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Inverting the CFT Bootstrap

Renato Gomes Ferreira Souza

Orientador

Pedro Vieira

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Resumo

O métodos numéricos utilizados em CFT Bootstrap são baseados nos trabalhos de Rattazzi et al., começando com um ansatz que satisfaz unitariedade podemos restringir vários aspectos de uma CFT genérica quando requeremos que a mesma obedeça crossing symmetry. Neste trabalho invertermos o processo, para começar iremos apresentar uma metodo para escrever expressões crossing simétricas e utilizando o formalismo do espaço alfa iremos extrair os coeficientes das OPEs, finalmente iremos restringir a dimensão de escala do primeiro escalar no espectro utilizando unitariedade. Ao contrário do método atual, o apresentado aqui será completamente analítico.

Palavras Chaves: Teoria Quântica de Campos; CFT; CFT Bootstrap.

Áreas do conhecimento: Física; Teoria Quântica de Campos; Teoria de Campos Conforme.

Abstract

The current numerical CFT Bootstrap is based on the works of Rattazzi et al., starting with a unitary ansatz it is possible to bound many properties of a generic CFT by requiring that the same expression obey crossing symmetry. In this work we will invert this process, to start we are going to present a way to write crossing symmetric expressions and using the alpha space formalism we will extract the exact OPE coefficients, finally we will bound the smallest scalar in the spectrum by demanding unitarity. Unlike the current setup the method presented in the main text is purely analytic, at the moment.

Keywords: Quantum Field Theory; CFT; CFT Bootstrap.

Knowledge areas: Physics; Quantum Field Theory; Conformal Field Theory.

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Chapter 1

Introduction

CFTs form a very peculiar and interesting set of theories as much for their content as their surprising appearances and applications in realistic phenomena in many dimensions. CFTs are theories that are invariant under conformal transformation, that is, transformations that leave the angle between any two curves invariant and the amount of symmetry present in these system are the biggest possible when not considering supersymmetry. Unlike other cases we may be used to, like in classical electromagnetism or general relativity, in which adding symmetry do make problems more tractable at the cost of representing less realistic scenarios, we seem to go in the opposite direction.

Both in QFT and statistical mechanics we stumble upon strange points in the space of possible theories, for QFT, and particular values for the thermodynamics variables, for statistical mechanics, that are invariant under scaling transformation, these represent fixed points in the RG and second order phase transition. What these points have in common are that both have an universal effect, being the UV or IR limit of many completely different QFT and the behaviour of many different statistical mechanics systems close to the critical point. What is quite remarkable is that both are usually CFTs even though at no point we need to introduce the extra symmetries needed to upgrade an scale invariance to full conformal invariance.

The applications of CFT don't end here but continue into newer ideas that are relevant these days, noticeable the AdS/CFT correspondence, a part of the Gauge/Gravity duality conjecture that links gauge theories and string theories, that have several applications outside its initial range. Although the body of application keeps growing, the content inside CFTs still interesting on its own. The huge amount of symmetries possessed by these theories are actually capable of completely constrain any n-point function without restricting us to a trivial QFT, not only that, but using these symmetries we are able to analyse these systems independent of the strength of interactions.

The idea behind CFT bootstrap is not new, the influential work done by Polyakov and Ferrara *et al.*, [1, 2], in the 70's took advantage of the extra sym-

metries in 2D CFTs, the Virasoro algebra, to analyse these theories using only its consistency conditions. The consistency condition generated in this process were already capable of finding the minimal models, however the lack of this extra algebra in higher dimensions impeded the same ideas to be readily applicable outside of 2D. It wasn't until Rattazzi *et al.*, [3], work bringing a new way of tackling the same problem we were able to extract important information out of the crossing equation and began mapping the space of possible CFTs in any dimension.

In this new method we start by writing a possible correlation function in terms of conformal blocks with positive coefficients, in this way we would already be satisfying reflection positivity, and we truncate the crossing equation, which consists of an infinite sum over every operators in our theory. Each operator can be represented as a vector in a finite dimensional space, and if the truncated sum was to be satisfied then these vector should sum to approximately zero. However if all vectors live in the same side of a hyperplane, or equivalently if there is a linear map that map all of them to a number with the same sign, we can conclude that it is impossible for them to sum to zero, and such this CFT cannot exist.

Since [3] many advancements were made, although the general idea was already present since the start it took a while until it was really possible to analyse any dimension, [4, 5], or to analyse mixed correlators, [6]. In this work we will be looking at the opposite direction, that is, instead of starting with a reflection positive ansatz and proceed to check whether or not the expression satisfy crossing symmetry, we will start with a crossing symmetric expression and check for unitarity. This idea is based on another type of bootstrap, the S-Matrix bootstrap, which similar to CFT in 2 dimensions, there are analytic methods to deal with some problems, but in higher dimensions numerical methods are the norm. The difference here is while there are ways to check unitarity no such expressions are known to exist in CFTs, with the closest in appearance being the inversion formulas.

This work is organized in the following way, in chapter 2 we start by presenting the behavior of a correlator function under the crossing symmetries followed by showing how we can choose different coordinate in such way that crossing symmetric ansatz becomes easy to write. During chapter 3 we go over the basic aspects of alpha space in 1D and higher dimensions that we will need for obtaining the conformal block decomposition. In chapter 4 we go back to the correlator and use the previously mention coordinates to write some simple 4-point functions

and look for known 4-point functions to get ideas on how to express more general correlators in terms of our new coordinates. Finally at chapter 5 we use our new knowledge to both analyse some simple correlation function and find how to write and analyse the more complicated cases.

While the main text follows only the main points, the appendices expand most of ideas and show possibilities and findings that are not explored in the text. In appendix A we go back to calculate the crossing functions, explain why we choose the functions we used, and present some other possibilities in using these functions. Appendix B talks about alpha space, in there we calculate expressions we will use close to the end of the main text, we explain a little more about alpha space in higher dimensions and present some functions that could be used in general dimensions. In appendix C we will look mainly at the expression calculated in appendix B, this will be enough to show that are many simple unitary 4-point function. In appendix D we will try to analyse more complicated expression with the help of the results in appendix C.

1.1 A quick review of CFT

Before continuing we will review some details related to CFTs and go over some of the well known results, starting with the definition of a CFT, a CFT is any QFT that is invariant under conformal transformations, that is, transformations that preserve angles, but not distances. There are two important pieces information that makes CFTs interesting to study, first if we refrain to use supersymmetry then it is the maximum amount of symmetries we can have without necessarily landing into the free theories, second due to the algebra generated by the conformal symmetries we are able to look at theories regardless of the strength of interaction.

In this review we will see the conformal transformations, the conformal algebra, the constrains in the 1, 2, 3 and higher point functions. For a more complete review of CFTs there are [7, 8], and for something more focused in bootstrap there is [9].

1.1.1 Conformal transformations

The conformal transformations can be expressed as those transformations that change the metric in the following form

$$g_{\mu\nu} \rightarrow g'_{\mu\nu} = \Omega(x)^2 g_{\mu\nu}, \quad (1.1)$$

this changes can be captured by some infinitesimal transformations of the form $x^\mu \rightarrow x'^\mu = x^\mu - \xi^\mu(x)$, which will lead to the following expression

$$\left(\partial_\mu \xi^\alpha + \partial_\nu \xi^\beta \right) g_{\alpha\beta} = \delta\omega^2 g_{\mu\nu}. \quad (1.2)$$

The case we are interested with are those that are conformally flat, that is, the metric is proportional to $\delta_{\mu\nu}$, or to $\eta_{\mu\nu}$ if we were looking at the Lorentzian case, because of this our expression simplify to

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \delta\omega^2 \delta_{\mu\nu}. \quad (1.3)$$

For dimension bigger than 2 the solutions have the general form

$$\xi^\mu = a^\mu + b^\mu_\nu x^\nu + c^\mu_{\nu\rho} x^\nu x^\rho, \quad (1.4)$$

with some conditions between the coefficients a , b and c , within these conditions there is a basis for the most general differential operator $\xi^\mu \partial_\mu$, most of which can easily interpret its physical meaning

$$p_\mu = \partial_\mu, \quad (1.5)$$

$$m_{\mu\nu} = x_\nu \partial_\mu - x_\mu \partial_\nu, \quad (1.6)$$

$$d = x^\mu \partial_\mu, \quad (1.7)$$

$$k_\mu = 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu. \quad (1.8)$$

For 2 dimensions any holomorphic function $f(z)$ generates a conformal transformation, but that can be more easily expressed by changing the metric to

$$ds^2 = dzd\bar{z} \rightarrow ds'^2 = f(z) \bar{f}(\bar{z}) dzd\bar{z}, \quad (1.9)$$

this will create the Virasoro algebra.

1.1.2 Conformal algebra

Similarly on how the Poincare symmetries impose constraints in a QFT the conformal symmetries will do the same in a CFT, not only that but the algebra have some interesting properties which give some major results for CFTs.

We define the abstract operator P_μ , $M_{\mu\nu}$, D and K_μ , which are related to the previously defined differential operators by $[M_{\mu\nu}, \mathcal{O}] = \text{Rep}_{\mathcal{O}}(m_{\mu\nu}) \mathcal{O}$, and so on, and let their commutator be the same as those, that is,

$$[M_{\mu\nu}, P_\rho] = \delta_{\nu\rho} P_\mu - \delta_{\mu\rho} P_\nu, \quad (1.10)$$

$$[M_{\mu\nu}, K_\rho] = \delta_{\nu\rho} K_\mu - \delta_{\mu\rho} K_\nu, \quad (1.11)$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = \delta_{\nu\rho} M_{\mu\sigma} - \delta_{\mu\rho} M_{\nu\sigma} + \delta_{\mu\sigma} M_{\nu\rho} - \delta_{\nu\sigma} M_{\mu\rho}, \quad (1.12)$$

$$[D, P_\mu] = P_\mu, \quad (1.13)$$

$$[D, K_\mu] = -K_\mu, \quad (1.14)$$

$$[K_\mu, P_\nu] = 2\delta_{\mu\nu} D - 2M_{\mu\nu}, \quad (1.15)$$

and all the others are zero. There are multiple ways to write this algebra including one that shows explicitly that it is isomorphic to $SO(2, d-1)$, however it will be more convenient to keep its form for the moment. The two most important parts here is the fact that $M_{\mu\nu}$ and D commute and fact that P_μ and K_μ act as creation and annihilation operators for D , respectively. In a conformal theory all operators needs to be part of some conformal representation and such all operators can be classified by their representation under rotations, their spin, and under dilatation, their scaling dimension, and for a unitary conformal theory all those representation have a primary operator, an operator that satisfy $[K_\mu, \mathcal{O}] = 0$, as such all correlation functions can be calculated from the correlation function of primaries.

1.1.3 Correlation functions

Unlike a General QFT the 1, 2 and 3-point functions in a CFT are fairly simple, they are completely determined by conformal transformation and their representation under rotation, this is because there is always a conformal transformation

that maps the three distinct points x_1, x_2 and x_3 to any three distinct points x'_1, x'_2 and x'_3 , so these quantities do not contain any dynamics.

We will show here the general form for the 1, 2 and 3-point function for scalars

$$\langle \mathcal{O}_i(x) \rangle = 0, \text{ if } \Delta_i \neq 0, \quad (1.16)$$

$$\langle \mathcal{O}_i^\dagger(x_i) \mathcal{O}_j(x_j) \rangle = \frac{\delta_{ij}}{|x_{ij}|^{2\Delta_i}}, \quad x_{ij} = x_i - x_j, \quad (1.17)$$

$$\langle \mathcal{O}_i(x_i) \mathcal{O}_j(x_j) \mathcal{O}_k(x_k) \rangle = \frac{c_{ijk}}{|x_{ij}|^{\Delta_i+\Delta_j-\Delta_k} |x_{ik}|^{\Delta_i+\Delta_k-\Delta_j} |x_{kj}|^{\Delta_k+\Delta_j-\Delta_i}}, \quad (1.18)$$

for operators with spin there will be more complicated structures related to the representations of the operators in the correlator. In general a correlator with only primary operator transform like

$$\langle \mathcal{O}_1^{N_1}(x_1) \dots \mathcal{O}_n^{N_n}(x_n) \rangle = \left| \frac{\partial x'}{\partial x} \right|_{x=x_1}^{\frac{\Delta_1}{D}} \dots \left| \frac{\partial x'}{\partial x} \right|_{x=x_n}^{\frac{\Delta_n}{D}} \mathcal{R}_{M_1}^{N_1} \dots \mathcal{R}_{M_n}^{N_n} \langle \mathcal{O}_1'^{M_1}(x_1) \dots \mathcal{O}_n'^{M_n}(x_n) \rangle, \quad (1.19)$$

with $\mathcal{R}$ being the spin representation of each operator. Unlike lower point function the conformal transformation cannot fully constrain the form of these functions, however CFTs present us with a bit more tools that will tie all of this information together. Unlike regular QFTs an OPE in a CFT does give us a convergent expansion, this allow us to rewrite any n-point function in term of n-1-point functions, so although the 4-point function and onward don't seem to be fixed at first they are fully dependent on the 2 and 3-points functions. As an example we have the following for a 4-point function

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle = \sum_i \sum_j c_{12i} c_{34j} C_{12i}(x_{12}, x_2) C_{34j}(x_{34}, x_4) \langle \mathcal{O}_i(x_2) \mathcal{O}_j(x_4) \rangle, \quad (1.20)$$

the sums run over every possible operator, both primaries and descendants, and by the form of the 2-point function it will be zero unless both of them contain the same conformal representation. All terms in the OPE can be calculated from the 3-point function and the sum over all operator in one representation is completely

fixed by conformal symmetry and are generally expressed in terms of conformal blocks, $\sum_{Rep_{\mathcal{O}_i}} C_{12i}(x_{12}, x_2) C_{34i}(x_{34}, x_4) \langle \mathcal{O}_i(x_2) \mathcal{O}_i(x_4) \rangle \sim G_{J,\Delta}^{\Delta_1-\Delta_2, \Delta_4-\Delta_3}(u, v)$

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle = \left| \frac{x_{24}}{x_{14}} \right|^{\Delta_1-\Delta_2} \left| \frac{x_{24}}{x_{14}} \right|^{\Delta_1-\Delta_2} \frac{\mathcal{G}(u, v)}{|x_{12}|^{\Delta_1+\Delta_2} |x_{34}|^{\Delta_3+\Delta_4}}, \quad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}, \quad (1.21)$$

$$\mathcal{G}(u, v) = \sum_{J,\Delta} p_{J,\Delta} G_{J,\Delta}^{\Delta_1-\Delta_2, \Delta_4-\Delta_3}(u, v). \quad (1.22)$$

To keep things concise instead of writing $\Delta_1 - \Delta_2, \Delta_4 - \Delta_3$ it will be kept implicit.

The information contained in the 2 and 3-point function form what is called the CFT data, so by knowing this data and the special functions related to conformal symmetry it would be possible to write any correlation function regardless of the strength of interactions.

1.1.4 4-point function and crossing symmetry

The focus on an 4-point function wasn't accidental, they will be the object of study during all this work, so let's check a general 4-point function in an arbitrary dimension. The first step would be to get rid of any irrelevant information, to do this we can use the conformal transformations to bring them to a canonical form, using the special conformal transformation it is possible to send x_4 to infinity, although this takes the correlation to zero, with proper normalization we can make it finite, now using rotations it is possible to bring the 3 points left to the plane 1-2, and after a translation have x_3 along the axis 1 and x_1 at the origin, lastly with dilatation its possible to leave it a unit distance from 0

$$\langle \mathcal{O}_1(0) \mathcal{O}_2(x_2''') \mathcal{O}_3(1) \mathcal{O}_4(\infty) \rangle. \quad (1.23)$$

Since x_2''' is in a plane the full 4-point function is parametrized by only 2 real parameters, the usual one used are the Dolan-Osborn variables, for Lorentzian signature there is $x_2^1 = \frac{x+z}{2}$ and $x_2^0 = \frac{x-z}{2}$ and for Euclidean $x_2^1 = \frac{x+z}{2}$ and $x_2^2 = \frac{x-z}{2i}$. For now on we will be working mainly in Euclidean signature.

In CFTs crossing symmetry is a statement about consecutive OPEs, because of the convergence of an OPE it shouldn't matter in which order the OPE is taken, the result needs to be the same. An OPE converges if it is possible to draw a circle that contains only the two operators inside, with this we can say that both the $\mathcal{O}_1, \mathcal{O}_2$ and $\mathcal{O}_2, \mathcal{O}_3$ converges if $x_2^2 \neq 0$ or $x_2^2 = 0$ and $0 < x_2^1 < 1$. Since the OPE can be easily expressed as a sum of conformal blocks, the way to compare all possible OPEs is to first use the conformal transformation to set the desired operator in the origin and at 1 and then use the OPE first with the operator in the origin then with the operator at 1 to obtain the decomposition

$$\begin{aligned} \langle \mathcal{O}_1(0) \mathcal{O}_2(x_2''') \mathcal{O}_3(1) \mathcal{O}_4(\infty) \rangle &= \sum_i c_{12i} C_{12i}(x_2''', x_i) \langle \mathcal{O}_i(x_i) \mathcal{O}_3(1) \mathcal{O}_4(\infty) \rangle \\ &= \sum_i \sum_j c_{12i} c_{i3j} C_{12i}(x_2''', x_i) C_{i3j}(x_{i3}, x_j) \langle \mathcal{O}_j(x_j) \mathcal{O}_4(\infty) \rangle. \end{aligned} \quad (1.24)$$

Restricting ourselves to 4 identical scalar and using the Dolan-Osborn variables

$$\begin{aligned} \langle \phi(0,0) \phi(x,z) \phi(1,1) \phi(\infty,\infty) \rangle &= \langle \phi(0,0) \phi(1-x, 1-z) \phi(1,1) \phi(\infty,\infty) \rangle \\ &= \left| \frac{1}{1-x} \right|^{2\Delta_\phi} \left\langle \phi(0,0) \phi\left(\frac{x}{x-1}, \frac{z}{z-1}\right) \phi(1,1) \phi(\infty,\infty) \right\rangle, \end{aligned} \quad (1.25)$$

since it is the stripped correlator, $\mathcal{G}$, the object that actually have a conformal block decomposition, then it is better to rewrite this in term of them

$$\mathcal{G}(x,z) = \left| \frac{x}{x-1} \right|^{2\Delta_\phi} \mathcal{G}(1-x, 1-z) = \mathcal{G}\left(\frac{x}{x-1}, \frac{z}{z-1}\right). \quad (1.26)$$

An important detail left out of these discussions is that crossing symmetric 4-point functions implies crossing symmetric n-point functions, since far away operators don't contribute to the OPE we could change (1.24) to

$$\begin{aligned} \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) X \rangle &= \sum_i c_{12i} C_{12i}(x_{12}, x_i) \langle \mathcal{O}_i(x_i) \mathcal{O}_3(x_3) X \rangle \\ &= \sum_i \sum_j c_{12i} c_{i3j} C_{12i}(x_{12}, x_i) C_{i3j}(x_{i3}, x_j) \langle \mathcal{O}_j(x_j) X \rangle, \end{aligned} \quad (1.27)$$

if all 4-point functions with $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ are crossing symmetric then the expression above also will be.

1.1.5 Radial quantization and the OPE convergence

In this part we are going to discuss the OPE convergence, prove that if all 4-point satisfy crossing symmetry, then any n-point function also will be, the starting point is what is called radial quantization. Unlike Lorentzian QFTs the Euclidean case does not have a time evolution as a result the states that live in a particular time slice $|\psi(t)\rangle$ or the evolution operator $U(t_1, t_2)$ don't have a natural analog, however it is possible to induce one. Since time doesn't exist in Euclidean space the flow of a fictitious time around a point could flow in any direction, as such it is possible to have time flow radially from a point, at each surface at fixed distance there will be Hilbert space where states $|\Psi(r)\rangle$ exists, and there will be an evolution operator $U(r_1, r_2)$ that allow us to transverse between Hilbert spaces.

Observables here are related to the amplitudes between two surfaces, $\langle \Psi_1(r_1) | \mathcal{O}_i(r_i, \vec{x}_i) \dots \mathcal{O}_j(r_j, \vec{x}_j) | \Psi_2(r_2) \rangle$, however the special part comes with the *state-operator correspondence*, it turns out that in a CFT it is possible to evolve the state back in "time" and condense it into an operator at the center

$$\langle \Psi_1(r_1) | \mathcal{O}_i(r_i, \vec{x}_i) \dots \mathcal{O}_j(r_j, \vec{x}_j) | \Psi_2(r_2) \rangle = \langle \Psi_1(r_1) | \mathcal{O}_i(r_i, \vec{x}_i) \dots \mathcal{O}_j(r_j, \vec{x}_j) \mathcal{O}_{\Psi_2}(0) | 0 \rangle. \quad (1.28)$$

In this formalism an OPE can be thought this way, starting with two operators close to the center, $\langle \dots \mathcal{O}_1(r_1, \vec{x}_1) \mathcal{O}_2(r_2, \vec{x}_2) | 0 \rangle$, by evolving the vacuum past the furthest of both, but before a third, then the correlator becomes $\langle \dots | \Psi \rangle$, by evolving back this state can be written as a new operator $\langle \dots \mathcal{O}_{\Psi}(0) | 0 \rangle$, lastly since in a CFT there is a basis of operator this $\mathcal{O}_{\Psi}(0)$ can be written as the familiar sum $\sum_i c_{12i} C_{12i}(\vec{x}_{12}, 0) \mathcal{O}_i(0)$. As this shows the OPE between two operators will converge if there is a circle that separate both operators from the rest, for the 4-point function in the canonical form it becomes clear where both OPEs converges, it is time to move to the n-point functions.

Let's start with the supposition that every 4-point function is crossing symmetric, that is, given a $\langle 0 | \mathcal{O}(r, \vec{x}) \mathcal{O}_1(r_1, \vec{x}_1) \mathcal{O}_2(r_2, \vec{x}_2) \mathcal{O}_3(r_3, \vec{x}_3) | 0 \rangle$, with generic operators $\mathcal{O}$, $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ whenever there is a region where two OPEs converge they will give the same value, now let's focus on the OPE between the three $\mathcal{O}_1$,

$\mathcal{O}_2$ and $\mathcal{O}_3$, changing the operator $\mathcal{O}$ to its state $\langle \Psi_{\mathcal{O}} |$ we can say that the 3-point function $\langle \Psi_{\mathcal{O}}(\tilde{r}) | \mathcal{O}_1(r_1, \vec{x}_1) \mathcal{O}_2(r_2, \vec{x}_2) \mathcal{O}_3(r_3, \vec{x}_3) | 0 \rangle$, $r_1 < \tilde{r} < r$, is crossing symmetric. Now consider the n-point function, X represent all the other operators with $r_i > r_1$, $\langle 0 | X \mathcal{O}_1(r_1, \vec{x}_1) \mathcal{O}_2(r_2, \vec{x}_2) \mathcal{O}_3(r_3, \vec{x}_3) | 0 \rangle$, by substituting all operators inside of X for a state $\langle \Psi_X(\tilde{r}) | \mathcal{O}_1(r_1, \vec{x}_1) \mathcal{O}_2(r_2, \vec{x}_2) \mathcal{O}_3(r_3, \vec{x}_3) | 0 \rangle$ the result is identical to the 3-point function above, and since $\Psi_{\mathcal{O}}$ is generic it includes Ψ_X , so any n-point function are crossing symmetric.

1.1.6 CFT bootstrap

An important point to be aware is that not all CFT data is consistent, as was said before the OPE in a CFT will give us a convergent expansion this allow us to compare two of these expansions, provided is in a place where both converges, to check if they give the same result, this is called crossing symmetry and if a n-point function fail this test it cannot be part of a consistent CFT. Another test that is interesting for a QFT is whether the expression is unitary, or reflection positive in the Euclidean case. The application of both constrains to a CFT data is called CFT bootstrap, in this way it is possible to carve the space of allowed theories or to find some universal properties of CFTs.

Using the conformal block decomposition in (1.26) the numerical bootstrap is capable of carving out the space of possible CFT data, in short, unitarity can be imposed by demanding that correlators of the form $\langle f | f \rangle$ have positive OPE coefficients $c_{1+2+i} + c_{i21}$, with this crossing symmetry can be related to a problem in vector spaces and linear functional allowing for a computer to analyse the problem and give rigorous bounds. By solving (1.26) at the start we can guarantee that crossing symmetry will be satisfied, the challenge becomes testing for unitarity.

1.1.7 1D CFT

Although the definition of CFT given a few paragraphs back only makes sense in 2D and higher it is possible to come up with a definition for 1D, a 1D CFT is a QFT that is invariant under translation, P , dilatation, D , and special conformal transformation, K . In practice 1D CFTs appears as defects that still are invariant under some conformal transformations, but they could be studied alone as a simpler case or a toy model.

Like higher dimensional CFTs the 1, 2 and 3-point function are known and for 4-point and higher functions we can use the OPE to reduce them to 2-point

functions, the only difference here is the crossing symmetry, unlike the previous case only two OPEs converge at a time

$$\mathcal{G}(x) = \left(\frac{x}{1-x}\right)^{2\Delta_\phi} \mathcal{G}(1-x). \quad (1.29)$$

In this work we will be focusing into 2D CFTs but the 1D case allow us to create the alpha space, which will be used to recover the data from our correlator.

Chapter 2

Crossing functions

The main idea behind bootstrap isn't to look at a particular theory but a way to look at the full space of possible theories, in any framework theories need to satisfy some self-consistent conditions and such not every theory will be valid, by reducing the observables to a set of physical data it is possible to carve the space of valid data, it is also possible to use additional conditions to restrict the space we are dealing with to a more interesting one.

While all these ideas could be used in a variety of situations the broad stroke approach they were presented isn't easy to apply to any, which set of data, if any, would give a good relation to observables? What would be the minimum set of conditions the system needs to satisfy to make it a valid theory? How these conditions carve out the space of valid data? All of these questions need to be answered case by case, however for CFTs some of them are quite straightforward.

For a CFT there are two conditions necessary for it to be a valid QFT, first is crossing symmetry, this one is completely essential and without it the CFT itself doesn't exist, the second is unitarity or reflection positivity in Euclidean signature, this one can be removed in some cases like statistical mechanics and other non-QFTs scenarios. CFTs have a few peculiarities when compared to other QFTs, due to the conformal transformations the 1, 2 and 3-point functions are completely fixed and the OPEs give a convergent expansion, because of this any higher point function can be reduced to a 2-point function, as such there is a natural set of data in a CFT, the information in the 2 and 3-point function.

Before asking how these two sets interact let's explain some of the points above, starting with the OPE. Whenever two operator are "close together", that is there is a sphere that separates both operators from the rest, then the substitution of the product $\mathcal{O}_i \times \mathcal{O}_j \sim \sum_k c_{ijk} \mathcal{O}_k$ inside a correlator will converge, minding the position of the last operators, the important point is that when two different OPEs converges then they should give the same result, this is crossing symmetry. Reflection positivity is quite simple, it simply ask if every correlator of the form $\langle f|f \rangle$ is non-negative. The last point is the CFT data themselves, the information

in the 2-point functions are the spectrum of the theory, the scaling dimensions and the spin representation of the operators, and in the 3-point function there is the structure constants c_{ijk} that appear in the OPE.

Now with the central ideas introduced the question remains, how these the set of conditions create bounds in the set of CFT data? For crossing symmetry alone its done by comparing whether the two OPE expansions give the same result, as can be seen this still very vague, for 4-point functions in particular

$$\langle \phi^4 \rangle \equiv \langle \phi(0,0) \phi(x,z) \phi(1,1) \phi(\infty,\infty) \rangle = \frac{\mathcal{G}(x,z)}{|x|^{2\Delta_\phi}}, \quad (2.1)$$

$\mathcal{G}$ can be expanded by doing the OPE between the first and second and then the third operator, and then swapping the operators with a conformal transformation and doing the same procedure, then forcing the two series to obey

$$\mathcal{G}(x,z) = \left| \frac{x}{1-x} \right|^{2\Delta_\phi} \mathcal{G}(1-x, 1-z), \quad \mathcal{G}(x,z) = \mathcal{G}\left(\frac{x}{x-1}, \frac{z}{z-1}\right). \quad (2.2)$$

Surprisingly reflection positivity is much more direct, any 2n-point function with the form $\langle f|f \rangle$ can be written as

$$\begin{aligned} \langle \mathcal{O}_1^\dagger(\vec{x}_1) \dots \mathcal{O}_1(\vec{x}_{2n}) \rangle &= \sum_j c_{1+2^\dagger \dots n^\dagger j^\dagger} c_{12 \dots nj} G_{J_j, \Delta_j}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{2n}) \\ &= \sum_{J, \Delta} p_{J, \Delta} G_{J, \Delta}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{2n}), \end{aligned} \quad (2.3)$$

for some coefficient $c_{12 \dots nj}$, $p_{J, \Delta}$ and special function $G_{J, \Delta}$, this expression will be unitary if the coefficients p are non-negative. Although we gave an answer to the question it didn't become any clearer how to combine both conditions, the work of Rattazzi *et. al.*, [3], gave the first step to combine both, out of the two equations in (2.2) the second is trivial since it can be solved block by block, the non-trivial one is the first

$$\sum_{J,\Delta} p_{J,\Delta} \frac{G_{J,\Delta}(x,z)}{|x|^{2\Delta_\phi}} = \sum_{J,\Delta} p_{J,\Delta} \frac{G_{J,\Delta}(1-x,1-z)}{|1-x|^{2\Delta_\phi}}, \quad (2.4)$$

$$\sum_{J,\Delta} p_{J,\Delta} F_{J,\Delta}^{\Delta_\phi}(x,z) = 0, \quad F_{J,\Delta}^{\Delta_\phi} = \frac{G_{J,\Delta}(x,z)}{|x|^{2\Delta_\phi}} - \frac{G_{J,\Delta}(1-x,1-z)}{|1-x|^{2\Delta_\phi}}. \quad (2.5)$$

The main idea in [3] was to change from the infinite dimensional vector space that is the function space into a finite one, $F_{J,\Delta}^{\Delta_\phi}(x,z) \rightarrow \vec{F}_{J,\Delta}^{\Delta_\phi}$, suppose there is a linear functional that satisfy $\kappa(\vec{F}_{J,\Delta}^{\Delta_\phi}) > 0$ for all spins bigger than zero and for scalars for scaling dimensions bigger than Δ^* , then due to unitarity, $p_{J,\Delta} \geq 0$, crossing symmetry cannot be satisfied if there isn't a scalar with $\Delta \leq \Delta^*$. With a computer and a good functional κ it is possible to search the space of functionals and come up with bounds for the data, an important point is that this algorithm only say whether a data is invalid, it cannot say if the others exist.

