

Angular-planar CMB power spectrum

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Gaussianity and statistical isotropy of the Universe are modern cosmology's minimal set of hypotheses. In this work we introduce a new statistical test to detect observational deviations from this minimal set. By defining the temperature correlation function over the whole celestial sphere, we are able to independently quantify both *angular* and *planar* dependence (modulations) of the CMB temperature power spectrum over different slices of this sphere. Given that planar dependence leads to further modulations of the usual angular power spectrum C_ℓ , this test can potentially reveal richer structures in the morphology of the primordial temperature field. We have also constructed an unbiased estimator for this angular-planar power spectrum which naturally generalizes the estimator for the usual C_ℓ 's. With the help of a chi-square analysis, we have used this estimator to search for observational deviations of statistical isotropy in WMAP's 5 year release data set (ILC5), where we found only slight anomalies on the angular scales $\ell = 7$ and $\ell = 8$. Since this angular-planar statistic is model-independent, it is ideal to employ in searches of statistical anisotropy (e.g., contaminations from the galactic plane) and to characterize non-Gaussianities.

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I. INTRODUCTION

Many efforts have been made toward understanding the statistical properties of the cosmic microwave background (CMB) temperature field in the past few years. The main motivation behind these efforts is that, in a homogeneous and isotropic universe in which inflation is driven by a single canonical scalar field, the primordial temperature field is set by Gaussian and statistically isotropic physical processes. Since nonlinear evolution destroys all these putative initial Gaussianities, we must search for any fundamental deviations from these statistical properties at early epochs and as close as possible to the linear regime. This makes the CMB the ideal physical observable to employ in searches of statistical anisotropies and non-Gaussianities. Any significant observational deviation from this picture could reveal something as yet unsuspected about the basic nature of our Universe.

While this program seems to be well motivated by itself, careful analysis of recent temperature maps obtained by the WMAP team [1–3] have hinted at some apparent anomalies—mainly in the low multipoles sector [4–8]. If one leaves aside for a moment the perennial problem of *a posteriori* statistics [9], these findings raise the possibility that the anomalies could be a first hint toward some new physics. Theoretical attempts to explain their origin include primordial magnetic fields [10], nontrivial cosmic topologies [11,12], globally anisotropic models of the early

Universe [13–16] as well as local manifestations of cosmic anisotropy [17], and even anisotropic models of dark energy [18,19]. Of course, there is a good chance that these anomalies are due to astrophysical effects [20] or even some residual instrumental cross-contamination, in which case our Universe can still be easily accommodated in the standard scenario. It is a question of utmost concern to decide whether these known anomalies (as well as others which may be found in the future) are isolated statistical flukes, or if they are due to new physical/astrophysical effects.

Despite its importance and the efforts spent on it, we still have no compelling explanation for the nature of the low- ℓ anomalies. The main difficulty is twofold: first, we still do not know how to optimally separate the question of Gaussianity from that of statistical isotropy (see however [21] for a first step in this direction.) It is thus possible that our Universe is Gaussian but statistically anisotropic, statistically isotropic but non-Gaussian, or even non-Gaussian *and* anisotropic. Second, if the Universe is neither Gaussian nor statistically isotropic, then it can be—from a statistical point of view—virtually anything: there is only one kind of Gaussianity and isotropy, but there are infinite ways to brake either one. The absence of theoretical guidelines will inevitably lead to an infinite number of models and no underlying symmetries, which still would mean that we could not account for confirmed anomalies.

This means we must analyze the problem in as much a model-independent manner as possible. We can, for example, start from the very basic definition of our statistical quantities (such as the two-point correlation function)

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and check whether they can be modified in a model-independent manner, basing our reasoning solely on physical symmetries and observational hints.

One such possibility is to consider the two-point temperature correlation function, $C(\hat{n}_1, \hat{n}_2)$, without some of the symmetries of the underlying space-time. Attempts in this direction have been made by Pullen and Kamionkowski [22], where the temperature correlation function is assumed to depend on the direction of any given unit vector in the celestial sphere, in such a way that one can search for power multipole moments in temperature maps. However, that approach consists of considering the temperature correlation function at zero lag, and thus it does not allow us to consider correlations between two different points in the sky.

Another possibility is to consider the correlation function in its full form, i.e., a function that depends on all pairs of independent unit vectors in the sphere S^2 . This idea, which was introduced by Hajian and Souradeep [23–25], consists in expanding the temperature correlation function in a *bipolar* spherical harmonic series in order to take into account its functional dependence. The authors then construct a bipolar power spectrum κ_ℓ which can account for deviations from statistical anisotropy if observations give us $\kappa_\ell > 0$ at a statistically significant level. Unfortunately, that approach is too generic: it is not clear what the associated statistical test is measuring, nor how one can motivate it in the absence of an underlying theoretical or phenomenological model.

In this work we also go back to the two-point correlation function $C(\hat{n}_1, \hat{n}_2)$, but instead ask whether it can depend not only on the separation angle between two given unit vectors, $\cos\vartheta = \hat{n}_1 \cdot \hat{n}_2$, but also on the orientation of the plane of the great circle defined by the unit vectors. Such a functional dependence can be unambiguously constructed once we realize that for any two unit vectors in the CMB sky, their angular separation and their associated plane are uniquely defined by their dot and cross products, respectively. The new planar dependence (on the direction defined by the normal to the planes of the two unit vectors) codifies modulations of the usual two-point correlation function as we rotate these planes while keeping the separation angles ϑ fixed.

We have also constructed, in a completely model-independent way, an angular-planar power spectrum and its associated unbiased estimator, which naturally generalizes the usual angular power spectrum C_ℓ , and for which we recover the known results in the limit of statistical isotropy.

Our approach has a strong observational motivation, which lies in the fact that some astrophysical planes, like the galactic and ecliptic ones, play an important role in CMB measurements and could still be manifested in the data if the foregrounds were improperly removed. One such example was possibly found in [26] where, besides

the alignment of the multipoles $\ell = 2$ and $\ell = 3$, the authors detected a strong correlation between these two and the ecliptic plane. The existence of a preferential plane could also be related to the so-called north-south asymmetry [5,8,27], in which case a plane could naturally separate regions of maximum and minimum temperature power. There exists also a third situation in which a physical plane can play an important role in cosmology, namely, the unavoidable presence of our galactic plane in all CMB measurements acts as an important source of astrophysical and foreground contamination. All these facts lead us to believe that a planar signature on the correlation function would be an important statistical property of the CMB, and is a potential test of its nature.

We have organized this work in the following way: we begin Sec. II with a brief description of the two-point correlation function and its general properties. After discussing some of its known generalizations, we extend our argument to include a planar dependence. In Sec. III we carry a multipolar decomposition of the correlation function with planar dependence and show how the resulting coefficients (i.e., the angular-planar power spectrum) are related to the usual temperature multipolar coefficients $a_{\ell m}$'s. This leads us to the question of how to build an unbiased estimator to measure planar signatures in temperature maps and, in particular, how this can be implemented with the help of a simple chi-square analysis. We illustrate, still in this section, the application of our statistics to the well-known Λ CDM concordance model, where we present some figures for the “best-fit” Λ CDM angular-planar power spectrum. In Sec. IV we use a chi-square test to search for planar signatures in the WMAP full-sky temperature maps, and show that the angular scales $\ell = 7$ and $\ell = 8$ seem to be slightly anomalous for a particular range of planar separation l . We conclude in Sec. V, where we also give some perspective of further developments.

