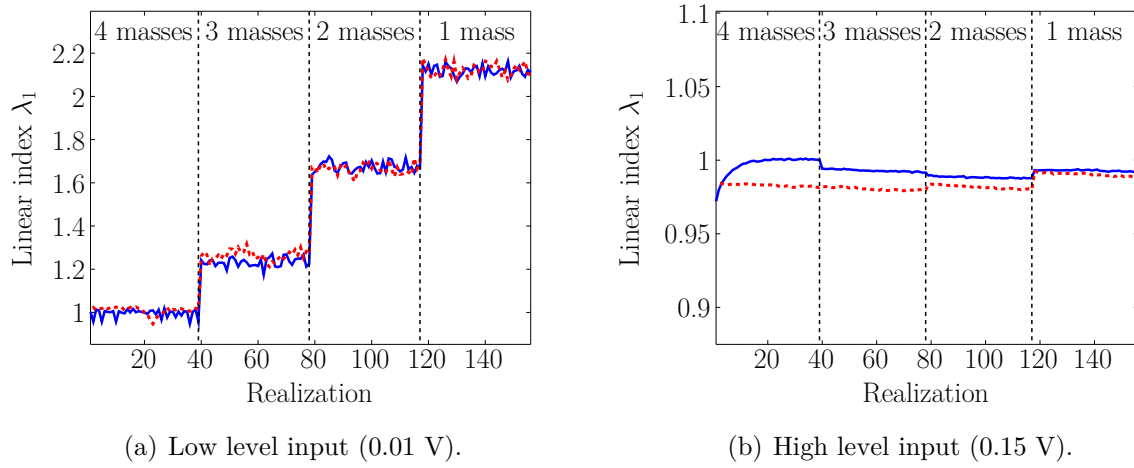


Source: Prepared by the author.

orders λ_1 and λ_3 were analyzed in order to observe differences between the performance of the linear and nonlinear models. This was also done for the linear regime with the input level of 0.01 V, and the nonlinear regime for 0.15 V. The indexes are illustrated in figure 35 for the linear index λ_1 , and figure 36 for the nonlinear index λ_3 , for both the low and high level input. In these figures the blue continuous line represents the indexes from states 1 to 4 during the decrease of the mass in the bolted connection. Meanwhile the red dashed lines presents the simulation of the repair of the system from the states 5 to 8. The vertical black dashed lines separates the indexes in different numbers of masses from 4 to 1 masses.

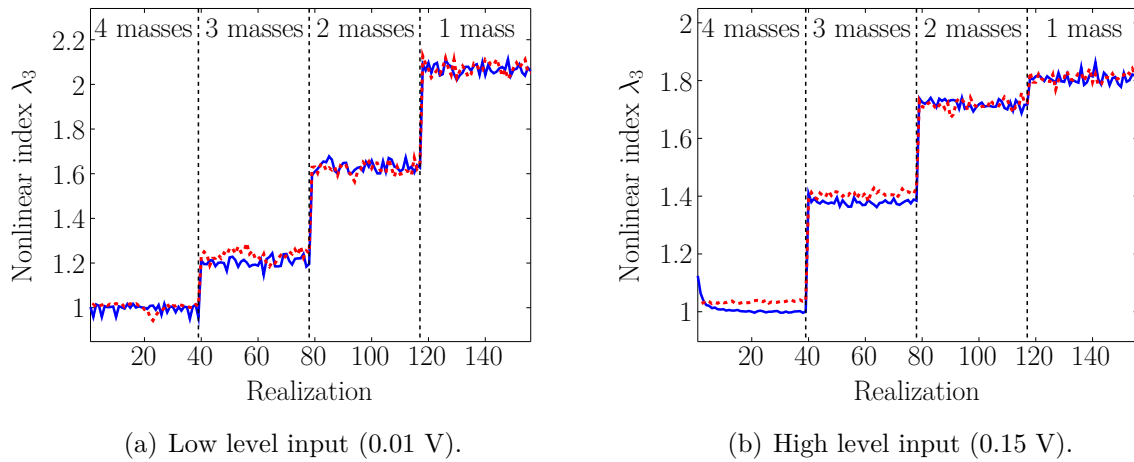
The linear damage index λ_1 presented in the figure 35(a) shows a clear separation between the structural states only in the case of low-level input 0.01 V. In this situation the linear model represented by the first Volterra kernels is already enough for an accurate detection of the increase in the prediction error due to the structural variation. Meanwhile when the input level is increased to 0.15 V, the nonlinear components of the response are more relevant as previously illustrated in figure 19. In this case the linear damage feature represented by the index λ_1 fails to distinguish the changes in the prediction error due to

Figure 35 – Linear index under two different input levels. The continuous line — is representing the indexes during the damage applications (states 1 to 4) and --- represents the indexes during the repair (states 5 to 8).



Source: Prepared by the author.

Figure 36 – Nonlinear index under two different input levels. The continuous line — is representing the indexes during the damage applications (states 1 to 4) and --- represents the indexes during the repair (states 5 to 8).



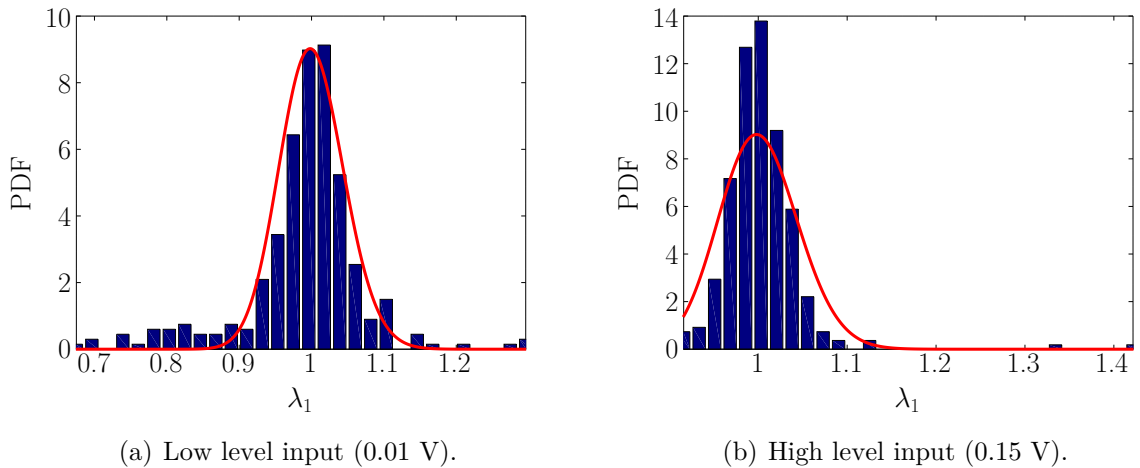
Source: Prepared by the author.

the nonlinearity and due to the removal of masses in the beam. This is observed in figure 35(b) as a lack of sensitivity in the index to the change in the structural state.

In the other hand, the nonlinear index λ_3 presented in the figure 36 can still separate the structural conditions in both the linear (0.01 V) and nonlinear regime of motion (0.15 V). This was already expected since the nonlinear higher-order terms are taken in account in the index λ_3 . Due to this, it is possible to see a clear difference on most of the realizations when comparing different states.

In order to have a statistical classification of the deviation with respect to the reference condition, it is possible to apply the statistical hypothesis test defined in the section 3.2. Before applying the test the theoretical probability density function (PDF) described by equation 28 was compared with the histogram of the experimentally calculated indexes. In order to have a more representative sample size the prediction errors were fractioned to increase the number of λ samples to 312 instead of 39. This was done in order o have a smoother histogram allowing to observe the trend of the experimental indexes to follow the theoretical distribution. The comparison between the theoretical distributions and the histogram of the experimental data are presented in figures 37 and 38 for the linear and nonlinear indexes respectively.