The aim in this project is to go the opposite route, by starting with an expression that already satisfy (2.2) there is no more a need to test if both conformal block decomposition match, what becomes important is to test the OPE coefficients $p_{J,\Delta}$. This method give a full expression for $\mathcal{G}$, not just a partial spectrum as before, so if it pass the unitarity test then it is a valid point in the data space.

2.1 Crossing symmetric bootstrap and the S-Matrix bootstrap

The idea of a crossing symmetric approach to the bootstrap is not new as other bootstrap programs uses similar ideas, as was shown in the last section trying to be general can be quite vague, so it would be useful to present the S-Matrix bootstrap as a concrete example, mainly for how similar their consistency conditions are. The set of conditions for a QFT is made of unitarity and crossing symmetry, unitarity states that no physical transition should have negative probability, this is clearly the same as the CFT $\langle f|f \rangle > 0$, and crossing symmetry is the relation obtain by switching incoming particles to outgoing antiparticles. The set of physical data here is quite different, it is the masses of the stable particles, their spin representation, the strength of interactions between them plus some extra information.

As in CFT the first observable to be tested are the 4-point functions, the $2 \rightarrow 2$

scattering, in those crossing symmetry can be satisfied by demanding an amplitude that is completely symmetric in relations to the Mandelstam variables s , t and u , however a problem with this is that only two variables are independent at a time, which leaves

$$\mathcal{M}(s, t) = \mathcal{M}(t, s) = \mathcal{M}\left(\sum_i m_i^2 - s - t, t\right). \quad (2.6)$$

Instead of dealing with the physical amplitude it is easier to expand the space by promoting u to a independent variable, this way there will be an amplitude in the extended space $\mathcal{M}(s, t, u)$ that is completely symmetric and the physical processes stays at the plane $s + t + u = \sum_i m_i^2$. The general form of a 4-point function is well known, it can always be divided in the poles representing the stable bound states and a mostly analytic part, as such

$$\mathcal{M}(s, t, u) = \sum_b \left(\frac{-g_b^2}{s - m_b^2} + \frac{-g_b^2}{t - m_b^2} + \frac{-g_b^2}{u - m_b^2} \right) + \mathcal{M}_{reg}(s, t, u). \quad (2.7)$$

Unitarity on the other hand can be found by decomposing the S-Matrix into partial waves, that is, instead of using the regular single particles modes $|\vec{p}_1, \vec{p}_2\rangle$, using the partial waves $|\vec{P}, E, l, m\rangle$

$$\langle \vec{p}_1, \vec{p}_2 | \vec{P}, E, l, m \rangle = N \delta(\vec{P} - \vec{p}_1 - \vec{p}_2) \delta(E - E_1 - E_2) Y_l^m \left(\frac{\vec{p}_1 - \vec{P}}{|\vec{p}_1 - \vec{P}|} \right). \quad (2.8)$$

By using the partial waves over the unitarity condition $S^\dagger S = \mathbb{1}$, or better yet $T^\dagger T = i(T^\dagger - T)$, there is

$$\langle \vec{P}_1, E_1, l_1, m_1 | T^\dagger T | \vec{P}_2, E_2, l_2, m_2 \rangle = i \langle \vec{P}_1, E_1, l_1, m_1 | T^\dagger - T | \vec{P}_2, E_2, l_2, m_2 \rangle, \quad (2.9)$$

defining

$$\left\langle \vec{P}_1, E_1, l_1, m_1 \left| T \right| \vec{P}_2, E_2, l_2, m_2 \right\rangle = i \delta \left(\vec{P}_1 - \vec{P}_2 \right) \delta \left(E_1 - E_2 \right) \delta_{l_1 l_2} \delta_{m_1 m_2} \left(1 + S_{l_1} \right), \quad (2.10)$$

which can be done by considering the Poincare invariance of QFTs then there is the follow inequality appears

$$|S_l| \leq 1, \quad (2.11)$$

$$S_J(s) = 1 + i \frac{(s-4)^{\frac{d-2}{2}}}{\sqrt{s}} \int_{-1}^1 dx \left(1 - x^2 \right)^{\frac{d-3}{2}} P_J(x) \mathcal{M}(s, t) \Big|_{t \rightarrow \frac{1}{2}(s-4)(x-1)}. \quad (2.12)$$

With the stage ready the same question emerges again, how to combine both conditions? The approach taken in [10, 11] was to expand the S-Matrix in a very interesting way

$$\mathcal{M}(s, t, u) = \sum_b \left(\frac{-g_b^2}{s - m_b^2} + \frac{-g_b^2}{t - m_b^2} + \frac{-g_b^2}{u - m_b^2} \right) + \sum_{a,b,c} A_{(a,b,c)} \rho_s^a \rho_t^b \rho_u^c, \quad (2.13)$$

an alternative look at unitarity could be done by looking at the discontinuity of the amplitude, [12], which happens at $s > \sum_i m_i^2$ plus crossing symmetry, however in the physical space the cut becomes very complicated, due to the third term in (2.6), but in the extended space it is very simple.

The extended space was designed to make crossing symmetry simple, just an exchange of s, t, u , but because the cut is given by $s > \sum_i m_i^2$ they completely separate, as such it is possible to treat each one on its own. The function ρ is the function that maps the complex plane with a cut at $s > \sum_i m_i^2$ into the unit disc, this way any regular function inside the disc ρ can be expressed as a series that will converge everywhere. For the numerical treatment its possible to cut some redundant $A_{a,b,c}$ and cut the sum to $a + b + c \leq \Lambda$, exploring the remaining $A_{a,b,c}$ allow us to show which data is possible and to look at some interesting features like what is the biggest interaction possible in the system, and so on.

To relate back to CFTs, unlike the QFT case the CFT crossing symmetry are not simple, but by taking a similar approach like creating an extended space with new variables it might be possible to write crossing symmetric expressions. For

unitarity, however, it becomes more complicated, the equivalents of (2.11, 2.12) are

$$\text{Res}_{\Delta=\Delta^*>J+D-2} (c(J, \Delta)) \geq 0, \quad (2.14)$$

$$c(J, \Delta) = \frac{\kappa_{J+\Delta}}{4} \int_0^1 \int_0^1 dz dx \mu(x, z) G_{\Delta+1-d, J+d-1}(x, z) d\text{Disc}[\mathcal{G}(x, z)], \quad (2.15)$$

because of the residue, which is a consequence of Δ being continuous, the integral needs to be analytic, which means that the stripped correlator needs to be exact. Perhaps there are other ways to deal with unitarity but as it stands it is not clear how to deal with a numerical version of the crossing symmetric bootstrap.

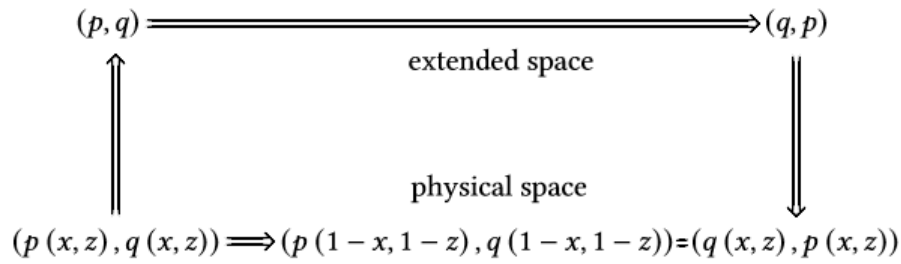
2.2 Crossing symmetry and the crossing functions

The first step in creating a crossing symmetric expression is by understanding the structure of crossing symmetry, although the CFT and the QFT have different transformations their structure are the same, they are both behave as S_3 . The linear nature of the QFT transformations make it quite clear, but the same can be shown for the CFT case by looking at the two generators $T_1 : (x, z) \mapsto (1-x, 1-z)$ and $T_2 : (x, z) \mapsto (\frac{x}{x-1}, \frac{z}{z-1})$, all possible permutation of the first, third and fourth operator in (2.1) are linked to a transformation created by using T_1 and T_2 . If everything works the idea is that there will be an extended space parametrized by p, q and r and an extended stripped correlator $\mathcal{G}(p, q, r)$, the difference for the QFT case is that the physical space is not parametrized by p and q , but by x and z , so the physical space is given by the slice $p(x, z), q(x, z)$ and $r(x, z)$ and the physical stripped correlator by $\mathcal{G}(p(x, z), q(x, z), r(x, z))$.

Just as the extended space was designed to make crossing symmetry simple in the QFT case, we should do the same here they should be permutations of the p, q and r , both generators satisfy the property $T_1^2 = T_2^2 = \mathbb{1}$ so their action in the extended space should maintain this property, as such the following action is appropriate

$$\begin{aligned}
T_1 &\implies p \iff q, \\
T_2 &\implies p \iff r,
\end{aligned} \tag{2.16}$$

and the others follow from here. There is a step remaining on the process, the relation between the physical and extended space is still unknown, but some information can be gained by analysing the physical and extended points using the paths shown in the picture bellow.



Let's start with a physical point, (x, z) , and its position in the extended space, $(p(x, z), q(x, z), r(x, z))$, in the first step let's promote it to an extended point, (p, q, r) , apply the transformation T_1 , (q, p, r) and restrict it back to the physical space, $(q(x, z), p(x, z), r(x, z))$. For the second step let's go back to the original point, (x, z) , and let's just apply T_1 , the new point is $(1-x, 1-z)$ and its position in the extended space is $(p(1-x, 1-z), q(1-x, 1-z), r(1-x, 1-z))$. Despite being written in a different way both points are the same, this means that there is some relations between these functions, in particular $p(x, z) = q(1-x, 1-z)$ and $r(x, z) = r(1-x, 1-z)$. By doing the same procedure for all possible transformations in the crossing symmetry the independent relations are

$$p(x, z) = q(1-x, 1-z), \tag{2.17}$$

$$p(x, z) = r\left(\frac{x}{x-1}, \frac{z}{z-1}\right), \tag{2.18}$$

$$p(x, z) = p\left(\frac{1}{x}, \frac{1}{z}\right). \tag{2.19}$$

Although these are the only necessary relations the crossing function needs to satisfy there are two auxiliary conditions that would keep things easier, the first

is symmetry in x and z and the second is analyticity in both variables, with the exception of a few cuts $\mathcal{G}(x, z)$ also satisfy those.

A quick touch into the Mellin amplitudes before moving on, unlike the regular correlator the variables in Mellin space are very similar to the Mandelstam variables, so it would be natural to ask if this bootstrap can be done there. In short there are two problems, first the analytic structure in a Mellin amplitude is much more complicated, [13], having to take care of the conformal block expansion right away, the second is that the full 4-point function is not only given by the Mellin amplitude but also by some subtractions, without being able to control those would make it harder to properly explore this possibility.

There are at least two ways to look at the conditions above, first is take (2.19) as saying that p have some information about $x, z = 1$, the second is it saying that p relates the region close to 0 to the one close to ∞ . The first approach can be related to meaningful expansions, imagine an expansion in p , $a_0 + a_1 p + a_2 p^2 + \dots$, if p is 0 at a random physical point (x_0, z_0) , then the expansion becomes ambiguous, it would be trying to tell us about what is happening in two distinct points at the same time, however if p is 0 at $(1, 1)$, not only the expansion becomes unambiguous but it is telling something about the operator that lives there. The second is more similar to the S-Matrix approach, to create the ρ variables it is necessary to know how the extend space looks like and this point of view helps to clarify that, the point which the ρ variables expand is unimportant since the space always could be deformed to expand around another one, given that the new point is not at the boundary.

If the aim is to have a good new parameter space, then there shouldn't be distinct points that give the exact same value, $(p(x_1, z_1), q(x_1, z_1), r(x_1, z_1)) = (p(x_2, z_2), q(x_2, z_2), r(x_2, z_2))$, taking that into consideration from the calculations done in appendix A the best solution will be

$$p(x, z) = \frac{(1-x)(1-z)}{2+2xz-x-z}, \quad (2.20)$$

$$q(x, z) = \frac{xz}{2+2xz-x-z}, \quad (2.21)$$

$$r(x, z) = \frac{1}{2+2xz-x-z}. \quad (2.22)$$

The last step will be to use these functions to construct the general stripped correlator, here is another point where the S-Matrix and CFTs differ, unlike the amplitudes $\mathcal{M}(s, t, u)$ that are completely symmetric under crossing symmetry, the correlator $\mathcal{G}(p, q, r)$ do not behave linearly as is shown in (2.2). The extra term $\left|\frac{x}{1-x}\right|^{2\Delta_\phi}$ appears as p and q are switched, first suggesting that $\mathcal{G}$ is symmetric in p and r , and second saying

$$\mathcal{G}(p, q, r) = \left|\frac{x}{1-x}\right|^{2\Delta_\phi} \mathcal{G}(q, p, r), \quad (2.23)$$

however the extra term can be rewritten as $\left(\frac{q}{p}\right)^{\Delta_\phi}$, giving the result

$$\mathcal{G}(p, q, r) = \left(\frac{q}{p}\right)^{\Delta_\phi} \mathcal{G}(q, p, r). \quad (2.24)$$

All of this can be re-expressed as

$$\mathcal{G}(p, q, r) = \frac{F(p, q, r)}{(pr)^{\Delta_\phi}}, \quad (2.25)$$

and as long F is a symmetric function, $\mathcal{G}$ will be crossing symmetric. With this the first part of the method is done, the next step will be to recover the CFT data from the stripped correlator and test it for unitarity. Unlike the methods created to deal with the crossing equation, currently there is no numerical test for unitarity, but there still is some analytic ones, the one that will be used is the alpha space created in [14, 15], before applying it we'll go over a review of the formalism.

Chapter 3

Review of Alpha Space

The alpha space formalism works by creating a complete basis of eigenfunctions and the inner product that project into this basis, this will allow us to properly obtain the conformal block decomposition from a 4-point function. Since this method was created for 1D CFTs it, and its generalization to higher dimensions, don't use the shadow blocks integrals as is done in [16], likewise it uses the stripped correlator instead of its double discontinuity.

3.1 Alpha space in 1D

In 1D CFTs the conformal blocks are eigenfunctions for the quadratic Casimir operator

$$C_2 k_h = z^2 \frac{\partial^2 (1-z) k_h}{\partial z^2} - (a+b) z^2 \frac{\partial k_h}{\partial z} - ab z k_h = h(h-1) k_h, \quad (3.1)$$

$$k_h = z^h {}_2F_1 \left(\begin{matrix} h+a, & h+b \\ & 2h \end{matrix} \middle| z \right), \quad (3.2)$$

and as a second order differential operator the Sturm-Liouville theorem tells how to construct a measure and indicates the possibility to find a basis with the eigenfunctions. From the OPE in 1D we already know that our stripped correlator is given by the sum

$$\mathcal{G}(z) = \sum_{h \in \mathbb{R}_{\geq 0}} \lambda_h^2 k_h(z), \quad (3.3)$$

with positive scaling dimension. At first this sum may seem similar to regular expansions, like a decomposition into some orthogonal polynomials

$$f(x) = \sum_n c_n P_n(x), \quad c_n = \langle f, P_n \rangle, \quad (3.4)$$

however, as we will see, there are many differences between the two. One of the problem here is that unlike the cases with discrete eigenvalue we need to be more careful when obtaining the basis since their inner product should be a delta function in the eigenvalue. Second, the continuous h means that we will obtain a density function in h instead of the discrete coefficients c_n , this means that if we were to find (3.3) from the inner product, like the c_n are, we should see a sum of delta functions, however this cannot happen because, as we will see, the k_h do not form the basis. To give an example a similar case happens when a function like $f(x) = \sum_n c_n \exp(ia_n x)$ pass through a Fourier transform

$$\tilde{f}(k) \sim \sum_n c_n \delta(k - a_n), \quad (3.5)$$

the coefficients c_n can be extracted since they do appear together with the correct delta function, after the proper normalization. To properly find the OPE coefficients we'll need to go further and start defining the alpha transform

$$\langle g, \Psi_\alpha \rangle = \hat{g}(\alpha) = \int_0^1 \frac{dz}{z^2} (1-z)^{a+b} g^*(z) \Psi_\alpha(z), \quad (3.6)$$

$$\Psi_\alpha(z) = z^{-a} {}_2F_1 \left(\begin{matrix} \frac{1}{2} + \alpha + a, & \frac{1}{2} - \alpha + a \\ 1 + a + b \end{matrix} \middle| \frac{z-1}{z} \right), \quad (3.7)$$

with $\alpha = h - \frac{1}{2}$, the Ψ_α functions will form the basis for 1D 4-point functions, not for all α , only imaginary ones. To see how this basis is formed and why the restriction to the imaginary α it is necessary expand Ψ_α in z , this functions have two distinct series starting with the powers $z^{\frac{1}{2} \pm \alpha}$. For the basis to make sense $\langle \Psi_\alpha, \Psi_\beta \rangle$ should be, without the normalization factor, represented by delta functions, however in practice this integral won't usually converge, if $\text{Re}[\alpha]$ or $\text{Re}[\beta] \neq 0$ the integral will diverge because the behavior at $z = 0$, in turn this tell us what is actually happening. As we took inspiration from the Fourier transform to try to understand how to recover the OPE coefficients we expected $\hat{g}(h)$ to

take the form $\sum_i \lambda_i^2 \delta(h - h_i)$, however with the correct basis the inverse transform $g(z) = \int_{i\mathbb{R}} d\alpha \frac{1}{2\pi i N(\alpha)} \hat{g}(\alpha) \Psi_\alpha$ expects us to recover a sum from an integral, the natural way to go is to use the residue theorem, doing the calculation carefully for $a = b = 0$

$$g(x) = \int_{i\mathbb{R}} \frac{d\alpha}{2\pi i} \frac{\hat{g}(\alpha)}{N(\alpha)} \Psi_\alpha(x) = \int [d\alpha] \frac{\hat{g}(\alpha)}{N(\alpha)} \Psi_\alpha(x), \quad (3.8)$$

$$N(\alpha) = \frac{Q(\alpha) Q(-\alpha)}{2}, \quad Q(\alpha) = \frac{2\Gamma(-2\alpha)}{\left(\Gamma\left(\frac{1}{2} - \alpha\right)\right)^2}. \quad (3.9)$$

From this expression the conformal block decomposition is obtained from the sum of the residues in the expression, and due to the fact that Ψ_α , $N(\alpha)$, $\hat{g}(\alpha)$ and the region of integration are invariant under $\alpha \mapsto -\alpha$ we can simplify the inverse transform to

$$g(x) = \int [d\alpha] \frac{2\hat{g}(\alpha)}{Q(-\alpha)} k_{\frac{1}{2}+\alpha}(x), \quad (3.10)$$

lastly, since $Q(-\alpha)$ and $k_{\frac{1}{2}+\alpha}(x)$ do not have any poles in the region $\text{Re}[\alpha] > -\frac{1}{2}$ and the contour is going in a clockwise direction we are left with the expression

$$g(x) = \sum_{\alpha=\alpha^*} \frac{-2\text{Res}_{\alpha=\alpha^*} [\hat{g}(\alpha)]}{Q(-\alpha)} k_{\frac{1}{2}+\alpha}(x). \quad (3.11)$$

While it works in generic cases, we need to be careful when dealing with $\alpha = 0$ and $\alpha = -\frac{1}{2}$.

Before continuing we address the convergence of the integral for the alpha density $\hat{g}$, the way this integral should be understood is through an analytic continuation in $x = 0$ and a cutoff close to $x = 1$. Because of the identity operator it is impossible for the integral to converge close to 0, however, by subtracting the divergent terms and adding them back through their analytic continued value we are able to make sense of one of the dangerous zone. The region close to 1 may converge or diverge depending of the 4-point function and its behavior in the t-channel, in the cases in which it diverges we can perform the integral until a cutoff and we need to subtract the divergent terms. Although it may seem complicated some calculations are done in the appendix B and hopefully they

should make it clear.

3.2 Alpha space in higher dimensions

As we go to higher dimensions we start to have two eigenvalues to worry about, one is related to the scaling dimension, while the other is related to the spin. A new problem is whether the alpha space really exists in higher dimensions, although we can define $\Psi_{\alpha,\beta}$, [15], in terms of higher dimension conformal blocks, [17], we don't have a closed expression outside some dimensions, but in the cases we have a closed expression we can confirm its existence. The alpha transform, if it exist, would be given by

$$\hat{g}(\alpha, \beta) = \int_0^1 \int_0^1 dz dx \mu(x, z) g(x, z) \Psi_{\alpha,\beta}(x, z), \quad (3.12)$$

$$\mu(x, z) = \frac{((1-x)(1-z))^{a+b}}{(xz)^2} \left| \frac{x-z}{xz} \right|^{d-2}, \quad (3.13)$$

with x and z independent of each other and the inverse transformation being

$$g(x, z) = \int [d\alpha] [d\beta] \frac{\hat{g}(\alpha, \beta)}{N(\alpha, \beta)} \Psi_{\alpha,\beta}(x, z). \quad (3.14)$$

Many of the previous comments are still valid here, like the one about the convergence of a 4-point function, the inner product between the $\Psi_{\alpha,\beta}$ and some care needs to be taken when dealing with some specific values of α and β . A new consideration for dimensions larger than 2 is in new poles entering in the expression for g , this could occur because there are poles in the conformal blocks when we dip bellow the unitary bounds, their existence were discussed in [18, 19], this is clearly expressed in the Lorentzian inversion formula, [16]. In the alpha space we don't need to worry about this region since these poles stay in the region α and/or $\beta < -\frac{1}{2}$, and we will never need to deform the contour beyond $-\frac{1}{2}$, the exception being for scalar operators, however in this cases we still shouldn't find any poles since we will be inside the unitary region for scalars.

The $\Psi_{\alpha,\beta}$ are a complicated sum of conformal blocks and their coefficients

are known up to normalization, we can use this fact to re-derive these functions using only the quadratic and quartic Casimir, however at the moment, unlike the complicated expressions for conformal blocks, their form is not known for arbitrary α , β and d . Still there are some cases known for arbitrary dimension, like those representing spin 0, 1 or conserved current, and they turned out to be quite simple despite involving conformal blocks with non-integer spin. The solution for scalar and conserved current was already presented in [15]

$$\Psi_{\frac{\Delta-2}{2}, \frac{\Delta-1}{2}}(x, z) = \mathcal{F}_{1,1,1}^{1,2,2} \left(\begin{array}{c|cc} 1 & \frac{\Delta}{2}, & \frac{3-\Delta}{2} \\ \frac{3}{2} & & 1 \end{array} \middle| \begin{array}{c|cc} \frac{\Delta}{2}, & \frac{3-\Delta}{2} \\ 1 & & \frac{z-1}{z}, \frac{x-1}{x} \end{array} \right), \quad (3.15)$$

the spin 1 and the conserved currents can equally be represented by either a Kampé de Fériet or a sum of them. As a speculation it may be possible to use these more restricted cases to calculate some OPE coefficients, note that the d -dimensional scalar can be exactly written as

$$\Psi_{\frac{\Delta+1-d}{2}, \frac{\Delta-1}{2}}(x, z) = \mathcal{F}_{1,1,1}^{1,2,2} \left(\begin{array}{c|cc} 1 & \frac{\Delta}{2}, & \frac{d-\Delta}{2} \\ \frac{d}{2} & & 1 \end{array} \middle| \begin{array}{c|cc} \frac{\Delta}{2}, & \frac{d-\Delta}{2} \\ 1 & & \frac{z-1}{z}, \frac{x-1}{x} \end{array} \right). \quad (3.16)$$

Chapter 4

Constructing 4-point functions

The idea behind this chapter is to give a look at some simple, and not so simple, 4-point function that we can create using the crossing functions, to check for important conditions and look at some know theories that produce them.

In chapter 2 we were able to show that using our crossing functions we could write a function that have the same symmetries as a stripped correlator

$$\mathcal{G}(z, x) = \frac{F(p(z, x), q(z, x), r(z, x))}{(p(z, x) r(z, x))^{\Delta_\phi}}, \quad (4.1)$$

this being the result of having a solution for the crossing conditions,

$$p(z, x) = q(1 - z, 1 - x), \quad p(z, x) = r\left(\frac{z}{z-1}, \frac{x}{x-1}\right), \quad p(z, x) = p\left(\frac{1}{z}, \frac{1}{x}\right), \quad (4.2)$$

that is simple enough for us to be able to rewrite $\frac{xz}{(1-x)(1-z)}$ in terms of p, q and r . There is one question that we want to start this chapter with, does any F give us a valid 4-point function? Another important question would be does any 4-point function can be written in the form given by (4.1), but in this case we don't have a clear solution.

4.1 Generalized polynomials

As its was shown in the last chapter the integrals in the alpha space formalism consist of integrating a striped correlator against a hypergeometric function, so to start analysing the possible F it would be better to try the simplest non-trivial cases before moving on. In a power expansion around a regular point only integers powers appears, however if its done around a singular point not only any value could be in the exponent but the is no guarantee that such power expansion will

exist. This comes to play when dealing with power expansions of p , q and r around 0, by the way these functions were constructed these expansions are at the singular points in a 4-point function, which can be seen from their explicit forms, because of this it is natural to expect expansions with arbitrary powers, any other behaviour is forbidden due to the OPE. In light of these facts the natural F to start is a generalized polynomial, that is, a function that looks like a polynomial, but with arbitrary powers

$$F(p, q, r) = p^a q^b r^c + \text{permutations of powers} \sim (a, b, c). \quad (4.3)$$

Calculating the contribution of the first term for the correlator gives the following expression

$$\frac{p^a q^b r^c}{p^\Delta r^\Delta} = (xz)^b ((1-x)(1-z))^{a-\Delta} (2+2xz-x-z)^{2\Delta-a-b-c}, \quad (4.4)$$

out of the three terms the first two behave nicely but the third one is filled with problems. In any 4-point function cuts and/or poles appear at 0, 1 and ∞ , these are behaviours related to two operator either approaching each other or becoming light-like, however when a , b and c are arbitrary the last term will create an infinite amount of points with new branch cuts or poles not related to any operator, this can be solved by introducing the constrain

$$2\Delta - a - b - c \in \mathbb{Z}_{\geq 0}. \quad (4.5)$$

When choosing an F there needs to be so care to make sure that the analytic structure makes sense for a 4-point function, for the generalized polynomials the condition above is enough, but it could be better, the extended space is made up of three coordinates, but the physical space needs only two variables, which allow for a relation between the three

$$p(x, z) + q(x, z) + r(x, z) = 1. \quad (4.6)$$

Putting both the condition and the relation together allow us to complement the exceeding power, if F had $2\Delta - a - b - c = n$ then the new $\tilde{F}$ defined as

$$\mathcal{G} = \frac{F(p, q, r)}{(pr)^\Delta} = \frac{F(p, q, r) (p + q + r)^n}{(pr)^\Delta} = \frac{\tilde{F}(p, q, r)}{(pr)^\Delta}, \quad (4.7)$$

would generate the same striped correlator in the physical space and would have $2\Delta - \tilde{a} - \tilde{b} - \tilde{c} = 0$, this means that any generalized polynomial could be written by a sum of terms with no exceeding powers, (a, b, c) .

To reiterate what was said previously not all F will give a valid 4-point function, not only that but the generalized polynomials are much more constrain than initially seen, while they are definitely simple they could, at least in principle, be used as a basis to approximate more complex correlation function, however due to the condition $a + b + c = 2\Delta$ this won't be possible. Sadly the analytic structure required is quite strong which makes finding a suitable expression hard, F like $F = \exp(-(p^2 + q^2 + r^2))$ will always have unphysical cuts, while others like $F = \sqrt{p^2 + q^2 + r^2}$ have even newer cuts due to the squared root, in short there needs more work to find a 4-point function that is not given by a generalized polynomial, for the moment let's look at some examples of these

$$\begin{aligned} (a, b, c) = \mathcal{G}(x, z) &= \frac{p^a q^b r^c + p^a q^c r^b + p^b q^a r^c + p^b q^c r^a + p^c q^a r^b + p^c q^b r^a}{(p(x, z) r(x, z))^\Delta} \quad (4.8) \\ &= ((1-x)(1-z))^{a-\Delta} (xz)^b + ((1-x)(1-z))^{a-\Delta} (xz)^c \\ &\quad + ((1-x)(1-z))^{b-\Delta} (xz)^a + ((1-x)(1-z))^{b-\Delta} (xz)^c \\ &\quad + ((1-x)(1-z))^{c-\Delta} (xz)^a + ((1-x)(1-z))^{c-\Delta} (xz)^b. \end{aligned}$$

Despite being extremely simple, expression like this appears in many theories, most of them trivial, but some actually interesting ones. The most simple of them is $(a, b, c) = (\Delta, \Delta, 0)$, this one appears in the generalized free boson, it is the $\langle \phi^4 \rangle$ correlation function. In a theory with multiples generalized free boson we can create $\mathcal{O} = \frac{1}{\sqrt{2N}} \sum_{i=1}^N : \phi_i^2 :$ and the $\langle \mathcal{O}^4 \rangle$ will be given by $(\Delta, \Delta, 0) + \frac{4}{N} \left(\Delta, \frac{\Delta}{2}, \frac{\Delta}{2} \right)$, in general the $\left\langle \left(\sum_{i=1}^N : \phi_i^n : \right)^4 \right\rangle$ will all be a generalized polynomials.

A second interesting family is the $(2\Delta, 0, 0)$, these are the cosine operators in the free boson, the 4-point functions that the bound goes to in the low scaling dimension limit.

If we pay attention to each term in the stripped correlator we'll notice that the

terms with q^a will generate a tower of conformal blocks starting with $G_{0,2a}$ which it means that unless one of the coefficients are zero the correlator is decoupled from the identity. This is not the first time something like this was observed, in [20] it was found a expression with similar properties, in fact their expression is given by $\left(\frac{2\Delta}{3}, \frac{2\Delta}{3}, \frac{2\Delta}{3}\right)$.

Another method to create a crossing symmetric function is through the Mellin amplitude, for these simple examples they are easy to calculate and could provide some new information. Following the procedure from [13] we are able to calculate the Mellin amplitude for the generalized polynomials, for all of them $M(\gamma_{12}, \gamma_{13}, \gamma_{14}) = 0$, this, however, do not tell us that these 4-point functions are trivial or that they are present only on trivial theories. For integer and half-integer scaling dimension some correlation function in the minimal models are given by these function.