II. TEMPERATURE CORRELATION FUNCTION

The main observable in the CMB is the temperature fluctuation field, ΔT . In its full generality, this field is a function of a position vector \hat{n} and of the time interval in which we measure this temperature—but in practice our measurements are made in time intervals which are negligible compared with the cosmological time scales. The field ΔT is a scalar, continuous function on the unit sphere, which means we can decompose it in the usual fashion, in terms of spherical harmonics:

$$\Delta T(\hat{n}) = \sum_{\ell, m} a_{\ell m} Y_{\ell m}(\hat{n}). \quad (1)$$

All information is therefore encrypted in the multipolar coefficients $a_{\ell m}$. Essentially all inflationary models predict these coefficients not as uniquely given, but rather as realizations of a random variable, in such a way that the physics is not in the $a_{\ell m}$'s themselves, but rather on their

statistical properties. Since, by construction, this field has zero expectation value, $\langle \Delta T \rangle = 0$, the two-point correlation function expresses the first nontrivial momenta of the underlying statistical properties of the physical field, and is given by

$$\begin{aligned} C(\hat{n}_1, \hat{n}_2) &\equiv \langle \Delta T(\hat{n}_1) \Delta T(\hat{n}_2) \rangle \\ &= \sum_{\ell_1, m_1} \sum_{\ell_2, m_2} \langle a_{\ell_1 m_1} a_{\ell_2 m_2}^* \rangle Y_{\ell_1 m_1}(\hat{n}_1) Y_{\ell_2 m_2}^*(\hat{n}_2). \end{aligned} \quad (2)$$

Alternatively, the *covariance matrix* above, $\langle a_{\ell_1 m_1} a_{\ell_2 m_2}^* \rangle$, gives all the information about the quadratic momenta of the underlying distribution. If the field ΔT is Gaussian, then this covariance matrix encloses *all* the information that is needed to describe the nature of the fluctuation field (1). In this work we shall restrict ourselves to a fiducial Gaussian model, for simplicity.

We note also that the separable nature of the definition (2) implies a reciprocity relation for the correlation function:

$$C(\hat{n}_1, \hat{n}_2) = C(\hat{n}_2, \hat{n}_1). \quad (3)$$

This symmetry must always be satisfied, regardless of the underlying physics.

A. Isotropic case

In a globally homogeneous and isotropic universe, the two-point correlation function of the temperature can only depend on the separation angle between the vectors \hat{n}_1 and \hat{n}_2 , that is

$$C(\hat{n}_1, \hat{n}_2) = C(\hat{n}_1 \cdot \hat{n}_2) = \sum_{\ell} \frac{2\ell + 1}{4\pi} C_{\ell} P_{\ell}(\hat{n}_1 \cdot \hat{n}_2). \quad (4)$$

Comparing this expression with Eq. (2), we notice that the covariance matrix becomes diagonal

$$\langle a_{\ell_1 m_1} a_{\ell_2 m_2}^* \rangle = C_{\ell_1} \delta_{\ell_1 \ell_2} \delta_{m_1 m_2}, \quad (5)$$

with the diagonal terms given by the *angular power spectrum*, C_{ℓ} . In principle the angular power spectrum suffices to describe the statistical properties of the temperature field (1). However, since we have only one universe to measure, and therefore only one set of $a_{\ell m}$'s, the average in (5) is poorly determined. The best we can do then is to take advantage of the ergodic hypothesis, which states that averaging over an ensemble can be treated as averaging over space, and hence to consider each of the $2\ell + 1$ real numbers in $a_{\ell m}$ as statistically independent, in such a way as to build a statistical estimator for the C_{ℓ} 's:

$$\hat{C}_{\ell} \equiv \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2.$$

Since $\langle \hat{C}_{\ell} \rangle = C_{\ell}$, this estimator is said to be unbiased. Also, because for a Gaussian field $\langle (\hat{C}_{\ell} - C_{\ell}) \times$

$(\hat{C}_{\ell'} - C_{\ell'}) \rangle \propto \delta_{\ell \ell'}$, this estimator has the least “cosmic variance.” \hat{C}_{ℓ} is, therefore, the best estimator that can measure the statistical properties of the multipolar coefficients $a_{\ell m}$ when both statistical isotropy and Gaussianity hold.

B. Some anisotropic cases

The first line in Eq. (4) for the temperature two-point correlation function is valid if and only if the Universe is statistically isotropic. This means that any functional dependence that does not reduce to a dependence on $\cos \vartheta = \hat{n}_1 \cdot \hat{n}_2$ will measure some deviation from statistical isotropy. There are infinite possible combinations of \hat{n}_1 and \hat{n}_2 that violate statistical isotropy. However, since the vectors \hat{n}_1 and \hat{n}_2 are constrained to have a common origin and size, symmetry and simplicity does not leave us many choices. One possibility is to consider these two vectors as being the same, in which case we are left with a correlation function of the form

$$C: S^2 \rightarrow \mathbb{R}, \quad (6)$$

and for which a decomposition similar to (1) exists. This form of the correlation function makes it suitable for searching for power multipole moments in CMB temperature and polarization maps, once we define a power multipole moment estimator [22]. On the other hand, this is also a correlation function at zero lag, so by construction it does not allow us to consider anisotropic correlations between different points in the sky.

A second possibility is to consider the correlation function as being the most general (but separable) function of two unit vectors that one can possibly have [23]:

$$C: S^2 \times S^2 \rightarrow \mathbb{R}. \quad (7)$$

This function admits a decomposition in terms of the bipolar spherical harmonics [23] which has the nice property of behaving—in many mathematical aspects—as the usual spherical harmonics. The main drawback of the decomposition (7), however, is that it carries too many degrees of freedom which, in the absence of a specific cosmological model, cannot be resolved with simple estimators. Therefore, these two approaches are either too simple or too generic to reveal deviations from statistical isotropy in a more model-independent way.

C. Anisotropy through planar dependence

The guiding principle used in the construction of (6) and (7) is rather general and based mainly on our prejudices about what statistical anisotropy should look like. However, in the absence of theoretical guidelines, we have to confine ourselves to the observations of the CMB temperature or, more specifically, to the signature of its known anomalies. One example is the role played by the galactic and ecliptic plane in the quadrupole-octupole/

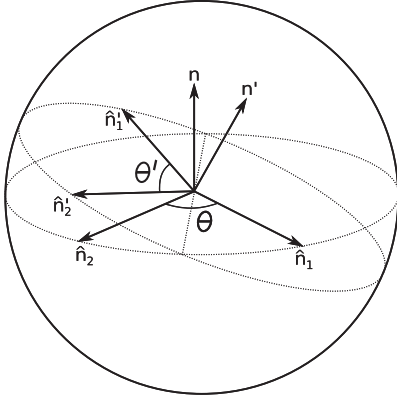


FIG. 1. Schematic representation of the functional dependence of the correlation function (8). In the anisotropic case we are considering, fields at the positions given by vectors living on different planes can have different correlations, regardless of their angular separation.

north-south anomalies [4–8], not to mention the importance of our galactic plane as a source of foreground contamination in the construction of cleaned CMB maps. The existence of a cosmic plane might even be a manifestation of some mirror symmetry [28]

In general, the simple fact that we are bound to make all our measurements inside our galactic plane suggests that the correlation between fields at two positions \hat{n}_1 and \hat{n}_2 might be sensitive not only to their separation angle but also to the orientation of the plane they live in, as is shown in Fig. 1.