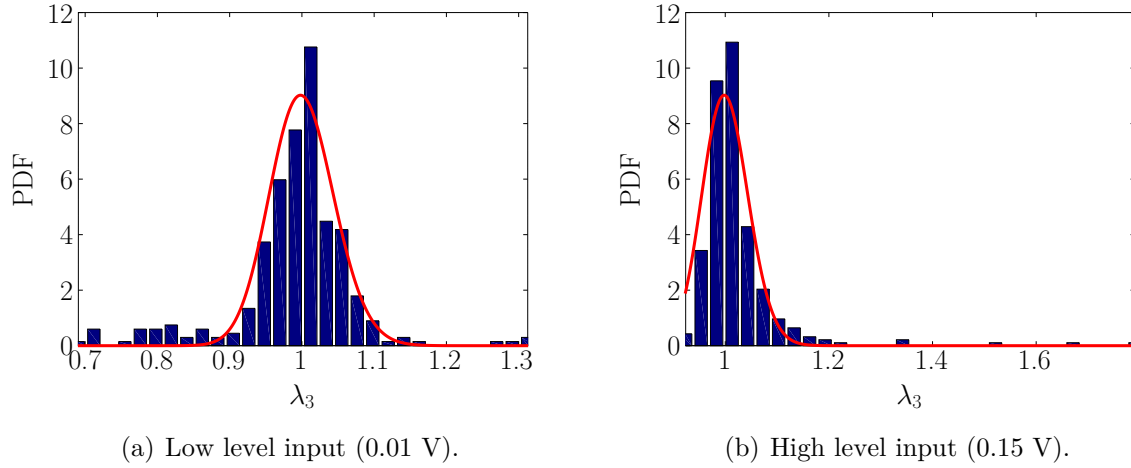
Figure 37 – Distribution of the experimental linear damage index compared to the theoretical distribution. The blue bar represents the experimental data and the continuous line — represents the theoretical distribution.



Source: Prepared by the author.

It is possible to observe that most of the indexes tend to follow the theoretical distribution where the only major exception is in the case of the linear index under the high level input. This is explained by the inadequate use of the linear model in the nonlinear regime that makes the data not to follow the distribution.

Figure 38 – Distribution of the experimental nonlinear damage index compared to the theoretical distribution. The blue bar represents the experimental data and the continuous line — represents the theoretical distribution.



Source: Prepared by the author.

Even in this case it was considered that the distribution of λ described by equation 28 was a good model of the statistical properties of the data. In the paper of Sohn and Farrar (2001), a correction on the degrees of freedom of the distribution was presented in order to take in account deviations of the experimental data with respect to the theoretical distribution. The results with the correction coefficients are suppressed in this thesis since little difference was obtained in these results.

Tables 12 and 13 summarize the results of the hypothesis tests for the indexes λ_1 and λ_3 for the linear and nonlinear regime respectively. The percentages of false alarms are the relative number of cases where the damage was detected in the structure with no damage, and the true detections are the cases where the damaged states were correctly detected. These values are written for significance levels of the test α of 2, 1 and 0.5% for comparison.

In the table 12 representing the performance of the indexes in the low-level input, no false alarms were observed and all the damaged cases, with less than 4 masses in the bolted connection, were correctly detected. This result was already observed in the figures 35 and 36 since the indexes in the reference condition and damaged states are well separated.

Table 13 however shows different results for the nonlinear regime. In this case, as it was previously said, the index λ_1 shows to be insensitive to structural variations, which is clearly showed by the 0% true detection rate. In the other hand, the index λ_3 detect 100% of the damaged states for the three values of α used. The drawback is that some

false alarms are present but they can be decreased from 34.6% to 3.85% by changing the significance level α from 2% to 0.5%. Decreasing the significance level in this hypothesis test is equivalent to use a tighter threshold value because α can be seen as a probability of false detections (BENDAT; PIERSOL, 2011). Since the values of λ_3 in the reference state 1 and in the damaged states are far from each other, using a very small value of α does not cause a decrease in the true detection rate.

Table 12 – Results of the hypothesis tests for different significance levels under low level input (0.01 V).

α [%]	Linear index (λ_1)		Nonlinear index (λ_3)	
	False alarm [%]	True detection [%]	False alarm [%]	True detection [%]
2	0	100	0	100
1	0	100	0	100
0.5	0	100	0	100

Source: Prepared by the author.

Table 13 – Results of the hypothesis tests for different significance levels under high level input (0.15 V).

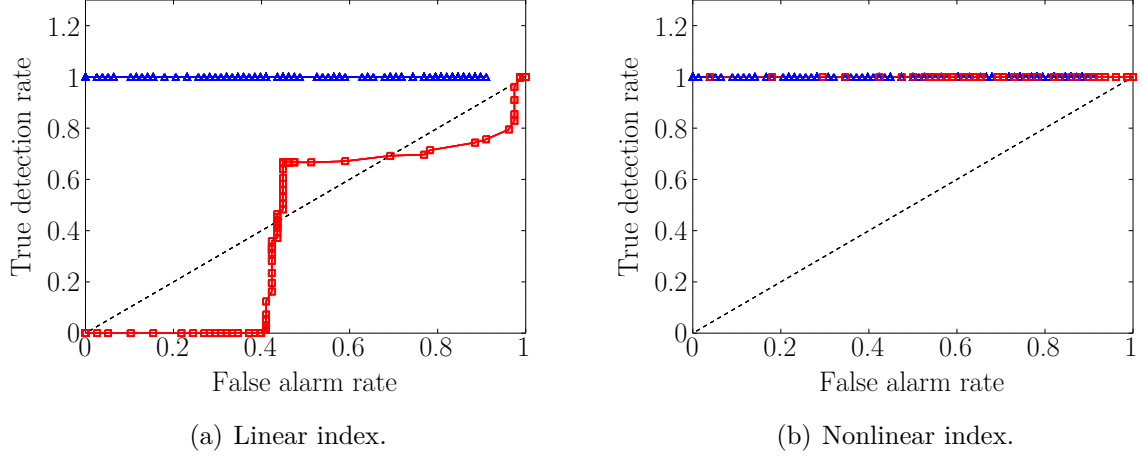
α [%]	Linear index (λ_1)		Nonlinear index (λ_3)	
	False alarm [%]	True detection [%]	False alarm [%]	True detection [%]
2	0	0	34.6	100
1	0	0	17.9	100
0.5	0	0	3.85	100

Source: Prepared by the author.

In order to analyze the variation of the classification index under different values of the significance level α it is possible to map this parameter in a receiver operating characteristic curve (ROC) (FARRAR; WORDEN, 2012a). This plot presents the false alarm rates against the corresponding true detection rate for different values of thresholds of the binary classification. In the case of this work the threshold is represented by the parameter α of the hypothesis test. The ROC curves are illustrated in figure 39 for the results of the linear and nonlinear indexes under the excitation levels of 0.01 and 0.15 V.

As previously stated, the linear index λ_1 shows to have a good detection rate only in the low level input, while it significantly loses performance when the nonlinear response is relevant. Meanwhile, the nonlinear index keeps higher values of detection rate even in the nonlinear regime. The only possible drawback is in the higher false alarms rate as previously mentioned but this value can be minimized by decreasing the significance level of the hypothesis test.

Figure 39 – ROC curves of the damage indexes. $-\triangle-$ is the curve for the low level input (0.01 V) and $-\square-$ is the curve for the high level input (0.15 V).



Source: Prepared by the author.

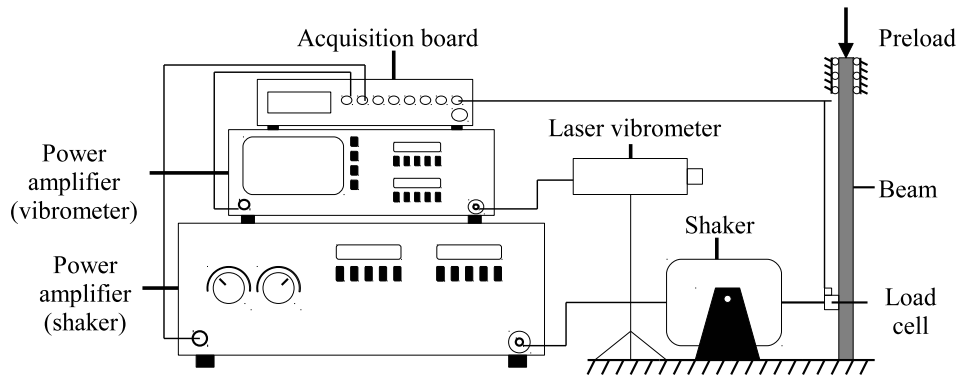
5.2 Parameter quantification in a nonlinear system

The parameter identification study was performed using the methodology described in the section 3.3. The structure that was investigated consisted in a clamped beam with dimensions of $460 \times 18 \times 2$ mm with a compressive preload applied in the top and regulated by a screw. The compressive force was adjusted to be just before the buckling load. The compressive load has the effect of decreasing the linear stiffness of the system allowing it to have more prominent nonlinear effects as stated by Tang et al. (2015) for the same kind of system. An electrodynamic shaker was attached 65 mm from the clamped end of the beam and a laser vibrometer was focused on the center of the beam measuring the velocities in this position. The power amplifier of the shaker was configured in current mode to keep constant the supplied electrical current for different input frequencies. The acquisition system and sensors and actuators used during these tests are:

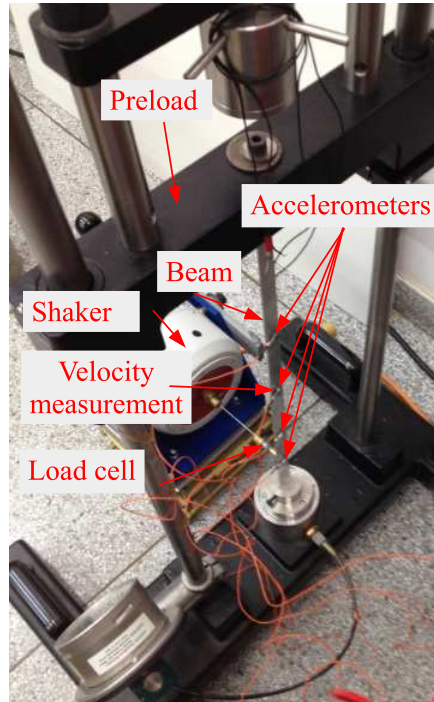
- TIRA Shaker TV51120-M;
- TIRA BAA 500 Power Amplifier;
- Polytec OFV-525/-5000-S modular laser vibrometer;
- Dytran load cell model 1022V (IEPE force sensor);
- m+p VibPilot acquisition system.