4.2 Generalized Minimal models

Moving on from a simple, yet not so realistic CFT, to a realistic but complicated one, in the minimal models one of the standards 4-point functions to be calculated is the $\langle \phi_{2,1}^4 \rangle$ in $\mathcal{M}_{p,q}$, $p, q \in \mathbb{Z}$, for the theories with $p = q \pm 1$ it is know that any correleation function is unitary, however given the explicit expression it is possible to extend it to $\mathcal{M}_{a,b}$ for arbitrary a and b and obtain the general expression

$$\frac{\left| {}_2F_1 \left(\begin{matrix} \beta, & 1-\beta \\ & 2-2\beta \end{matrix} \middle| x \right) \right|^2 + \frac{\sin(3\pi\beta)}{\sin(\pi\beta)} \left| x^{2\beta-1} \frac{\Gamma(3\beta-1)\Gamma(2-2\beta)}{\Gamma(1-\beta)\Gamma(2\beta)} {}_2F_1 \left(\begin{matrix} \beta, & 3\beta-1 \\ & 2\beta \end{matrix} \middle| x \right) \right|^2}{|1-x|^{-\beta}}, \quad (4.9)$$

with $\beta = \frac{b}{a}$, $\Delta = \frac{3\beta-2}{2}$ and $\Delta_1 = \text{Min}(4\beta-2, 4) = \text{Min}\left(\frac{8\Delta+2}{3}, 4\right)$.

These are the generalized Minimal models and they form a shocking contrast when compared to the other 4-point functions we have, these expressions can be written with p, q and r using Appell F_4 , but to check the symmetry it is needed to use some of the F_4 identities, that being said we will look at some cases where it simplifies. Before showing these cases it is interesting to notice that changing $\beta \rightarrow \frac{1}{\beta}$ is equivalent to the change $\phi_{2,1} \rightarrow \phi_{1,2}$ in the same theory.

The simplest ones are the $\langle \epsilon^4 \rangle$ in the Ising model, the $c = 1$ Minimal Model

and the case with $\Delta = \frac{3}{2}$, these are, respectively

$$\frac{p^2 + q^2 + r^2 - (pq + qr + rp)}{pr}, \quad (4.10)$$

$$\frac{1}{2p^{\frac{1}{2}}r^{\frac{1}{2}}}, \quad (4.11)$$

$$\frac{(2-p)(2-q)(2-r)}{36p^{\frac{3}{2}}r^{\frac{3}{2}}}. \quad (4.12)$$

In these cases the hypergeometric functions completely simplify to a polynomial due to beta assuming the form $\beta = \frac{n}{3}$, it is easy to see that they do allow to be written as a simple function of p, q and r . A similar pattern continue when they simplify to something more complicated, here we show the cases $\beta = \frac{1}{4}$, the $\langle \sigma^4 \rangle$ in the Ising model and the $\langle \sigma'^4 \rangle$ in the tricritical Ising model

$$\sqrt{2} \frac{p^{-\frac{1}{2}}q^{-\frac{1}{2}}r^{-\frac{1}{2}}}{p^{-\frac{3}{4}}r^{-\frac{3}{4}}} \sqrt{\frac{p^{\frac{1}{2}} + q^{\frac{1}{2}} + r^{\frac{1}{2}}}{p^{\frac{1}{4}}r^{\frac{1}{4}}}}, \quad (4.13)$$

$$\frac{\sqrt{r^{\frac{1}{2}} + q^{\frac{1}{2}} + p^{\frac{1}{2}}}}{\sqrt{2} p^{\frac{1}{8}}r^{\frac{1}{8}}}, \quad (4.14)$$

$$\frac{\left(2 \left(p^{\frac{3}{2}} + q^{\frac{3}{2}} + r^{\frac{3}{2}}\right) + \frac{3}{2}p^{\frac{1}{2}}q^{\frac{1}{2}}r^{\frac{1}{2}} - \left(pr^{\frac{1}{2}} + qr^{\frac{1}{2}} + pq^{\frac{1}{2}} + rq^{\frac{1}{2}} + rp^{\frac{1}{2}} + qp^{\frac{1}{2}}\right)\right) \sqrt{r^{\frac{1}{2}} + q^{\frac{1}{2}} + p^{\frac{1}{2}}}}{2\sqrt{2} p^{\frac{7}{8}}r^{\frac{7}{8}}}. \quad (4.15)$$

Even in these new examples the function F turnout to be very similar to the generalized polynomials, however it is interesting that it has a square root term because as was said terms like these tend to not respect the analytic structure of a 4-point function, so there must be something here.

After testing some simple 4-point functions we can say that not all F create a valid correlator, the reason being the creation of unphysical cuts, but with care we are able to avoid it. We are now ready to go to the last step, there we will discuss about the methods we have to check the unitarity of our functions and apply them.

Chapter 5

Unitarity checks

In the previous chapter we were able to create some simple 4-point functions, however for the most part we do not know whether they are unitary or not, as of now there are only, speaking in broad brushes, two ways to check the unitarity of our expressions, the first one are analytic methods and the second are testing necessary conditions. The inversion formulas are a part of the analytic methods, with those we are able to obtain the exact value of OPE coefficients, however they do require full knowledge of the exact behaviour close to 0 and 1 to work properly. In the necessary condition group we have the double discontinuity and approximated expansion around 0, in any unitary QFT the double discontinuity should be positive and with a partial expansion it is possible to obtain some OPE coefficients, the problem here is that they are not enough to fully constrain our correlators to unitary theories, but if we were trying to use numerical expressions this would be the better path to follow.

For this chapter we will be using the alpha space in 2D CFTs, we will start with the generalized polynomials, before moving to more complicated cases, we go over more examples in the appendix D. To be more clear in this chapter we are not only interested in knowing whether an expression is unitary, but also in explore the space of unitary 4-point functions scavenging for interesting properties while we are on it, here we will be looking for the scalar gap.

5.1 Generalized Polynomials

Although the generalized polynomials do form simple 4-point functions it is not possible to test the most general case possible, partially because, as will be shown, the OPE coefficients are already quite complicated, to be more manageable the first step will be to look at the simplest case possible, a generalized polynomial with a single (a, b, c) , before going to a slightly more general case.

Single term generalized polynomials

Between the simple (a, b, c) there are two cases of interest, the unusual ones have a, b and c non-zero, they will create 4-point functions decoupled from the identity, these will be explored first, while the others will have the identity operator. In the generic cases there will be three different series that don't overlap, however since the terms that generate each series can be obtained by swapping parameters there is only need to focusing in one. The full polynomial is given by

$$\mathcal{G}(x, z) = \frac{p^a q^b r^c + \text{permutations}}{p^{\frac{a+b+c}{2}} r^{\frac{a+b+c}{2}}}, \quad (5.1)$$

but from the 6 terms in this expression each series is created by 2

$$\frac{p^a q^b r^c + p^c q^b r^a}{p^{\frac{a+b+c}{2}} r^{\frac{a+b+c}{2}}}, \quad (5.2)$$

the start of the expansion will be determined by the power in q and they won't overlap unless the are parameters that differ by an integer. The x and z needed for the alpha space can be recovered by using (2.20 – 2.22)

$$\frac{p^a q^b r^c + p^c q^b r^a}{p^{\frac{a+b+c}{2}} r^{\frac{a+b+c}{2}}} = \frac{(xz)^b ((1-x)(1-z))^a + (xz)^b ((1-x)(1-z))^c}{((1-x)(1-z))^{\frac{a+b+c}{2}}}, \quad (5.3)$$

because the 2D conformal blocks are just a product of 1D conformal blocks there are two different ways to obtain the decomposition, first is to use the results from 1D, [14], in x and z and the multiply both and the second is just to use the 2D alpha space. Although it doesn't really matter the way since both cases give the exact same expression the hypergeometric in the 1D results will appear so many times that it is better to analyse its on its own.

$$(x)^l (1-x)^k = \sum_{n=0}^{\infty} \frac{(-1)^n (l)_n^2}{n! (2l-1+n)_n} {}_3F_2 \left(\begin{matrix} -n, & 2l-1+n, & -k \\ & l, & l \end{matrix} \middle| 1 \right) k_{l+n}(x). \quad (5.4)$$

As can be seen whenever the power l is zero this expression becomes undefined,

this is a problem that plague most expressions with the identity operator, for this reason it was better to begin with the case with a, b and c non-zero. All of these problem are only apparent since these expression do have a conformal block decomposition, there are two ways to get the actual answer, the first apply if there is a parameter in the expression that allow for it to decouple from the identity, in this case after finishing calculating the general decomposition take the limit to the desired expression, if it doesn't have such parameter multiply the expression by $\frac{p^\epsilon q^\epsilon r^\epsilon}{p^{3\epsilon/2} r^{3\epsilon/2}}$ and follow the previous step.

The generalized hypergeometrics found above are not some random functions, they are actually a type of continuous Hahn polynomial

$$p_n(a, b, c, d | t) = {}_3F_2 \left(\begin{matrix} -n, & n + a + b + c + d - 1, & a + t \\ & a + c, & a + d \end{matrix} \middle| 1 \right), \quad (5.5)$$

these well known polynomials have many properties that will help us to analyse all OPEs. Looking back at the decomposition in (5.4) it is natural to define the polynomial

$$f_a^n(t) = \frac{(a)_n^2}{(2a - 1 + n)_n} {}_3F_2 \left(\begin{matrix} -n, & n + 2a - 1, & \frac{a}{2} - t \\ & a, & a \end{matrix} \middle| 1 \right), \quad (5.6)$$

using some common properties of this hypergeometric, check appendix C for a more detailed account, it is possible to prove that this function is even or odd in t when n is even or odd, the degree of the polynomial is n and the value at $t = 0$ can be shown to be non-negative when $a \geq 0$. An even stronger set of results is that all coefficients are non-negative if $a \geq 0$, with this $f_a^n(t)$ is always positive if $t > 0$ and if $|t| \geq |s|$ then $|f_a^n(t)| \geq |f_a^n(s)|$.

The normalization being consider for the 2D conformal block is

$$G_{J,\Delta}(x, z) = \frac{k_{\frac{J+\Delta}{2}}(x) k_{\frac{\Delta-J}{2}}(z) + k_{\frac{J+\Delta}{2}}(z) k_{\frac{\Delta-J}{2}}(x)}{1 + \delta_{J,0}}, \quad (5.7)$$

with this the conformal block decomposition for one of the series in the generalized polynomial is

$$\frac{p^a q^b r^c + p^c q^b r^a}{p^{\frac{a+b+c}{2}} r^{\frac{a+b+c}{2}}} = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{(-1)^{n+m} G_{m-n, 2b+n+m}(x, z)}{n!m!} \left(\quad \right) \quad (5.8)$$

$$f_b^n \left(\frac{a-c}{2} \right) f_b^m \left(\frac{a-c}{2} \right) + f_b^n \left(\frac{c-a}{2} \right) f_b^m \left(\frac{c-a}{2} \right) =$$

$$\sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \left(1 + (-1)^{n+m} \right) \frac{G_{m-n, 2b+n+m}(x, z)}{n!m!} f_b^n \left(\left| \frac{a-c}{2} \right| \right) f_b^m \left(\left| \frac{a-c}{2} \right| \right).$$

In the last expression it is clear that the OPE coefficients are either positive, for even spins, or 0, for odd spins, as such all (a, b, c) are actually unitary, of course that needs to be some care since whenever a, b and c are bigger than zero there is no identity, as can be seen by the tower of conformal blocks

$$G_{m-n, 2a+n+m}(x, z), \quad G_{m-n, 2b+n+m}(x, z), \quad G_{m-n, 2c+n+m}(x, z). \quad (5.9)$$

This may sound to good to be true, from the works of Rattazzi *et al.*, [3], there is a maximal scalar gap $b(\Delta)$, so that no unitary theory would exist for gaps beyond $b(\Delta)$, the plot of the cases with $a, b, c > 0$ show that there isn't actually a problem.

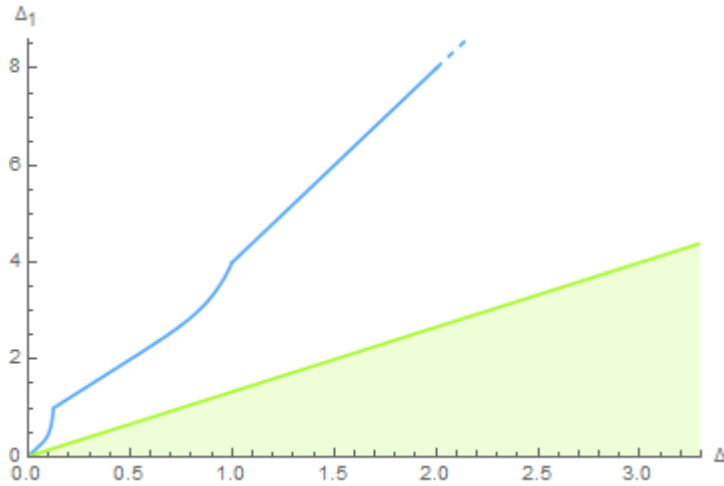


Figure 5.1: Here we have a crude approximation of the boundary in 2D, in blue, and which region our single polynomials can cover, for now. to have a better look at the boundary I would suggest the first picture in [21].

As a reminder all the considerations above are, in principle, valid only for

cases with $a, b, c \neq 0$, for the cases with at least one equal to zero there needs to be some care. First of all, it turns out that all properties previously discussed for $f_a^n(t)$ survive when taking the limit to f_0^n for each individual n separately, this means that like the cases above all $(0, b, c)$ are unitary, including setting b and/or c to 0. In practice only the limits for $n = 0$ and $n = 1$ needs to be taken separately, in these cases there are some $\Gamma(0)$ that do appear after the limit, all the others can be taken together either as a regularized hypergeometric or as is shown bellow

$$(1-x)^t = k_0(x) - tk_1(x) + \sum_{n=0}^{\infty} \frac{(-1)^n t^2 (2)_n^2}{(n+2)!(3+n)_n} {}_3F_2 \left(\begin{matrix} -n, & 3+n, & 1-t \\ & 2, & 2 \end{matrix} \middle| 1 \right) k_{2+n}(x). \quad (5.10)$$

Just like before there is no need to worry about the plot of the scalar gap, however it does allow to fix what could have been considered a problem in the previous case. Consider two unitary 4-point functions, $\langle \phi^4 \rangle$ and $\langle \psi^4 \rangle$, coming from operators with the same dimension, $\Delta_\phi = \Delta_\psi$, then $A \langle \phi^4 \rangle + B \langle \psi^4 \rangle$ is again a valid 4-point function of a hypothetical operator $\mathcal{O}$ with dimension $\Delta_{\mathcal{O}} = \Delta_\phi$, so if one of them doesn't have the identity but the other has then the sum will have it, and as long as $A, B > 0$ the result will be unitary and the gap will be the minimum of the two.

Before finishing this subsection there are two interesting cases for the gap in the $(0, b, c)$, the first comes from the fact that there is always the t^2 in expression (5.10), since $t = \left\lfloor \frac{b-c}{2} \right\rfloor$, if $b = c$ there is only one term left, this case the gap becomes $2b$ while $\Delta = b$, looking at the generalized polynomial, $(0, b, b)$ it is clear that it is the generalized free boson. If $t \neq 0$ then there is the generic expression bellow

$$\mathcal{G} = G_{0,2a} + G_{0,2b} + G_{0,2c} + \text{Coeff } G_{0,2a+2} + \dots, \quad (5.11)$$

when a is taken to 0 the gap jumps to $\text{Min}(2b, 2c, 2)$, however there is nothing impeding b to also go to 0, therefore the gap becomes $\text{Min}(2c, 2)$ and $\Delta = \frac{c}{2}$, so for small enough c the gap becomes $\Delta_1 = 4\Delta$, again as was said in the last chapter this generalized polynomial, $(0, 0, c)$, are the cosine operator for the free boson. As is shown in the plot bellow both of these special cases clearly pop up from the rest, and as was said 2 paragraphs above if we sum the $(0, \Delta, \Delta)$ with some of the (a, b, c) that doesn't have the identity all of the previous region are now not only

unitary, but also have the identity operator in their spectrum.

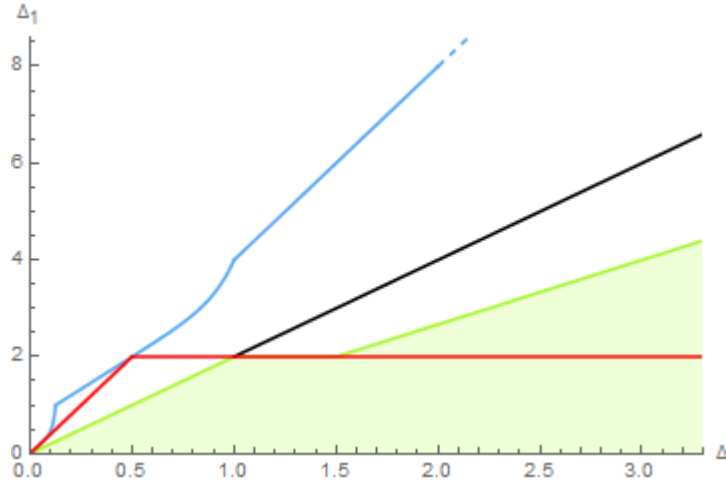


Figure 5.2: This is all the region that can be reached with a single polynomial term, adding a second allow all of the green region to have an identity operator in their spectrum.

Multiple generalized polynomials

It is time to move from the simple single term generalized polynomials to more complicated ones, as was shown in the previous section the OPE coefficients for these correlation functions are already quite complicated and trying to deal with sum of them won't be easy. A big problem when trying to deal with the most general generalized polynomial is the fact that, outside of the restriction $2\Delta - a - b - c = 0$, there are an infinite amount of ways to arrange a , b and c , which would be too hard to take care of, in practice the best way to follow is to instead try to find the set of interesting correlator, in our case the ones with large scalar gap. In the previous section there was the first example of a correlator that still is interesting after the sum, the sum of a 4-point function without an identity and one with it allow for the identity to be added without problem, in short what makes a multi term generalized polynomial interesting is its less rigid OPE, as (5.8) shows the OPE structure of a (a, b, c) is extremely rigid, but by summing and subtracting extra terms it is possible to add and delete operators without hurting either unitarity or crossing symmetry. Again, because of the arbitrary a , b and c the aim is not to show every possible interesting correlator, but to give some ideas on how to construct some of those.

Although not explicitly expressed there were already examples shown that it

is possible to increase the gap by adding more terms, out of them the only one to still be unitary was (4.10), the $\langle \epsilon^4 \rangle$ in the Ising model turns out to be given by a generalized polynomial and as is known it sits close to a kink in the bound for the gap. As was mentioned before the idea is not to look at the most general case, but to give an idea on how to construct some interesting cases, a good way to follow is to have a quick analysis of a two term generalized polynomial since it already shows most of the problems which lies ahead.

Starting with the most general case, $A(a, b, c) + B(d, e, f)$, $a \geq b \geq c$ and $d \geq e \geq f$, the aim is to maximize the scalar gap, if either $c > f$ or $f > c$, then the only solution would be to have A or B be 0 which is uninteresting, as such $c = f$. Applying $a + b + c = 2\Delta = d + e + f$, then $a = d + \delta$, $b = e - \delta$ for some δ , by having $A = -B \neq 0$ the gap increases to $\text{Min}(2a, 2d, 2b, 2e, 2c + 2)$, however while (a, b, c) have three series that don't interact with each other this case have five, this means that for a generic δ two of those series will be purely negative, even with proper δ , if $d - c$ or $e - c$ is an integer, at least one of the series will be purely negative for arbitrary a and b . Here are some lessons to take from the two term case, first when adding a new term is important to either check to make sure its series interact with other existing series or make sure its coefficient never becomes otherwise the expression is automatically non-unitary, second if a set of parameters differ by integers these series will overlap, this is what happen in the Ising model $(2, 0, 0) - (1, 1, 0)$. A third lesson is hidden in the fact that either A or B could be negative, if both lessons were already taken into account then there would be multiple hypergeometric functions in the same series, which one should be taken to be negative to make the expression unitary? The properties previously discussed about $f_a^n(t)$ are not enough to know, but those are not the only ones at our disposal, as is shown in appendix C the polynomials f have a clear asymptotic behaviour which would tell if any of the two cases could be unitary. To give an example here is a case that, like (a, b, c) , have only three distinct series

$$A(0, b, c) + B(1, b - 1, c) + C(1, b, c - 1) + D(0, b - 1, c + 1) + \quad (5.12) \\ E(0, b + 1, c - 1) + F(2, b - 1, c - 1), \quad 1 < b, c < 2.$$

What is special about this example? Well, notice that it follow both lessons, not only the parameters are grouped in sets in which the terms just change by an integer, but also they were chosen carefully to make sure every term have a series

interacting with at least another one. This example is a bit of an overkill, there were all the possibilities given the constraint of the parameters, so for the first step there will be some simplification. The step 0 will be $(0, b, c)$, since $1 < b, c < 2$ the scalar gap is 2, for the first step there will be the minimum amount of terms from the expression above that keep the lessons

$$A(0, b, c) + B(1, b - 1, c) + D(0, b - 1, c + 1), \quad 1 < b, c < 2. \quad (5.13)$$

The next step for this example would be to use (5.8) to decompose said expression, there are three conditions that needs to be satisfied, the first is the normalization of the 4-point function, or the coefficient of $G_{0,0}$ satisfying $2A + 2D = 1$, and the second and third are annihilation of $G_{0,2b-2}$: $D + B = 0$ and of $G_{0,2}$: $4B + (b - c)^2 A + (b - c - 2)^2 D = 0$. After all of this the gap went from 2 to $\text{Min}(2b, 2c)$, now the only step left is to check whether this expression is unitary, this won't be shown here because it is time to move to another important case, what would happen if the parameters cannot be changed by an integer?

Sometimes the expressions being analysed will have parameters that cannot be changed by an integer amount, that may be because of the region being analysed or because the Δ is too low to admit such changes, in this cases there is no universal rule proceed, only trial and error, the worst case would be

$$-A(a, b, c) + B(a, b + \delta_1, c - \delta_1) + C(a + \delta_2, b, c - \delta_2) + D(a + \delta_3, b - \delta_3, c). \quad (5.14)$$

A quick glance at the expression above reveal to us that only (a, b, c) could have a negative coefficient, that is because we have new series $a + \delta_2, b + \delta_1$ and so on, that could spoil unitarity right away. The expression above may seem like it could work, but in all the cases where it could be interesting and it is unitary it will bring the gap down. A better example here is

$$A(0, 0, 2b + 1) + B(0, b, b + 1) - C(1, b, b), \quad 0 < b < 1, \quad (5.15)$$

the gap here is $2b$, and by the first lesson A should never be negative since no series overlap with the one created by $2b + 1$, unlike the previous expression this is unitary with gap $2b + 2$ as is worked in appendix D.

The picture below shows which regions it is known to be covered by the generalized polynomial, note that this doesn't mean that these are the only regions, or this are the closet that can reached to the boundary using the polynomials, but only what is known at the moment.

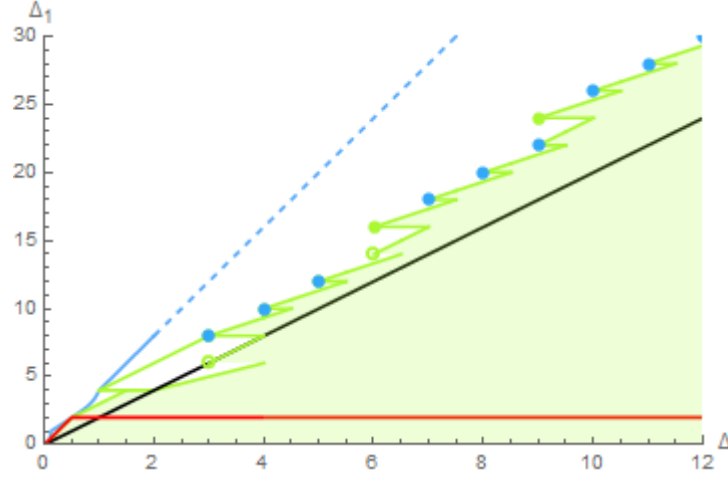


Figure 5.3: Regions we can cover using a more complicated generalized polynomials, for $\Delta \geq 3$ only integers Δ with the blue dot and the line below it have an identity operator above the black line.

5.2 Non-Polynomial 4-Point functions

Unlike the previous case, there isn't much to say here, as was shown even in the simplest cases the OPE coefficients were already complicated, not only that but the lack of different species of non-polynomial functions will not help in constructing this region. The aim here will be to talk a little of what can be learn looking at $p^{\frac{1}{2}} + q^{\frac{1}{2}} + r^{\frac{1}{2}}$ and the generalized minimal models and finding their gap.

$p^{\frac{1}{2}} + q^{\frac{1}{2}} + r^{\frac{1}{2}}$ and its cousins

One of big surprises the checking the generalized minimal models was finding that even some of those complicated 4-point functions have a very simple form when expressed in terms of p, q and r , one of those expressions was $\sqrt{p^{\frac{1}{2}} + q^{\frac{1}{2}} + r^{\frac{1}{2}}}$, so a natural question is whether this can be extended to other generalized polynomials. In short the answer is no, the analytic structure of a 4-point function requires that any cut needs to be connected to $x, z = 1$ and $x, z = 0$, however a general generalized polynomial will have random curves in $\mathbb{C}^2$ where its value

are a negative real number, this would translate into unphysical cuts appearing in the physical space.

A way to move forward is to impose that such generalized polynomial won't ever be negative reals inside the root, as will be shown this means that the function is a constant. If $f(x, z)$ are analytic everywhere but are never equal to some values $\{t_n\}$, then fix $x = x_0$, now $f(x_0, z) = g(z)$ is a regular 1-variable analytic function, if $g(z)$ is never $\{t_n\}$, then $k_m(z) = g(z) - t_m$ is never zero, or $h_m(z) = \frac{1}{k_m(z)}$ never diverges, if there is only one element in $\{t_n\}$, then $h_m(z)$ could be an entire function, but if not it can only be a constant due to Liouville theorem, since this is true for every x_0 , then $f(x, z) = f(x)$, repeating the argument and f can only be a constant.

There is a clear way to circumvent this limitation, by using a function that is not analytic everywhere, since the square root maps the complex plane into the region $\text{Re}[z] \geq 0$, then the sum of three square roots $p^{\frac{1}{2}} + q^{\frac{1}{2}} + r^{\frac{1}{2}}$ cannot be a negative real number and such $\sqrt{p^{\frac{1}{2}} + q^{\frac{1}{2}} + r^{\frac{1}{2}}}$ won't ever have extra cuts in the physical space due to the overall root. Some change in the previous expression could cause problems, one example would be $\sqrt{p^{\frac{1}{2}+\delta} + q^{\frac{1}{2}+\delta} + r^{\frac{1}{2}+\delta}}$, if δ is positive there would be a window to have negative reals inside the overall root.

For an example I would suggest plotting the real and imaginary part of the expression above, $((1-x)(1-z))^{\frac{1}{2}+\delta} + (xz)^{\frac{1}{2}+\delta} + 1$, supposing we already got rid of the $2xz + 2 - x - z$, for $\delta = 0.05$, $z = 10i$ and $x = x + iy$ with $-5 < x, y < 5$, once with the overall square root and once without it. In both plots we have two cuts emerging from $x = y = 0$ and $x = 1, y = 0$, these are regions in $\mathbb{C}^2$ where two operators are becoming light-like separated, but in the square root plot there is a third cut that shouldn't exist. This is a way to show why $\delta = 0$ is special, but not why there is no $p^a + q^a + r^a$, $a < \frac{1}{2}$.

A hidden property of the square root, actually any root, is that they will create an infinite tower of series of conformal blocks related to the powers $a, 2a, 3a$ and so on, the only way these series can interact is if they become separated by an integer value at some point. If a is irrational it will never happen and eventually negative terms will appear no matter if it is an overall squared root or any other root, so they could not exist by themselves, if a is rational then the infinite tower condenses into a finite number of families that only interact within themselves, the problem is that for an overall power of $\frac{1}{2}$ the third term is already negative, so unless there is at most two families they again cannot exist by themselves, this condition is only satisfied by $a = \frac{1}{2}$.

For the cases with a irrational there is still hope that by cleverly adding or multiplying an extra term they could give a unitary expression, if a is rational then there are other possibilities like having the overall power b be big enough, whether or not this procedure works there were still multiple functions that weren't even considered, some examples are $\left(p^{\frac{1}{2}} + q^{\frac{1}{2}}\right)^b$ + permutations, $\left((pq)^{\frac{1}{2}} + (qr)^{\frac{1}{2}} + (rp)^{\frac{1}{2}}\right)^b$, and so on. For now there is nothing that select $\sqrt{p^{\frac{1}{2}} + q^{\frac{1}{2}} + r^{\frac{1}{2}}}$ over the other possibilities, however there is not to say that there is no difference between them. The way the Ising model can be written as a sum of hypergeometric function may be unique and it help to simplify the analysis, it'll be shown that the minimal models OPE coefficient can be written as Wilson Polynomials, ${}_4F_3$. Other simplifications include a general hypergeometric times a generalized polynomial becoming a Kampé de Fériet, a hypergeometric with two variables and a generalization of Appell functions, while the others will have to rely on expansions using non-symmetric generalized polynomial to create expansions that do make sense.

Possible expansions

An important point to make as quick as possible is that expansions are really hard, not only there is no control over how the function actually looks at long distances, which is necessary when imposing that the function have the correct analytic structure, but also any clear and manifest symmetry will, usually, be drowned out. Starting off with an example that showcase all of these problem, here is (4.14)

$$\sqrt{\frac{r^{\frac{1}{2}} + q^{\frac{1}{2}} + p^{\frac{1}{2}}}{2 p^{\frac{1}{4}} r^{\frac{1}{4}}}} = \sqrt{\frac{r^{\frac{1}{2}} + p^{\frac{1}{2}}}{2 p^{\frac{1}{4}} r^{\frac{1}{4}}}} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \left(\frac{q^{\frac{1}{2}}}{r^{\frac{1}{2}} + p^{\frac{1}{2}}}\right)^n, \quad q^{\frac{1}{2}} \leq r^{\frac{1}{2}} + p^{\frac{1}{2}}, \quad (5.16)$$

from the expansion it is neither clear that the function is crossing symmetric nor that it behaves as expected for a 4-point function. Although there are certainly big drawbacks there still are pretty good reasons to study these expansions, the main reasons at the moment are, first when dealing with complicated expressions the alpha transform can become extremely hard, by using an expansion it may be possible to deal with these issues. Second these expansions are familiar, the series in the regions $q \ll p \ll r$ and $p \ll q \ll r$ reorganize themselves into $\frac{q}{r}$ and $\frac{p}{r}$, these are the familiar expansions around 0 and 1 respectively, as such their

behaviour and convergence are already well known in the physical space.