Such a planar dependence can be included in the correlation function if we realize that two unit vectors on the sphere S^2 uniquely define both a separation angle ϑ and a direction \hat{n} perpendicular to the great circle (or plane) where they live. We are then left with a new possibility for the functional dependence of the two-point correlation function:

$$C(\hat{n}_1, \hat{n}_2) = C(\hat{n}_1 \times \hat{n}_2), \quad (8)$$

which corresponds formally to a function of the form $C: D^3 \rightarrow \mathbb{R}$, where D^3 is the set of all points (x, y, z) such that $x^2 + y^2 + z^2 \leq 1$ [29]. Defining $\mathbf{n} \equiv \hat{n}_1 \times \hat{n}_2$, the above expression can be further decomposed in spherical coordinates as follows:

$$C(\mathbf{n}) = \sum_{\ell} \sum_{l,m} \frac{2\ell + 1}{\sqrt{4\pi}} C_{\ell}^{lm} P_{\ell}(\cos \vartheta) Y_{lm}(\hat{n}), \quad l \in 2\mathbb{N}, \quad (9)$$

where

$$|\mathbf{n}| = \sin \vartheta, \quad \hat{n} = \{\theta, \phi\}.$$

Notice that

$$\begin{aligned} \hat{n}_1 \cdot \hat{n}_2 &= \cos \vartheta \\ &= \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2), \end{aligned} \quad (10)$$

where (θ_i, ϕ_i) are the angles defined by the vectors \hat{n}_i .

Some comments on the decomposition (9) are in order. First, we note that there is an intrinsic ambiguity in the sense of the vector \mathbf{n} (as we might as well have defined $\mathbf{n} \equiv \hat{n}_2 \times \hat{n}_1$), which is obviously inherited from the ambiguity in the definition of the normal to a plane. This ambiguity can be avoided if we restrict the sum in l to even values, which is what we will do from now on. Note that such a restriction arises naturally as a consequence of the reciprocity relation, Eq. (3). Second, for $\ell = 0$ we recover (6) and therefore all the analysis made in [22] arises as a special case here.

III. ANGULAR-PLANAR POWER SPECTRUM

The multipolar C_{ℓ}^{lm} coefficients in Eq. (9) correspond to a generalization of the usual angular power spectrum C_{ℓ} 's. In fact, they can be seen as a spherical harmonic decomposition of the angular power spectrum, if it suffers modulations as we sweep planes on the sphere. The function $C_{\ell}(\hat{n})$ for a given ℓ is

$$C_{\ell}(\hat{n}) = \sqrt{4\pi} \sum_{l,m} C_{\ell}^{lm} Y_{lm}(\hat{n}), \quad l \in 2\mathbb{N}.$$

Clearly, the monopole of $C_{\ell}(\hat{n})$ (the average over the whole sphere) is the usual angular power spectrum, $C_{\ell}^{00} = C_{\ell}$, and the higher multipoles measure modulations of the spectrum.

Since we are restricting our analysis to the Gaussian case, the set of coefficients C_{ℓ}^{lm} completely characterizes the two-point correlation function. Still, what is accessible through observations are temperature maps which we can use to try to estimate the correlation function. In this respect the multipolar coefficients C_{ℓ}^{lm} would be of limited interest, unless we can relate them directly to our observables. It would be interesting if we could, for example, relate these coefficients to the covariance matrix $\langle a_{\ell_1 m_1} a_{\ell_2 m_2}^* \rangle$ by equating expressions (9) and (2), as is usually done. However, this procedure is far from being trivial, since the complicated coupling of the angles ϑ , θ , and ϕ defined in (10) make it difficult to use the usual orthogonality relations to isolate the C_{ℓ}^{lm} 's.

Fortunately, as we show below, we can estimate the C_{ℓ}^{lm} 's if we use the invariance of the scalar product $\hat{n}_1 \cdot \hat{n}_2$ and choose our coordinate system in order to integrate out the ϑ dependence. Once this is done, we make a passive rotation of the coordinate system and then we integrate over the remaining angles θ and ϕ , which then are given precisely by the Euler angles used in the rotation. The details are rather technical and can be found in the Appendix. The final expression is

$$\frac{\mathcal{C}_\ell^{lm}}{\sqrt{2l+1}} = 2\pi \sum_{\ell_1, m_1} \sum_{\ell_2, m_2} \langle a_{\ell_1 m_1} a_{\ell_2 m_2} \rangle \begin{pmatrix} l & \ell_1 & \ell_2 \\ m & m_1 & m_2 \end{pmatrix} I_{\ell_1 \ell_2}^{l, \ell}, \quad (11)$$

where the 6-index expression in parentheses is the Wigner 3J symbol, and

$$I_{\ell_1 \ell_2}^{l, \ell} \equiv \sum_m (-1)^m \lambda_{\ell_1 m} \lambda_{\ell_2 m} \begin{pmatrix} l & \ell_1 & \ell_2 \\ 0 & m & -m \end{pmatrix} \times \int_0^\pi d(-\cos\vartheta) P_\ell(\cos\vartheta) e^{im\vartheta}, \quad (12)$$

where $\lambda_{\ell, m}$ are a set of coefficients resulting from the ϑ integration, which vanish unless $\ell_i + m = \text{even}$ (see the Appendix for more details.)

It is easy to show that expression (11) induces no coupling between the eigenvalues ℓ and l , as expected, since the length of the vector \mathbf{n} is completely independent of its orientation. There are, however, subtle couplings present in (11) which do make a difference when we apply it to real data. This is due to the Legendre polynomial in the integral (12), which selects only those values of ℓ_1 and ℓ_2 which have the same parity as the angular momentum ℓ . Moreover, the 3J symbols appearing in (11) give different weights to the triple (l, ℓ_1, ℓ_2) depending on the parity of (ℓ_1, ℓ_2) and, as a consequence, we can expect typical oscillations in any function of (11) that we may build when plotted as a function of ℓ . This will be shown explicitly in the next section, when we apply these tools to the WMAP 5 year data.

Expression (11) does not take into account the fact that real data are not given exactly by (1), but rather by a pixelated temperature map which is a combination of the true cosmological signal, plus instrumental noise and residual foreground contamination. Schematically, the temperature of the map in each pixel i is given by

$$\Delta T^{\text{map}}(\hat{\mathbf{n}}_i) = \Delta T^S(\hat{\mathbf{n}}_i) + \Delta T^N(\hat{\mathbf{n}}_i) + \Delta T^R(\hat{\mathbf{n}}_i).$$

Typically, the cosmological signal ΔT^S is smoothed out by a Gaussian beam $W(\hat{\mathbf{n}}_i)$ of finite width which, in harmonic space, is given by $W_\ell = \exp(-\ell^2 \sigma_b^2)$, where $\sigma_b = \theta_{\text{fwhm}}/\sqrt{8 \ln 2}$ and θ_{fwhm} is the beam full width at half maximum. For the V-band frequency map of the WMAP experiment, $\theta_{\text{fwhm}} = 0.35^\circ$, which implies a minimum $\ell_{\text{min}} \gtrsim 390$ for which the effect of a beam smoothing will be important, much higher than the low- ℓ regions where known anomalies were reported. Thus, for the sake of simplicity we will neglect the effect of the beam in this work. Also, for the $\ell \lesssim 390$ region, cosmic variance is known to dominate the source of error over instrumental noise, and therefore we can neglect the latter as well. On the other hand, the residual foreground can be an important source of contamination, and therefore deserves a careful analysis which is beyond the scope of the present work. In a companion paper we carry a more rigorous analysis of

planar signature in CMB data in which the effect of the residual foreground will be estimated [30].