Figure 40 shows the experimental setup used in this section together with the nonlinear structure to be tested.

Figure 40 – Experimental setup of the clamped beam subjected to axial compression.



(a) Schematic diagram.



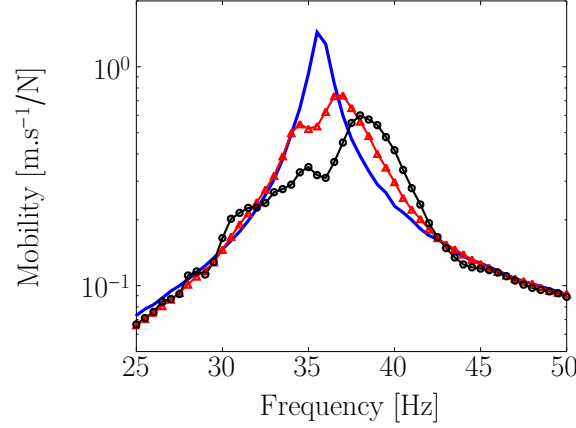
(b) Picture of the structure.

Source: Prepared by the author.

Figure 41 illustrates the FRF of the system to a chirp input that swept the range from 20 to 50 Hz, with input levels applied in the shaker of 0.01, 0.05 and 0.10 V. The systems shows to have a hardening effect as the displacement in the beam is increased since the natural frequency moves to higher values. One can also observe differences in the peak amplitude in the FRF.

Figure 42 shows the response of the system to a stepped sine excitation in the pre-stressed beam. The amplitude of oscillation is showed divided by the voltage level since

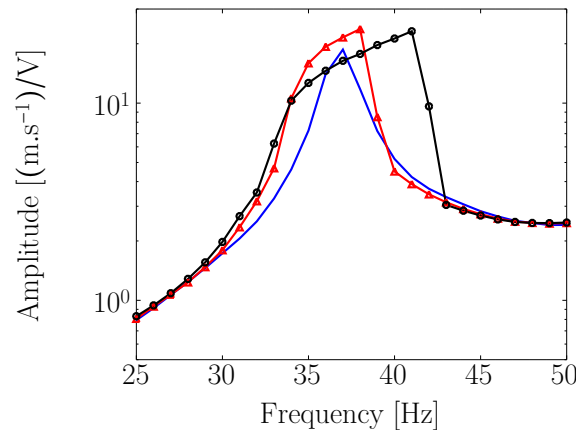
Figure 41 – Experimental FRF for the pre-compressed beam. The continuous line — was calculated for a low input level (0.01 V), -△- has a medium input level (0.05 V) and -○- has a high input level (0.10 V).



Source: Prepared by the author.

this electric parameter was easier to control during the experiment than the force itself due to the shaker-structure interaction. The beam was excited from 25 to 50 Hz with steps of 1 Hz in three different levels: 0.01, 0.14 and 0.20 V. Similar to the FRF in the figure 41 it is possible to observe the peak of response moving to higher values of frequency. It is also possible to see a sudden fall in the amplitude of response that represents the jump-down phenomenon.

Figure 42 – Experimental stepped sine response for the pre-compressed beam. The continuous line — was calculated for a low input level (0.01 V), -△- has a medium input level (0.14 V) and -○- has a high input level (0.20 V).



Source: Prepared by the author.

The dataset with a chirp input with 0.01 and 0.10 V applied in the shaker were used to calculate an approximation for the Volterra kernels and are summarized in the table 14.

Table 14 – Input signal used for the identification of the Volterra kernels representing the pre-compressed beam.

Parameter	Value
Input signal type	Chirp
Frequency range [Hz]	20 – 50
Number of samples	4096
Sweep rate [Hz/s]	7.5
Input levels [V]	0.01 V (low) and 0.10 V (high)

Source: Prepared by the author.

The convergence analysis of the model showed that an order $J_1 = 2$, $J_2 = 2$ and $J_3 = 8$ was enough to model the behavior of the structure. The parameters of the optimization algorithm and the values used for the Kautz orthonormal representation of the Volterra kernels are summarized in the tables 15 and 16 respectively. The main diagonals of the first three kernels considered in this study are illustrated in figure 43.

Table 15 – Optimization parameters for the calculation of the Kautz parameters in the pre-compressed beam.

Parameter of the optimization	Value
Algorithm	SQP
Objective function	NRMSE
Number of parameters	4 ($[\omega_2, \zeta_2, \omega_3, \zeta_3]$)
Maximum number of iterations	1000
Termination tolerance on the variable	10^{-10}
Termination tolerance on the objective function	10^{-10}
Initial guess	[35.77 Hz, 1.63 %, 35.77 Hz, 1.63 %]
Lower search bound	[28.61 Hz, 0.1 %, 28.61 Hz, 0.1 %]
Upper search bound	[42.92 Hz, 5.0 %, 42.92 Hz, 5.0 %]

Source: Prepared by the author.

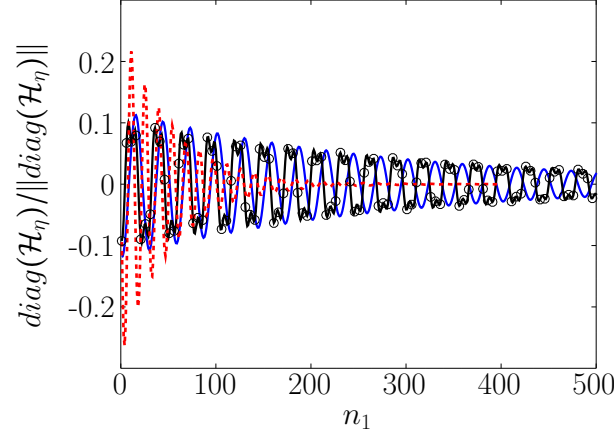
Table 16 – Information for the identification of the Kautz functions of the pre-compressed beam.

ω_1 [Hz]	ζ_1 [%]	ω_2 [Hz]	ζ_2 [%]	ω_3 [Hz]	ζ_3 [%]
35.77	1.63	35.67	4.31	36.76	0.44

Source: Prepared by the author.

The estimated kernels can be used as a reference of the experimental structure in order to obtain a parametric model of this system. With the observed effects in the structure it is possible to assume a gray-box model by assuming a Duffing-type of oscillator with polynomial stiffness to take in account the observed hardening effect. Note that the

Figure 43 – Identified Volterra kernels representing the pre-compressed beam. The continuous line — is the normalized 1st kernel ($\mathcal{H}_1/\|\mathcal{H}_1\|$), --- is the normalized main diagonal of the 2nd kernel ($\text{diag}(\mathcal{H}_2)/\|\text{diag}(\mathcal{H}_2)\|$) and -o- is the normalized main diagonal of the 3rd kernel ($\text{diag}(\mathcal{H}_3)/\|\text{diag}(\mathcal{H}_3)\|$).