Something interesting to point out about the expansion above is that the horrible term $2xz + 2 - x - z$ is cancelled term by term, not only that but there is no bad cut or pole in each term, this should always happens since the region $q \ll p, r$ is exactly the physical region $x \ll z$, or $z \ll x$, which do allow for an expansion in the conformal cross ratios u and v . The pole structure in $\hat{g}$ is completely determined by the behavior of g close to zero, this is the way these expansion tell us exactly which conformal blocks to expect in the spectrum. To continue with the example it is still hard to do the alpha transform, expanding in p fixes the problem

$$\sqrt{\frac{p^{\frac{1}{2}} + q^{\frac{1}{2}} + r^{\frac{1}{2}}}{2 p^{\frac{1}{4}} r^{\frac{1}{4}}}} = \sqrt{\frac{r^{\frac{1}{4}}}{2 p^{\frac{1}{4}}} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \left(\frac{q^{\frac{1}{2}}}{r^{\frac{1}{2}}}\right)^n \sum_{m=0}^{\infty} \binom{\frac{1}{2} - n}{m} \left(\frac{p^{\frac{1}{2}}}{r^{\frac{1}{2}}}\right)^m}, \quad (5.17)$$

when calculating the OPE coefficients, however, the new sum over m will be posing new problems, the sum will have to be performed for every coefficient, but at the end of appendix D there are some tricks to deal with it.

Before ending this note on expansions there is one last point to say, not all expansions are so wild to the point of being useless to define a 4-point-function, the simplest "example" would be

$$\sum_n a_n z^n \rightarrow \sum_n a_n \left(x + \frac{1}{x}\right)^n, \quad (5.18)$$

here it is clear that the function is symmetric under $x \mapsto \frac{1}{x}$, similar takes could be done with a crossing symmetric expression to either build F or the $\mathcal{F}$ defined in appendix A, taking the care to not upset the analytic structure. While this first idea maintain the analyticity, a numerical-friendly expansion is another possibility, in this case both the crossing symmetry and the analytic structure can be addressed by introducing another set of p, q and r , this approach is touched in appendix A.

Generalized Minimal Models

Instead of the expression in (4.9) it will be more convenient to use a slightly generalization for the 4-point function, first multiply (4.9) by $\left(\frac{2+2\Delta-3\beta}{3}, \frac{2+2\Delta-3\beta}{3}, \frac{2+2\Delta-3\beta}{3}\right)$ and then change the parameter β with $2 - k = 3\beta$, now the correlation function can be decoupled from the identity which makes calculation easier

$$\begin{aligned}
& \left| x^{\frac{k+2\Delta}{3}} (1-x)^{\frac{k-\Delta}{3}} {}_2F_1 \left(\begin{matrix} k, \frac{1+k}{3} \\ \frac{2+2k}{3} \end{matrix} \middle| x \right) \right|^2 + \\
& \frac{\sin(\pi(2-k))}{\sin(\frac{2-k}{3}\pi)} \left| x^{\frac{1-k+2\Delta}{3}} (1-x)^{\frac{1-k-\Delta}{3}} \frac{\Gamma(1-k)\Gamma(\frac{2+2k}{3})}{\Gamma(\frac{1+k}{3})\Gamma(\frac{4-2k}{3})} {}_2F_1 \left(\begin{matrix} 1-k, \frac{2-k}{3} \\ \frac{4-2k}{3} \end{matrix} \middle| x \right) \right|^2.
\end{aligned} \tag{5.19}$$

These hypergeometrics have some nice change of variables, the one that is interesting here is the following

$${}_2F_1 \left(\begin{matrix} a, b \\ 2b \end{matrix} \middle| x \right) = (1-x)^{-\frac{a}{2}} {}_2F_1 \left(\begin{matrix} \frac{a}{2}, b - \frac{a}{2} \\ b + \frac{1}{2} \end{matrix} \middle| -\frac{x^2}{4-4x} \right), \tag{5.20}$$

this change allow us to have a nicer expansion since the function inside the hypergeometric have a special 1D conformal block decomposition. Since the second term can be obtained from $k \mapsto 1-k$ and multiplying the proper constants it is enough to calculate the first term.

$$\begin{aligned}
& x^{\frac{k+2\Delta}{3}} (1-x)^{\frac{k-\Delta}{3}} {}_2F_1 \left(\begin{matrix} k, \frac{1+k}{3} \\ \frac{2+2k}{3} \end{matrix} \middle| x \right) = x^{\frac{k+2\Delta}{3}} (1-x)^{\frac{-k-2\Delta}{6}} {}_2F_1 \left(\begin{matrix} \frac{k}{2}, \frac{2-k}{6} \\ \frac{5+2k}{6} \end{matrix} \middle| -\frac{x^2}{4-4x} \right) \\
& = \left(\frac{x^2}{1-x} \right)^{\frac{k+2\Delta}{6}} \sum_{n=0}^{\infty} \frac{\left(\frac{k}{2}\right)_n \left(\frac{2-k}{6}\right)_n}{n! \left(\frac{5+2k}{6}\right)_n} \left(-\frac{1}{4} \frac{x^2}{1-x} \right)^n.
\end{aligned} \tag{5.21}$$

Jumping a few steps, done in appendix D, the OPE coefficients are, like the generalized polynomials, given by a hypergeometric polynomial

$$c_n = \frac{4^{-n} \left(\frac{k+2\Delta}{6} \right)_n^2}{n! \left(n - \frac{1}{2} + \frac{k+2\Delta}{3} \right)_n} P_n \left(\frac{1+k}{6}, \frac{2\Delta-1}{6}, \frac{2\Delta-1}{6}, \frac{4+k}{6} \middle| \frac{2k-1}{6} \right), \tag{5.22}$$

$$x^{\frac{k+2\Delta}{3}} (1-x)^{\frac{k-\Delta}{3}} {}_2F_1 \left(\begin{matrix} k, \frac{1+k}{3} \\ \frac{2+2k}{3} \end{matrix} \middle| x \right) = \sum_{n=0}^{\infty} c_n k_{\frac{k+2\Delta}{6}+2n}(x), \quad (5.23)$$

$$P_n(a, b, c, d | t) = {}_4F_3 \left(\begin{matrix} -n, n+a+b+c+d-1, a+t, a-t \\ a+b, a+c, a+d \end{matrix} \middle| 1 \right). \quad (5.24)$$

Like it was said before, because of the identity operator a new term was multiplied so that the coefficients are well defined. These functions are called Wilson polynomials and like the Hahn polynomials that appeared in the generalized polynomials they do have recursion relations, unlike it though they are more complicated, so there won't be an exact region where the unitary solutions are located. To recover the generalized minimal models the new term $\left(\frac{2+2\Delta-3\beta}{3}, \frac{2+2\Delta-3\beta}{3}, \frac{2+2\Delta-3\beta}{3} \right)$ should disappear, this can be achieved in the limits $k = -2\Delta$, this limit was explored in [22].

Because $c_0(k, \Delta) = 1$ there is no unitary solutions when $\frac{\sin(\pi(2-k))}{\sin(\frac{2-k}{3}\pi)}$ is negative, moreover, $k + 2\Delta$, $1 - k + 2\Delta$, $c_1(k, \Delta)$ and $c_1(1 - k, \Delta)$ should be non-negative. The region formed by this condition is presented below, testing for $n \leq 10$ shows no sign of changes, so it is possible that this is the unitary region, the gap can be easily calculated in terms of k .

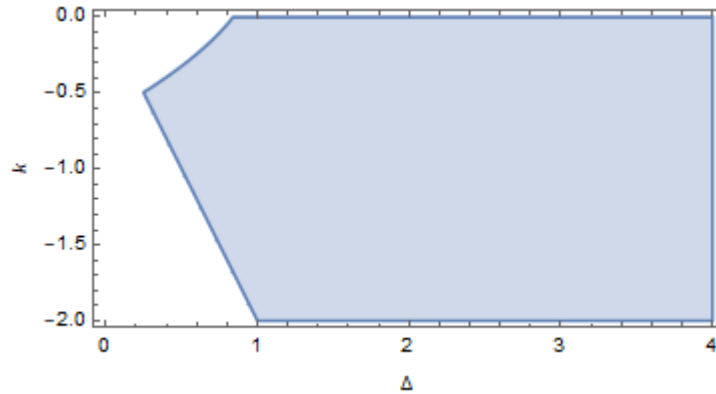


Figure 5.4: The unitary region for equation (5.19) is the blue region.

As a way to put the last two topics together suppose there were no hypergeometric expression of $\langle \sigma^4 \rangle$ only an expansion in terms of p , q and r

$$\frac{\sqrt{p^{\frac{1}{2}} + q^{\frac{1}{2}} + r^{\frac{1}{2}}}}{\sqrt{2} p^{\frac{1}{8}} r^{\frac{1}{8}}} = \frac{r^{\frac{1}{8}}}{\sqrt{2} p^{\frac{1}{8}}} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \left(\frac{q^{\frac{1}{2}}}{r^{\frac{1}{2}}}\right)^n \sum_{m=0}^{\infty} \binom{\frac{1}{2} - n}{m} \left(\frac{p^{\frac{1}{2}}}{r^{\frac{1}{2}}}\right)^m \quad (5.25)$$

$$= \frac{r^{\frac{1}{8}}}{\sqrt{2} p^{\frac{1}{8}}} \sum_{n,m=0}^{\infty} \frac{\frac{1}{2}!}{n!m! \left(\frac{1}{2} - n - m\right)!} \frac{q^{\frac{n}{2}} p^{\frac{m}{2}}}{r^{\frac{n+m}{2}}}. \quad (5.26)$$

From here the conformal block decomposition can be easily calculated, but before having a definitive answer it is necessary perform the infinite sum in m

$$\sum_{n,m=0}^{\infty} \frac{\frac{1}{2}!}{n!m! \left(\frac{1}{2} - n - m\right)!} \sum_{k \geq l} \frac{(-1)^{k+l} G_{k-l,n+k+l}}{\sqrt{2} k!l!} f_{\frac{n}{2}}^k \left(\frac{n}{4} + \frac{m}{2} - \frac{1}{8}\right) f_{\frac{n}{2}}^l \left(\frac{n}{4} + \frac{m}{2} - \frac{1}{8}\right). \quad (5.27)$$

Focusing on the OPE coefficients for $G_{0,0}$, $G_{1,1}$ and $G_{2,2}$

$$\{0,0\} : \sum_{m=0}^{\infty} \frac{\frac{1}{2}!}{\sqrt{2} m! \left(\frac{1}{2} - m\right)!} = \sum_{m=0}^{\infty} \frac{(-1)^m \left(-\frac{1}{2}\right)_m}{\sqrt{2} m!} = \frac{\sqrt{2}}{\sqrt{2}} = 1, \quad (5.28)$$

$$\begin{aligned} \{1,1\} : \sum_{m=0}^{\infty} \frac{\frac{1}{2}!}{\sqrt{2} m! \left(\frac{1}{2} - m\right)!} \left(\frac{m}{2} - \frac{1}{8}\right) &= -\frac{1}{8} + \sum_{m=0}^{\infty} \frac{(-1)^m \left(-\frac{1}{2}\right)_m}{\sqrt{2} m!} \frac{m}{2} \\ &= -\frac{1}{8} + \frac{1}{4\sqrt{2}} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2}\right)_m}{m!} = -\frac{1}{8} + \frac{1}{8} = 0, \end{aligned} \quad (5.29)$$

$$\{2,2\} : \sum_{m=0}^{\infty} \frac{1}{2} \frac{\frac{1}{2}!}{\sqrt{2} m! \left(\frac{1}{2} - m\right)!} \left(\frac{m(m-1)}{4} + \frac{m}{8} + \frac{1}{64}\right) = \quad (5.30)$$

$$\begin{aligned} \frac{1}{128} + \frac{1}{64} + \sum_{m=0}^{\infty} \frac{(-1)^m \left(-\frac{1}{2}\right)_m}{\sqrt{2} m!} \frac{m(m-1)}{8} &= \frac{3}{128} - \frac{1}{32\sqrt{2}} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{3}{2}\right)_m}{m!} = \\ \frac{3}{128} - \frac{1}{128} &= \frac{1}{64}' \end{aligned}$$

a quick comparison with the previous result with the Wilson polynomials show that they both agree perfectly, the term in (5.19) that generate these conformal blocks is the first, as such

$$\sum_{m \geq n} c_n c_m \frac{k_{\frac{k+2\Delta}{6}+2n}(x) k_{\frac{k+2\Delta}{6}+2m}(z) + k_{\frac{k+2\Delta}{6}+2m}(x) k_{\frac{k+2\Delta}{6}+2n}(z)}{1 + \delta_{0,J}} = \sum_{m \geq n} c_n c_m G_{2m-2n, \frac{k+2\Delta}{3}+2m+2n'} \quad (5.31)$$

taking the limit $k + 2\Delta = 0$ finally give the results

$$\{0, 0\} : 1, \quad (5.32)$$

$$\{1, 1\} : 0, \quad (5.33)$$

$$\{2, 2\} : \frac{\Delta(\Delta+1)}{2(5-4\Delta)} \xrightarrow{\Delta=\frac{1}{8}} \frac{1}{64}. \quad (5.34)$$

To end this chapter it would be interesting to mention some different ways we can test unitarity. One such method is to check the double discontinuity of a 4-point function, if we were to use it in the generalized minimal models we would know that the cases outside $3l \leq \Delta \leq 1 + 3l, l \in \mathbb{Z}_{\geq 0}$ are non-unitary, from here we could use the expansion to complement the analysis. If we were talking strictly about the generalized minimal models this would be pointless since it is much easier to use the Wilson polynomials, but an expression like

$$\frac{pq + qr + rp}{pr} {}_0F_1 \left(- \middle| \frac{(pqr)^{\frac{1}{6}}}{p^{\frac{1}{2}} + r^{\frac{1}{2}} + q^{\frac{1}{2}}} \right), \quad (5.35)$$

would be really hard to check, so the double discontinuity could save us time by eliminating chunks and after that we focus in analyse the remaining possible regions. A second and more crude way to go would be to match the conformal blocks numerically, since our current unitary test are the inversion formulas or positive coefficients this would be the only numerical way we have to check our functions.

Chapter 6

Summary

Going back to the beginning we used the S-Matrix bootstrap as a guide to find a way to invert the current CFT bootstrap. The main difficulty in writing a crossing symmetric function in CFT when compared to a S-Matrix was in how crossing symmetry was presented in both cases, in S-Matrix all it does is a permutation in the Mandelstam variables s , t and u , while in CFT we had a complicated sets of maps. To deal with this problem we found a different set of coordinates p , q and r that allowed crossing symmetry to be as simple as it could be, and such we could write our crossing symmetric function in a simple way

$$\mathcal{G}(p, q, r) = \frac{F(p, q, r)}{(pr)^\Delta}. \quad (6.1)$$

Soon we found that this expression wasn't enough, not all F s will give us a 4-point function, the problem being due to the very specific analytic structure $\mathcal{G}$ have. With crossing symmetry conquered the only step needed was a way to test unitarity, for this we used the formalism created in [14, 15], which allowed us to extract the full conformal block decomposition of our expressions.

To test this method we created a set of simple functions, the generalized polynomials, and applied the full machinery in the appendices to extract the OPE coefficients, and although the coefficients were complicated we were able to use the properties of the Hahn's polynomials to show that all (a, b, c) are unitary. An interesting quantity in CFT bootstrap is the size of the gap between the identity and the first scalar, from the beginning in Rattazzi work it was shown that there is a limit of how big this gap can be, but when using the (a, b, c) to explore the same quantity we had some disappointing results. For the most part in Fig. 5.2 we are quite far from the bound, however when we allow a slightly more complicated generalized polynomial we were able to fill another big chunk of the region in Fig. 5.3, although simple the generalized polynomials seem to be able to cover a significant area of this plot.

There are two direction we can take if we want to use this to continue looking for the bound, we can continue to explore the possible F s analytically, either looking for new functions or using the ones we already have, we could focus in other quantities like the next gap or we could move to numerical methods, the problem being the lack of numerical methods that test unitarity. We also need a new set of p, q and r that are friendly to numerical methods but we go over this in the next appendix.

Appendix A

Calculating the crossing functions

The main objectives we will be trying to accomplish in this appendix is to calculate possible crossing functions and looking for promising cases. First of all let's us look at the crossing conditions

$$p(x, z) = q(1 - x, 1 - z), \quad (\text{A.1})$$

$$p(x, z) = r\left(\frac{x}{x-1}, \frac{z}{z-1}\right), \quad (\text{A.2})$$

$$p(x, z) = p\left(\frac{1}{x}, \frac{1}{z}\right). \quad (\text{A.3})$$

These relations are useful if we are looking for ways to create functions that are completely crossing symmetric, that is, $\mathcal{M}(x, z) = \mathcal{M}(1 - x, 1 - z) = \mathcal{M}\left(\frac{x-1}{x}, \frac{z-1}{z}\right)$, however the stripped correlator, $\mathcal{G}$ satisfy a more complicated relation. Being able to relate $\mathcal{G}$ to a completely symmetric function like it is done in the main text may be rare, however we still have a second option which we will look later.

A.1 simple crossing function

Here we will, using some assumptions, calculate the functions used in the main text, some of the conditions were already stated, we will look for functions symmetric in x and z and analytic in both variables for most of $\mathbb{C}^2$. The previous idea was to have a series relating $1 - z$ or $1 - x$ and p , and so on, this can be encoded as $p \rightarrow 0$ when z or $x \rightarrow 1$, however we want to update it to $p(x, z) = 0 \leftrightarrow x$ or $z = 1$ and $\frac{1}{p}$ is analytic everywhere else.

The reason for these conditions are, first, we avoid possible ambiguities when we talk about the region around $p = 0$, if we had multiple regions where p goes to zero we wouldn't be able to easily differentiate between them. Also, we usually

want to impose behaviour close to x or $z = 0$ and by crossing symmetry it translate to x or $z = 1$, if we had multiple places with $p = 0$ we would influence them. Lastly $\mathcal{G}$ have a very particular analytic structure, we want $\frac{1}{p}$ to be analytic outside x or $z = 1$ as a way to not upset this structure.

First let's look at which structure we have. If we start with operators in an Euclidean space in the position $(0, 0)$, $\vec{v} = (v^0, v^1)$, $(0, 1)$ and ∞ and we choose the direction $(0, 1)$ to be the time direction we would see the following, for $0 < v^1 < 1$

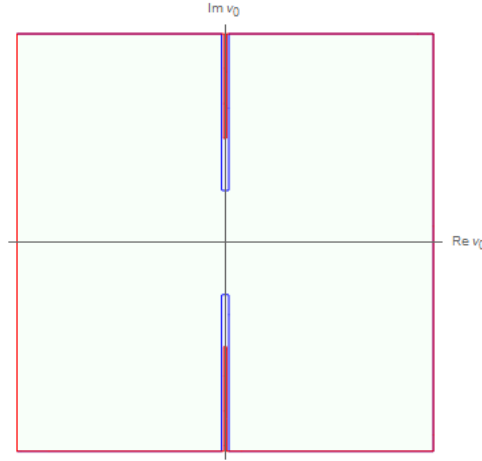


Figure A.1: The cuts occur when the operators become light-like.

We have four cuts present, they are related to the lightcone of each operator, that is, the cuts are present in the positions where $\vec{v}^2 < 0$ or $(1 - \vec{v})^2 < 0$ and each side of the cut relates to a different ordering of the operators in the correlator. This image is only valid if we keep v^1 real, so we could ask what happens when v^1 go outside of the intended region, for this we remind ourselves that we start in a Euclidean space, so we can take time to be in an arbitrary direction $(\sin(\alpha), \cos(\alpha))$. In these cases we could rotate our space in such a way that we obtain time being in the original direction $(1, 0)$, here we already know that we have a cut for v'^0 complex and v'^1 real when $\vec{v} = \vec{v}'^2 < 0$, but if we rotate back it indicates a cut for complex $v^1 = |v'^1| e^{\pm i\alpha}$. We can affirm that the positions of these cuts should be at $\vec{v}^2 < 0$ or $(1 - \vec{v})^2 < 0$, this translate to the regions $xz < 0$ and $(1 - z)(1 - x) < 0$ in the Dolan-Osborn variables. To have a good 4-point function we shouldn't have cuts or poles outside of these regions.

Now we are going to look at $\frac{1}{p}$, by keeping z fixed we have a usual complex function analytic in $\tilde{\mathbb{C}} - \{1\}$. The lack of cuts around 1 tell us that it is possible to write $\frac{1}{p}$ as a Laurent series arbitrary close to 1 and the radius of convergence is $R = \infty$ since our function is analytic everywhere else

$$\frac{1}{p(x, z = fix)} = \sum_{n=-N}^{\infty} k_n (1-x)^n. \quad (\text{A.4})$$

Since this series converge everywhere we can say that far from $x = 1$ we can approximate $\frac{1}{p}$ by the non-negative powers of the series, the new function

$$f(x) = \sum_{n=0}^{\infty} k_n (1-x)^n, \quad (\text{A.5})$$

will have the same analytic structure as $\frac{1}{p}$ far from $x = 1$, however by our assumptions f will not have any poles, including at $x = \infty$, the only function that is analytic everywhere in the Riemann sphere is the constant function, therefore

$$\frac{1}{p(x, z = fix)} = \sum_{n=-N}^0 k_n (1-x)^n. \quad (\text{A.6})$$

Due to the symmetric assumption under x and z we can take p to be

$$p(x, z) = \frac{(1-x)^N (1-z)^N}{\sum_{n,m=0}^N k_{n,m} x^n z^m}, \quad (\text{A.7})$$

the symmetry tell us that $k_{n,m} = k_{m,n}$ and the last condition here are those from the crossing functions, the coefficients will satisfy

$$k_{n,m} = k_{N-n, N-m}. \quad (\text{A.8})$$

Let's think about $\mathcal{G}$ for a moment, if we find a way to use some crossing functions P , Q and R to write a stripped correlator, we will only be able to write those correlator that have the same properties of our crossing functions, that is, if two non-equivalent pairs, $(x_1, z_1) \neq (x_2, z_2) \neq (z_1, x_1)$, give the same value for all crossing function then so would happen in the stripped correlator. An example for this is the following crossing function

$$P = \frac{(1-x)^2 (1-z)^2}{xz}, \quad Q = \frac{x^2 z^2}{(1-x)(1-z)}, \quad R = \frac{1}{xz(1-x)(1-z)}, \quad (\text{A.9})$$

if we try to look at $P = Q = R = 1$ we have the solutions

$$x = e^{\pm \frac{i\pi}{3}}, \quad z = e^{\pm \frac{i\pi}{3}}, \quad (\text{A.10})$$

from the 4 solutions 3 are non-equivalent, therefore if we were to use it we would only be able to create stripped correlators that satisfy

$$\mathcal{G}\left(e^{\frac{i\pi}{3}}, e^{\frac{i\pi}{3}}\right) = \mathcal{G}\left(e^{-\frac{i\pi}{3}}, e^{-\frac{i\pi}{3}}\right) = \mathcal{G}\left(e^{\frac{i\pi}{3}}, e^{-\frac{i\pi}{3}}\right). \quad (\text{A.11})$$

If we want to write a general correlator we cannot have multiple solution for the same (p, q, r) . Trying to solve the three equations at the same time would generally result in no solutions, but in the cases where it has some we can expect many solutions for large N . We want the minimum amount of non-equivalent solutions, so we should try to go as low as possible, because this we are going to look at the solutions with $N = 1$

$$p = \frac{(1-x)(1-z)}{k_{0,0}(1+xz) + k_{1,0}(x+z)}. \quad (\text{A.12})$$

In these cases we have two nice properties, first, if we try to solve for p and q there is only one pair (x, z) up to the x and z symmetry, this already solve the previous problem, the second property is that all of them have satisfy an expression of the form $a + b(p + q + r) + c(pq + qr + rp) + d pqr = 0$, this linear relation in r tell us what is the physical region in this extended space.

The main problem we have in writing a stripped correlator is the fact that it is not completely crossing symmetric since

$$\mathcal{G}(x, z) = \left(\frac{xz}{(1-x)(1-z)} \right)^\Delta \mathcal{G}(1-x, 1-z). \quad (\text{A.13})$$

Despite the complicated denominators the extra term can be counter using our crossing functions

$$\frac{q((k_{0,0} + 2k_{1,0})p + 1)}{p((k_{0,0} + 2k_{1,0})q + 1)} = \frac{xz}{(1-z)(1-x)}, \quad k_{0,0} + k_{1,0} \neq 0, \quad (\text{A.14})$$

if we ask the denominator of p to be invariant under $(x, z) \mapsto (1 - x, 1 - z)$ we will find a function proportional to the expressions in the main text, $k_{0,0} + 2k_{1,0} = 0$. It may look like we could write a stripped correlator as

$$\mathcal{G}(x, z) = \frac{F(p, q, r)}{(pr(1 + (k_{0,0} + 2k_{1,0})q))^{\Delta}}, \quad (\text{A.15})$$

and if we were only looking for a function that has the same properties under change of variable as $\mathcal{G}$ this would be the end. Like the expression in the main text there are unphysical cuts/poles that would be generated by the crossing functions that need to be eliminated. Although it should be possible to write any generalized polynomial using the general expression we will need to be aware of all the unphysical terms in order to get a valid 4-point function, to solve this it is better to go on and choose $k_{0,0} + 2k_{1,0} = 0$, since we do not lose any $\mathcal{G}$ in this choice.

A.2 not-so-simple crossing functions

From the simple solution there were three main problems that we struggled in finding good functions, the first was the multiple pairs that gave the same value for (p, q, r) , the second was in the fact that $\mathcal{G}$ is not completely symmetric and the last were the unwanted poles and cuts generated by the denominators. The second in particular is easily seen when p is very complicated, in these cases the factor $\frac{xz}{(1-x)(1-z)}$ becomes very difficult to account for.

By allowing cuts and other behaviours we start to enter a dangerous territory, that is because we will need to be mindful of problems in $\mathbb{C}^2$ like $(\sqrt{xz})^2 = xz$, $\sqrt{(xz)^2} \neq xz$ and similar, but it can be worth it since we may find new ways to deal with the unphysical cuts. A bad possibility for most cases is to simply put a root in p or a root in the numerator and denominator of p , mainly because $\sqrt{ab} \neq \sqrt{a}\sqrt{b}$, this wouldn't let us get rid of the problematic cuts and may even cause the new p to not be a crossing function, so it is better to look for original functions instead of working with previous ones, an example of a complicated crossing function would be

$$p(x, z) = (xz)^{a/2} {}_2F_1 \left(\begin{matrix} a & b \\ 2b \end{matrix} \middle| 1 - xz \right). \quad (\text{A.16})$$

This is very different from the previous p , first of all this is not associated to any series, in fact this is more similar to the second option where p seems to relate to both 0 and ∞ , second just as was said before there is no easy way to write a $\mathcal{G}(p, q, r)$ because we don't know how to write $\frac{xz}{(1-x)(1-z)}$. It is possible that in these complicated crossing functions there are many solutions for a single (p, q, r) , however we do have some ways out of this problem as long as we are not looking for a way to create a 4-point function everywhere. If we are defining it inside the causal diamond we wouldn't see the difference between $\sqrt{\frac{((1-x)(1-z))^2}{xz}}$ and $\frac{(1-x)(1-z)}{\sqrt{xz}}$, so some of these cases may not have any complication when it comes to numerical techniques. Also because of crossing symmetry if our crossing functions define a good and unique function inside $\left\{x \mid \left(|x| \leq 1 \wedge \operatorname{Re}[x] \geq \frac{1}{2} \wedge \operatorname{Im}[x] \geq 0\right) \mid \left(|x| < 1 \wedge \operatorname{Re}[x] > \frac{1}{2} \wedge \operatorname{Im}[x] < 0\right)\right\} \times \mathbb{C}$ we can define the whole function using crossing symmetry.

The main reason to search for different crossing functions are for possible numerical setups, looking at [11] we see that using numerical techniques together with an expansion that worked everywhere it was possible to get bounds in the S-Matrix, if we were to use such ideas in CFT bootstrap we couldn't use the functions present in the main text. The rule $a + b + c = 2\Delta$ and similar are related to the cuts in our function, in an expansion we cannot get close to the cuts easily to make sure our function makes sense, but if we remove any such rule we could do an expansion in ρ_p, ρ_q and ρ_r and create a valid 4-point function in a region without needing some extra complication coming from checking the analytic structure of our function.

A.3 a second possibility

As we've been mentioning every other sentence, $\mathcal{G}$ is not totally symmetric and this have been causing some trouble when trying to deal with more complicated crossing functions, however the conformal block decomposition is not the only way to break the 4-point function in kinematics and dynamical parts, a second possibility is

$$\langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \phi_4(x_4) \rangle = \mathcal{F}(u, v) \prod_{i < j}^4 |x_{ij}|^{\sum_{k=1}^4 \frac{\Delta_k}{3} - \Delta_i - \Delta_j}. \quad (\text{A.17})$$

Unlike $\mathcal{G}$, $\mathcal{F}$ is completely crossing symmetric, this may be a way to, together with more complicated crossing function, create more interesting 4-point function. For the record although it is possible to write $\mathcal{F}$ only in terms of crossing functions, when we recover $\mathcal{G}$ what we will have is a function of both crossing functions and x and z . Since our p above doesn't have any unwanted cut we could use $\mathcal{F}$ as a way to generate a 4-point function, however it will be interesting to have a simpler crossing function without any unphysical cuts, let's ask what would be the simplest function satisfying $f(z) = f\left(\frac{1}{z}\right)$ and that maps the negative reals, and preferably only the negative reals, to the negative reals.