A. Statistical estimators and χ^2 analysis

We now would like to use expression (11) to examine the observed Universe. We start by noting that in the limit of statistical isotropy (SI), that is, when $\langle a_{\ell_1 m_1} a_{\ell_2 m_2} \rangle = C_{\ell_1} \delta_{\ell_1 \ell_2} \delta_{m_1 m_2}$, expression (11) reduces to

$$\mathcal{C}_\ell^{lm} \stackrel{\text{(SI)}}{=} C_\ell \delta_{l0} \delta_{m0}. \quad (13)$$

Conversely, if the only nonzero \mathcal{C}_ℓ^{lm} 's are given by $l = m = 0$, then $\mathcal{C}_\ell^{00} = C_\ell$. Therefore, statistical isotropy is achieved if and only if the \mathcal{C}_ℓ^{lm} 's are of the form (13), and any observational deviation from this relation would be an indication of statistical anisotropy.

However, we only get to observe one universe, and this makes the “cosmic sample variance” a severe restriction that we have to live with. This means that if we want to know, let us say, the mean value and variance of the \mathcal{C}_ℓ^{lm} 's, we will have to build statistical functions which can only *estimate* these properties, just like it is done with the fundamental quantities $a_{\ell m}$ and the associated estimators \hat{C}_ℓ (see the discussion in Sec. II A).

In other words, in order to evaluate the statistical properties of the \mathcal{C}_ℓ^{lm} 's we will have to treat them as our new “fundamental” quantities, which will be determined exclusively as a function of the $a_{\ell m}$'s. As a consequence, we will redefine expression (11) as

$$\mathcal{C}_\ell^{lm} \rightarrow 2\pi \sqrt{2l+1} \sum_{\ell_1, m_1} \sum_{\ell_2, m_2} a_{\ell_1 m_1} a_{\ell_2 m_2} \begin{pmatrix} l & \ell_1 & \ell_2 \\ m & m_1 & m_2 \end{pmatrix} I_{\ell_1 \ell_2}^{l, \ell}, \quad (14)$$

and will treat the coefficients \mathcal{C}_ℓ^{lm} as uniquely given once we have a map. Of course, expression (14) is nothing more than the unbiased estimator of the angular-planar power spectrum (11) and—as long as cosmic variance is an issue—this “second order” approach we are adopting here (i.e., the prescription of adopting this estimator of the correlation function as our fundamental quantity, rather than the temperature field) is the best we can do when searching for statistical deviations of isotropy. In theory, it is also possible to use the CMB polarization induced by galactic clusters to probe different surfaces where CMB photons last scattered, and to use such independent measurements as a way to alleviate cosmic variance [31,32]. However, the gain in terms of a reduced variance is still limited.

Having these limitations in mind, we can now ask: how good does a theoretical model of anisotropy, $\mathcal{C}_\ell^{\text{th}, lm}$, fit the observational data once it is given by (14)? To answer this question we can use the well-known chi-square (χ^2) goodness-of-fit test, which in our case can be written in the following “generalized” form:

$$(\chi^2)_\ell^l \equiv \sum_{m=-l}^l \frac{|C_\ell^{lm} - C_\ell^{\text{th},lm}|^2}{(\sigma_\ell^{lm})^2},$$

in which the scales l and ℓ are seen as independent degrees of freedom, and where σ_ℓ^{lm} is just the standard deviation of the difference $(C_\ell^{lm} - C_\ell^{\text{th},lm})$. Although this expression can be readily applied to any theoretical model of anisotropy, in practice it is better to work with its reduced version

$$(\chi_\nu^2)_\ell^l \equiv \frac{1}{2l+1} \sum_{m=-l}^l \frac{|C_\ell^{lm} - C_\ell^{\text{th},lm}|^2}{(\sigma_\ell^{lm})^2}, \quad (15)$$

which is just the chi-square function divided by the $2l+1$ planar degrees of freedom.

Expression (15) will be the starting point of our statistical analysis, which we will pursue in detail in Sec. IV. Before we move on, it is important to choose a particular cosmological model of anisotropy against which we want to compare our estimator (14).

B. Λ CDM model

The most important model to be analyzed using the estimator (14) is, of course, the concordance Λ CDM model which was confirmed with striking accuracy by the 5 year release data set of the WMAP team [2]. For this model, statistical isotropy holds and, as we have shown, any multipolar coefficient C_ℓ^{lm} with nonzero planar dependence should be identically zero in this case [see expression (13)]. Therefore, for this model we can take

$$C_\ell^{\text{th},lm} \equiv 0, \quad (16)$$

where it should be clear that we are only considering the cases with $l \geq 2$ (as we will do from now on). Now, if the data under analysis are really Gaussian and SI, then their

covariance matrix can be explicitly calculated:

$$\begin{aligned} \mathcal{M}_\ell^{ll'mm'} &\equiv \langle (C_\ell^{lm} - C_\ell^{\text{th},lm})^* (C_\ell^{l'm'} - C_\ell^{\text{th},l'm'}) \rangle \\ &= \langle (C_\ell^{lm})^* C_\ell^{l'm'} \rangle = 8\pi^2 \sum_{\ell_1, \ell_2} C_{\ell_1} C_{\ell_2} (I_{\ell_1 \ell_2}^{l, \ell})^2 \delta_{ll'} \delta_{mm'}, \end{aligned}$$

where we have used (5) and the null hypothesis (16).

This covariance matrix has some interesting properties: first, we note that the planar degrees of freedom in (15) are really independent in this case. Moreover, the variance $(\sigma_\ell^{lm})^2 = \mathcal{M}_\ell^{llmm}$ becomes m -independent:

$$(\sigma_\ell^{lm})^2 \rightarrow (\sigma_\ell^l)^2 = 8\pi^2 \sum_{\ell_1, \ell_2} C_{\ell_1} C_{\ell_2} (I_{\ell_1 \ell_2}^{l, \ell})^2. \quad (17)$$

Second, its diagonal terms (i.e., σ_ℓ^l) are completely determined by the angular power spectrum C_ℓ , up to some geometrical coefficients which arise as a consequence of the way in which we split our CMB sky. This makes it possible to give a visual interpretation of the angular-planar power spectrum C_ℓ^{lm} , similar to that of the C_ℓ 's. For that, let us introduce the *reduced angular-planar spectrum*:

$$H_\ell^l \equiv \sqrt{\frac{2\ell+1}{2}} \sigma_\ell^l, \quad (18)$$

which has a simple interpretation when compared to the usual angular spectrum, because $H_\ell^0 = \sqrt{(2\ell+1)/2} \sigma_\ell^0 = C_\ell$, as can be easily shown using Eqs. (17) and (A5).