Source: Prepared by the author.

assumed model is representing a system composed by the shaker and the structure itself. Asymmetric effects were ignored for this model since they showed not to be relevant in the response of the considered model:

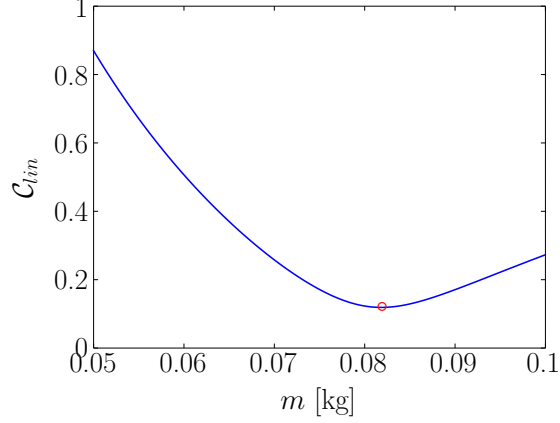
$$m\ddot{x} + c\dot{x} + k_1x + k_3x^3 = f(t) \quad (34)$$

For the extraction of the Volterra kernels representing this parametric model, the motion equation was numerically integrated with the Newmark average acceleration method adapted for nonlinear polynomial systems and solved with a Newton-Raphson procedure (NEWMARK, 1959; JACOB; EBECKEN, 1994). The same sampling frequency and input signals as in the experimental data were used to integrate the motion equation. With the velocity obtained in the solution of the differential equation and the measured input signal, the kernels representing the Duffing oscillator were calculated. These Volterra kernels can be compared with the reference ones using the objective functions previously presented in equations 32 and 33.

For the calculation of parameters of the underlying linear model (m , c and k_1), the low input level dataset (0.01 V) was employed. The natural frequency and damping ratio of the first Volterra kernel were considered to be the true values of the Duffing oscillator in equation 34. With this consideration it was possible to map the values of the mass coefficient m with the linear objective function ($\mathcal{C}_{lin}$) based on the residue of the first Volterra kernel (equation 32). One should note that although the Volterra

series methodology is used here, more conventional ways can be used to obtain the linear parameters. Figure 44 shows $\mathcal{C}_{lin}$ as a function of the mass coefficient with the minimum found illustrated as a red circle.

Figure 44 – Linear objective function as a function of the mass coefficient.

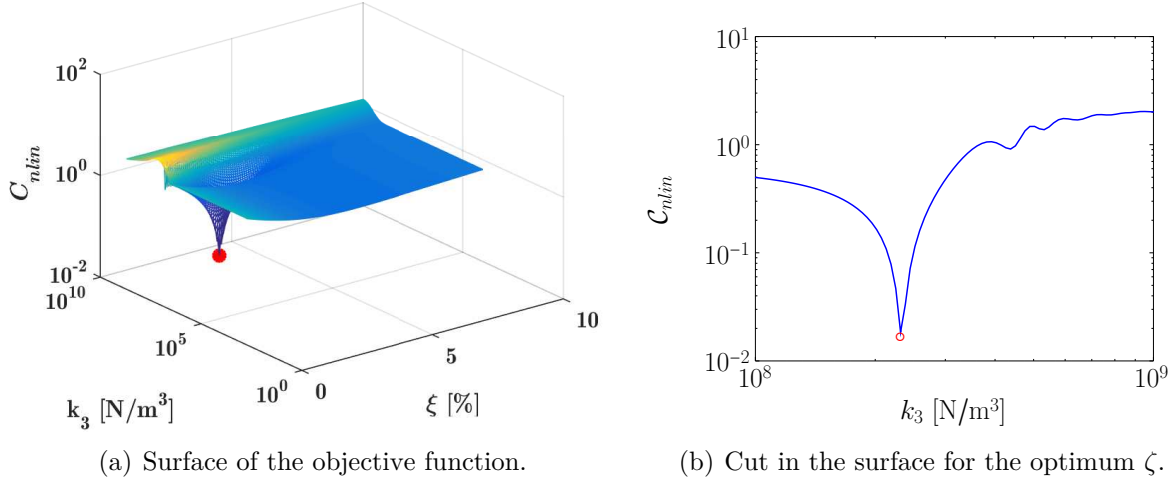


Source: Prepared by the author.

After the identification of the linear parameters of the motion equation it is possible to calculate the cubic stiffness of the oscillator. The nonlinear metric based on the deviations of the second and third-order kernels was mapped for the higher input levels of 0.05 and 0.10 V where the nonlinear behavior started to be more visible in the system. However, the damping ratio of the Duffing oscillator (ξ) showed to vary in the mapping of $\mathcal{C}_{nlin}$ as the input levels was increased. In this sense, ξ was also included as a parameter to be updated with the use of the nonlinear objective function. Figures 45 and 46 shows the objective function surface as a function of k_3 and ξ with a red circle indicating the optimum value found for the two higher input levels.

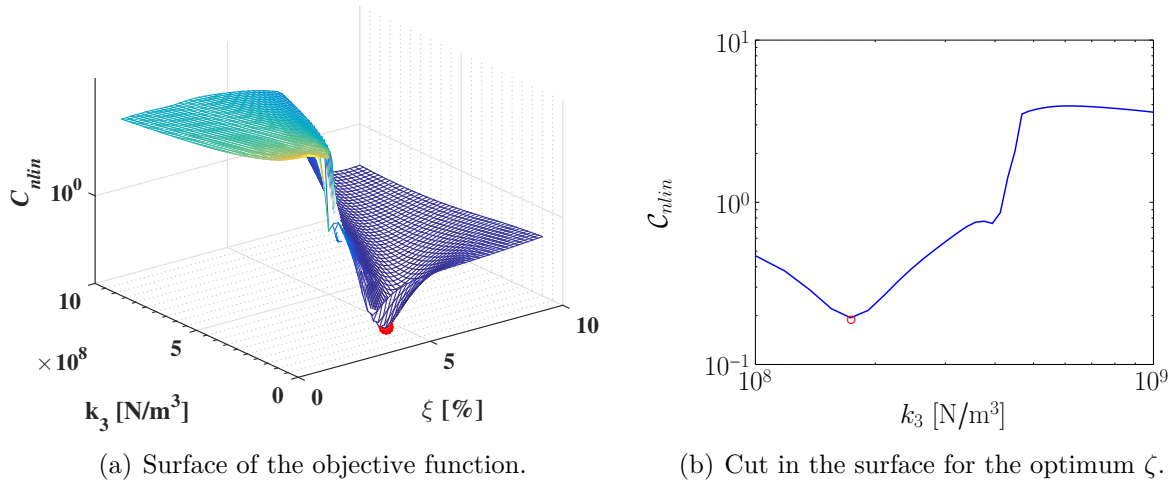
The parameters found for the Duffing equation for each input level are summarized in table 17. Although some variation was found in k_3 the value maintained the same order of magnitude. The damping ratio has a big increase through the three different input levels, it is likely that the damping itself was not correctly modeled and a nonlinear model can be necessary.

Figure 45 – Nonlinear objective function in relationship to the k_3 and ζ for the medium input level data (0.05 V).



Source: Prepared by the author.

Figure 46 – Nonlinear objective function in relationship to the k_3 and ζ for the high input level data (0.10 V).



Source: Prepared by the author.

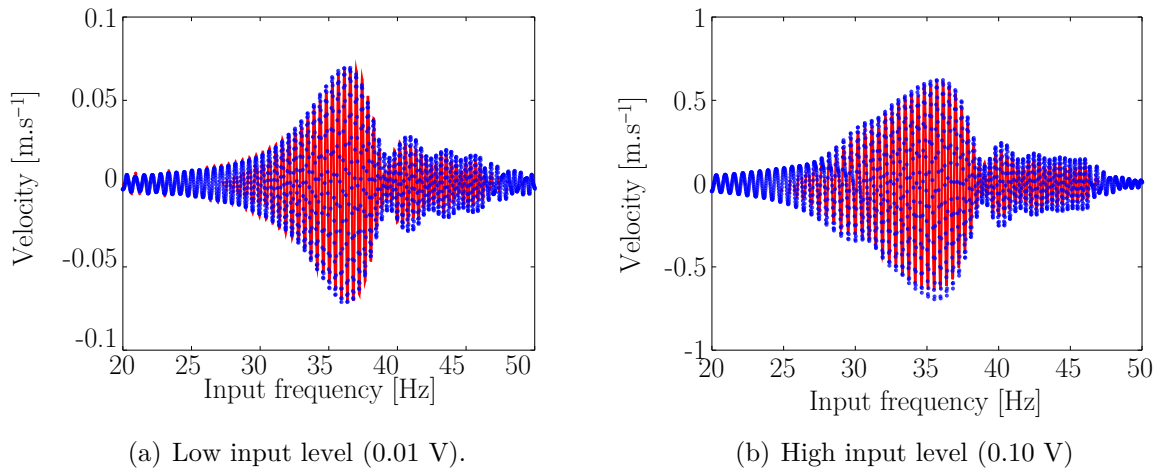
Table 17 – Identified parameters for the motion equation.