We can take the line from 0 to -1 to map to $-\infty$ or $-\infty$ to 0, choosing the first option we have something of the form

$$f(z) = \frac{g(z)}{(z+1)^2} = \frac{z^2 g\left(\frac{1}{z}\right)}{(z+1)^2} = f\left(\frac{1}{z}\right), \quad (\text{A.18})$$

a good candidate for f is

$$f(z) = \frac{z}{(z+1)^2}, \quad (\text{A.19})$$

if we look closely we will notice that the only the negative reals map to the negative reals, so we can guarantee that p^a will make sense to all a , something that we cannot guarantee from the previous case. In the end

$$p(x, z) = f(xz), \quad q(x, z) = f((1-x)(1-z)), \quad r(x, z) = f\left(\frac{xz}{(1-x)(1-z)}\right), \quad (\text{A.20})$$

we can define the ρ functions as

$$\rho_p = \frac{\frac{1}{2} - \sqrt{p}}{\frac{1}{2} + \sqrt{p}}, \quad (\text{A.21})$$

this allow us to write a general expansion for $\mathcal{F}$ that converge for every x and z not in the cut

$$\mathcal{F} = \sum_{n,m,o=0} A_{(n,m,o)} \rho_p^n \rho_q^m \rho_r^o. \quad (\text{A.22})$$

Of course the solutions for a given trio (p, q, r) is not unique, however this particular functions may be interesting for possible numerical techniques inside

the causal diamond.

We could ask whether (A.22) does make sense, sure the expansion should represent a function without poles or cuts in the physical space, but how do we know it will converge in the extended space, maybe there is no way to extend the function and have it be analytic at the same time. Well it turns out we have a theorem called Cartan's theorem B that guarantee that an analytic extensions from Stein manifold to another Stein manifold always exist. $\mathcal{F}$ is an analytic function defined in a polydisc, which is a Stein manifold, and we are extending to another polydisc, it make sense to say that $\mathcal{F}$ will converge as long as all ρ s have absolute value less than 1.

A.4 Other representations of S_3

At the start of this appendix we've shown that the region where $\mathcal{G}$ is defined everywhere, but $xz \leq 0$, $(1-x)(1-z) \leq 0$ and $\frac{xz}{(1-x)(1-z)} \leq 0$. From here there seems to exist better functions to parametrize our correlator in such a way we would had a good function everywhere. What exactly is the problem with

$$\frac{1 - \sqrt{xz}}{1 + \sqrt{xz}}, \quad \frac{1 - \sqrt{(1-x)(1-z)}}{1 + \sqrt{(1-x)(1-z)}}, \quad \frac{1 - \sqrt{\frac{xz}{(1-x)(1-z)}}}{1 + \sqrt{\frac{xz}{(1-x)(1-z)}}}, \quad (\text{A.23})$$

they seem to be the perfect counterpart to the ρ s defined in the S-Matrix bootstrap. They would generate a unique solution for each pair (x, z) , they would parametrize the function everywhere and they would generate every function defined in the domain, so what went wrong? It turns out nothing, we were the ones who limit ourselves at one representation of S_3 .

The way we defined how crossing symmetry would act on the extended space is equivalent to choosing a 3 dimensional representation of S_3 , let me show this

$$\begin{aligned} \mathbb{1} : (x, z) &\mapsto (x, z), \quad T_1 : (x, z) \mapsto (1-x, 1-z), \quad T_2 : (x, z) \mapsto \left(\frac{x}{x-1}, \frac{z}{z-1} \right), \\ &(\text{A.24}) \\ C_1 : (x, z) &\mapsto \left(\frac{1}{1-x}, \frac{1}{1-z} \right), \quad C_2 : (x, z) \mapsto \left(\frac{x}{x-1}, \frac{z}{z-1} \right), \quad C_3 : (x, z) \mapsto \left(\frac{1}{x}, \frac{1}{z} \right). \end{aligned}$$

Here we have all the transformations allowed by crossing symmetry in the x and z variables, what we want, however, is for it to act on p, q and r as simple permutation, so we are left with

$$\mathbb{1} : (p, q, r) \mapsto (p, q, r), \quad T_1 : (p, q, r) \mapsto (q, p, r), \quad T_2 : (p, q, r) \mapsto (r, q, p), \quad (\text{A.25})$$

$$C_1 : (p, q, r) \mapsto (q, r, p), \quad C_2 : (p, q, r) \mapsto (r, p, q), \quad C_3 : (p, q, r) \mapsto (p, r, q).$$

We can see it quite clearly that $T_1 \equiv (12)$, $T_2 \equiv (13)$ and so on, we could even write the explicit matrices each one correspond to, but this is not the most general 3 dimensional representation we could have, of course the others will give us something more complicated conditions but we still should look for them

$$\mathbb{1} : (p, q, r) \mapsto (p, q, r), \quad T_1 : (p, q, r) \mapsto (s_1 q, s_1 p, s_2 r), \quad T_2 : (p, q, r) \mapsto (s_3 r, s_4 q, s_3 p), \quad (\text{A.26})$$

$$C_1 : (p, q, r) \mapsto (a_1 q, a_2 r, a_3 p), \quad C_2 : (p, q, r) \mapsto (a_4 r, a_5 p, a_6 q), \quad C_3 : (p, q, r) \mapsto (s_5 p, s_6 r, s_6 q).$$

Since every transformation can be reached using T_1 and T_2 every constant should depend only on s_1, s_2, s_3 and s_4 . The way we calculate the coefficients is as follows $C_1 = T_1 \circ T_2$, therefore $p \circ C_1 = s_1 q \circ T_2 = s_1 s_4 q$, $q \circ C_1 = s_1 s_3 r$ and $r \circ C_1 = s_2 s_3 p$, and so on. In short we have

$$\begin{aligned} a_1 &= s_1 s_4, \quad a_2 = s_1 s_3, \quad a_3 = s_2 s_3 \\ a_4 &= s_3 s_2, \quad a_5 = s_4 s_1, \quad a_6 = s_3 s_1 \\ s_5 &= s_2, \quad s_6 = s_1 s_2 s_3. \end{aligned} \quad (\text{A.27})$$

We still need to use some properties of the elements of S_3 to fully constrain the possible values, we have $T_1^2 = T_2^2 = C_3^2 = C_1^3 = C_2^3 = \mathbb{1}$, or $s_i^2 = 1$ and $a_1 a_2 a_3 = a_4 a_5 a_6 = 1$. All of these conditions only tell us that $s_2 = s_4$ and $s_i = \pm 1$, it may seem like we have 8 options to choose, but due to the fact that we could change C_3 by any of the T s we only have 2 distinct structures, the $2 \otimes 1$ and the $2 \otimes \bar{1}$, given by $s_2 = \pm 1$, the others s only give us the relative signs which are arbitrary. If we check our functions we have

$$p = \frac{1 - \sqrt{xz}}{1 + \sqrt{xz}}, \quad q = \frac{1 - \sqrt{(1-x)(1-z)}}{1 + \sqrt{(1-x)(1-z)}}, \quad r = \frac{1 - \sqrt{\frac{xz}{(1-x)(1-z)}}}{1 + \sqrt{\frac{xz}{(1-x)(1-z)}}}, \quad (\text{A.28})$$

these are in the representation with $s_1 = s_3 = -s_2 = 1$, in turn if we were to write $\mathcal{F}$ with them, as in (A.22), we would have the relations $A_{a,b,c} = (-1)^b A_{c,b,a} = (-1)^c A_{b,a,c}$. Note that any other permutation need to be build from these two, which means $A_{a,b,c} = (-1)^{a+b+c} A_{a,c,b}$

A.5 A brief dive into 1D CFTs

The same ideas we been using for higher dimensions CFTs we can apply into the 1D case with minor modifications. First of all, crossing symmetry is given by S_2 , this means that there is only one interesting representation for this case which give us $\mathbb{1} : (p, q) \mapsto (p, q)$ and $T_1 \equiv (12) : (p, q) \mapsto (q, p)$. The stripped correlator only needs to satisfy $\mathcal{G}(x) = \left(\frac{x}{1-x}\right)^{2\Delta} \mathcal{G}(1-x)$, and as we see it depends in only one variable x and is defined in $\mathbb{C} - \{-\infty, 0\} - [1, \infty\}$. The only crossing condition is

$$p(1-x) = q(x), \quad (\text{A.29})$$

there are two solutions that are interesting, the first is the trivial one $p(x) = 1-x$ and $q(x) = x$ and the other one is $\rho_p(x) = \frac{1-\sqrt{2-2x}}{1+\sqrt{2-2x}}$ and $\rho_q(x) = \frac{1-\sqrt{2x}}{1+\sqrt{2x}}$. In the first we can write any correlator as

$$\mathcal{G}(x) = \frac{F(p, q)}{p^{2\Delta}}, \quad (\text{A.30})$$

and allow for analytic expressions, while the second one give us

$$\mathcal{F}(x) = \sum_{n,m=0} A_{(n,m)} \rho_p^n \rho_q^m, \quad (\text{A.31})$$

and allow us to write any 4-point function.

Appendix B

Conformal block decomposition using alpha space

In the main text we used some expressions for the OPE coefficient of our 4-point functions, there we went into a quick review of the alpha space formalism, but now we are going to look at it in a bit more detail and we are going to use it to recover the expressions in the main text.

In the first part we will be looking at the one dimensional case, here we can calculate all the basics, we are going to start by finding Ψ_α out of the solutions for the quadratic Casimir, we are going to show that both the transformation and its inversion are well-defined and we are going to calculate the conformal block decomposition for terms that appear in the generalized polynomials.

For the second part we will be looking for a higher dimensional generalization for the same transformation, however unlike the 1D case we cannot prove its existence outside of some particular cases.

B.1 1D alpha space

In the alpha space formalism we are trying to find a basis for 4-point functions, unlike higher dimensional case the shadow operator formalism don't work here since the integral never converges. Nevertheless we are able to define any 4-point function as a function over the interval $[0, 1]$ using the Dolan-Osborn coordinates and we still have a conformal block decomposition coming from the OPE. The blocks satisfy a quadratic differential equation, so the next idea would be to try to use Sturm-Liouville theorem to find such basis.

The problem here is that unlike cases with discrete spectrum we should expect an integral under the eigenvalue, similar to a Fourier transform, and not a sum over functions, like a sum over spherical harmonics. So at first the transformation won't give us the OPE coefficients we will need to work a bit until we can get

there.

The differential equation given by the quadratic Casimir operator is

$$z^2 \frac{\partial^2 (1-z) k_h^s}{\partial z^2} - (a+b) z^2 \frac{\partial k_h^s}{\partial z} - ab z k_h^s = h(h-1) k_h^s. \quad (\text{B.1})$$

It is not hard to calculate the measure associated with the equation above, which is $\mu(z) = \frac{(1-z)^{a+b}}{z^2}$, we can rewrite our expression as

$$\mu^{-1} \frac{\partial \left(\mu z^2 (1-z) \frac{\partial k_h^s}{\partial z} \right)}{\partial z} - ab z k_h^s = h(h-1) k_h^s, \quad (\text{B.2})$$

and because of Strum-Liouville we expect an inner product of the form

$$\langle f, g \rangle = \int_0^1 dz \mu(z) f^*(z) g(z). \quad (\text{B.3})$$

As was said the integrals in (B.3) should give us a delta function when f and g are part of the basis, however we don't know of any function with this property. If we look at the conformal blocks

$$k_h^s = z^h {}_2F_1 \left(\begin{matrix} h+a, & h+b \\ 2h \end{matrix} \middle| z \right), \quad (\text{B.4})$$

we have a divergence at $z=1$, and for high enough a and b , $\langle k_h^s, k_{h'}^s \rangle$ will diverge, a way to deal with this is to look for a solution that don't behave so bad close to $z = 1$. There is a symmetry between the eigenvalues, the transformation $h \mapsto 1-h$ keeps the differential equation invariant, this means that $Ak_h^s + Bk_{1-h}^s$ will be another eigenfunction. Using the properties of hypergeometric functions we can find what is the behaviour of this new solution close to $z = 1$ and at which values of A and B it will behave well, by normalizing it in such a way that $\Psi_h(1) = 1$ we have

$$\Psi_h(z) = z^{-a} {}_2F_1 \left(\begin{matrix} h+a, & a+1-h \\ 1+a+b \end{matrix} \middle| \frac{z-1}{z} \right). \quad (\text{B.5})$$

However the usual way to write the eigenvalue is $h = \frac{1}{2} + \alpha$, since it will be symmetric under $\alpha \mapsto -\alpha$,

$$\Psi_\alpha(z) = z^{-a} {}_2F_1 \left(\begin{matrix} \alpha+a, & a-\alpha \\ 1+a+b \end{matrix} \middle| \frac{z-1}{z} \right). \quad (\text{B.6})$$

This expression can be obtained by setting the constants to

$$\Psi_\alpha(z) = \frac{1}{2} \left[Q_s(\alpha) k_{\frac{1}{2}+\alpha}^s(z) + Q_s(-\alpha) k_{\frac{1}{2}-\alpha}^s(z) \right], \quad (\text{B.7})$$

$$Q_s(\alpha) = \frac{2\Gamma(-2\alpha)\Gamma(1+a+b)}{\Gamma\left(\frac{1}{2}+a-\alpha\right)\Gamma\left(\frac{1}{2}+b-\alpha\right)}, \quad (\text{B.8})$$

we could ask now that our function is always well behaved close to $z = 1$ if we can find a function that will have a delta function as its inner product with itself, but when we look at (B.7) we can see that the behaviour close to zero is $z^{\frac{1}{2}\pm\alpha}$. If we were to calculate $\langle \Psi_\alpha, \Psi_\beta \rangle$ the behaviour of the whole integrand would be a sum of all the possibilities in $z^{-1\pm\alpha\pm\beta}$, we would need $\text{Re}[-1\pm\alpha\pm\beta] > -1$ for the integral to converge, which is impossible if we let $\alpha, \beta \in \mathbb{R}$. The analog of this problem in the Fourier transform is

$$\langle e^{-ik'x}, e^{-ikx} \rangle = \int_{-\infty}^{\infty} \frac{dx}{2\pi i} e^{ik'^*x} e^{-ikx}, \quad (\text{B.9})$$

this should be a delta, but if the imaginary part of $k'^* - k$ is non-zero, it will always diverge, this forces k and k' to be reals and, if needed, we can extend the validity of the inverse transform by deforming the integration contour. Back to alpha space, if we move to complex α and β we can find a convergent expression as long as $\text{Re}[\alpha] = \text{Re}[\beta] = 0$ and $\text{Im}[\alpha \pm \beta] \neq 0$. As we will show now the inner product between these functions do behave as a sum of delta functions

$$\langle \Psi_\alpha, \Psi_\beta \rangle \propto \delta(\alpha + \beta) + \delta(\alpha - \beta). \quad (\text{B.10})$$

For a purely imaginary α , $\Psi_\alpha^* = \Psi_\alpha$, to perform the calculation we start by rewriting the first Ψ_α as its Mellin-Barns representation

$$\langle \Psi_\alpha, \Psi_\beta \rangle = \int_0^1 dz \mu(z) \Psi_\beta(z) \int [ds] z^{-a} \frac{\Gamma(-s) \Gamma\left(\frac{1}{2} \pm \alpha + a + s\right) \Gamma(1 + b + a)}{\Gamma\left(\frac{1}{2} \pm \alpha + a\right) \Gamma(1 + b + a + s)} \left(\frac{1-z}{z}\right)^s. \quad (\text{B.11})$$

Here we are showing two notations we are going to be using, first is when we have product of functions where a term changes sign $f(a+b)f(a-b)$ we will write it as $f(a \pm b)$, the second is when we have a complex integral going parallel to the whole imaginary axis we will write

$$\int_{-i\infty-\epsilon}^{i\infty-\epsilon} \frac{ds}{2\pi i} = \int [ds]. \quad (\text{B.12})$$

For the second Ψ we create a small deformation

$$= \int_0^1 dz \mu(z) \int [dt] \frac{\Gamma(-t) \Gamma\left(\frac{1}{2} \pm \beta + a + t\right) \Gamma(1 + b + a + \epsilon)}{\Gamma\left(\frac{1}{2} \pm \beta + a\right) \Gamma(1 + b + a + t + \epsilon)} \left(\frac{1-z}{z}\right)^t \quad (\text{B.13})$$

$$\int [ds] z^{-2a+\epsilon} \frac{\Gamma(-s) \Gamma\left(\frac{1}{2} \pm \alpha + a + s\right) \Gamma(1 + b + a)}{\Gamma\left(\frac{1}{2} \pm \alpha + a\right) \Gamma(1 + b + a + s)} \left(\frac{1-z}{z}\right)^s,$$

$$= \int [dt] [ds] \frac{\Gamma(-t) \Gamma\left(\frac{1}{2} \pm \beta + a + t\right) \Gamma(1 + b + a + \epsilon)}{\Gamma\left(\frac{1}{2} \pm \beta + a\right) \Gamma(1 + b + a + t + \epsilon)} \quad (\text{B.14})$$

$$\frac{\Gamma(-s) \Gamma\left(\frac{1}{2} \pm \alpha + a + s\right) \Gamma(1 + b + a)}{\Gamma\left(\frac{1}{2} \pm \alpha + a\right) \Gamma(1 + b + a + s)} \int_0^1 dz \mu(z) z^{-2a+\epsilon} \left(\frac{1-z}{z}\right)^{s+t}.$$

The z integral will not converge because we inverted the order of integral, in its original form the poles in the s and t integrals would pick very specific powers

to create a series that will make the original z integral to converge. However by doing an analytic continuation it will be enough since the s and t integral won't be influenced by the extra poles coming from the analytic continued result.

$$\int [dt] [ds] \frac{\Gamma(-t) \Gamma\left(\frac{1}{2} \pm \beta + a + t\right) \Gamma(1 + b + a + \epsilon)}{\Gamma\left(\frac{1}{2} \pm \beta + a\right) \Gamma(1 + b + a + t + \epsilon)} \quad (\text{B.15})$$

$$\frac{\Gamma(-s) \Gamma\left(\frac{1}{2} \pm \alpha + a + s\right) \Gamma(1 + b + a) \Gamma(a + b + t + s + 1) \Gamma(-s - t - 1 - 2a + \epsilon)}{\Gamma\left(\frac{1}{2} \pm \alpha + a\right) \Gamma(1 + b + a + s) \Gamma(b - a + \epsilon)},$$

we can do the t integral using Barnes's second lemma

$$\int [ds] \frac{\Gamma(-s) \Gamma\left(\frac{1}{2} \pm \alpha + a + s\right) \Gamma\left(\epsilon - \frac{1}{2} \pm \beta - a - s\right) \Gamma(1 + b + a) \Gamma(1 + b + a + \epsilon)}{\Gamma(\epsilon - s) \Gamma\left(\frac{1}{2} \pm \alpha + a\right) \Gamma\left(\frac{1}{2} \pm \beta + b + \epsilon\right)}. \quad (\text{B.16})$$

At this point almost all the ϵ are unnecessary, the $\frac{1}{\Gamma(\epsilon - s)}$ is telling that in the limit the poles generated by $\Gamma(-s)$ will have residue 0 and the ones summing constants won't influence the results, the only ones that are important are the ones that make sure that when $\alpha = \beta$ there are no double poles. This becomes clearer when we think about the delta function as limit of an analytic function $2\pi i \delta(x - x_0) \equiv \text{disc} \left[\frac{1}{x - x_0} \right] = \lim_{\epsilon \rightarrow 0} \left[\frac{1}{(x - x_0 + \epsilon)} - \frac{1}{(x - x_0 - \epsilon)} \right]$, at least if we have a contour passing through x_0 going up the imaginary axis.

$$\frac{(\Gamma(1 + b + a))^2}{\Gamma\left(\frac{1}{2} \pm \alpha + a\right) \Gamma\left(\frac{1}{2} \pm \beta + b\right)} \int [ds] \Gamma\left(\frac{1}{2} \pm \alpha + a + s\right) \Gamma\left(\epsilon - \frac{1}{2} \pm \beta - a - s\right), \quad (\text{B.17})$$

this last integral can be done using Barnes's first lemma

$$\frac{(\Gamma(1 + b + a))^2 \Gamma(\epsilon + \alpha + \beta) \Gamma(\epsilon - \alpha + \beta) \Gamma(\epsilon + \alpha - \beta) \Gamma(\epsilon - \alpha - \beta)}{\Gamma\left(\frac{1}{2} \pm \alpha + a\right) \Gamma\left(\frac{1}{2} \pm \beta + b\right) \Gamma(2\epsilon)}. \quad (\text{B.18})$$

The question to be asked here is whether or not it behaves like a delta function, because of $\frac{1}{\Gamma(2\epsilon)}$ the single poles disappear in the limit $\epsilon \rightarrow 0$. We should look for the places with double poles in the expression

$$\Gamma(\epsilon + \alpha + \beta) \Gamma(\epsilon - \alpha + \beta) \Gamma(\epsilon + \alpha - \beta) \Gamma(\epsilon - \alpha - \beta), \quad (\text{B.19})$$

since they are the only places we could get a delta, that is, $\alpha = \pm\beta$. Looking at $\alpha = \beta - \delta$ we have

$$\frac{\Gamma(\epsilon + 2\beta) \Gamma(\epsilon + \delta) \Gamma(\epsilon - \delta) \Gamma(\epsilon - 2\beta)}{\Gamma(2\epsilon)} \approx \quad (\text{B.20})$$

$$\Gamma(2\beta) \Gamma(-2\beta) \frac{2\epsilon}{(\epsilon + \delta)(\epsilon - \delta)} = \Gamma(2\beta) \Gamma(-2\beta) \left(\frac{1}{\epsilon + \delta} - \frac{1}{\delta - \epsilon} \right). \quad (\text{B.21})$$

From here we can say that

$$\langle \Psi_\alpha, \Psi_\beta \rangle = \frac{Q_s(\pm\alpha)}{2} \left(\frac{\delta(\alpha - \beta) + \delta(\alpha + \beta)}{2} \right). \quad (\text{B.22})$$

Now that we've shown that Ψ inner product behave as a delta function we will recover the conformal block decomposition from a stripped correlator. Once we used alpha space formalism on a correlator we can use the inverse transformation

$$g(z) = \int [d\alpha] \frac{\hat{g}(\alpha)}{N_s(\alpha)} \Psi_\alpha(z), \quad N_s(\alpha) = \frac{Q_s(\pm\alpha)}{2}, \quad (\text{B.23})$$

to find the conformal block decomposition, it is best seen when we rewrite the integral as

$$g(z) = \int [d\alpha] \frac{2\hat{g}(\alpha)}{Q_s(-\alpha)} k_{\frac{1}{2}+\alpha}^s(z). \quad (\text{B.24})$$

We can rewrite it in this way simply because both the interval and every function inside is even in α , now the only place we need to check for poles are

in $\hat{g}(\alpha)$ since neither $k_{\frac{1}{2}+\alpha}^s(z)$ and $\frac{1}{Q_s(-\alpha)}$ have poles for $\alpha > 0$, that being said there is some care that needs to be taken when we are looking for operators with dimension $\frac{1}{2}$, $\alpha = 0$, or the identity, $\alpha = -\frac{1}{2}$.

Now that we know how to obtain a conformal block decomposition from a 4-point function we will use it to calculate a particular case, $x^k(1-x)^l$.

$$\hat{g}(\alpha) = \int_0^1 dz \frac{(1-z)^{a+b}}{z^2} z^k (1-z)^l z^{-a} {}_2F_1 \left(\begin{matrix} \frac{1}{2} + \alpha + a, & \frac{1}{2} - \alpha + a \\ 1 + a + b \end{matrix} \middle| \frac{z-1}{z} \right), \quad (\text{B.25})$$

$$= \int_0^1 dz \int [ds] \frac{\Gamma(-s) \Gamma\left(\frac{1}{2} \pm \alpha + a + s\right) \Gamma(1 + a + b)}{\Gamma\left(\frac{1}{2} \pm \alpha + a\right) \Gamma(1 + a + b + s)} z^{k-a-s-2} (1-z)^{l+a+b+s}. \quad (\text{B.26})$$

Depending on the values of a , b and l this integral may not converge, the way to deal with this we need to introduce a cutoff for the integral close to $z = 1$ and take the terms that converges.

$$= \int [ds] \int_0^{1-\epsilon} dz \frac{\Gamma(-s) \Gamma\left(\frac{1}{2} \pm \alpha + a + s\right) \Gamma(1 + a + b)}{\Gamma\left(\frac{1}{2} \pm \alpha + a\right) \Gamma(1 + a + b + s)} z^{k-a-s-2} (1-z)^{l+a+b+s}, \quad (\text{B.27})$$

$$= \int [ds] \int_0^1 du (1-\epsilon) \frac{\Gamma(-s) \Gamma\left(\frac{1}{2} \pm \alpha + a + s\right) \Gamma(1 + a + b)}{\Gamma\left(\frac{1}{2} \pm \alpha + a\right) \Gamma(1 + a + b + s)} \quad (\text{B.28})$$

$$(1-\epsilon)^{k-a-s-2} u^{k-a-s-2} (1-(1-\epsilon)u)^{l+a+b+s}, \quad (\text{B.29})$$

$$= \int [ds] \frac{\Gamma(-s) \Gamma\left(\frac{1}{2} \pm \alpha + a + s\right) \Gamma(1 + a + b)}{\Gamma\left(\frac{1}{2} \pm \alpha + a\right) \Gamma(1 + a + b + s)} (1 - \epsilon)^{k-a-s-1} \quad (\text{B.30})$$

$$\frac{\Gamma(k - a - s - 1) \Gamma(1)}{\Gamma(k - a - s)} {}_2F_1 \left(\begin{matrix} k - a - s - 1, & -l - a - b - s \\ k - a - s \end{matrix} \middle| 1 - \epsilon \right). \quad (\text{B.31})$$

We can use the properties of hypergeometric functions to rewrite it separating the divergent part and the convergent using

$${}_2F_1 \left(\begin{matrix} a, & b \\ c \end{matrix} \middle| 1 - z \right) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} {}_2F_1 \left(\begin{matrix} a, & b \\ a + b - c + 1 \end{matrix} \middle| z \right) + \quad (\text{B.32})$$

$$z^{c-a-b} \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} {}_2F_1 \left(\begin{matrix} c - a, & c - b \\ c - a - b + 1 \end{matrix} \middle| z \right).$$

When $\text{Re}[1 + l + a + b + s] > 0$ the second term goes to zero as we take ϵ to zero, however if it is non-positive it will diverge as either a power law or as a logarithm, therefore we can always ignore the second term, the first will always goes to one

$$= \int [ds] \frac{\Gamma(-s) \Gamma\left(\frac{1}{2} \pm \alpha + a + s\right) \Gamma(1 + a + b)}{\Gamma\left(\frac{1}{2} \pm \alpha + a\right) \Gamma(1 + a + b + s)} \quad (\text{B.33})$$

$$\frac{\Gamma(k - a - s - 1) \Gamma(1)}{\Gamma(k - a - s)} \frac{\Gamma(k - a - s) \Gamma(1 + l + a + b + s)}{\Gamma(1) \Gamma(k + b + l)}, \quad (\text{B.34})$$

$$= \frac{\Gamma(1 + a + b)}{\Gamma(k + b + l) \Gamma\left(\frac{1}{2} \pm \alpha + a\right)} G_{3,3}^{2,3} \left(1 \middle| \begin{matrix} \frac{1}{2} - a - \alpha, & \frac{1}{2} - a + \alpha, & -a - b - l \\ 0, & k - a - 1, & -a - b \end{matrix} \right). \quad (\text{B.35})$$

Something to notice were is that if we had pretended that our expression would converge from the start we would have the exactly same expression at the end.

What we can do instead is to simply analytically continue whenever we have a divergence at $z = 1$, if we don't have a parameter that allow us to do this analytical continuation we can multiply the expression by $\frac{(pqr)^{\frac{2(x-\Delta)}{3}}}{(pr)^{x-\Delta}}$, any unitary, and a lot of non-unitary, expressions will converge in $0 \leq x < 1$, after that we need to do the continuation until $x = \Delta$.

Although we could stop here, the poles in α are not clear, we can make them appear by decomposing the G function into hypergeometrics

$$A_{3,3,1}^{2,3} {}_3F_2 \left(\begin{matrix} \frac{1}{2} + a + \alpha, & \frac{1}{2} + a - \alpha, & 1 + a + b + l \\ & 2 + a - k, & 1 + a + b \end{matrix} \middle| 1 \right) + \quad (B.36)$$

$$A_{3,3,2}^{2,3} {}_3F_2 \left(\begin{matrix} k - \frac{1}{2} + \alpha, & k - \frac{1}{2} - \alpha, & k + l + b \\ & k - a, & k + b \end{matrix} \middle| 1 \right),$$

$$A_{3,3,1}^{2,3} = \frac{\Gamma(k - a - 1) \Gamma\left(\frac{1}{2} + a \pm \alpha\right) \Gamma(1 + a + b + l)}{\Gamma(1 + a + b)}, \quad (B.37)$$

$$A_{3,3,2}^{2,3} = \frac{\Gamma(1 - a - k) \Gamma\left(k - \frac{1}{2} \pm \alpha\right) \Gamma(k + l + b)}{\Gamma(k + b)}. \quad (B.38)$$

We can finally write an expression for the alpha transform of our correlator with the poles clearly expressed.

$$\hat{g}(\alpha) = \frac{\Gamma(k - a - 1) \Gamma(1 + a + b + l)}{\Gamma(k + b + l)} {}_3F_2 \left(\begin{matrix} \frac{1}{2} + a + \alpha, & \frac{1}{2} + a - \alpha, & 1 + a + b + l \\ & 2 + a - k, & 1 + a + b \end{matrix} \middle| 1 \right) + \quad (B.39)$$

$$\frac{\Gamma(1 - a - k) \Gamma\left(k - \frac{1}{2} \pm \alpha\right) \Gamma(1 + a + b)}{\Gamma(k + b) \Gamma\left(\frac{1}{2} \pm \alpha + a\right)} {}_3F_2 \left(\begin{matrix} k - \frac{1}{2} + \alpha, & k - \frac{1}{2} - \alpha, & k + l + b \\ & k - a, & k + b \end{matrix} \middle| 1 \right).$$

After all our work we were able to get a sum of hypergeometric from which we can see that only one term will create the poles we are looking for, the $\Gamma\left(k - \frac{1}{2} - \alpha\right)$, however we may be worried that despite our best efforts this expression still don't converge in some regions. If $-k - l - a + 1 \leq 0$ neither

hypergeometric converge, but we will see that the sum will, using theorem (2.4.4) in [23] we can show that this sum is equal to

$$\hat{g}(\alpha) = \frac{\Gamma\left(k - \frac{1}{2} \pm \alpha\right) \Gamma(1 + a + b + l)}{\Gamma(k + b + l) \Gamma(k + a)} {}_3F_2 \left(\begin{matrix} \frac{1}{2} + a + \alpha, & \frac{1}{2} + a - \alpha, & -l \\ k + a, & 1 + a + b \end{matrix} \middle| 1 \right). \quad (\text{B.40})$$

Unlike previous expression this converges if $k + l + b > 0$, since for 4-point functions that need to satisfy reflection positivity $a = b$ we will always cover the whole space, and for the last one we have a case symmetric in a and b we could use the theorem (3.3.5) from the same book to find

$$\hat{g}(\alpha) = \frac{\Gamma\left(k - \frac{1}{2} \pm \alpha\right) \Gamma(1 + a + b)}{\Gamma(k + b) \Gamma(k + a)} {}_3F_2 \left(\begin{matrix} k - \frac{1}{2} + \alpha, & k - \frac{1}{2} - \alpha, & -l \\ k + a, & k + b \end{matrix} \middle| 1 \right), \quad (\text{B.41})$$

Which converges for $1 + l + a + b > 0$.