In Fig. 2 we show some plots of the reduced spectrum H_ℓ^l , both as a function of l and ℓ . Notice that, as a result of our planar splitting of the CMB sky, the low- ℓ sector of the spectrum H_ℓ^l is suppressed when we consider planes separated by smaller angles (bigger values of l). This is a consequence of the nontrivial coupling of the moments l ,

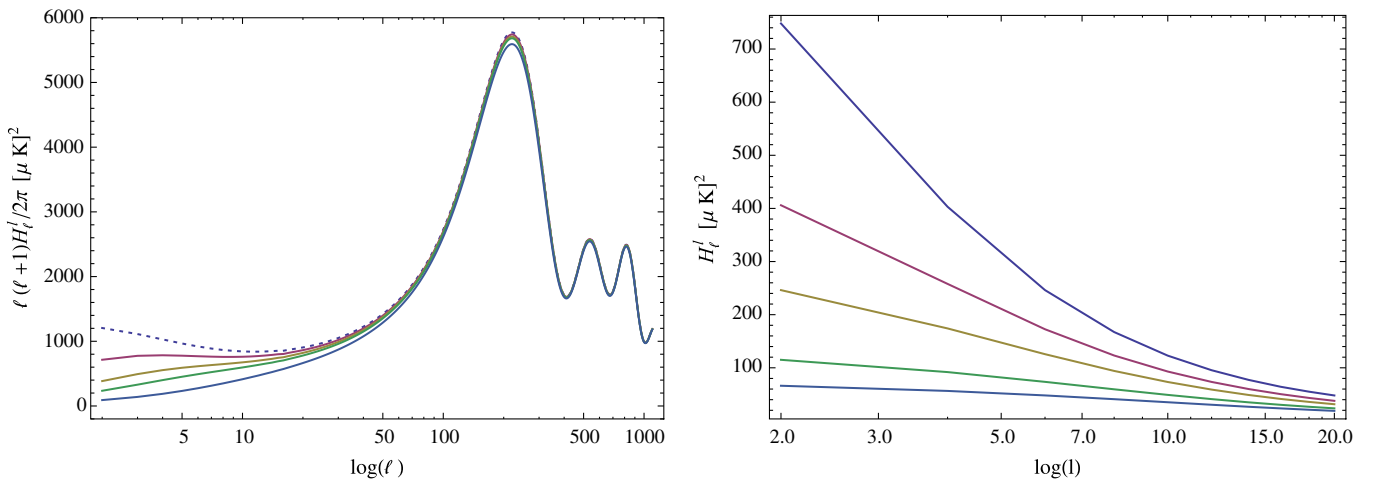


FIG. 2 (color online). Reduced angular-planar power spectrum H_ℓ^l for a fiducial Λ CDM model. The left panel shows the spectrum as a function of ℓ for the particular values (from top to bottom) $l = (0, 2, 4, 6, 12)$; the case $l = 0$ (dotted line) represents the standard angular power spectrum C_ℓ . The right panel shows the same spectra, but now as a function of l , for the particular values $\ell = (2, 3, 4, 6, 8)$.

ℓ_1 , and ℓ_2 : since the C_ℓ 's are roughly given by a monotonically decreasing sequence, and since $|l - \ell_1| \leq \ell_2 \leq l + \ell_1$, bigger values of l make the moment ℓ_2 probe deeper and deeper regions of the Sachs-Wolfe plateau. This suppression reaches cosmological scales up to the first acoustic peak, after which the planar dependence becomes negligible.

IV. χ^2 TEST OF STATISTICAL ANISOTROPY

We now come back to the question of how the angular-planar power spectrum C_ℓ^{lm} fits the observed Universe. We begin by showing that if we want to compare our data against the standard Λ CDM universe, then the chi-square function (15) becomes a very simple expression. As we have shown in the preceding section, for this model, $\sigma_\ell^{lm} = \sigma_\ell^l$ and $C_\ell^{th,lm} = 0$. Therefore (15) simplifies to

$$(\chi_\nu^2)_\ell^l = \frac{1}{2l+1} \sum_{m=-l}^l \frac{|C_\ell^{obs,lm}|^2}{(\sigma_\ell^l)^2}, \quad (19)$$

where $C_\ell^{obs,lm}$ is calculated by applying the estimator (14) to the data given by $a_{\ell m}^{obs}$. It is now clear that if the data under analysis are really Gaussian and statistically isotropic, then it should be true that

$$\langle (\chi_\nu^2)_\ell^l \rangle = \frac{1}{2l+1} \sum_{m=-l}^l \frac{\langle (C_\ell^{obs,lm})^* C_\ell^{obs,lm} \rangle}{(\sigma_\ell^l)^2} = 1.$$

This means that a positive test of planarity will be quantified by how far our chi-square function deviates from unity. We can do even better and define a new function as

$$\bar{\chi}_\ell^l \equiv (\chi_\nu^2)_\ell^l - 1, \quad (20)$$

which, if significantly different from zero, will point toward anisotropy.

It should be stressed that, for a given CMB map, the chi-square analysis must be done entirely in terms of that map's data. Indeed, any arbitrary introduction of a fiducial bias in (19) (for example, by calculating σ_ℓ^l using $C_\ell^{\Lambda\text{CDM}}$) would only include our *a priori* prejudices about what the map's anisotropies should look like. The angular spectrum C_ℓ , being by construction a measure of statistical isotropy, can only be said to be small/big when compared to a particular cosmological model (for example, the Λ CDM model). Consequently, an anomalous detection of C_ℓ is by no means a measure of statistical anisotropy, and it is this value that should be used to calculate σ_ℓ^l if we want to find deviations of isotropy, regardless of how high/low it is. Note also that while the function $\bar{\chi}_\ell^l$ has some "isotropy variance" which could be computed for the Λ CDM model from first principles, in practice it is much easier to simulate many realizations of a Gaussian and isotropic random field to obtain that variance.

Finally, we would like to mention that although each number $\bar{\chi}_\ell^l$ is an individual measure of anisotropy (i.e., planarity), a consistently biased set of values over a range of l 's or ℓ 's can also be seen as an indication of anisotropy, even if all individual $\bar{\chi}_\ell^l$'s in that range are well within their variance limits.

Following the prescription outlined above, we applied the estimator (20) to the 5 year WMAP full-sky data (also known as the ILC5 map) where, for practical reasons, we

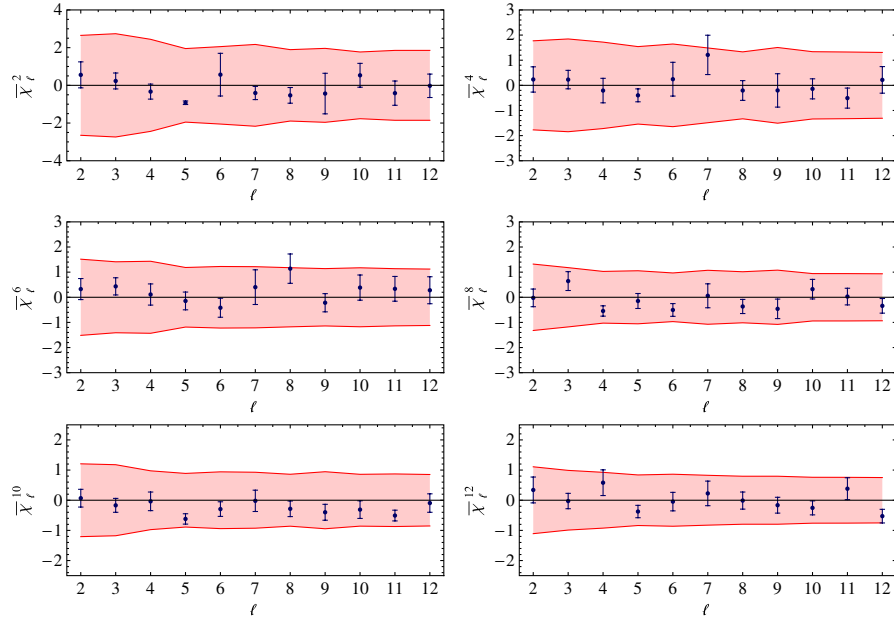


FIG. 3 (color online). Anisotropic angular-planar estimator applied to the WMAP ILC5 data. The panel shows $\bar{\chi}_\ell^l$ as a function of ℓ , for the particular values $l = (2, 4, 6, 8, 10, 12)$. The shaded region and error bars represent 2σ cosmic variance and systematical (foreground) contaminations, respectively.