Input level [V]	m [kg]	ξ [%]	$k_1 \times 10^3$ [N/m]	$k_3 \times 10^8$ [N/m ³]
0.01	0.082	1.6	4.13	
0.05	0.082	3.3	4.13	2.13
0.10	0.082	4.7	4.13	1.73

Source: Prepared by the author.

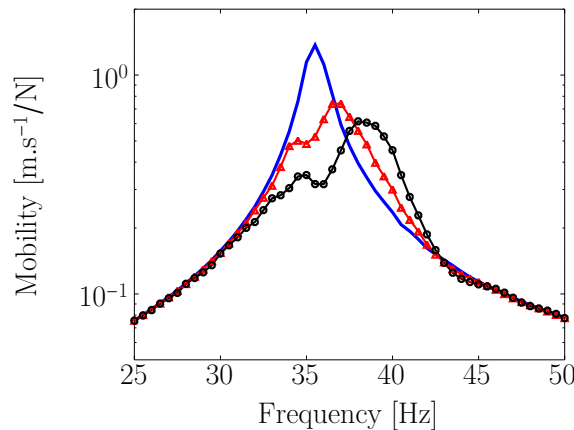
Figures 47, 48 and 49 shows the response of the gray-box model with the direct comparison between the time-domain signals, the FRFs and the response to the stepped sine input respectively. These figures shows that the model can reproduce the same type of behavior observed in the experimental data. The only major difference that can be seen in the response to the stepped sine input since the model is not subjected to the shaker-structure interaction that affects the measured response. Other issue in the identified model, as it was previously stated, is about the model of the damping in the system that showed to behave in a nonlinear way as well. This should be taken in account in the future although the modeling of energy dissipation is not yet fully understood.

Figure 47 – Output of the identified grey-box model. The continuous line — represents the response of the model and, $\circ$ is the measured response.



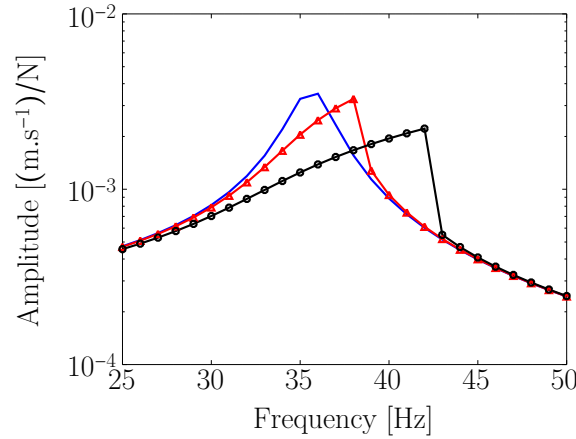
Source: Prepared by the author.

Figure 48 – Experimental FRF for the pre-compressed beam. The continuous line — was calculated for a low input level (0.01 V), $-\triangle-$ has a medium input level (0.05 V) and $-\circ-$ has a high input level (0.10 V).



Source: Prepared by the author.

Figure 49 – Experimental stepped sine response for the pre-compressed beam. The continuous line — was calculated for a low input level (0.01 V), $\text{--}\triangle\text{--}$ has a medium input level (0.14 V) and $\text{--}\circ\text{--}$ has a high input level (0.20 V).



Source: Prepared by the author.

5.3 Conclusions

This chapter presented the application of Volterra series for the problems of damage detection and quantification of parameters for initially nonlinear systems. The methodology of damage detection presented in this thesis was applied in a nonlinear magneto-elastic system in order to detect small mass variations introduced in the system by removing the nuts in a bolted connection. The proposed index was based on the detection of changes on the prediction error of the reference Volterra model representing the healthy structure. An hypothesis test was applied in the experimental indexes in order to test the classification capability of the linear and nonlinear versions of the index. While the linear index showed to fail to detect structural variations during the nonlinear regime of motion, the nonlinear index was still able to detect these variations. However, it is likely that for stronger levels of nonlinearity the nonlinear index will have problems to correctly classify the structure since more terms in the Volterra series expansion might be necessary. Also, environmental effects (e.g. variations in the temperature and humidity) that are not taken in account in the reference model can still be a problem causing errors in the damage detection.

The parameter identification study was done in a clamped beam under compressive preload. The analysis of the system behavior showed a hardening effect and the jump effect during the stepped sine testing. With this, a gray-box model was assumed to represent the first mode of the shaker-beam system as a symmetric Duffing oscillator where the nonlinearity is given by a cubic polynomial term in the stiffness. The parameters

of the oscillator were estimated by minimizing the residue between the experimental reference Volterra kernels and the ones extracted from the Duffing oscillator. The linear parameters were estimated using the input and output in the system during the linear regime of motion with a very low input level. After that, the nonlinear stiffness and the damping ratio were updated for the higher levels of input. Since the damping had to be updated when the excitation was increased, it is possible that a nonlinearity in the damping is also manifesting in the system and the linear viscous damping might not be the best assumption.

The results obtained in this chapter tried to make clear the needs of considering the inherent nonlinearities in the structures to be monitored and modeled. For the damage detection algorithm, it is important to take in account apparent variations in the behavior of the system that comes from the system itself so that it is really possible to detect structural variations. In a similar way, proper parametric models have also to be assumed in order to reproduce nonlinear effects as the hardening behavior and the jump effect.

6 Final remarks

This final chapter presents the closure of this thesis. The main conclusions about the results obtained in this work are summarized in the section 6.1 and a few suggestions for future works are presented in the section 6.2.

6.1 Main conclusions

In this work, methodologies were presented for the application of Volterra series for problems of damage detection and quantification of parameters. The formulation for the calculation of Volterra kernels and the application of the model as a SHM tool were presented respectively in chapters 2 and 3. The proposed strategies were then illustrated in the chapters 4 and 5 for experimental setups with datasets from modal tests.

Considering the obtained results the following conclusions were obtained:

- The Volterra model using the orthonormal representation of the kernels showed to be a more practical way to estimate this model minimizing the overparameterization issues. The separation of the linear and nonlinear components of the response represented the increase in the nonlinear behavior with higher levels of displacement in the studied systems, as well as the harmonic generation in the response. However, this system identification technique is still not able to fully represent all the rich nonlinear dynamics, specially considering more complex effects as jumps and chaotic responses, or even discontinuous nonlinearities;
- As a damage indicator, the indexes based on the Volterra model allowed to show the differences of performance between linear and nonlinear metrics during the nonlinear regime of the structure. The need for nonlinear models in damage detection was made clear by these results since classical linear indicators can have a lack of sensitivity in the monitoring of nonlinear systems;
- The objective function formulated to be minimized in problems of parameter iden-

tification showed to be able to find suitable parameters that reproduce at least qualitatively the behavior of the experimental setup. The methodology based on the Volterra kernels allowed to calculate a linear and nonlinear versions of the metric in order to obtain a parametric gray-box model that reproduced the jump effect in the response of the setup. However, variations in the damping of the structure with the increase in the input level brings the fact that the assumption of linear viscous damping was not totally satisfactory;

- The mentioned limitations of the Volterra series are still a problem to be solved since nonlinear effects that are not reproducible by this model could still be viewed as outliers generated by structural variations. Improvements in the Volterra model can be done trying to solve these limitations and more robust tools have to be investigated in order to properly treat the nonlinear phenomena.

6.2 Suggestions for future works

Analyzing this work it is possible to define a few future research topics in order to improve the current results and also explore new applications:

- Evaluate the Volterra series expansion with higher-order terms to understand the necessary number of terms to represent the nonlinear response. Also a better understanding of the use of orthonormal functions in the representation of the kernels and a development of a faster way to define these functions can improve the use of the Volterra model;
- Reformulate the parameter identification scheme avoiding the integration of the motion equation in every iteration that the kernels are calculated for the model to be updated. One possibility is to use an harmonic probing method to evaluate the kernels of the parametric model and compare those with the Volterra kernels estimated with the experimental time-series;
- Updating of nonlinear finite element models using objective functions computed by Volterra series (BUSSETTA et al., 2015);
- There are a large uncertainty to define the Kautz parameters in the Volterra series. Additionally, the Volterra kernels also have several uncertainties and are assumed to be deterministic in the present work. It is necessary to approach the Volterra

kernels as random variables and to perform an uncertainty quantification (VILLANI et al., 2015);

- Include the possibility of variation of the Kautz parameters for different levels of displacement of the system using, for example, expressions of natural frequency and damping ratios calculated using perturbation analysis (KOVACIC; BRENNAN, 2011);
- Use the structure of the Volterra series expansion to represent nonlinear systems without the knowledge of the excitation (SCUSSEL; da SILVA, 2015);
- Application of the Volterra kernels as a model for the vibration control of nonlinear structures.