As was said before the OPE coefficients are related to the poles in $\Gamma\left(k - \frac{1}{2} - \alpha\right)$ which are in $\alpha = n + k - \frac{1}{2}$ and have residue $\frac{(-1)^{n+1}}{n!}$ therefore we have the following decomposition

$$x^k (1 - x)^l = \sum_{n=0}^{\infty} - \frac{2 \text{Res}_{\alpha=n+k-\frac{1}{2}}(\hat{g}(\alpha))}{Q_s\left(-n - k + \frac{1}{2}\right)} k_{n+k}^s(x), \quad (\text{B.42})$$

with the minus sign coming from the way we close our contour. Since $a = b$ we will be using the last form. Although each $\hat{g}$ has a very defined region where it converge the poles are polynomials, they always converge and we can use previous theorems to relate the three expressions everywhere

$$x^k (1 - x)^l = \sum_{n=0}^{\infty} \frac{(-1)^n (k + a)_n (k + b)_n}{n! (2k - 1 + n)_n} {}_3F_2 \left(\begin{matrix} -n, & 2k - 1 + n, & -l \\ k + a, & k + b \end{matrix} \middle| 1 \right) k_{n+k}^s(x). \quad (\text{B.43})$$

We won't be analysing this series now, we will do it in the next appendix,

however it is important to know how to deal with the more problematic cases, since every unitary 4-point function with identical operators have the identity.

For conformal blocks with scaling dimension higher than $h = \frac{1}{2}$ we won't have any problems using the expression above, however, there are three cases where this condition fails but we still can have unitary solutions, the first case is $h = \frac{1}{2}$. Expressions still converge when doing the alpha space integral, our problem lies in two different places, first of all looking at the $\hat{g}$ we will have a $\Gamma(\pm\alpha)$ in the expression have a double pole right in the contour of integration, this is countered by the zero $\frac{1}{\Gamma(2\alpha)}$ which appear in $\frac{1}{Q(-\alpha)}$. This solve one of the problems, but it doesn't tell us which value to give to the expression, the full residue or the principal value? We can check this by looking at a simple case

$$x^{\frac{1}{2}} \stackrel{?}{=} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_n^2}{n! (n)_n} k_{n+\frac{1}{2}}^s(x), \quad a = b = 0, \quad (\text{B.44})$$

we could ask whether the equation makes sense or not, by expanding the first term we will see that the coefficients will agree, so we need to take the full residue to obtain the correct expression.

The second interesting case occur when $0 < h < \frac{1}{2}$, here the integral do not converge anymore this is the result of the previous pole crossing the integration contour, to better define the integral in this case we can deform the contour in such a way that it pick up the correct poles in a counterclockwise direction and to avoid the mirrored pole. Outside of this small problem we don't need to make any modifications to our expression, so we move to the last case $h = 0$, here we will encounter multiple problems, the first is another double pole due to $\Gamma\left(-\frac{1}{2} \pm \alpha\right)$, set $\alpha = -\frac{1}{2}$ and we see a $\Gamma(0) \Gamma(-1)$. The way to recover the OPE in this case is by doing the calculation for $h = \epsilon$ and then taking the limit $\epsilon \rightarrow 0$ at the end. If we have a more general expression a foolproof way to deal with this is to multiply by an expression that behaves as x^ϵ for small x and goes to 1 when we take the limit to zero, an example would be the generalized polynomial $(\epsilon, \epsilon, \epsilon)$ which maintain the crossing symmetry, after that we calculate the terms we need and take the limit at the end.

B.2 Alpha space in higher dimensions

As we move to higher dimensions we won't be able to solve it in the way we did in 1D, this is mainly because of two reasons, the first is that the calculations done above needed the explicit form of Ψ , this won't be possible in higher dimensions, the second is that the measure becomes more complicated, even in cases where we can write an explicit expression the measure can be too complicated to be integrated by the methods we've been using.

First of all we need to know whether we can find a similar function to Ψ_α in higher dimensions. Unlike the 1D case the Casimir depends on two eigenvalues, these could be thought as scaling dimension and spin, but like in the previous case there is a more symmetric way of writing it. Although the general expression for conformal blocks is complicated, we can use its asymptotic behaviour to determine whether we can create a function that is well-behaved everywhere in the causal diamond. The function that accomplishes it is the following

$$\Psi_{\alpha,\beta} = \frac{1}{4} (Q(\alpha, \beta) \tilde{G}_{\alpha,\beta} + Q(-\alpha, \beta) \tilde{G}_{-\alpha,\beta} + Q(\alpha, -\beta) \tilde{G}_{\alpha,-\beta} + Q(-\alpha, -\beta) \tilde{G}_{-\alpha,-\beta}), \quad (\text{B.45})$$

$$Q(\alpha, \beta) = \left(\frac{2\Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{d}{2} + a + b\right)}{\Gamma(d-1) \Gamma(1+a+b)} \right) Q_s(\alpha) Q_s(\beta) \gamma(\alpha + \beta), \quad (\text{B.46})$$

$$\gamma(t) = \frac{\Gamma(d-1) \Gamma(-t)}{2\Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{d}{2} - 1 - t\right)}, \quad (\text{B.47})$$

with α and β being relating to the usual eigenvalues as

$$J = 1 - \frac{d}{2} + \beta - \alpha, \quad \Delta = \frac{d}{2} + \beta + \alpha. \quad (\text{B.48})$$

The conformal blocks used here, $\tilde{G}$, are normalized in a unusual way, there is always a relation between the conformal blocks with spin J and $2 - d - J$, this normalization make it symmetric between the two.

The relation between this normalization and the one used in [16] is the

following

$$\tilde{G}_{\alpha,\beta} = \gamma(\alpha - \beta) G_{\alpha,\beta}. \quad (\text{B.49})$$

Just like in 1D we would define our alpha space transform using Ψ , the only difference here is the presence of two integrals

$$\hat{g}(\alpha, \beta) = \int_0^1 \int_0^1 dx dz \mu(x, z) g(x, z) \Psi_{\alpha,\beta}(x, z), \quad (\text{B.50})$$

$$\mu(x, z) = \frac{((1-x)(1-z))^{a+b}}{x^2 z^2} \left| \frac{x-z}{xz} \right|^{d-2}. \quad (\text{B.51})$$

In cases like 2D, 4D and 6D which we can write the basis in a closed form we are able to prove that Ψ inner product with itself behaves like a delta function, regardless of that if it is assumed that the alpha space exist in all dimensions the inverse transformation would be

$$g(x, z) = \int [d\alpha] [d\beta] \frac{\hat{g}(\alpha, \beta) \tilde{\Psi}_{\alpha,\beta}}{N(\alpha, \beta)} = \int [d\alpha] [d\beta] \frac{\hat{g}(\alpha, \beta) \tilde{G}_{\alpha,\beta}}{Q(-\alpha, -\beta) \gamma(\alpha - \beta) \gamma(\beta - \alpha)}, \quad (\text{B.52})$$

$$N(\alpha, \beta) = Q(\alpha, \beta) Q(-\alpha, -\beta) \gamma(\alpha - \beta) \gamma(\beta - \alpha). \quad (\text{B.53})$$

The way we recover the conformal block decomposition for g is identical as in 1D we need to find the residues in the expression and sometimes we need to deform the contour beyond $\alpha > 0$ and $\beta > 0$, but we need to be careful when dealing with dimensions bigger than two, there the behaviour of the unitary bounds differs when comparing scalars and spinning operators.

In a unitary 4-point function cannot have poles in arbitrary α and β , we need to satisfy the unitary bounds and need to be a single valued function in the Euclidean case. We might expect poles only for conformal blocks of positive integer spins, but this isn't the case, as was said before there is a relation between $G_{J,\Delta}$ and $G_{2-d-J,\Delta}$, which means that we have a mirrored region.

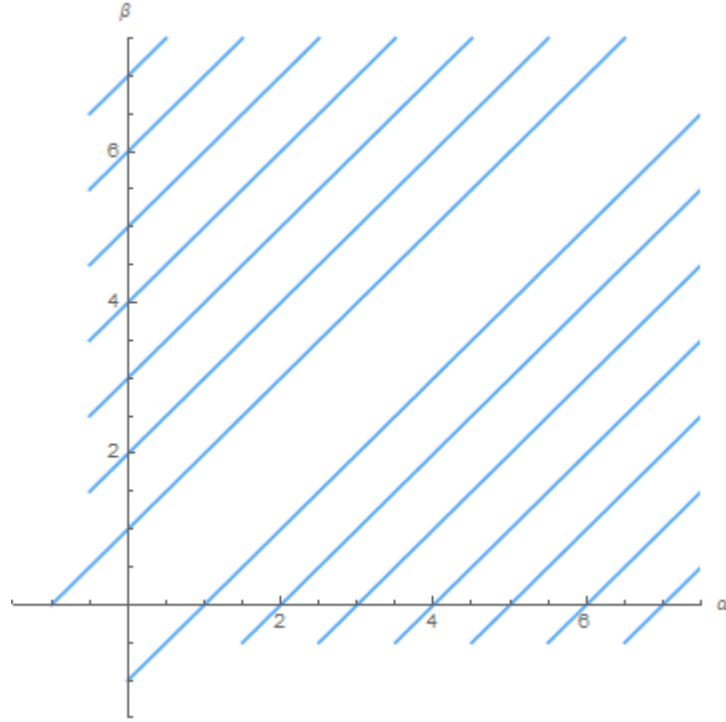


Figure B.1: Physical region and its mirrored, $J \mapsto 2 - d - J$, region for $d = 4$.

The poles in the mirrored region are just copies of the ones in the physical region with the same residue, and due to the relation between $\tilde{G}_{J,\Delta}$ and $\tilde{G}_{2-d-J,\Delta}$ we can use them to double those in the physical region.

Unlike in the previous section we won't prove our statements or calculate some alpha space transform, instead we will show a series that every Ψ in higher dimension satisfy and show some solutions. We can rewrite the quadratic and quartic Casimir in the variable's $\tilde{x} = \frac{x-1}{x}$ and $\tilde{z} = \frac{z-1}{z}$ to obtain an expansion for our Ψ

$$C_2 = \mathcal{D}_z + \mathcal{D}_x + (d-2) \frac{xz}{x-z} \left((1-x) \frac{\partial}{\partial x} - (1-z) \frac{\partial}{\partial z} \right), \quad (\text{B.54})$$

$$c_2 = \alpha^2 + \beta^2 - \frac{d^2 - 2d + 2}{2}, \quad (\text{B.55})$$

$$\mathcal{D}_z = z^2 \frac{\partial}{\partial z} \left((1-z) \frac{\partial}{\partial z} \right), \quad (\text{B.56})$$

$$C_4 = \left(\frac{xz}{z-x} \right)^{d-2} (\mathcal{D}_z - \mathcal{D}_x) \left(\frac{xz}{z-x} \right)^{2-d} (\mathcal{D}_z - \mathcal{D}_x), \quad (\text{B.57})$$

$$c_4 = \alpha^4 + \beta^4 - 2 \left(\frac{d-2}{2} \right)^2 (\alpha^2 + \beta^2) + \left(\frac{d-2}{2} \right)^4 - 2\alpha^2\beta^2, \quad (\text{B.58})$$

after a change of variables we will obtain a recursion relation for the coefficients

$$\Psi_{\alpha,\beta}(x, z) = \sum_{n,m=0}^{\infty} k_{n,m} \left(\frac{x-1}{x} \right)^n \left(\frac{z-1}{z} \right)^m, \quad k_{0,0} = 1, \quad k_{n,m} = k_{m,n}, \quad (\text{B.59})$$

$$\begin{aligned} 4m^2 k_{n,m} = & 4(n+2)^2 k_{n+2,m-2} + 4(m-n-2)(d+m+n-2) k_{n+1,m-1} \\ & + \left(4n(n+d-1) + 4m(m-1) - 4(\alpha^2 + \beta^2) + 1 + (d-1)^2 \right) k_{n,m-1} \\ & - \left(4(m-2)(m+d-3) + 4(n+1)(n+2) - 4(\alpha^2 + \beta^2) + 1 + (d-1)^2 \right) k_{n+1,m-2}, \end{aligned} \quad (\text{B.60})$$

$$4k_{n,1} = -4(n+1)(n+d-1)k_{n+1,0} + \left(4n(n+d-1) - 4(\alpha^2 + \beta^2) + 1 + (d-1)^2 \right) k_{n,0}. \quad (\text{B.61})$$

Using the symmetry between the indices we can find $k_{1,0}$, by setting $n = 0$ in the equation above

$$k_{1,0} = \frac{(d^2 - 2d + 2 - 4(\alpha^2 + \beta^2))}{4d}, \quad (\text{B.62})$$

$$Ak_{n+2,0} + Bk_{n+1,0} + Ck_{n,0} = 0, \quad (\text{B.63})$$

$$\begin{aligned} A &= 4(n+2)(n+d-1) \left(n + \frac{d}{2}\right) \left(n+1 + \frac{d}{2}\right), \\ B &= - \left(n + \frac{d}{2}\right)^2 \left(-6 - 4(\alpha^2 + \beta^2) + d(6+d) + 8n(n+d)\right), \\ C &= 4 \left(n + \frac{d-1}{2} - \alpha\right) \left(n + \frac{d-1}{2} + \alpha\right) \left(n + \frac{d-1}{2} - \beta\right) \left(n + \frac{d-1}{2} + \beta\right). \end{aligned}$$

Although these equations are complicated we can solve it exactly in at least three cases, the first one is for scalars

$$\Psi_{\frac{\Delta+1-d}{2}, \frac{\Delta-1}{2}}(x, z) = \mathcal{F}_{1,1,1}^{1,2,2} \left(\begin{array}{c|c|c} 1 & \frac{\Delta}{2}, \frac{d-\Delta}{2} & \frac{\Delta}{2}, \frac{d-\Delta}{2} \\ \frac{d}{2} & 1 & 1 \end{array} \middle| \frac{z-1}{z}, \frac{x-1}{x} \right), \quad (\text{B.64})$$

while the second is for conserved currents, $\Delta - J = d - 2$

$$\Psi_{-\frac{1}{2}, J+\frac{d-3}{2}}(x, z) = \mathcal{F}_{2,0,0}^{2,1,1} \left(\begin{array}{c|c|c} 1-J, J+d-2 & \frac{d-2}{2} & \frac{d-2}{2} \\ d-2, \frac{d}{2} & - & - \end{array} \middle| \frac{z-1}{z}, \frac{x-1}{x} \right), \quad (\text{B.65})$$

and the third is the spin-1 solution, which we won't be showing here, not only that but we can solve (B.63) in the general case

$$\Psi_{\frac{\Delta-J+1-d}{2}, \frac{\Delta+J-1}{2}}(x, 1) = \quad (\text{B.66})$$

$$\sum_{n,m=0}^{\infty} \frac{\left(\frac{\Delta-J}{2}\right)_{n+m} \left(\frac{d-\Delta-J}{2}\right)_{m+n} (d-2)_{2n+m} \left(-\frac{J}{2}\right)_n \left(\frac{1-J}{2}\right)_n}{n!m! \left(\frac{d}{2}\right)_{n+m} \left(\frac{\Delta-J}{2}\right)_n \left(\frac{d-\Delta-J}{2}\right)_n \left(\frac{d-1}{2}\right)_n (d-2)_{n+m}} \left(-\frac{x-1}{x}\right)^n \left(\frac{x-1}{x}\right)^m.$$

We can see the same structure, $\sum^{floor(\frac{J}{2})}$, that appear in the general conformal blocks, so we might have a similar expression in the general solution.

Before closing off this appendix we will go over one particular detail of the generalized polynomials in higher dimensions, the reason for the rigid OPE structure that we observe can be traced back to the 1D alpha space, there we have poles appearing from a Gamma function. Since we can rewrite the even dimension conformal blocks as a sum of products of 1D conformal blocks we expect the poles in a generalized polynomials to be generated by a $\Gamma(t_1 - \alpha) \Gamma(t_2 - \beta)$, from here we can already see this rigid Regge-like trajectories appearing, but what happens in odd dimensions? Odd dimensions behave differently under alpha space, unlike the even case one to the poles will come from the denominator $N(\alpha, \beta)$, looking like $\Gamma(-J) \Gamma(J + d - 2)$, this is the term that forces the operators to stay at integer spins. We could expect something different to occur in this case, but if we use [24] to break the conformal blocks in d dimensions into a sum over the ones in $d - 1$ dimensions we can guarantee that we have the same behaviour

$$G_{J,\Delta}^{(d)} = \sum_{j \equiv J \pmod{2n}=0}^J \sum_{n=0}^{\infty} \mathcal{A}_{j,n}(J, \Delta) G_{j,\Delta+2n}^{(d-1)}. \quad (\text{B.67})$$

All $\mathcal{A}$ are positive if $G_{J,\Delta}^{(d)}$ are inside the unitary bounds, which means that a unitary expression in higher dimensions are still unitary in lower ones. Looking back at Fig. B.1 we can see a little problem, because of the rigid OPE we cannot have generalized polynomials starting at the tail created by the difference between the unitary bounds for scalars and spinning operators, that is, $\frac{d-2}{4} \leq a, b, c < \frac{d-2}{2}$. In two dimensions we have the operator of spin two $G_{2,2a+2}^{(2D)}$, but the expression above say that it will also need to exist in all higher dimensions. This forces all unitary generalized polynomials to be either a generalized free boson $(\Delta, \Delta, 0)$ or a sum of the form $(\Delta, \Delta, 0) + Extra$, with each term in $Extra$ being a (a, b, c) with $a, b, c \geq \frac{d-2}{2}$, notice that we need at least $\Delta = \frac{3d-6}{4}$ to find the second kind.

Appendix C

Properties of the generalized polynomials

In the last appendix we used the alpha space formalism to find the conformal block decomposition of terms that appear in the generalized polynomials, here we will look at the properties of these OPE coefficients and show that they are always non-negative.

The previous calculations tell us

$$x^a (1-x)^b = \sum_{n=0}^{\infty} \frac{(-1)^n (a)_n^2}{n! (2a-1+n)_n} {}_3F_2 \left(\begin{matrix} -n, & 2a-1+n, & -b \\ & a, & a \end{matrix} \middle| 1 \right) k_{a+n}(x), \quad (\text{C.1})$$

this hypergeometric have some nice properties that we will be using to analyse the OPE coefficients, the first here is, [\[25\]](#),

$${}_3F_2 \left(\begin{matrix} -n, & b, & c \\ & d, & e \end{matrix} \middle| 1 \right) = \frac{(d-b)_n}{(d)_n} {}_3F_2 \left(\begin{matrix} -n, & b, & e-c \\ & b-d-n+1, & e \end{matrix} \middle| 1 \right), \quad (\text{C.2})$$

$$\begin{aligned} {}_3F_2 \left(\begin{matrix} -n, & 2a-1+n, & -b \\ & a, & a \end{matrix} \middle| 1 \right) &= \frac{(1-a-n)_n}{(a)_n} {}_3F_2 \left(\begin{matrix} -n, & 2a-1+n, & a+b \\ & a, & a \end{matrix} \middle| 1 \right) \\ &= (-1)^n {}_3F_2 \left(\begin{matrix} -n, & 2a-1+n, & a+b \\ & a, & a \end{matrix} \middle| 1 \right). \end{aligned} \quad (\text{C.3})$$

This indicate a more natural way to look at the hypergeometric

$$f_a^n(t) = \frac{(a)_n^2}{(2a-1+n)_n} {}_3F_2 \left(\begin{matrix} -n, & 2a-1+n, & \frac{a}{2}-t \\ & a, & a \end{matrix} \middle| 1 \right), \quad (\text{C.4})$$

the reason for a minus sign for t chosen here is not obvious if we look at the definition of the continuous Hahn polynomials, but if we calculate the first two non-trivial n we'll start to see why

$$f_a^1(t) = t, \quad f_a^2(t) = t^2 + \frac{a^2}{4(2a+1)}. \quad (\text{C.5})$$

Not only we will have functions that are even or odd depending on n , but we can make these first two polynomials have positive coefficients, this will be enough to show that every other one also have positive coefficients. Besides that this form have some other properties that we will explore bellow. Since f can be represent as a continuous Hahn polynomial,

$$p_n(t) = \frac{(a+c)_n (a+d)_n}{(n+a+b+c+d-1)_n} {}_3F_2 \left(\begin{matrix} -n, & n+a+b+c+d-1, & a-t \\ & a+c, & a+d \end{matrix} \middle| 1 \right), \quad (\text{C.6})$$

we can use its recursion relation, [26],

$$p_{n+1}(t) = p_n(t)(t-a-h_n-C_n) - A_{n-1}C_n p_{n-1}(t) \quad (\text{C.7})$$

$$A_n = -\frac{(n+a+b+c+d-1)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)(2n+a+b+c+d)}, \quad (\text{C.8})$$

$$C_n = \frac{n(n+b+c-1)(n+b+d-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)}, \quad (\text{C.9})$$

to find when f_a^n is positive. If we set $a = b = c = d = \frac{h}{2}$, then we will have the recursion

$$f_h^{n+1}(t) = t f_h^n(t) + \frac{n(n+h-1)^2(n+2h-2)}{4(2n+2h-1)(2n+2h-3)} f_h^{n-1}(t), \quad n \geq 0, \quad (\text{C.10})$$

all the recursion coefficients are non-negative when $t \geq 0$ and $h > 0$, this coupled with the fact that f_h^1 and f_h^2 are non-negative in the same interval is enough to show that f will be non-negative for all n . However we can go a bit further, suppose that t and δ are positive reals, then

$$\begin{aligned} f_h^{n+1}(t+\delta) - f_h^{n+1}(t) &= \delta f_h^n(t+\delta) + t(f_h^n(t+\delta) - f_h^n(t)) \\ &+ \frac{n(n+h-1)^2(n+2h-2)}{4(2n+2h-1)(2n+2h-3)} (f_h^{n-1}(t+\delta) - f_h^{n-1}(t)), \end{aligned} \quad (\text{C.11})$$

we know that the first term is positive, but we still don't know about the terms in the parenthesis. For f_h^1 and f_h^2 we know that these values are positive, this is, again, enough to say that the subtraction will be non-negative when $h > 0$. This is pretty useful when dealing with a sum of generalized polynomials, since if we find the dominant f have a coefficient bigger than the sum of the negative ones we would be able to say that the expression is positive.

However this won't be enough when analysing more complicated correlators, for an example imagine we have the following decomposition

$$\sum_{n=0}^{\infty} \frac{1}{n!} ((C - \epsilon) f_a^n(t+1) - C f_a^n(t) + \epsilon f_a^n(0)) k_{a+n}(t), \quad t, C > 0, \quad (\text{C.12})$$

although we know that the first polynomial is the leading term, its coefficient is slightly smaller than the negative one, how can we know whether this expression is valid in a unitary 4-point function?

Even with the equation (C.11) above, if the following limit is true

$$\lim_{n \rightarrow \infty} \frac{f_a^n(t+1)}{f_a^n(t)} = 1, \quad (\text{C.13})$$

then for large odd n this equation becomes negative, but if we have

$$\lim_{n \rightarrow \infty} \frac{f_a^n(t+1)}{f_a^n(t)} = \infty, \quad (\text{C.14})$$

the slightly difference won't really matter. We need to find a way to know what kind of limit we are dealing with, otherwise we will need to work with the harsh

supposition in (C.13). We can have an idea of what is happening if we choose an appropriated t

$$\frac{f_a^n\left(\frac{a}{2}+1\right)}{f_a^n\left(\frac{a}{2}\right)} = 1 + \frac{n(2a-1+n)}{a^2}, \quad (\text{C.15})$$

$$\frac{f_a^n\left(\frac{a}{2}+2\right)}{f_a^n\left(\frac{a}{2}+1\right)} = \frac{1 + \frac{2n(2a-1+n)}{a^2} + \frac{n(n-1)(2a-1+n)(2a+n)}{a^2(a+1)^2}}{1 + \frac{n(2a-1+n)}{a^2}}, \quad (\text{C.16})$$

these two examples will behave as n^2 for large n , the limit is clear here, however we are looking at the cases where the hypergeometric simplifies, we should look for a better way to find this limit, preferably with a precise asymptotic. Since we already know that $f_a^n(t)$ is a polynomial in t , we can write it as

$$f_a^n(t) = \sum_{k=0}^{\text{floor}\left(\frac{n}{2}\right)} c_a^n(k) t^{n-2k}, \quad (\text{C.17})$$

we could use the recursion relations of f to calculate the coefficients

$$c_a^{n+1}(k) = c_a^n(k) + \frac{n(n+a-1)^2(n+2a-2)}{4(2n+2a-1)(2n+2a-3)} c_a^{n-1}(k-1), \quad (\text{C.18})$$

and with all the information above we can tell that these coefficients are positive. Although we could calculate them exactly, we will just approximate its value for large n with only the leading term,

$$c_a^{n+1}(k) \approx c_a^n(k) + \frac{n^2}{16} c_a^{n-1}(k-1), \quad (\text{C.19})$$

$$c_a^n(0) = 1, \quad c_a^n(1) \approx \frac{n^3}{48}, \quad c_a^n(2) \approx \frac{n^6}{4208}, \quad \dots, \quad (\text{C.20})$$

but this is valid only when n is much larger than $2k$, a clear departure appears for $n = 2k$

$$c_a^{2k}(k) = \frac{\left(\frac{1}{2}\right)_k \left(\frac{a}{2}\right)_k^2 \left(\frac{2a-1}{2}\right)_k}{4^k \left(\frac{2a+1}{4}\right)_k \left(\frac{2a-1}{4}\right)_k}. \quad (\text{C.21})$$

This rapidly increase between consecutive coefficients will quickly drown out any t we choose, however suppose at high n one coefficient, independent of n , grows so fast that it completely dominate over any other, we would have the following approximation

$$f_a^n(t) \approx ct^{k^*}, \quad (\text{C.22})$$

or

$$\frac{f_a^n(t)}{f_a^n(s)} \approx \left(\frac{t}{s}\right)^{k^*}, \quad (\text{C.23})$$

which is completely incompatible with the previous limit.