TABLE I. Full-sky WMAP maps used in our analysis to estimate the error bars of Fig. 3. The data points in this (and subsequent) figure was calculated using only the ILC5 map [1,34].

Full-sky maps	References
Hinshaw <i>et al.</i>	[1,34]
de Oliveira-Costa <i>et al.</i>	[35]
Kim <i>et al.</i>	[36]
Park <i>et al.</i>	[37]
Delabrouille <i>et al.</i>	[38]

have restricted our analysis to the range of values $\ell \in [2, 12]$ and $l \in [2, 12]$ (notice that the momenta l can only assume even values). Our results are presented in Fig. 3, where we keep the momenta l fixed, and vary the momenta related to angular separation, ℓ . As discussed

before, for this range of values cosmic variance dominates over other sources of noise. We estimated the effects of cosmic variance by running a simulation of 10^3 realizations of this estimator, using the best-fit (theoretical) scalar C_ℓ 's made available in [33]; this corresponds to the shaded area in Fig. 3.

It is also important to explain that, while the data points in Fig. 3 were calculated using the ILC5 map *alone*, we have also included in our analysis a rough estimate of the possible residual foreground contamination present in the data. This was done by computing the sample variances of the full-sky maps shown in Table I, which were then used as error bars. In other words, the error bars in Fig. 3 *do not* account for instrumental noise, which is believed to be under control at these angular scales.

Theses figures present some peculiarities: first, we notice that the magnitudes of the error bars oscillate for the smallest values of ℓ . As we mentioned in Sec. III, this is

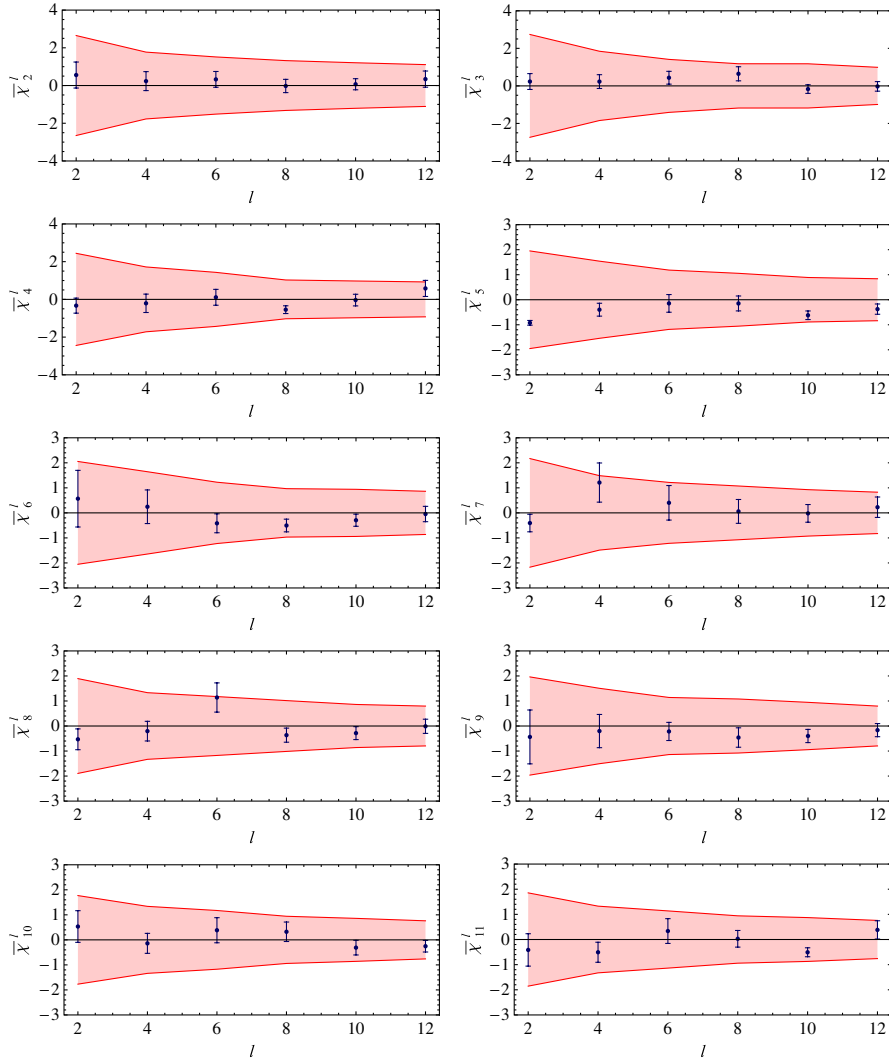


FIG. 4 (color online). Anisotropic angular-planar estimator applied to the WMAP ILC5 data. The panel shows $\bar{\chi}_\ell^l$ as a function of l , for the particular values $\ell = (2, 3, 4, 5, 6, 7, 8, 9, 10, 11)$. Note that for fixed l , the error bars do not oscillate (see the text for more details).

partially a consequence of the $3J$ symbols, which are weights appearing in the definition of the anisotropic power spectra and whose effect is to couple differently odd and even multipoles. The second peculiarity is that, in all these figures, the modulations of the quadrupole moment $\ell = 2$ are entirely consistent with zero. This result suggests that the low value of the quadrupole C_2 is perhaps not a consequence of statistical anisotropy, at least for the test we are considering here. Note also that the octupole $\ell = 3$, which has been reported as unusually planar by some groups, grows slightly from $l = 4$ to $l = 8$, although it is compatible with cosmic variance in all the planar range considered.

In what concerns deviations of isotropy, our analysis shows that the most “anomalous” scales are in the sectors $(l, \ell) = (4, 7)$ and $(6, 8)$, where we can see that the points $\bar{\chi}_7^4$ and $\bar{\chi}_8^6$ are only marginally allowed by the 2σ cosmic variance area.

In order to make the visualization of the above figures easier, we repeat the analysis but now keeping the angular separation ℓ fixed and varying the planar separation l . The result is shown in Fig. 4. Notice that the planar modulations of the quadrupole $\ell = 2$ are consistently positive, but always compatible with zero. We can also see in these figures the growing behavior in the octupole $\ell = 3$ from $l = 4$ to $l = 8$ as mentioned before.

V. CONCLUSIONS

We have investigated the minimum statistical framework of modern cosmology by enlarging the domain of the two-point correlation function to admit not only the usual angular dependence, but a directional (planar) dependence as well. Our observable, the anisotropic angular-planar power spectrum, can account not only for the usual angular separation between any two spots in a CMB map, but also for any planar signature that this map might have. Besides having a strong observational motivation, an interesting feature of this approach is that it leads naturally to an unbiased estimator of statistical anisotropy, in the same spirit as is done with the multipolar temperature coefficients $a_{\ell m}$ ’s and their associated estimator \hat{C}_ℓ . As an example of its use, we have applied this estimator to a concrete model of cosmology, i.e., the Λ CDM model, where we have shown that under the hypothesis of Gaussianity and statistical isotropy, the angular-planar power spectra have zero mean, but of course, nonzero covariance.