A few of the suggested research subjects are ongoing topics in the investigations of a doctorate student and a postdoctoral researcher.

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Appendix A - Details on the identification of Volterra kernels

This appendix shows in detail the construction of the matrices for the least-square estimation of the Volterra kernels in the orthonormal basis $\mathcal{B}_\eta(i_1, \dots, i_\eta)$ which was briefly presented in chapter 2. The procedure for the identification of Volterra kernels in two separated steps is also presented.

Identification of the orthonormal Volterra kernels

According to the Volterra theory, the k -th sample of the η -order response ($y_\eta(k)$) of the Volterra model can be given by (SCHETZEN, 1980):

$$y_\eta(k) = \sum_{n_1=0}^{N_\eta-1} \cdots \sum_{n_\eta=0}^{N_\eta-1} \mathcal{H}_\eta(n_1, \dots, n_\eta) \prod_{i=1}^{\eta} u(k - n_i) \quad (\text{A.1})$$

where $\mathcal{H}_\eta(n_1, \dots, n_\eta)$ is the η -order Volterra kernel in the physical basis, N_η is the memory length of the η -order kernel and $u(k - n_i)$ is the input signal delayed in n_i samples. This expression can instead be written in the orthonormal basis:

$$y_\eta(k) \approx \sum_{i_1=0}^{J_\eta-1} \cdots \sum_{i_\eta=0}^{J_\eta-1} \mathcal{B}_\eta(i_1, \dots, i_\eta) \prod_{j=1}^{\eta} l_{\eta, i_j}(k) \quad (\text{A.2})$$

where J_η is the number of orthonormal functions used to represent the kernel $\mathcal{H}_\eta(n_1, \dots, n_\eta)$, $\mathcal{B}_\eta(i_1, \dots, i_\eta)$ is the orthonormal projection of the kernel and $l_{\eta, i_j}(k)$ is the input signal $u(k)$ filtered by the i_j -th orthonormal filter ψ_{η, i_j} .

To demonstrate the identification of the kernels, it was considered a model of order $\eta = 3$ without loss of generality since for higher orders the procedure follows the same idea. In this case, it is possible to obtain the first three components of the Volterra expansion with the equation A.2:

$$y_1(k) = \sum_{i_1=0}^{J_1-1} \mathcal{B}_1(i_1) l_{1, i_1}(k) \quad (\text{A.3})$$

$$y_2(k) = \sum_{i_1=0}^{J_2-1} \sum_{i_2=0}^{J_2-1} \mathcal{B}_2(i_1, i_2) l_{2,i_1}(k) l_{2,i_2}(k) \quad (\text{A.4})$$

$$y_3(k) = \sum_{i_1=0}^{J_3-1} \sum_{i_2=0}^{J_3-1} \sum_{i_3=0}^{J_3-1} \mathcal{B}_3(i_1, i_2, i_3) l_{3,i_1}(k) l_{3,i_2}(k) l_{3,i_3}(k) \quad (\text{A.5})$$

where $y_1(k)$, $y_2(k)$ and $y_3(k)$ are the first, second and third-order responses respectively, $\mathcal{B}_1(i_1)$, $\mathcal{B}_2(i_1, i_2)$ and $\mathcal{B}_3(i_1, i_2, i_3)$ are the first, second and third-order orthonormal kernels, and $l_{1,i_1}(k)$, $l_{2,i_1}(k)$, $l_{3,i_1}(k)$, $l_{2,i_2}(k)$, $l_{3,i_2}(k)$ and $l_{3,i_3}(k)$ are the input signals filtered by the orthonormal functions. The Volterra model can be written in the matrix format to a better understanding of the calculation of the kernels. In this case, the collection of the first-order response of the Volterra model $y_1(k)$ written in the vector format can be expressed as:

$$\mathbf{y}_1 = \mathbf{\Gamma}_1 \mathbf{\Phi}_1 \quad (\text{A.6})$$

where $\mathbf{y}_1$ is the vector with the collection of the k samples of the linear response:

$$\mathbf{y}_1 = \begin{bmatrix} y_1(0) \\ y_1(1) \\ \vdots \\ y_1(k) \end{bmatrix} \quad (\text{A.7})$$

and $\mathbf{\Gamma}_1$ is the matrix with the input signal filtered by the i_j -th orthonormal functions representing the first Volterra kernel. The k -th row of the matrix $\mathbf{\Gamma}_1$ can be written as:

$$\mathbf{\Gamma}_1(k, :) = \begin{bmatrix} l_{1,0}(k) & l_{1,1}(k) & \cdots & l_{1,i_j}(k) \end{bmatrix} \quad (\text{A.8})$$

and $\mathbf{\Phi}_1$ is a vector with the collection of the samples of the first-order orthonormal kernel:

$$\mathbf{\Phi}_1 = \begin{bmatrix} \mathcal{B}_1(0) \\ \mathcal{B}_1(1) \\ \vdots \\ \mathcal{B}_1(i_1) \end{bmatrix} \quad (\text{A.9})$$

In a similar way for the second order response, $y_2(k)$ can be gathered in the vector format. The matrix form of the Volterra series expansion is still possible even for the nonlinear terms because the Volterra model is linear with respect to the parameters (AGUIRRE, 2007). The vector of the second-order response $\mathbf{y}_2$ can be given by :

$$\mathbf{y}_2 = \mathbf{\Gamma}_2 \mathbf{\Phi}_2 \quad (\text{A.10})$$

where the vector of the second-order response is written as:

$$\mathbf{y}_2 = \begin{bmatrix} y_2(0) \\ y_2(1) \\ \vdots \\ y_2(k) \end{bmatrix} \quad (\text{A.11})$$

the k - th row of the matrix $\mathbf{\Gamma}_2$ with the filtered input signal can be given by:

$$\mathbf{\Gamma}_2(k, :) = \begin{bmatrix} l_{2,0}(k)^2 & 2l_{2,1}(k)l_{2,0}(k) & l_{2,1}(k)^2 & \cdots & 2l_{2,i_j}(k)l_{2,0}(k) & \cdots & l_{2,i_j}(k)^2 \end{bmatrix} \quad (\text{A.12})$$

and the vector with the unique terms of the second kernel is given by:

$$\mathbf{\Phi}_2 = \begin{bmatrix} \mathcal{B}_2(0,0) \\ \mathcal{B}_2(1,0) \\ \mathcal{B}_2(1,1) \\ \vdots \\ \mathcal{B}_2(i_1,0) \\ \vdots \\ \mathcal{B}_2(i_1,i_2) \end{bmatrix} \quad (\text{A.13})$$

One should note here that not all the terms of the whole kernel $\mathcal{B}_2(i_1, i_2)$ are contained in the vector $\mathbf{\Phi}_2$ due to the consideration of symmetry of the second-order kernel. This property allows to use only the unique terms of the kernel $\mathcal{B}_2(i_1, i_2)$ that in this case are the main diagonal and the terms below the main diagonal of the matrix representing $\mathcal{B}_2(i_1, i_2)$. In this representation the off-diagonal terms have to be multiplied by 2 as it is possible to observe in equation A.12 since these terms appear twice in $\mathcal{B}_2(i_1, i_2)$.

Finally for the third-order response $y_3(k)$ the samples gathered in the vector $\mathbf{y}_3$ can be given as:

$$\mathbf{y}_3 = \mathbf{\Gamma}_3 \mathbf{\Phi}_3 \quad (\text{A.14})$$

where this vector is written as:

$$\mathbf{y}_3 = \begin{bmatrix} y_3(0) \\ y_3(1) \\ \vdots \\ y_3(k) \end{bmatrix} \quad (\text{A.15})$$

the k -th row of the matrix with the filtered input signal $\mathbf{\Gamma}_3$ is given by:

$$\mathbf{\Gamma}_3(k, :) = \begin{bmatrix} l_{3,0}(k)^3 & 3l_{3,1}(k)l_{3,0}(k)^2 & 3l_{3,1}(k)^2l_{3,0}(k) & l_{3,1}(k)^3 & 6l_{3,2}(k)l_{3,1}(k)l_{3,0}(k) & \cdots \\ \cdots & l_{3,2}(k)^3 & \cdots & 3l_{3,i_j}(k)l_{3,0}(k)^2 & \cdots & l_{3,i_j}(k)^3 \end{bmatrix} \quad (\text{A.16})$$

and the vector with the unique terms of the third-order orthonormal Volterra kernel is:

$$\mathbf{\Phi}_3 = \begin{bmatrix} \mathcal{B}_3(0,0,0) \\ \mathcal{B}_3(1,0,0) \\ \mathcal{B}_3(1,1,0) \\ \mathcal{B}_3(1,1,1) \\ \mathcal{B}_3(2,0,0) \\ \mathcal{B}_3(2,1,0) \\ \vdots \\ \mathcal{B}_3(2,2,2) \\ \vdots \\ \mathcal{B}_3(i_1,0,0) \\ \vdots \\ \mathcal{B}_3(i_1,i_2,i_3) \end{bmatrix} \quad (\text{A.17})$$

As considered before, the symmetry of the kernels allows to represent only the unique terms in $\mathbf{\Phi}_3$. For this is necessary to considerate all the permutations of the indexes $i_1, i_2, \dots, i_\eta$ in the input matrix. Because of this in equation A.16 a few terms are multiplied by 3 or 6 depending on how many permutations of the indexes are available.