Looking at our initial examples (C.15, C.16), it seen like the actual value of the limit (C.23) behave like n^{2t-2s} , we can prove it using a Rodrigues-like formula for the continuous Hahn polynomials, again from [26],

$$f_a^n(t) = \frac{\left(\frac{\delta}{\delta t}\right)^n \left(\Gamma\left(\frac{a+n}{2} + t\right)\right)^2 \left(\Gamma\left(\frac{a+n}{2} - t\right)\right)^2}{(n+2a-1)_n \left(\Gamma\left(\frac{a}{2} + t\right)\right)^2 \left(\Gamma\left(\frac{a}{2} - t\right)\right)^2}, \quad (\text{C.24})$$

with

$$\left(\frac{\delta}{\delta t}\right) p(t) = p\left(t + \frac{1}{2}\right) - p\left(t - \frac{1}{2}\right). \quad (\text{C.25})$$

For a more explicit expression we have

$$\frac{\sum_{k=0}^n (-1)^k \binom{n}{k} \left(\Gamma\left(\frac{a}{2} + n - k + t\right)\right)^2 \left(\Gamma\left(\frac{a}{2} + k - t\right)\right)^2}{(n+2a-1)_n \left(\Gamma\left(\frac{a}{2} + t\right)\right)^2 \left(\Gamma\left(\frac{a}{2} - t\right)\right)^2}. \quad (\text{C.26})$$

To calculate the limit we should look at the dominant terms

$$f_a^n(t) \sim \frac{1}{(n+2a-1)_n} \left(\frac{(\Gamma(\frac{a}{2} + n + t))^2}{(\Gamma(\frac{a}{2} + t))^2} + (-1)^n \frac{(\Gamma(\frac{a}{2} + n - t))^2}{(\Gamma(\frac{a}{2} - t))^2} \right), \quad (C.27)$$

$$f_a^n(t) \sim \frac{e^{-n} n^{n+2a-1-1/2}}{e^{-2n} (2n)^{2n+2a-1-1/2}} \left(\frac{2\pi e^{-2n} n^{2n+a+2t-1}}{(\Gamma(\frac{a}{2} + t))^2} + (-1)^n \frac{2\pi e^{-2n} n^{2n+a-2t-1}}{(\Gamma(\frac{a}{2} - t))^2} \right), \quad (C.28)$$

$$f_a^n(t) \sim \frac{2\pi e^{-n} n^{n+2a-1}}{2^{2n+2a-1-1/2}} \left(\frac{n^{2t-a}}{(\Gamma(\frac{a}{2} + t))^2} + (-1)^n \frac{n^{-2t-a}}{(\Gamma(\frac{a}{2} - t))^2} \right), \quad (C.29)$$

$$\frac{f_a^n(t)}{n!} \sim \frac{\sqrt{2\pi} n^{2a-\frac{3}{2}}}{2^{2n+2a-\frac{3}{2}}} \left(\frac{n^{2t-a}}{(\Gamma(\frac{a}{2} + t))^2} + (-1)^n \frac{n^{-2t-a}}{(\Gamma(\frac{a}{2} - t))^2} \right). \quad (C.30)$$

These are the leading terms which will dominate when t is positive or negative, the reason we can believe in this behaviour is because the series do behave like an asymptotic series when n goes to infinity, or with the correct ratio we have a convergent series

$$\frac{f_a^n(t)}{\phi_0(n)} - \sum_{k=0}^{N < \text{floor}(\frac{n}{2})} \frac{\phi_k(n)}{\phi_0(n)} = \mathcal{O}\left(\frac{\phi_{N+1}(n)}{\phi_0(n)}\right) (n \rightarrow \infty), \quad (C.31)$$

$$\begin{aligned} \phi_k(n) = \frac{(-1)^k \binom{n}{k}}{(n+2a-1)_n} & \left(\frac{(\Gamma(\frac{a}{2} + n - k + t) \Gamma(\frac{a}{2} + k - t))^2}{(\Gamma(\frac{a}{2} + t) \Gamma(\frac{a}{2} - t))^2} + \right. \\ & \left. (-1)^n \frac{(\Gamma(\frac{a}{2} + n - k - t) \Gamma(\frac{a}{2} + k + t))^2}{(\Gamma(\frac{a}{2} + t) \Gamma(\frac{a}{2} - t))^2} \right). \end{aligned} \quad (C.32)$$

To show this we start looking at the following term, assuming $t > 0$

$$\left| \frac{\phi_{k+1}(n)}{\phi_k(n)} \right| = \frac{n-k}{k+1} \left(\frac{\frac{a}{2} + k - t}{\frac{a}{2} + n - k + t} \right)^2 \left| \frac{1 + (-1)^n \left(\frac{\Gamma(\frac{a}{2} + n - k - 1 - t) \Gamma(\frac{a}{2} + k + t + 1)}{\Gamma(\frac{a}{2} + n - k - 1 + t) \Gamma(\frac{a}{2} + k + 1 - t)} \right)^2}{1 + (-1)^n \left(\frac{\Gamma(\frac{a}{2} + n - k - t) \Gamma(\frac{a}{2} + k + t)}{\Gamma(\frac{a}{2} + n - k + t) \Gamma(\frac{a}{2} + k - t)} \right)^2} \right|, \quad (\text{C.33})$$

the only complicated terms appearing here are the ones like

$$\frac{\Gamma(\frac{a}{2} + n - k - t) \Gamma(\frac{a}{2} + k + t)}{\Gamma(\frac{a}{2} + n - k + t) \Gamma(\frac{a}{2} + k - t)} = \frac{(\frac{a}{2} + k - t)_{2t}}{(\frac{a}{2} + n - k - t)_{2t}}, \quad (\text{C.34})$$

given our condition $k < \text{floor}(\frac{n}{2})$ we know that $\frac{a}{2} + n - k - t > \frac{a}{2} + k - t$. Given how fast the $\Gamma(t)$ increases for t mildly bigger than 1 it is natural to say that the expression above is smaller than 1. The term goes closer to 1 when k goes to $(\frac{n}{2})$, this is not a problem for n even, but it could be for n odd, it won't be since the term in the numerator of (C.33) would be closer to zero than the one in the denominator. We can conclude that for n even the big term in (C.33) is bound by 2 and for n odd by 1, so

$$\left| \frac{\phi_{k+1}(n)}{\phi_k(n)} \right| = \mathcal{O} \left(\frac{n-k}{k+1} \left(\frac{\frac{a}{2} + k - t}{\frac{a}{2} + n - k + t} \right)^2 \right), \quad (\text{C.35})$$

$$\left| \frac{\phi_k(n)}{\phi_0(n)} \right| = \mathcal{O} \left(\left| \frac{(-n)_k}{k!} \left(\frac{(\frac{a}{2} - t)_k}{(-\frac{a}{2} - n - t)_k} \right)^2 \right| \right). \quad (\text{C.36})$$

This would be fine if we were dealing with a simple function, however we have an infinite sum, so we also have to make sure that the sum have the same asymptotic, in fact we will show that it converges

$$\left| \sum_k \frac{\phi_k(n)}{\phi_0(n)} \right| \leq \sum_k \left| \frac{\phi_k(n)}{\phi_0(n)} \right| \sim \sum_k \left| \frac{(-n)_k}{k!} \left(\frac{(\frac{a}{2} - t)_k}{(-\frac{a}{2} - n - t)_k} \right)^2 \right|. \quad (\text{C.37})$$

For finite n the sum goes from $k = 0$ to $k = \text{floor}(\frac{n}{2})$, to check whether the sum converges we will need to look only at the kn limit, $h = \frac{1}{2}$, at large n we

have the k -th term behaving as

$$\left| \frac{h^{hn+a-2t-\frac{3}{2}} n^{a-2t-\frac{3}{2}}}{(1-h)^{(h-1)n-a-2t-\frac{1}{2}}} \right|. \quad (\text{C.38})$$

We can use the root test to show that the equivalent sequence is convergent

$$\lim_{n \rightarrow \infty} \left| \frac{h^{hn+a-2t-\frac{3}{2}} n^{a-2t-\frac{3}{2}}}{(1-h)^{(h-1)n-a-2t-\frac{1}{2}}} \right|^{\frac{1}{n}} = \left| \frac{h^h}{(1-h)^{(h-1)}} \right|^{\frac{h-1}{2}} \frac{1}{2}. \quad (\text{C.39})$$

The last step here is to show that the initial sum also converge. The big-O notation tell us that, at least close to the asymptotic region, we have the relation

$$\left| \frac{\phi_{k+1}(n)}{\phi_k(n)} \right| \leq M_k \frac{n-k}{k+1} \left(\frac{\frac{a}{2} + k - t}{\frac{a}{2} + n - k + t} \right)^2, \quad (\text{C.40})$$

$$\left| \frac{\phi_k(n)}{\phi_0(n)} \right| \leq \left(\prod_{j=0}^k M_j \right) \left| \frac{(-n)_k}{k!} \left(\frac{(\frac{a}{2} - t)_k}{(-\frac{a}{2} - n - t)_k} \right)^2 \right|, \quad (\text{C.41})$$

if we still want to use the root test we need to consider the extra term $\prod M_j$. The only behaviour that matter here is the limit when j goes to infinity, we know that for even j $1 < M_j < 2$ and for odd, $0 < M_j < 1$, but if we look closely at each expression we will notice that the complicated term in the numerator will go closer and closer to the one in the denominator as k goes to $\frac{n}{2}$, so we can guarantee that in each M we have $0 < \text{denominator} \approx \text{numerator} < 1$, this way we can say that $M_\infty \approx 1$, the root test would give us

$$\lim_{n \rightarrow \infty} \left| \frac{\phi_{\frac{n}{2}}(n)}{\phi_0(n)} \right|^{\frac{1}{n}} = \frac{M_\infty}{2} < 1. \quad (\text{C.42})$$

Going back to the original series we find

$$\frac{f_a^n(t)}{\phi_0(n)} - \sum_{k=0}^{N < \text{floor}(\frac{n}{2})} \frac{\phi_k(n)}{\phi_0(n)} = \frac{\phi_{N+1}(n)}{\phi_0(n)} \sum_{k=N+1}^{\text{floor}(\frac{n}{2})} \frac{\phi_k(n)}{\phi_{N+1}(n)} = \frac{\phi_{N+1}(n)}{\phi_0(n)} C_n. \quad (\text{C.43})$$

Since we know that C_n is finite in the limit we can say that the asymptotic behaviour presented by the series is valid.

It would be fair to ask what happens at $t = 0$, since it has a very different behaviour at this point, if we restrict ourselves to even n then there is no difference between $t = 0$ and $t \neq 0$ since the extra term in (C.33) disappear, and for odd n it is always zero, another interesting property of these polynomial are the forward shift operator, [26],

$$\frac{\delta}{\delta t} f_a^n(t) = f_a^n\left(t + \frac{1}{2}\right) - f_a^n\left(t - \frac{1}{2}\right) = n f_{a+1}^{n-1}(t), \quad (\text{C.44})$$

this transformation generate the same consequences as the parity of f^n but in 4D, that is, the parity causes odd spins to be zero in 2D while this relation causes odd spins to be zero in 4D, it is easier to see when using $\Psi_{\alpha,\beta}$ build out of 1D conformal blocks. It is also really useful to deal with the subdominant terms in the expansion when needed.

We've been avoiding $h = 0$ until now for a quite good reason, according to (C.4) f_0 is not defined for any n or t , however we can talk about the conformal block decomposition of $(1-t)^b$ without any problems, the way we treat this problem is by understanding f_0 as $\lim_{a \rightarrow 0^+} f_a$.

$$f_0^0(t) = 1, \quad f_0^1(t) = t, \quad f_0^2(t) = t^2, \quad (\text{C.45})$$

$$f_0^{n+1}(t) = t f_0^n(t) + \frac{n(n-1)^2(n-2)}{4(2n-1)(2n-3)} f_0^{n-1}(t), \quad n \geq 0, \quad (\text{C.46})$$

for a closed form we have

$$f_0^{n+2}(t) = t^2 \frac{(2)_n^2}{(3+n)_n} {}_3F_2\left(\begin{matrix} -n, & 3+n, & 1-t \\ & 2 & 2 \end{matrix} \middle| 1\right), \quad n \geq 0. \quad (\text{C.47})$$

These considerations are enough to show that all generalized polynomials are unitary, up to decoupling from the identity. A 4-point function generated by a single (a, b, c) is

$$\begin{aligned}
 \mathcal{G}(x, z) = & ((1-x)(1-z))^{a-\Delta} (xz)^b + ((1-x)(1-z))^{a-\Delta} (xz)^c \\
 & + ((1-x)(1-z))^{b-\Delta} (xz)^a + ((1-x)(1-z))^{b-\Delta} (xz)^c \\
 & + ((1-x)(1-z))^{c-\Delta} (xz)^a + ((1-x)(1-z))^{c-\Delta} (xz)^b.
 \end{aligned} \tag{C.48}$$

This expression will be formed, in general, by three non-overlapping series, taking only one of them we have the sum

$$\begin{aligned}
 & (xz)^a \left(((1-x)(1-z))^{\frac{c-a-b}{2}} + ((1-x)(1-z))^{\frac{b-a-c}{2}} \right) = \\
 & \sum_{m \geq n}^{\infty} \frac{(-1)^{n+m}}{n!m!} \left(f_a^n \left(\frac{b-c}{2} \right) f_a^m \left(\frac{b-c}{2} \right) + f_a^n \left(\frac{c-b}{2} \right) f_a^m \left(\frac{c-b}{2} \right) \right) G_{m-n, 2a+m+n} = \\
 & \sum_{m \geq n}^{\infty} \frac{1 + (-1)^{n+m}}{n!m!} f_a^n \left(\left| \frac{b-c}{2} \right| \right) f_a^m \left(\left| \frac{b-c}{2} \right| \right) G_{m-n, 2a+m+n}.
 \end{aligned} \tag{C.49}$$

This sum is clearly zero when $n + m$ is odd, this happen to be the cases with J is odd, to see how to get the third line we need to check when $n + m$ is even, in these cases either both f is already positive or, if one f is negative then both f are, making the multiplication positive, therefore the sum will always be non-negative.

There is a asymptotic behaviour that are interesting to calculate here before we move on, k is a fixed integer and $t, s > 0$

$$\frac{f_a^n(t)}{f_{a-k}^{n+k}(s)} \sim \frac{n^{n+2t+a-1}}{e^{-k} (n+k)^{n+2s+a-1}} \frac{\left(\Gamma \left(\frac{a-k}{2} + s \right) \right)^2}{\left(\Gamma \left(\frac{a}{2} + t \right) \right)^2} \sim n^{2t-2s} \frac{\left(\Gamma \left(\frac{a-k}{2} + s \right) \right)^2}{\left(\Gamma \left(\frac{a}{2} + t \right) \right)^2}, \tag{C.50}$$

$$\frac{f_a^n(t) (n+k)!}{f_{a-k}^{n+k}(s) n!} \sim n^{2t-2s+k} \frac{\left(\Gamma \left(\frac{a-k}{2} + s \right) \right)^2}{\left(\Gamma \left(\frac{a}{2} + t \right) \right)^2}. \tag{C.51}$$

This type of relation is important to check unitarity when there are multiple series overlapping.

Appendix D

unitary solutions

For this last appendix we have an aim to study the 4-point functions we can create using our crossing functions, using the properties from the simple generalized polynomials we will be able to create more interesting unitary polynomials, we are also going to look a bit to the generalized minimal models and other possibilities. The knowledge acquired about the asymptotic behaviour will be crucial here, but before that we need to look at which operators that appear in a simple generalized polynomial, since it will affect the asymptotic conditions we need to have when dealing with a sum of polynomials.

D.1 unitary generalized polynomials

For a general (a, b, c) , the conformal block decomposition is made up of three different series of the form

$$\sum_{m \geq n}^{\infty} \frac{1 + (-1)^{n+m}}{n!m!} f_a^n \left(\left| \frac{b-c}{2} \right| \right) f_a^m \left(\left| \frac{b-c}{2} \right| \right) G_{m-n, 2a+m+n}, \quad (\text{D.1})$$

the way the coefficients are written already tell us that no operators with odd spin will enter in this OPE, by mapping all the even spins we are able to see which operators contribute in the generic case

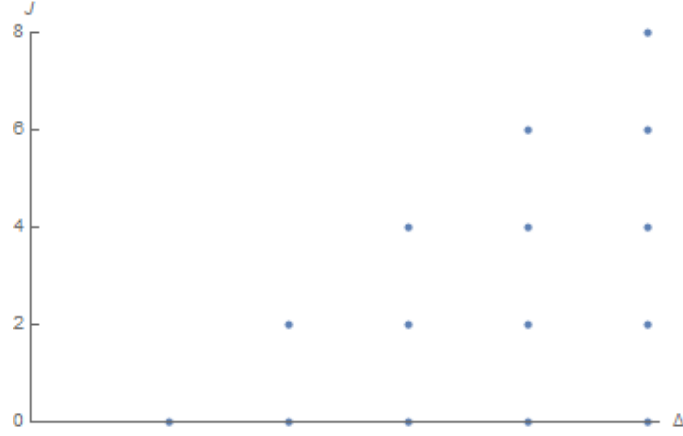


Figure D.1: This is the OPE form each series in a generic (a, b, c) .

There are cases in which this picture have more operators than what would exist, when $b = c$ in the expression we have a different behaviour, from the previous appendix we know that the functions f^n are even or odd depending on n and such we don't need to have $n + m$ odd, if one of them is odd the whole expression is zero, this creates the following image

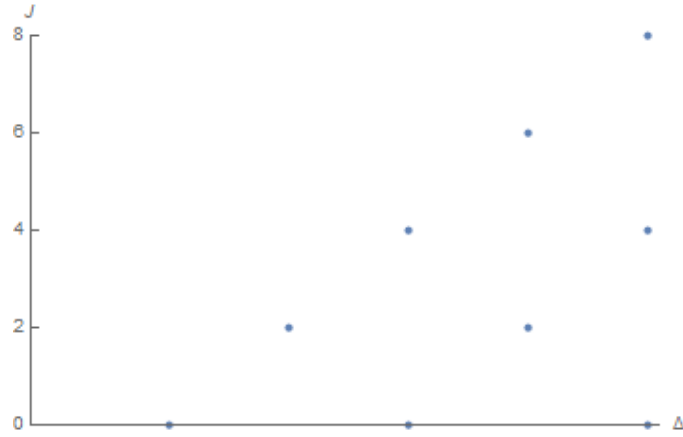


Figure D.2: Here we have the specific case when $t = 0$.

This image is characteristic of 4-point functions that can be broken in the form $g(x)g(z)$, or at least the terms that generates this decomposition. It also have some consequences, due to the holes that appear in the OPE we cannot consider terms like this when talking about the asymptotic behaviour, unless all terms are like this.

The last step here is to look in which cases series overlap with each other. In the generic case the scalars that appear in the series are of the form $2a + 2k$, so if two series overlap it happens because $2a + 2k = 2b$, the question becomes, which

one will dominate at large n , from the last equation in the previous appendix we can say that it will fully depend on the value of x and y .

$$\frac{f_{a+k}^{n-k}(x) n!}{f_a^n(y) (n-k)!} \sim n^{2x-2y+k} \frac{(\Gamma(\frac{a}{2} + y))^2}{(\Gamma(\frac{a+k}{2} + x))^2}. \quad (\text{D.2})$$

We are going to move to some examples, the idea is to show that we can get closer to the boundary by string together multiple (a, b, c) . By looking at these examples we will get an idea on how we could go about doing it in general, the $\langle \epsilon^4 \rangle$ in the 2D Ising model show us how we can start

$$\langle \epsilon^4 \rangle_{\text{Ising}} = (0, 0, 2) - (0, 1, 1). \quad (\text{D.3})$$

The conformal block decomposition for this expression is given by

$$(0, 0, 2) : \sum_{m \geq n} \frac{G_{m-n, 4+m+n}}{n!m!} f_2^n(0) f_2^m(0) + \sum_{m \geq n} \frac{1 + (-1)^{n+m}}{n!m!} f_0^n(1) f_0^m(1) G_{m-n, m+n}, \quad (\text{D.4})$$

$$(0, 1, 1) : 1 + \sum_{m \geq n} \frac{1 + (-1)^{n+m}}{n!m!} f_1^n\left(\frac{1}{2}\right) f_1^m\left(\frac{1}{2}\right) G_{m-n, 2+m+n}, \quad (\text{D.5})$$

the only problem in this expression is the sum of $f_1\left(\frac{1}{2}\right)$, we know that we cannot rely on $f_2(0)$ for the asymptotic so we will check $f_0(1)$, plugging into the asymptotic expression above we find

$$\frac{f_1^{n-1}\left(\frac{1}{2}\right) n!}{f_0^n(1) (n-1)!} \sim n^{1-2+1} \frac{(\Gamma(\frac{0}{2} + 1))^2}{(\Gamma(\frac{0+1}{2} + \frac{1}{2}))^2} = 1. \quad (\text{D.6})$$

If the ratio between the coefficients of $(2, 0, 0)$ and $(1, 1, 0)$ were bigger than 1 in absolute value we could say that the expression is positive, however they are equal, we will need to know the subleading terms to be able to reach any conclusion. The better way to deal with the subleading terms is through equation

(C.44), notice that according to (C.51) $\lim_{n \rightarrow \infty} \frac{f_a^n(t)}{n f_{a+1}^{n-1}(t - \frac{1}{2})} = 1$, so the subleading terms reorganize themselves as another f

$$f_0^n(1) = n f_1^{n-1}\left(\frac{1}{2}\right) + f_0^n(0), \quad n \geq 0, \quad (\text{D.7})$$

since $f_0^n(0) \geq 0$ we can say that $f_0^n(1) \geq n f_1^{n-1}\left(\frac{1}{2}\right)$, so our expression is unitary, in fact $f_0^n(0)$ is special, we will come back to it later.

Unlike the simple polynomials, we need to get creative when trying to create a more general generalized polynomials, since we don't have a fixed method to find them I'll be presenting two other examples and the idea behind them.

The first one is the following polynomial $A(0, b, c) + B(1, b-1, c) + D(0, b-1, c+1)$, when we start with $(0, b, c)$ we will have a scalar with dimension 2, unless $b = c$, so if $b, c > 1$ we will always have the gap being 2, since I'm trying to not add any other series I'll be choosing the parameters that differ by an integer. This is not the most general term that we could get by subtracting 1 from our variables, but notice that now we have to deal with $b-1$, if we did the same with c we would need to pay attention to another one, so we are keeping it to the minimum that might still work. Let me present the following table

$A(0, b, c) + B(1, b-1, c) + D(0, b-1, c+1), \quad b, c > 1$								
0	$\frac{\frac{b-c}{2}}{A}$	$\frac{\frac{ b-c-2 }{2}}{D}$	$b-1$	$\frac{\frac{c+1}{2}}{D}$	$\frac{\frac{c-1}{2}}{B}$	c	$\frac{\frac{b}{2}}{A}$	$\frac{\frac{ b-2 }{2}}{B}$
1	$\frac{\frac{ b-c-1 }{2}}{B}$	-	b	$\frac{\frac{c}{2}}{A}$	-	$c+1$	$\frac{\frac{b-1}{2}}{D}$	-
0	$A \text{ or } D > 0$		$b-1$	$D > 0$		c	$A > 0$	
1	$\left(\frac{b-c}{2}\right) A \text{ or } \left(\frac{ 2-b+c }{2}\right) D + B > 0$		b	$\left(\frac{c+1}{2}\right) D + A > 0$		$c+1$	$\left(\frac{b}{2}\right) A + D > 0$	
0	$A \text{ or } D + B > 0$		$b-1$	$A + D > 0$		c	$A + D > 0$	

Although it may seem a little complicated we will go over every line in here, the first one is

0	$\frac{\frac{b-c}{2}}{A}$	$\frac{\frac{ b-c-2 }{2}}{D}$	$b-1$	$\frac{\frac{c+1}{2}}{D}$	$\frac{\frac{c-1}{2}}{B}$	c	$\frac{\frac{b}{2}}{A}$	$\frac{\frac{ b-2 }{2}}{B}$
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What we are saying is that we have the following sums

$$\begin{aligned}
 & \sum_{m \geq n} \frac{1 + (-1)^{n+m}}{n!m!} \left[A G_{m-n, m+n} f_0^n \left(\frac{b-c}{2} \right) f_0^m \left(\frac{b-c}{2} \right) + \right. \\
 & \quad D G_{m-n, m+n} f_0^n \left(\left\lfloor \frac{b-c-2}{2} \right\rfloor \right) f_0^m \left(\left\lfloor \frac{b-c-2}{2} \right\rfloor \right) + D G_{m-n, 2b-2+n+m} f_{b-1}^n \left(\frac{c+1}{2} \right) f_{b-1}^m \left(\frac{c+1}{2} \right) \\
 & \quad B G_{m-n, 2b-2+n+m} f_{b-1}^n \left(\frac{c-1}{2} \right) f_{b-1}^m \left(\frac{c-1}{2} \right) + A G_{m-n, 2c+n+m} f_c^n \left(\frac{b}{2} \right) f_c^m \left(\frac{b}{2} \right) \\
 & \quad \left. B G_{m-n, 2c+n+m} f_c^n \left(\left\lfloor \frac{b-2}{2} \right\rfloor \right) f_c^m \left(\left\lfloor \frac{b-2}{2} \right\rfloor \right) \right]. \tag{D.8}
 \end{aligned}$$

As we can see it is much simpler to write in table form, the only thing to keep an eye out is if we have a $\frac{0}{G}$, in this case we don't have the factor $1 + (-1)^{n+m}$, just look at the previous example. The second line works identical to the first

1	$\frac{\lfloor \frac{b-c-1}{2} \rfloor}{B}$	-		b	$\frac{\frac{c}{2}}{A}$	-		$c+1$	$\frac{\frac{b-1}{2}}{D}$	-
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$$\begin{aligned}
 & \sum_{m \geq n} \frac{1 + (-1)^{n+m}}{n!m!} \left[B G_{m-n, 2+m+n} f_1^n \left(\left\lfloor \frac{b-c-1}{2} \right\rfloor \right) f_1^m \left(\left\lfloor \frac{b-c-1}{2} \right\rfloor \right) + \right. \\
 & \quad A G_{m-n, 2b+m+n} f_b^n \left(\frac{c}{2} \right) f_b^m \left(\frac{c}{2} \right) + D G_{m-n, 2c+2+n+m} f_{c+1}^n \left(\frac{b-1}{2} \right) f_{c+1}^m \left(\frac{b-1}{2} \right) \left. \right]. \tag{D.9}
 \end{aligned}$$

If we had more terms we could have more lines of this kind, but for this example these are the only ones we've got. One last note before going to the next type of line, notice that the different types of columns represent different expansions that doesn't interact with each other, we have one for 0 and 1, we have another for $b-1$ and b and a third one for c and $c+1$.

The third line is a completely different species of the ones we seen before, what we are actually looking at is a partial limit. In the previous expressions we had the sum with $m \geq n$, if we were looking at the asymptotic limit we could have m and n going to infinity at different speeds and our term should still be positive, but we could just take m to infinity while keeping n finite, this if the condition shown here. Now we also need to consider each column, in the first one we have the series

$$\sum_{m \geq n} \frac{1 + (-1)^{n+m}}{n!m!} \left[A G_{m-n, m+n} f_0^n \left(\frac{b-c}{2} \right) f_0^m \left(\frac{b-c}{2} \right) + \right. \quad (D.10)$$

$$D G_{m-n, m+n} f_0^n \left(\left| \frac{b-c-2}{2} \right| \right) f_0^m \left(\left| \frac{b-c-2}{2} \right| \right)$$

$$\left. B G_{m-n, 2+m+n} f_1^n \left(\left| \frac{b-c-1}{2} \right| \right) f_1^m \left(\left| \frac{b-c-1}{2} \right| \right) \right],$$

for the second we have

$$\sum_{m \geq n} \frac{1 + (-1)^{n+m}}{n!m!} \left[D G_{m-n, 2b-2+n+m} f_{b-1}^n \left(\frac{c+1}{2} \right) f_{b-1}^m \left(\frac{c+1}{2} \right) \right. \quad (D.11)$$

$$\left. B G_{m-n, 2b-2+n+m} f_{b-1}^n \left(\frac{c-1}{2} \right) f_{b-1}^m \left(\frac{c-1}{2} \right) + A G_{m-n, 2b+m+n} f_b^n \left(\frac{c}{2} \right) f_b^m \left(\frac{c}{2} \right) \right],$$

and for the third

$$\sum_{m \geq n} \frac{1 + (-1)^{n+m}}{n!m!} \left[A G_{m-n, 2c+n+m} f_c^n \left(\frac{b}{2} \right) f_c^m \left(\frac{b}{2} \right) + B G_{m-n, 2c+n+m} f_c^n \left(\left| \frac{b-2}{2} \right| \right) f_c^m \left(\left| \frac{b-2}{2} \right| \right) \right. \quad (D.12)$$

$$\left. + D G_{m-n, 2c+2+n+m} f_{c+1}^n \left(\frac{b-1}{2} \right) f_{c+1}^m \left(\frac{b-1}{2} \right) \right].$$

Let's us look at the limits $G_{m,m}$, $G_{m,2b-2+m}$ and $G_{m,2c+m}$ with m going to infinity, in these cases only one coefficient will dominate, I'll carefully go over the $G_{m,2b-2+m}$ case and the same idea can be applied for the others.

$$\sum_{m \geq n} \frac{1 + (-1)^{n+m}}{n!m!} \left[D G_{m-n, 2b-2+n+m} f_{b-1}^n \left(\frac{c+1}{2} \right) f_{b-1}^m \left(\frac{c+1}{2} \right) + \right. \quad (D.13)$$

$$\left. B G_{m-n, 2b-2+n+m} f_{b-1}^n \left(\frac{c-1}{2} \right) f_{b-1}^m \left(\frac{c-1}{2} \right) + A G_{m-n, 2b+m+n} f_b^n \left(\frac{c}{2} \right) f_b^m \left(\frac{c}{2} \right) \right],$$

out of here only two terms can generate the limit we are looking for, this could be seen already from the table since only the first line will contribute for this limit

$$\frac{1 + (-1)^m}{m!} \left[D G_{m,2b-2+m} f_{b-1}^0 \left(\frac{c+1}{2} \right) f_{b-1}^m \left(\frac{c+1}{2} \right) + \right. \quad (D.14)$$

$$\left. B G_{m,2b-2+m} f_{b-1}^0 \left(\frac{c-1}{2} \right) f_{b-1}^m \left(\frac{c-1}{2} \right) \right],$$

whichever term dominates it needs to be positive. Looking at the asymptotic behavior for f we know that the one that dominates is the one with bigger $|t|$, since $c+1 > c-1$, $f_{b-1}^n \left(\frac{c+1}{2} \right)$ dominates, which means that D needs to be positive for the expression to be unitary. Following this procedure we obtain the other two conditions, for $G_{m,2c+m}$ A needs to be positive and for $G_{m,m}$ it will depend on the value of $b-c$, in short we have.

$$\boxed{0 \mid A \text{ or } D > 0 \mid \mid b-1 \mid D > 0 \mid \mid c \mid A > 0}$$

For the fourth line we have the same idea but for the limits $G_{m,2+m}$, $G_{m,2b+m}$ and $G_{m,2c+2+m}$, again we will go over the $G_{m,2b+m}$.

$$\frac{1 + (-1)^{1+m+1}}{(m+1)!} \left[D G_{m,2b+m} f_{b-1}^1 \left(\frac{c+1}{2} \right) f_{b-1}^{m+1} \left(\frac{c+1}{2} \right) \right] \quad (D.15)$$

$$+ \frac{1 + (-1)^m}{m!} \left[A G_{m,2b+m} f_b^0 \left(\frac{c}{2} \right) f_b^m \left(\frac{c}{2} \right) \right],$$

I've already took the term that was subdominant before, now we ask what is the limit of the ratio $\frac{f_{b-1}^{m+1} \left(\frac{c+1}{2} \right)}{(m+1) f_b^m \left(\frac{c}{2} \right)}$, plugging into (C.51) and we will find 1, so we have

$$\frac{1 + (-1)^m}{m!} \left[D G_{m,2b+m} f_{b-1}^1 \left(\frac{c+1}{2} \right) \right] + \frac{1 + (-1)^m}{m!} \left[A G_{m,2b+m} f_b^0 \left(\frac{c}{2} \right) \right] > 0, \quad (D.16)$$

$$D f_{b-1}^1 \left(\frac{c+1}{2} \right) + A f_b^0 \left(\frac{c}{2} \right) > 0 \implies \left(\frac{c+1}{2} \right) D + A > 0. \quad (D.17)$$

Doing the same for the other two we have the fourth line

1	$\left(\frac{b-c}{2}\right) A \text{ or } \left(\frac{ 2-b+c }{2}\right) D + B > 0$	b	$\left(\frac{c+1}{2}\right) D + A > 0$	$c+1$	$\left(\frac{b}{2}\right) A + D > 0$
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In the last line we do take both n and m to infinity independent of each other, so we have $G_{m-n,m+n}$, $G_{m-n,2b-2+m+n}$ and $G_{m-n,2c+m+n}$ to look for, back at the $G_{m-n,2b-2+m+n}$ limit we have

$$\begin{aligned} & \frac{1 + (-1)^{n+1+m+1}}{(n+1)!(m+1)!} \left[D G_{m-n,2b+m+n} f_{b-1}^{n+1} \left(\frac{c+1}{2}\right) f_{b-1}^{m+1} \left(\frac{c+1}{2}\right) \right] \\ & + \frac{1 + (-1)^{m+n}}{n!m!} \left[A G_{m-n,2b+m+n} f_b^n \left(\frac{c}{2}\right) f_b^m \left(\frac{c}{2}\right) \right] > 0. \end{aligned} \quad (\text{D.18})$$

First we take m to infinity

$$\frac{1}{(n+1)!} \left[D G_{m-n,2b+m+n} f_{b-1}^{n+1} \left(\frac{c+1}{2}\right) \right] + \frac{1}{n!} \left[A G_{m-n,2b+m+n} f_b^n \left(\frac{c}{2}\right) \right] > 0, \quad (\text{D.19})$$

and then n , so we have

$$[D G_{m-n,2b+m+n}] + [A G_{m-n,2b+m+n}] > 0 \implies A + D > 0, \quad (\text{D.20})$$

applying the same limits to the other two we have the last line.