By means of a simple chi-square analysis, we have also applied our estimator of planar anisotropy to the WMAP ILC5 data, where we found that the planar modulations of the quadrupole $\ell = 2$ are compatible with the null hypothesis over the range of planar momenta $l \in [2, 12]$ we probed. Our results suggest that the low value of the quadrupole C_2 is perhaps due to some local physics, and

not to deviations of statistical isotropy, at least as far as planar modulations are concerned.

Our analysis has also shown that the angular scales $\ell = 7$ and $\ell = 8$ suffer some degree of modulation around the planar scales $l = 4$ and $l = 6$, respectively. This could be an indication of some foreground contamination coming from a planar region of typical size $\Delta l = 4 \sim 6$. However, a complete treatment of the sources of errors and the effect of masks is needed before we can reach a more definitive conclusion—that analysis is under way [30].

From a theoretical perspective, our techniques can be readily applied to any particular model of inflation predicting a specific anisotropic shape for the matter power spectrum. Because of the generality and simplicity of our formulas, the angular-planar power spectrum can also be used to analyze CMB polarization. Other possible applications include stacked maps of cosmic structure, such as the galaxy cluster catalog 2Mass [39].

We finally mention that, although in this work we have focused on testing isotropy while keeping within the Gaussian framework, our tools can also be used to search for deviations from Gaussianity in a completely model-independent way.

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APPENDIX A

1. Derivation of (11)

We will present here the details of the derivation of expression (11). We start by equating expressions (9) and (2)

$$\begin{aligned} \sum_{\ell} \sum_{l,m} \frac{2\ell+1}{\sqrt{4\pi}} C_{\ell}^{lm} P_{\ell}(\cos\vartheta) Y_{lm}(\hat{\mathbf{n}}) \\ = \sum_{\ell_1, m_1} \sum_{\ell_2, m_2} \langle a_{\ell_1 m_1} a_{\ell_2 m_2}^* \rangle Y_{\ell_1 m_1}(\hat{\mathbf{n}}_1) Y_{\ell_2 m_2}(\hat{\mathbf{n}}_2). \end{aligned} \quad (\text{A1})$$

As mentioned in the main text, the inversion of C_{ℓ}^{lm} as a function of the $a_{\ell m}$ ’s is not a trivial task, since the vector \mathbf{n} depends nonlinearly on the vectors $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$. The easiest way to achieve this goal is to pick up a coordinate system where only the ϑ dependence (i.e., the modulus of the vector \mathbf{n}) is present. After integrating it out, we rotate our coordinate system using three Euler angles to recover the (θ, ϕ) dependence, which can then be integrated with the

help of some Wigner matrices identities. We therefore start by positioning the vectors \hat{n}_1 and \hat{n}_2 in the xy plane, i.e., we chose $\hat{n}_1 = (\pi/2, \phi_1)$, $\hat{n}_2 = (\pi/2, \phi_2)$. By (10) we then have $\cos\vartheta = \cos(\phi_1 - \phi_2)$. Using the relation [40]

$$Y_{\ell m}(\pi/2, \phi) = \lambda_{\ell m} e^{im\phi}, \quad \text{where } \lambda_{\ell m} = \begin{cases} (-1)^{(\ell+m)/2} \sqrt{\frac{2\ell+1}{4\pi}} \frac{(\ell+m-1)!!}{(\ell+m)!!} \frac{(\ell-m-1)!!}{(\ell-m)!!} & \text{if } \ell + m \in 2\mathbb{N} \\ 0 & \text{otherwise} \end{cases} \quad (\text{A2})$$

and $\mathbf{n} = (\sin\vartheta, 0, 0)$, we can integrate the ϑ dependence on both sides of (A1). This gives us

$$\frac{1}{\sqrt{\pi}} \sum_{l,m} C_{\ell}^{lm} Y_{lm}(0, 0) = \sum_{\ell_1, m_1} \sum_{\ell_2, m_2} \langle a_{\ell_1 m_1} a_{\ell_2 m_2}^* \rangle I_{\ell_1 m_1 \ell_2 m_2}^{\ell}, \quad (\text{A3})$$

where we have introduced the following definition:

$$I_{\ell_1 m_1 \ell_2 m_2}^{\ell} \equiv -\lambda_{\ell_1 m_1} \lambda_{\ell_2 m_2} \int_0^{\pi} P_{\ell}(\cos(\phi_1 - \phi_2)) \times e^{i(m_1 \phi_1 - m_2 \phi_2)} d(\cos(\phi_1 - \phi_2)). \quad (\text{A4})$$

We need now to integrate out the θ and ϕ dependence in the right-hand side of (A3) which was hidden due to our choice of a particular coordinate system. In order to do that, we keep the vectors \hat{n}_1 and \hat{n}_2 fixed and make a rotation of our coordinate system using three Euler angles $\omega = \{\alpha, \beta, \gamma\}$. This rotation changes the coefficients C_{ℓ}^{lm} 's and $a_{\ell m}$'s according to

$$a_{\ell m} = \sum_{m'} D_{mm'}^{\ell}(\omega) \tilde{a}_{\ell m'}, \quad C_{\ell}^{lm} = \sum_{m'} D_{mm'}^{\ell}(\omega) \tilde{C}_{\ell}^{lm'},$$

where \tilde{C}_{ℓ}^{lm} and $\tilde{a}_{\ell m}$ are the multipolar coefficients in the

new coordinate system and where $D_{mm'}^{\ell}(\omega)$ are the elements of the Wigner rotation matrix. The advantage of positioning the vectors \hat{n}_1 and \hat{n}_2 in the plane xy is that now the angles θ and ϕ are given precisely by the Euler angles β and γ , regardless of the value of α

$$\begin{aligned} \sum_{l,m} C_{\ell}^{lm} Y_{lm}(0, 0) &= \sum_{l,m'} \tilde{C}_{\ell}^{lm'} \left(\sum_m D_{mm'}^{\ell}(\alpha, \beta, \gamma) Y_{lm}(0, 0) \right) \\ &= \sum_{l,m'} \tilde{C}_{\ell}^{lm'} Y_{l,-m}(\beta, \gamma), \end{aligned}$$

where in the last step we have used $Y_{lm}(0, 0) = \sqrt{(2l+1)/4\pi} \delta_{m0}$. Therefore, in our new coordinate system we have (dropping the “ \sim ” in our notation)

$$\begin{aligned} \frac{1}{2\pi} \sum_{l,m} C_{\ell}^{lm} D_{0m}^{\ell}(\omega) \sqrt{2l+1} \\ = \sum_{\ell_1, m_1} \sum_{\ell_2, m_2} \langle a_{\ell_1 m_1} a_{\ell_2 m_2}^* \rangle \sum_{m'_1 m'_2} I_{\ell_1 m'_1 \ell_2 m'_2}^{\ell} D_{m'_1 m'_2}^{\ell_1}(\omega) D_{m'_2 m'_1}^{\ell_2}(\omega). \end{aligned}$$