With the definitions of the response vectors it is possible to calculate the total response $\mathbf{y}$ for the model considered to be truncated in the order $\eta = 3$:

$$\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2 + \mathbf{y}_3 \quad (\text{A.18})$$

the matrix with the input signal filtered by the filters representing the first, second and third-order kernels $\mathbf{\Gamma}$ is given by:

$$\mathbf{\Gamma} = \begin{bmatrix} \mathbf{\Gamma}_1 & \mathbf{\Gamma}_2 & \mathbf{\Gamma}_3 \end{bmatrix} \quad (\text{A.19})$$

and the vector with the unique terms of the first three Volterra kernels $\mathbf{\Phi}$ is given by:

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}_1 \\ \mathbf{\Phi}_2 \\ \mathbf{\Phi}_3 \end{bmatrix} \quad (\text{A.20})$$

and the total response of the system is a simple multiplication:

$$\mathbf{y} = \mathbf{\Gamma}\mathbf{\Phi} \quad (\text{A.21})$$

With this formulation the vector with the unknown orthonormal Volterra kernels $\mathbf{\Phi}$ can be approximated by the least-squares approximation:

$$\mathbf{\Phi} = (\mathbf{\Gamma}^T \mathbf{\Gamma})^{-1} \mathbf{\Gamma}^T \mathbf{y} \quad (\text{A.22})$$

where now $\mathbf{y}$ should be interpreted as the vector with the measured response of the system and $\mathbf{\Gamma}$ is the matrix with the filtered experimental input. This methodology considers that the system excites the components of the response until the order $\eta = 3$. In this sense the user has to be sure that the nonlinearities of the structure are really being excited by the input signal $u(k)$.

Example of the construction of the $\mathbf{\Gamma}$ and $\mathbf{\Phi}$ matrices

For a simple example of the construction of the $\mathbf{\Gamma}$ and $\mathbf{\Phi}$ matrices, it is possible to consider the case where the kernels are truncated in the third-order response ($\eta = 3$) and each kernel is expanded with a pair of orthonormal Kautz functions ($J_1 = J_2 = J_3 = 2$). In this case the arrays representing the first three orthonormal projections of the Volterra kernels can be represented as:

$$\mathcal{B}_1(i_1) = \begin{bmatrix} \mathcal{B}_1(0) \\ \mathcal{B}_1(1) \end{bmatrix} \quad (\text{A.23})$$

$$\mathcal{B}_2(i_1, i_2) = \begin{bmatrix} \mathcal{B}_2(0, 0) & \mathcal{B}_2(0, 1) \\ \mathcal{B}_2(1, 0) & \mathcal{B}_2(1, 1) \end{bmatrix} \quad (\text{A.24})$$

$$\begin{aligned} \mathcal{B}_3(i_1, i_2, 0) &= \begin{bmatrix} \mathcal{B}_3(0, 0, 0) & \mathcal{B}_3(0, 1, 0) \\ \mathcal{B}_3(1, 0, 0) & \mathcal{B}_3(1, 1, 0) \end{bmatrix} \\ \mathcal{B}_3(i_1, i_2, 1) &= \begin{bmatrix} \mathcal{B}_3(0, 0, 1) & \mathcal{B}_3(0, 1, 1) \\ \mathcal{B}_3(1, 0, 1) & \mathcal{B}_3(1, 1, 1) \end{bmatrix} \end{aligned} \quad (\text{A.25})$$

Note that the first kernel $\mathcal{B}_1(i_1)$ is represented by a row vector with two terms, the second kernel $\mathcal{B}_2(i_1, i_2)$ is a 2×2 matrix and the third-order kernel $\mathcal{B}_3(i_1, i_2, i_3)$ is an array with three dimensions of order $2 \times 2 \times 2$ being represented by two layers of 2×2 matrices.

With these first three terms in the Volterra expansion the k -th sample of the responses can be written as:

$$y_1(k) = \mathcal{B}_1(0)l_{1,0}(k) + \mathcal{B}_1(1)l_{1,1}(k) \quad (\text{A.26})$$

$$\begin{aligned} y_2(k) &= \mathcal{B}_2(0, 0)l_{2,0}(k)^2 + \mathcal{B}_2(0, 1)l_{2,0}(k)l_{2,1}(k) + \mathcal{B}_2(1, 0)l_{2,1}(k)l_{2,0}(k) + \dots \\ &\dots + \mathcal{B}_2(1, 1)l_{2,1}(k)^2 \end{aligned} \quad (\text{A.27})$$

$$\begin{aligned} y_3(k) &= \mathcal{B}_3(0, 0, 0)l_{3,0}(k)^3 + \mathcal{B}_3(0, 0, 1)l_{3,0}(k)^2l_{3,1}(k) + \dots \\ &\dots + \mathcal{B}_3(0, 1, 0)l_{3,0}(k)^2l_{3,1}(k) + \mathcal{B}_3(0, 1, 1)l_{3,0}(k)l_{3,1}(k)^2 + \dots \\ &\dots + \mathcal{B}_3(1, 0, 0)l_{3,1}(k)l_{3,0}(k)^2 + \mathcal{B}_3(1, 0, 1)l_{3,1}(k)^2l_{3,0}(k) + \dots \\ &\dots + \mathcal{B}_3(1, 1, 0)l_{3,1}(k)^2l_{3,0}(k) + \mathcal{B}_3(1, 1, 1)l_{3,1}(k)^3 \end{aligned} \quad (\text{A.28})$$

One can note that higher-order responses can have many terms compared to the linear response. As previously stated, if the symmetry property is considered for the higher-order kernels, all the permutations of indexes $i_1, i_2, \dots, i_\eta$ will produce the same values in the η -th kernel $\mathcal{B}_\eta(i_1, i_2, \dots, i_\eta)$. For the example here presented this means that $\mathcal{B}_2(0, 1) = \mathcal{B}_2(1, 0)$ for the second-order kernel, and $\mathcal{B}_3(0, 0, 1) = \mathcal{B}_3(0, 1, 0) = \mathcal{B}_3(1, 0, 0)$ and $\mathcal{B}_3(0, 1, 1) = \mathcal{B}_3(1, 0, 1) = \mathcal{B}_3(1, 1, 0)$ for the third-order projection. This allows to

simplify the equations A.27 and A.28 of $y_2(k)$ and $y_3(k)$ respectively:

$$y_2(k) = \mathcal{B}_2(0, 0)l_{2,0}(k)^2 + 2\mathcal{B}_2(0, 1)l_{2,0}(k)l_{2,1}(k) + \mathcal{B}_2(1, 1)l_{2,1}(k)^2 \quad (\text{A.29})$$

$$\begin{aligned} y_3(k) &= \mathcal{B}_3(0, 0, 0)l_{3,0}(k)^3 + 3\mathcal{B}_3(0, 0, 1)l_{3,0}(k)^2l_{3,1}(k) + \dots \\ &\dots + 3\mathcal{B}_3(0, 1, 1)l_{3,0}(k)l_{3,1}(k)^2 + \mathcal{B}_3(1, 1, 1)l_{3,1}(k)^3 \end{aligned} \quad (\text{A.30})$$