0	$A \text{ or } D + B > 0$	$b-1$	$A + D > 0$	c	$A + D > 0$
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In short the table is a compact way to write both the conformal block decomposition and some important conditions that needs to be satisfied in order for the expression be unitary.

Going back to the problem there are three equations we need to solve, first of all we want to get rid of the scalar with dimension 2, this was the sole purpose to introduce the other terms, second there is a new scalar of dimension $2b-2$ and we don't want it, lastly we will normalize the 4-point function, let's calculate some OPE coefficients.

$$G_{0,0} : 2A + 2D = 1, \quad (D.21)$$

$$G_{0,2} : (b-c)^2 A + (b-c-2)^2 D + 4B = 0, \quad (D.22)$$

$$G_{0,2b-2} : 2D + 2B = 0, \quad (D.23)$$

$$A = \frac{c+4-b}{8}, \quad (D.24)$$

$$B = \frac{c-b}{8}, \quad (D.25)$$

$$D = \frac{b-c}{8}. \quad (D.26)$$

We will start looking at the two simplest cases the series created by $b-1$ and the one created by c , we need $A > 0$ and $D > 0$ to have the correct first asymptotic behaviour, the third line, in the limit m goes to infinity, so $0 < b-c < 4$. Since both are positive all the other asymptotic behaviours are fine for these two series, we just need $A+B \geq 0$ so the first scalar in c is positive, so $0 < b-c < 2$, this is enough to say that the whole series span by c and $b-1$ is non-negative.

Now comes the hard part, the series starting at 0, to simplify we will set $y = b-c$, $0 < y < 2$. To know who dominates we need to see which t is bigger, if $y > 1$ then $f_0^n(\frac{y}{2})$ dominates, if not $f_0^n\left(\left|\frac{y-2}{2}\right|\right)$ does. The problem appears at the second asymptotic behaviour, the fourth line, we need $\frac{|y-2|}{2}D + B > 0$, but $B+D=0$, so when D dominates the expression is non-unitary. If $y > 1$ then we have $\frac{y}{2}A + B > 0 \rightarrow \frac{y}{2}\frac{4-y}{8} - \frac{y}{8} > 0$, which is positive for the values given, the limit for n and m going to infinity is $A+B > 0$ which we already imposed.

We dealt with all the asymptotic behaviours, we just need to test a few finite cases to show that it is indeed unitary.

When $b, c > 2$ the gap becomes the line 4 for every Δ , we could add a two extra terms to pass the generalized free boson, however we will keep it here. For the second example we have $A(0,0,2b+1) + B(0,b,b+1) + C(1,b,b)$, a problem with the generalized polynomials is the fact that $a+b+c=2\Delta$, for low dimensions we cannot get enough possibilities when trying to shift the parameters by an integer, so we will need to get creative with the parameters.

$A(0,0,2b+1) + B(0,b,b+1) + C(1,b,b), \quad b > 0$								
0	$\frac{2b+1}{2}$	$\frac{1}{2}$	b	$\frac{b+1}{2}$	$\frac{1-b}{2}$	$2b+1$	$\frac{0}{A}$	-
1	$\frac{0}{C}$	-	$b+1$	$\frac{b}{2}$	-	-	-	-
0	$A > 0$		b	$B > 0$		$2b+1$	$A > 0$	
1	$A > 0$		$b+1$	$B > 0$		-	-	
0	$A > 0$		b	$B > 0$		$2b+1$	$A > 0$	

Notice that only one term spam the series $2b+1$, this means that its coefficient needs to be positive no matter what.

$$G_{0,0} : 2A + 2B = 1, \quad (D.27)$$

$$G_{0,2} : 2 \left(\frac{2b+1}{2} \right)^2 A + \frac{B}{2} + C = 0, \quad (D.28)$$

$$G_{0,2b} : 2B + 2C = 0, \quad (D.29)$$

$$A = \frac{1}{8b^2 + 8b + 4}, \quad (D.30)$$

$$B = \frac{(2b+1)^2}{8b^2 + 8b + 4}, \quad (D.31)$$

$$C = -\frac{(2b+1)^2}{8b^2 + 8b + 4}. \quad (D.32)$$

In this case all the asymptotic behaviours are harmless, and the only series that could cause a problem is the one starting at 0, so we just need to check a few finite cases, which will be quite simple to do. Looking at (C.44) we will have

$$f_0^{n+1} \left(\frac{1}{2} \right) - f_0^{n+1} \left(-\frac{1}{2} \right) = (n+1) f_1^n(0), \quad (D.33)$$

for n odd we gain no information, but for even we can get rid of the minus sign to find

$$f_0^{n+1} \left(\frac{1}{2} \right) = \frac{n+1}{2} f_1^n(0), \quad (\text{D.34})$$

so we can rewrite the sum of these two terms for n, m even as

$$\begin{aligned} B \sum_{m \geq n}^{\infty} & \left(\frac{2}{(n+1)!(m+1)!} f_0^{n+1} \left(\frac{1}{2} \right) f_0^{m+1} \left(\frac{1}{2} \right) - \frac{1}{n!m!} f_1^n(0) f_1^m(0) \right) G_{m-n, 2+m+n} \\ & = -B \sum_{m \geq n}^{\infty} \frac{2}{(n+1)!(m+1)!} f_0^{n+1} \left(\frac{1}{2} \right) f_0^{m+1} \left(\frac{1}{2} \right) G_{m-n, 2+m+n}. \end{aligned} \quad (\text{D.35})$$

In the end the unitarity of this expression rely on the test

$$f_0^{n+1} \left(\frac{2b+1}{2} \right) f_0^{m+1} \left(\frac{2b+1}{2} \right) - (2b+1)^2 f_0^{n+1} \left(\frac{1}{2} \right) f_0^{m+1} \left(\frac{1}{2} \right) > 0, \quad (\text{D.36})$$

which we can simplify by asking whether $f_0^{n+1} \left(\frac{2b+1}{2} \right) - (2b+1) f_0^{n+1} \left(\frac{1}{2} \right) \geq 0$ for even n . Testing until $n = 14$ give only non-negative values, at this point the subdominant f is around 1% from the asymptotic behavior, while the dominant f is well above it, so we can say that for $b > 0$ the expression is unitary. With these two simple examples we extended the unitary region, the result can be seen in the picture bellow

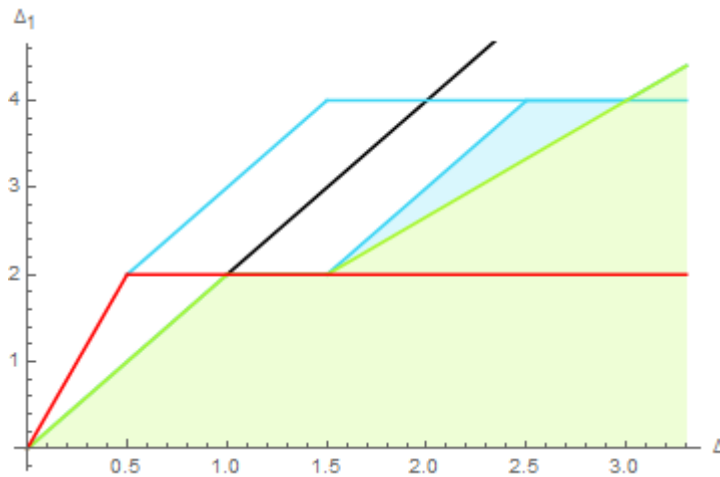


Figure D.3: The new unitary solutions are presented in blue.

Usually when it comes to generalized polynomials that are connected to the identity there exists scalars with scaling dimension equal to every even integers, however in a few particular cases it is possible to eliminate a big part of these operators in a very clean way, all due to the behaviour of $f_0^n(0)$. The equation (C.44) allow us to break an f into two, however since $f_0^n(0) = \delta_0^n$ we have the particular equation $f_0^n(1) = n f_1^{n-1}\left(\frac{1}{2}\right)$ for $n \geq 1$. Imagine we have for a moment $A(0, b, b+2) + B(0, b+1, b+1) + C(1, b, b+1)$, looking only at the series starting at 0 we would have

$$B + \sum_{m \geq n} \frac{1 + (-1)^{n+m}}{n!m!} \left[A f_0^n(1) f_0^m(1) G_{m-n, m+n} + C f_1^n\left(\frac{1}{2}\right) f_1^m\left(\frac{1}{2}\right) G_{m-n, 2+n+m} \right], \quad (\text{D.37})$$

if we set $A = -C$ we will have a huge cancellation leaving us with sum

$$B + \sum_m \frac{1 + (-1)^m}{m!} A f_0^m(1) G_{m, m}. \quad (\text{D.38})$$

If we were to try the same with f_a^n for arbitrary a we would always have some mixed terms $f_a^n f_{a+1}^{m-1}$, for some values of b we do get a unitary solution, but what was most interesting is that the same equations that lead to the cancellation of all those terms also get rid of the scalar with dimension b . This trend continues when we move up the ladder, if we had $A(0, b, b+4) + B(0, b+1, b+3) + C(0, b+2, b+2) + D(1, b, b+3) + E(1, b+1, b+2) + F(2, b, b+2)$ we can again find the same cancellations for the series starting at 0 and at b .

D.2 generalized minimal models

In the main text we talked about a particular 4-point function in the generalized minimal models, the $\langle \phi_{2,1}^4 \rangle$. At the start we are going to recalculate the OPE coefficients in two different ways, the first way was done in [22] and will give the coefficients in terms of Wilson polynomials and the second, using the alpha space, will give us a more complicated expression, but with its advantages.

All Minimal models have the identity operator, because of this we will stead be dealing with the small variation found in section 5. The expression we had before was

$$\left(\frac{x^2}{1-x}\right)^{\frac{k+2\Delta}{6}} \sum_{n=0}^{\infty} \frac{\left(\frac{k}{2}\right)_n \left(\frac{2-k}{6}\right)_n}{n! \left(\frac{5+2k}{6}\right)_n} \left(-\frac{1}{4} \frac{x^2}{1-x}\right)^n, \quad (\text{D.39})$$

luckily the x expression can be decomposed in a special form

$$\sum_{n=0}^{\infty} \frac{\left(\frac{k}{2}\right)_n \left(\frac{2-k}{6}\right)_n}{n! \left(\frac{5+2k}{6}\right)_n} \left(-\frac{1}{4}\right)^n \sum_{l=0}^{\infty} \frac{\left(2n + \frac{k+2\Delta}{3}\right)_l^2}{l! \left(4n + \frac{2k+4\Delta}{3} + l - 1\right)_l} \quad (\text{D.40})$$

$${}_3F_2 \left(\begin{matrix} -l, & 4n + \frac{2k+4\Delta}{3} + l - 1, & n + \frac{k+2\Delta}{6} \\ & 2n + \frac{k+2\Delta}{3}, & 2n + \frac{k+2\Delta}{3} \end{matrix} \middle| 1 \right) k_{l+2n+\frac{k+2\Delta}{3}}(x).$$

The hypergeometric appearing above can be computed using Watson's sum, this expression is non-zero only for even l , to recover the decomposition of the whole expression we change l and the order of summation to keep the eigenvalue constant

$$\sum_{\substack{l=0 \\ 0 \leq n \leq l}}^{\infty} \frac{\left(\frac{k}{2}\right)_n \left(\frac{2-k}{6}\right)_n \left(2n + \frac{k+2\Delta}{3}\right)_{2l-2n}^2}{(-4)^n n! \left(\frac{5+2k}{6}\right)_n (2l-2n)! \left(2n + \frac{2k+4\Delta}{3} + 2l - 1\right)_{2l-2n}} \quad (\text{D.41})$$

$${}_3F_2 \left(\begin{matrix} 2n - 2l, & 2l + 2n + \frac{2k+4\Delta}{3} - 1, & n + \frac{k+2\Delta}{6} \\ & 2n + \frac{k+2\Delta}{3}, & 2n + \frac{k+2\Delta}{3} \end{matrix} \middle| 1 \right) k_{2l+\frac{k+2\Delta}{3}},$$

to finish this calculation we just need to write the ${}_3F_2$ is a better way

$${}_3F_2 \left(\begin{matrix} 2n - 2l & 2l + 2n + 4g - 1 & n + g \\ & 2n + 2g & 2n + 2g \end{matrix} \middle| 1 \right) = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(2n + 2g) \Gamma(1 - n - g) \Gamma\left(n + g + \frac{1}{2}\right)}{\Gamma\left(n - l + \frac{1}{2}\right) \Gamma(n + l + 2g) \Gamma(1 - l - g) \Gamma\left(g + l + \frac{1}{2}\right)} \quad (\text{D.42})$$

$$= \frac{2^{2g+2l-1} \Gamma\left(\frac{1}{2}\right) \Gamma(2n + 2g)^2 \Gamma(1 - n - g) \Gamma\left(n + 2g + l - \frac{1}{2}\right)}{\Gamma\left(n - l + \frac{1}{2}\right) \Gamma(2n + 2l + 4g - 1) \Gamma(1 - l - g) \Gamma(n + g) \Gamma\left(g + l + \frac{1}{2}\right)}.$$

Multiplying this expression with the terms that contain l we obtain

$$\begin{aligned}
 & \frac{2^{2g+2l-1} \Gamma\left(\frac{1}{2}\right) \Gamma(2l+2g)^2 \Gamma(1-n-g) \Gamma\left(n+2g+l-\frac{1}{2}\right)}{\Gamma\left(n-l+\frac{1}{2}\right) \Gamma(4l+4g-1) \Gamma(1-l-g) \Gamma(n+g) \Gamma\left(g+l+\frac{1}{2}\right) (2l-2n)!} \\
 & \quad (D.43) \\
 & = \frac{(-1)^{n+l} 2^{2g+2l-1} \Gamma\left(\frac{1}{2}\right) \Gamma(2l+2g)^2 \Gamma(l+g) \Gamma\left(n+2g+l-\frac{1}{2}\right)}{\Gamma\left(n-l+\frac{1}{2}\right) \Gamma(4l+4g-1) \Gamma(n+g)^2 \Gamma\left(g+l+\frac{1}{2}\right) (2l-2n)!} \\
 & = \frac{(-1)^{n+l} 2^{2g+2l-1} \Gamma(l+g)^2 \Gamma\left(l+g+\frac{1}{2}\right)^2 \Gamma(l+g) \Gamma\left(n+2g+l-\frac{1}{2}\right)}{\Gamma\left(n-l+\frac{1}{2}\right) \Gamma\left(2l+2g-\frac{1}{2}\right) \Gamma(2l+2g) \Gamma(n+g)^2 \Gamma\left(g+l+\frac{1}{2}\right) (2l-2n)!},
 \end{aligned}$$

what we need to do now is get rid of all terms with $-n$ or $-l$, and if possible try to disentangle both variables so the sum becomes easier to express in a closed form

$$\begin{aligned}
 & = \frac{(g)_l^2 \left(2g+l-\frac{1}{2}\right)_n (-1)^{n+l} 2^{2g+2l-1} \Gamma\left(l+g+\frac{1}{2}\right) \Gamma(l+g)}{\left(l+2g-\frac{1}{2}\right)_l (g)_n^2 \Gamma\left(n-l+\frac{1}{2}\right) \Gamma(2l+2g) (2l-2n)!} \quad (D.44) \\
 & = \frac{(g)_l^2 \left(2g+l-\frac{1}{2}\right)_n}{\left(l+2g-\frac{1}{2}\right)_l (g)_n^2} \frac{(-1)^{n+l} \sqrt{\pi}}{\Gamma\left(n-l+\frac{1}{2}\right) (2l-2n)!} \\
 & = \frac{(g)_l^2 \left(2g+l-\frac{1}{2}\right)_n}{\left(l+2g-\frac{1}{2}\right)_l (g)_n^2} \frac{(-1)^{n+l} 2^{2n-2l} \pi}{\Gamma\left(n-l+\frac{1}{2}\right) (l-n)! \Gamma\left(l-n+\frac{1}{2}\right)} \\
 & = \frac{(g)_l^2 \left(2g+l-\frac{1}{2}\right)_n 2^{2n-2l}}{\left(l+2g-\frac{1}{2}\right)_l (g)_n^2 (l-n)!} = \frac{(g)_l^2 \left(2g+l-\frac{1}{2}\right)_n (-1)^n (-l)_n 2^{2n-2l}}{\left(l+2g-\frac{1}{2}\right)_l (g)_n^2 l!}.
 \end{aligned}$$

After all this tinkering the full expression becomes

$$\sum_{l=0}^{\infty} \sum_{n=0}^l \frac{\left(\frac{k}{2}\right)_n \left(\frac{2-k}{6}\right)_n}{(-4)^n n! \left(\frac{5+2k}{6}\right)_n} \frac{\left(\frac{k+2\Delta}{6}\right)_l^2 \left(\frac{k+2\Delta}{3} + l - \frac{1}{2}\right)_n}{\left(l + \frac{k+2\Delta}{3} - \frac{1}{2}\right)_l \left(\frac{k+2\Delta}{6}\right)_n^2} \frac{(-1)^n (-l)_n 2^{2n-2l}}{l!} k_{2l+\frac{k+2\Delta}{3}}(x), \quad (\text{D.45})$$

the sum in n can be rewritten as a hypergeometric function, which in closer inspection turns out to be a Wilson polynomial

$$\sum_{l=0}^{\infty} \frac{\left(\frac{k+2\Delta}{6}\right)_l^2}{\left(l + \frac{k+2\Delta}{3} - \frac{1}{2}\right)_l (4)^l l!} {}_4F_3 \left(\begin{matrix} -l, & l + \frac{k+2\Delta}{3} - \frac{1}{2}, & \frac{k}{2}, & \frac{2-k}{6} \\ & \frac{k+2\Delta}{6}, & \frac{k+2\Delta}{6}, & \frac{5+2k}{6} \end{matrix} \middle| 1 \right) k_{2l+\frac{k+2\Delta}{3}}(x). \quad (\text{D.46})$$

This expression is quite handy, we are able to condense the OPE coefficients into a known polynomials, however the only reason we've obtained it is because we are dealing only with minimal models. In the same way we can sum and multiply generalized polynomials we can mix the two to create a new 4-point function, an example was presented in equation (4.15), there we multiplied a polynomial with the Ising model, and we know that it is unitary since it is a 4-point function in the tricritical Ising model.

To deal with a generic generalized polynomial times a minimal model we need to calculate the alpha space density of

$$g(x) = x^a (1-x)^b {}_2F_1 \left(\begin{matrix} c, & d \\ e \end{matrix} \middle| x \right), \quad (\text{D.47})$$

we can quickly calculate it by using the Mellin-Barnes representation of ${}_2F_1$

$$g(x) = \int [ds] \frac{\Gamma(-s) (c)_s (d)_s}{(e)_s} (-1)^s x^{a+s} (1-x)^b, \quad (\text{D.48})$$

$$\hat{g}(\alpha) = \int [ds] \frac{\Gamma\left(a+s-\frac{1}{2} \pm \alpha\right) \Gamma(-s) (c)_s (d)_s}{(\Gamma(a+s))^2 (e)_s} (-1)^s {}_3F_2 \left(\begin{matrix} a+s-\frac{1}{2} + \alpha, & a+s-\frac{1}{2} - \alpha, & -b \\ & a+s, & a+s \end{matrix} \middle| 1 \right) \quad (\text{D.49})$$

$$\begin{aligned} &= \sum_{n=0}^{\infty} \int [ds] \frac{\Gamma\left(a-\frac{1}{2} \pm \alpha\right) \Gamma(-s) \left(a-\frac{1}{2} \pm \alpha\right)_{n+s} (c)_s (d)_s (-b)_n}{n! (\Gamma(a))^2 ((a)_{n+s})^2 (e)_s} (-1)^s \\ &= \frac{\Gamma\left(a-\frac{1}{2} \pm \alpha\right)}{(\Gamma(a))^2} \sum_{n,m=0}^{\infty} \frac{\left(a-\frac{1}{2} \pm \alpha\right)_{n+m} (c)_m (d)_m (-b)_n}{n! m! ((a)_{n+m})^2 (e)_m}. \end{aligned}$$

We can now write the conformal block decomposition for our mix

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (a)_n^2}{n! (2a-1+n)_n} \mathcal{F}_{2,1,0}^{2,2,1} \left(\begin{matrix} -n, & 2a+n-1 \\ a, & a \end{matrix} \middle| \begin{matrix} c, & d \\ e \end{matrix} \middle| \begin{matrix} -b \\ - \end{matrix} \middle| 1, 1 \right) k_{2a+n}(x), \quad (\text{D.50})$$

we are going to present two examples of mixes between a generalized polynomial and the Ising model, first we are going to use a cosine as our polynomial

$$\mathcal{G} = (2\Delta, 0, 0) \frac{\sqrt{p^{\frac{1}{2}} + q^{\frac{1}{2}} + r^{\frac{1}{2}}}}{(pr)^{\frac{1}{8}}}. \quad (\text{D.51})$$

Although this expression starts far from the boundary it is in the line $4\left(\Delta - \frac{1}{8}\right)$, so if this line is unitary and continues forever it would be a strong lower bound for the maximum gap possible. Since we are multiplying two unitary 4-point functions we can guarantee that the result is unitary, however this expression will flatten after $\Delta = \frac{1}{4}$.

To show this we consider two arbitrary unitary 4-point functions, the first

comes from an operator of scaling dimension Δ_1 and has a gap of h_1 and the second comes from a Δ_2, h_2 . We can imagine a theory with operators $\mathcal{O}_1 \times \mathcal{O}_2$, and n -point function being a product of the n -point functions of both theories. This means that the 4-point function $\langle \phi^4 \rangle$, $\phi = \phi_1 \times \phi_2$, will be the product of both 4-point functions, which is exactly what we wanted, the scaling dimension of ϕ is the sum of ϕ_1 and ϕ_2 , and the gap will be given by an operator of the form $1 \times \phi_2$ or $\phi_1 \times 1$, $h = \text{Min}(h_1, h_2)$. If the first theory have the OPE coefficients C_{ijk}^1 and the second C_{ijk}^2 we would have in the product theory $\phi_i^1 \times \phi_{i'}^2(z) \phi_j^1 \times \phi_{j'}^2(0) \sim \sum_{k,k'} C_{ijk}^1 C_{i'j'k'}^2 \phi_k^1 \times \phi_{k'}^2(z)$ so two unitary theories gives another one. Note that we are talking about 2D theories here, mainly because the spin group is $U(1)$, if we were in higher dimensions $\mathcal{O}_1^{\mu_1 \dots \mu_n} \times \mathcal{O}_2^{\nu_1 \dots \nu_m}$ won't belong to a irreducible representation so we will have to break it which *may* result in some negative coefficients. As a result the 4-point functions that are decoupled from the identity have a new behaviour in 2D, an almost unitary correlation function, the only source of non-unitarity is the lack of identity, with (Δ_1, h_1) and another with (Δ_2, h_2) will create a $(\Delta_1 + \Delta_2, h_1 + h_2)$.

Our initial curve may have flatten at $(\frac{3}{8}, 1)$, but we can still continue by choosing another polynomial, in this case we have, just for the example

$$\mathcal{G} = \left(\frac{9}{16} (1, 0, 0) - \frac{1}{8} \left(\frac{1}{2}, \frac{1}{2}, 0 \right) \right) \frac{\sqrt{p^{\frac{1}{2}} + q^{\frac{1}{2}} + r^{\frac{1}{2}}}}{(pr)^{\frac{1}{8}}}, \quad (\text{D.52})$$

this will only be a point, instead of a line, but if we calculate the gap we will notice that it is situated at $(\frac{5}{8}, 2)$. The only way this can be possible is if our polynomial is non-unitary. This is not enough to say that either the 4-point function or theories that contain (D.52) are non-unitary since we have a counter example for both, again the equation (4.15), although the product theory certainly is.

We could do the same for others generalized polynomials, or other theories in general, to continue to create new, and possibly interesting, unitary 4-point functions.

D.3 OPE coefficients from an expansion

As was shown back in chapter 5 there is so much the generalized polynomials can accomplish, however by using an expansion in the $q \ll p \ll r$ regime it

is possible to express any stripped correlator as a series over $\frac{q}{r}$ and $\frac{p}{r}$. As was argued there these expansions are related to the u and v expansions of the stripped correlator, as such it is valid to say that the extra term $2xz + 2 - x - z$ disappear term by term, because of this the expansions in q will look like the following

$$\mathcal{G}(p, q, r) = \frac{f_0(p, r) + q^{\frac{\tau_1}{2}} f_{\tau_1}(p, r) + q^{\frac{\tau_2}{2}} f_{\tau_2}(p, r) + \dots}{(pr)^\Delta}. \quad (\text{D.53})$$

Focusing in the term $q^{\frac{\tau_n}{2}} f_{\tau_n}(p, r)$ and expanding in p

$$\frac{q^{\frac{\tau_n}{2}} f_{\tau_n}(p, r)}{(pr)^\Delta} = \sum_m \frac{q^{\frac{\tau_n}{2}} A_m^{\tau_n} \left(\frac{p}{r}\right)^{\xi_m} r^{2\Delta - \frac{\tau_n}{2}}}{(pr)^\Delta} = \sum_m A_m^{\tau_n} \left(\frac{q}{r}\right)^{\frac{\tau_n}{2}} \left(\frac{p}{r}\right)^{\xi_m} \left(\frac{r}{p}\right)^\Delta, \quad (\text{D.54})$$

given what was said before this is the only possibility for the expansion in p , the conformal block decomposition for this expression is

$$\begin{aligned} \frac{q^{\frac{\tau_n}{2}} f_{\tau_n}(p, r)}{(pr)^\Delta} &= \sum_m \sum_{k \geq l} A_m^{\tau_n} \frac{(-1)^{k+l} G_{k-l, \tau_n+k+l}(x, z)}{k!l!} \\ &\quad f_{\frac{\tau_n}{2}}^k \left(\frac{\tau_n}{4} + \xi_m - \Delta\right) f_{\frac{\tau_n}{2}}^l \left(\frac{\tau_n}{4} + \xi_m - \Delta\right). \end{aligned} \quad (\text{D.55})$$

Although this expression looks complicated there are ways to simplify it, notice that the sum in m are a sum of the form

$$\sum_m A_m^{\tau_n} \text{Polynomial of } \left(\frac{\tau_n}{4} + \xi_m - \Delta\right), \quad (\text{D.56})$$

this can be done by using an exponential

$$\sum_m A_m^{\tau_n} \exp\left(w \left(\frac{\tau_n}{4} + \xi_m - \Delta\right)\right) = \exp\left(w \left(\frac{\tau_n}{4} - \Delta\right)\right) \sum_m A_m^{\tau_n} \exp(w(\xi_m)), \quad (\text{D.57})$$

defining $\frac{f_{\tau_n}(p, r)}{r^{2\Delta - \frac{\tau_n}{2}}} = g_{\tau_n}\left(\frac{p}{r}\right)$, then

$$k_{\tau_n}(w) = \sum_m A_m^{\tau_n} \exp\left(w\left(\frac{\tau_n}{4} + \xi_m - \Delta\right)\right) = \exp\left(w\left(\frac{\tau_n}{4} - \Delta\right)\right) g_{\tau_n}(\exp(w)). \quad (\text{D.58})$$

With k_τ defined any sum of the form $\sum_m A_m^{\tau_n}$ Polynomial of $(\frac{\tau_n}{4} + \xi_m - \Delta)$ can be solved by doing some derivatives in $k_\tau(w)$ and setting w to 0, so the previous sum can be rewritten as

$$\frac{q^{\frac{\tau_n}{2}} f_{\tau_n}(p, r)}{(pr)^\Delta} = \sum_{k \geq l} \frac{(-1)^{k+l} G_{k-l, \tau_n+k+l}(x, z)}{k!l!} \left[f_{\frac{\tau_n}{2}}^k(\partial_w) f_{\frac{\tau_n}{2}}^l(\partial_w) k_{\tau_n}(w) \right] \Big|_{w \rightarrow 0}, \quad (\text{D.59})$$

this way bu knowing $f_{\tau_n}(p, r)$ it is possible to crate a new function $k_{\tau_n}(w)$ that control the decomposition with its Taylor series. To test it let's using the f_0 from the Ising model, $f_0(p, r) = \sqrt{\frac{p^{\frac{1}{2}} + r^{\frac{1}{2}}}{2}}$,

$$g_0\left(\frac{p}{r}\right) = \frac{f_0(p, r)}{r^{\frac{1}{4}}} = \sqrt{\frac{1}{2} + \frac{1}{2}\left(\frac{p}{r}\right)^{\frac{1}{2}}} \quad (\text{D.60})$$

$$k_0(w) = e^{-\frac{w}{8}} g_0(e^w) = e^{-\frac{w}{8}} \sqrt{\frac{1}{2} + \frac{1}{2}e^{\frac{w}{2}}} = \sqrt{\text{Cosh}\left(\frac{w}{4}\right)}. \quad (\text{D.61})$$

Testing the 0-twist operators, they will be given by

$$\sum_k \frac{(-1)^k G_{k,k}(x, z)}{k!} \left[f_0^k(\partial_w) \sqrt{\text{Cosh}\left(\frac{w}{4}\right)} \right] \Big|_{w \rightarrow 0}, \quad (\text{D.62})$$

all the odd spins will disappear from the expansion since Cosh is an even function, taking k from 0 to 4 we find $\left\{1, 0, \frac{1}{64}, 0, \frac{9}{40960}\right\}$ which match with the values found from the Wilson polynomials. Since the expansion for the Ising model do contain an $f_2(p, q)$ to test the operators with twist 2 these new terms would need to be considered.

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