We may now isolate C_{ℓ}^{lm} using the identities [40]

$$\begin{aligned} \int d\omega D_{m_1 m'_1}^{\ell_1}(\omega) D_{m_2 m'_2}^{\ell_2}(\omega) &= \frac{8\pi^2}{2l_1+1} \delta_{l_1 l_2} \delta_{m_1 m_2} \delta_{m'_1 m'_2} \\ \int d\omega D_{m'_1 m_1}^{\ell_1}(\omega) D_{m'_2 m_2}^{\ell_2}(\omega) D_{m'_3 m_3}^{\ell_3}(\omega) &= 8\pi^2 \begin{pmatrix} l_1 & l_2 & l_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \end{aligned}$$

where $d\omega = \sin\beta d\beta d\alpha d\gamma$, to obtain

$$\frac{1}{\sqrt{2l+1}} C_{\ell}^{lm} = 2\pi \sum_{\ell_1, m_1} \sum_{\ell_2, m_2} \langle a_{\ell_1 m_1} a_{\ell_2 m_2}^* \rangle \sum_{m'_1 m'_2} I_{\ell_1 m'_1 \ell_2 m'_2}^{\ell} (-1)^{m_2+m'_2+m} \begin{pmatrix} \ell_1 & \ell_2 & l \\ m'_1 & -m'_2 & 0 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & l \\ m_1 & -m_2 & -m \end{pmatrix}.$$

If we now do the redefinitions

$$-m_2 \rightarrow m_2, \quad -m \rightarrow m, \quad C_{\ell}^{lm} \rightarrow (-1)^m C_{\ell}^{l,-m}$$

and note that the first 3J symbol above is identically zero unless $m'_1 = m'_2$, we obtain finally (11).

2. Useful identities

We present here some useful identities related to the 3J symbols.

(i) Isotropic limit:

$$\begin{pmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \end{pmatrix} = \frac{(-1)^{l_2-m_1}}{\sqrt{2l_1+1}} \delta_{l_1 l_2} \delta_{m_1, -m_2}.$$

(ii) Parity and permutations:

$$\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} = \begin{pmatrix} l & l_1 & l_2 \\ m & m_1 & m_2 \end{pmatrix} = (-1)^{l_1+l_2+l} \begin{pmatrix} l_2 & l_1 & l \\ m_2 & m_1 & m \end{pmatrix} = (-1)^{l_1+l_2+l} \begin{pmatrix} l_1 & l_2 & l \\ -m_1 & -m_2 & -m \end{pmatrix}.$$

(iii) Orthogonality:

$$\begin{aligned} \sum_{m_1=-l_1}^{l_1} \sum_{m_2=-l_2}^{l_2} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l'_3 \\ m_1 & m_2 & m'_3 \end{pmatrix} &= \frac{\delta_{l_3 l'_3} \delta_{m_3 m'_3}}{2l_3 + 1} \\ \sum_{l_1=|l_2-l_3|}^{l_2+l_3} \sum_{m_1=-l_1}^{l_1} (2l+1) \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_4 \\ m_1 & m'_2 & m'_3 \end{pmatrix} &= \delta_{m_2 m'_2} \delta_{m_3 m'_3} \\ \sum_{m=-l}^l (-1)^{l-m} \begin{pmatrix} l & l & \ell \\ m & -m & 0 \end{pmatrix} &= \sqrt{2l+1} \delta_{\ell,0}. \end{aligned}$$

The last expression is particularly useful in the derivation of (13).

3. Some properties of the integral (12)

The geometrical coefficients $I_{\ell_1 \ell_2}^{l, \ell}$ defined in (12) have many interesting properties which can be explored in order to speed up numerical computation of (11). First, we note that it is symmetric under permutation of ℓ_1 and ℓ_2

$$\begin{aligned} I_{\ell_1 \ell_2}^{l, \ell} &= \sum_m I_{\ell_1 m \ell_2 m}^{\ell} (-1)^m \begin{pmatrix} \ell_1 & \ell_2 & l \\ m & -m & 0 \end{pmatrix} = \sum_m I_{\ell_2 m \ell_1 m}^{\ell} (-1)^{m+\ell_1+\ell_2+l} \begin{pmatrix} \ell_2 & \ell_1 & l \\ -m & m & 0 \end{pmatrix} \\ &= \sum_m I_{\ell_2 m \ell_1 m}^{\ell} (-1)^{m+2(\ell_1+\ell_2+l)} \begin{pmatrix} \ell_2 & \ell_1 & l \\ m & -m & 0 \end{pmatrix} = I_{\ell_2 \ell_1}^{l, \ell}. \end{aligned}$$

Some of the other properties are a consequence of the integral $I_{\ell_1 m \ell_2 m}^{\ell}$ defined in (A4). We may note, for example, that, due to the symmetry of the $\lambda_{\ell m}$ coefficient defined in (A2), we will have

$$I_{\ell_1 \ell_2}^{l, \ell} = 0, \quad \text{for any } \{(\ell_1, \ell_2) \in \mathbb{N} | \ell_1 + \ell_2 = \text{odd}\}.$$

Furthermore, the $\lambda_{\ell m}$ coefficients restrict the m summation above to their values which obey $m + \ell_1 + \ell_2 = \text{even}$. If we further notice that (A4) is proportional to the integral of an integral of the form $\int_{-1}^1 P_{\ell}(\cos\theta) \cos m\theta d\theta$, and that this integral is zero unless $\ell + m = \text{even}$, we conclude that

$$I_{\ell_1 \ell_2}^{l, \ell} = 0, \quad \text{for any } \{(\ell_1, \ell_2, \ell) \in \mathbb{N} | \ell_1 + \ell_2 + \ell = \text{odd}\}.$$

Besides, using the fact that the integral $\int_{-1}^1 P_{\ell}(\cos\theta) \times \cos m\theta d\theta$ is zero for any $m < \ell$, we find

$$I_{\ell_1 \ell_2}^{l, \ell} = 0, \quad \text{for any } \{(\ell_1, \ell_2, \ell) \in \mathbb{N} | \ell_1 < \ell, \ell_2 < \ell\}.$$

We finally comment on the special case where $l = 0$, for which we have

$$I_{\ell_1 \ell_2}^{0, \ell} = \frac{(-1)^{\ell_1}}{\sqrt{2\ell_1 + 1}} \left(\sum_m I_{\ell_1 m \ell_1 m}^{\ell} \right) \delta_{\ell_1 \ell_2}.$$

However

$$\begin{aligned} \sum_{m=-\ell'}^{\ell'} I_{\ell' m \ell' m}^{\ell} &= \int_0^{\pi} P_{\ell}(\cos\vartheta) \left(\sum_{m=-\ell'}^{\ell'} \frac{(2\ell' + 1)}{4\pi} \frac{(\ell' + m - 1)!!}{(\ell' + m)!!} \frac{(\ell' - m - 1)!!}{(\ell' - m)!!} e^{im\vartheta} \right) d(-\cos\vartheta), \\ \ell' + m = \text{even} &= \frac{2\ell' + 1}{4\pi} \int_{-1}^1 P_{\ell}(x) P_{\ell'}(x) dx = \frac{1}{2\pi} \delta_{\ell \ell'}. \end{aligned}$$

where in the derivation above we have made use of the Fourier series expansion of the Legendre polynomial. So we conclude that

$$I_{\ell_1 \ell_2}^{0, \ell} = \frac{(-1)^{\ell_1}}{2\pi\sqrt{2\ell_1 + 1}} \delta_{\ell \ell_1} \delta_{\ell_1 \ell_2}, \quad (\text{A5})$$

which is needed in the derivation of (13) and (18).

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