With the equations A.26, A.29 and A.30 it is then possible to write the matrices $\mathbf{\Gamma}_1$, $\mathbf{\Gamma}_2$ and $\mathbf{\Gamma}_3$ as:

$$\mathbf{\Gamma}_1 = \begin{bmatrix} l_{1,0}(0) & l_{1,1}(0) \\ l_{1,0}(1) & l_{1,1}(1) \\ \vdots & \vdots \\ l_{1,0}(k) & l_{1,1}(k) \end{bmatrix} \quad (\text{A.31})$$

$$\mathbf{\Gamma}_2 = \begin{bmatrix} l_{2,0}(0)^2 & 2l_{2,0}(0)l_{2,1}(0) & l_{2,1}(0)^2 \\ l_{2,0}(1)^2 & 2l_{2,0}(1)l_{2,1}(1) & l_{2,1}(1)^2 \\ \vdots & \vdots & \vdots \\ l_{2,0}(k)^2 & 2l_{2,0}(k)l_{2,1}(k) & l_{2,1}(k)^2 \end{bmatrix} \quad (\text{A.32})$$

$$\mathbf{\Gamma}_3 = \begin{bmatrix} l_{3,0}(0)^3 & 3l_{3,0}(0)^2l_{3,1}(0) & 3l_{3,0}(0)l_{3,1}(0)^2 & l_{3,1}(0)^3 \\ l_{3,0}(1)^3 & 3l_{3,0}(1)^2l_{3,1}(1) & 3l_{3,0}(1)l_{3,1}(1)^2 & l_{3,1}(1)^3 \\ \vdots & \vdots & \vdots & \vdots \\ l_{3,0}(k)^3 & 3l_{3,0}(k)^2l_{3,1}(k) & 3l_{3,0}(k)l_{3,1}(k)^2 & l_{3,1}(k)^3 \end{bmatrix} \quad (\text{A.33})$$

The matrices $\mathbf{\Phi}_1$, $\mathbf{\Phi}_2$ and $\mathbf{\Phi}_3$ with the orthonormal kernels are then written as:

$$\mathbf{\Phi}_1 = \begin{bmatrix} \mathcal{B}_1(0) \\ \mathcal{B}_1(1) \end{bmatrix} \quad (\text{A.34})$$

$$\mathbf{\Phi}_2 = \begin{bmatrix} \mathcal{B}_2(0, 0) \\ \mathcal{B}_2(0, 1) \\ \mathcal{B}_2(1, 1) \end{bmatrix} \quad (\text{A.35})$$

$$\Phi_3 = \begin{bmatrix} \mathcal{B}_3(0,0,0) \\ \mathcal{B}_3(0,0,1) \\ \mathcal{B}_3(0,1,1) \\ \mathcal{B}_3(1,1,1) \end{bmatrix} \quad (\text{A.36})$$

With these expressions of Γ_1 , Γ_2 and Γ_3 , and Φ_1 , Φ_2 and Φ_3 it is possible to build the matrices Γ and Φ by using the equations A.19 and A.20:

$$\Gamma = \begin{bmatrix} l_{1,0}(0) & l_{1,1}(0) & l_{2,0}(0)^2 & 2l_{2,0}(0)l_{2,1}(0) & l_{2,1}(0)^2 \\ l_{1,0}(1) & l_{1,1}(1) & l_{2,0}(1)^2 & 2l_{2,0}(1)l_{2,1}(1) & l_{2,1}(1)^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ l_{1,0}(k) & l_{1,1}(k) & l_{2,0}(k)^2 & 2l_{2,0}(k)l_{2,1}(k) & l_{2,1}(k)^2 \\ l_{3,0}(0)^3 & 3l_{3,0}(0)^2l_{3,1}(0) & 3l_{3,0}(0)l_{3,1}(0)^2 & l_{3,1}(0)^3 \\ \dots & l_{3,0}(1)^3 & 3l_{3,0}(1)^2l_{3,1}(1) & 3l_{3,0}(1)l_{3,1}(1)^2 & l_{3,1}(1)^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{3,0}(k)^3 & 3l_{3,0}(k)^2l_{3,1}(k) & 3l_{3,0}(k)l_{3,1}(k)^2 & l_{3,1}(k)^3 \end{bmatrix} \quad (\text{A.37})$$

$$\Phi = \begin{bmatrix} \mathcal{B}_1(0) \\ \mathcal{B}_1(1) \\ \mathcal{B}_2(0,0) \\ \mathcal{B}_2(0,1) \\ \mathcal{B}_2(1,1) \\ \mathcal{B}_3(0,0,0) \\ \mathcal{B}_3(0,0,1) \\ \mathcal{B}_3(0,1,1) \\ \mathcal{B}_3(1,1,1) \end{bmatrix} \quad (\text{A.38})$$

Although a very simple example was presented in this section, a similar procedure can be done in the cases of problems with higher number of orthonormal functions used to represent the Volterra kernels.

Calculation of the kernels in two steps

Another important possibility with the Volterra model is to use a dataset where the

nonlinear response is nearly zero. With this linear dataset it is possible to calculate the linear kernel by finding Φ_1 with the input matrix Γ_1 and the linear output vector $\mathbf{y}_1$:

$$\Phi_1 = (\Gamma_1^T \Gamma_1)^{-1} \Gamma_1^T \mathbf{y}_1 \quad (\text{A.39})$$

With the knowledge of the first kernel in Φ_1 , it is possible to estimate the remaining higher-order kernels using a new nonlinear dataset. Since the linear kernel is already known in this stage it is possible to estimate $\mathbf{y}_1$ for the nonlinear dataset. In this case only the higher-order components remain:

$$\mathbf{y}_{\text{nl}} = \mathbf{y}_2 + \mathbf{y}_3 \quad (\text{A.40})$$

In this case it is possible to estimate only the nonlinear kernels by defining:

$$\Gamma_{\text{nl}} = \begin{bmatrix} \Gamma_2 & \Gamma_3 \end{bmatrix} \quad (\text{A.41})$$

and:

$$\Phi_{\text{nl}} = \begin{bmatrix} \Phi_2 \\ \Phi_3 \end{bmatrix} \quad (\text{A.42})$$

Then it is possible to estimate the nonlinear kernels with the least-squares estimation:

$$\Phi_{\text{nl}} = (\Gamma_{\text{nl}}^T \Gamma_{\text{nl}})^{-1} \Gamma_{\text{nl}}^T \mathbf{y}_{\text{nl}} \quad (\text{A.43})$$

This other way of estimating the Volterra kernels can be seen as a two-steps procedure. This can be done when it is possible to measure a dataset in the structure where the behavior is nearly linear. In this case the linear kernel is estimated in one step and in the second step the nonlinear kernels are calculated. This procedure has the advantage of adding more information to the identification of the system regarding the linear behavior of the structure which ensures the user that the linear behavior is properly separated from the nonlinear components.

Appendix B - Publications of the author during the doctorate studies

A few examples and applications performed during the doctorate studies with the subject of this thesis were published in the following conference papers:

- Shiki et al. (2012a): "Ajuste de Modelos em Estruturas Mecânicas Não-Lineares Através da Decomposição em Modos Próprios Ortogonais";
- Shiki et al. (2012b): "Identification of Parameters in Nonlinear Structures Using Discrete Volterra Series";
- Shiki et al. (2012c): "Model updating of the non-linear vibrating structures through Volterra series and proper orthogonal decomposition";
- Scussel et al. (2013): "Identification of Mechanical Systems Through Volterra Series - Study of Benchmark Cases";
- Shiki et al. (2013): "Nonlinear Mechanical Systems Identification Through Wiener/Volterra Series";
- Shiki et al. (2013): "Identification of Mechanical Systems With Local Nonlinearities Through Discrete-Time Volterra Series and Kautz Functions";
- Hansen et al. (2014a): "Structural Health Monitoring in a Buckled Beam using Volterra Series";
- Hansen et al. (2014b): "Non-parametric identification of a non-linear buckled beam using discrete-time Volterra series";
- Shiki et al. (2014a): "Characterization of the nonlinear behavior of a F-16 aircraft using discrete-time Volterra series";
- Shiki et al. (2014b): "Nonlinear features identified by Volterra series for damage detection in a buckled beam";

- Shiki et al. (2014c): "Parameter identification of a nonlinear beam with hardening behavior using Volterra series";
- Bussetta et al. (2015): "Identification of nonlinear finite element model through discrete-time Volterra series."

And in the following journal papers:

- Shiki et al. (2013): "Damage Detection in Nonlinear Structures using Discrete-Time Volterra Series";
- Shiki et al. (2014): "Identification of nonlinear structures using discrete-time Volterra series";
- Shiki and da Silva (2015): "Evaluation of damage indexes for structural health monitoring in nonlinear mechanical systems based on Volterra series".

The following paper was accepted for publication and it is being reviewed: "On the application of discrete-time Volterra series for the damage detection problem in initially nonlinear structures" submitted to the SHM Journal in a partnership with Prof. Dr. Michael D. Todd from the University of California, San Diego (United States).

Also, the following paper was sent for publication and it is being reviewed for a re-submission: "Nonlinear dynamic analysis of mechanical systems using discrete-time Volterra series" submitted to the International Journal of Nonlinear Mechanics in a partnership with the Eng. Dr. Fabio L. M. dos Santos and the Eng. Dr. Bart Peeters of Siemens Industry Software in Leuven (Belgium).