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Persistence of periodic orbits from planar piecewise linear Hamiltonian differential systems with two or three zones

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Ronisio Moises Ribeiro

Persistence of periodic orbits from planar piecewise linear Hamiltonian differential systems with two or three zones

Tese apresentada como parte dos requisitos para obtenção do título de Doutor em Matemática, junto ao Programa de Pós-Graduação em Matemática, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista “Júlio de Mesquita Filho”, Câmpus de São José do Rio Preto.

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A minha mãe Sônia;

A minha família e amigos;

A meus mestres;

Dedico.

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"Feliz aquele que transfere o que sabe e aprende o que ensina."

(Cora Coralina, 1983, p.136)

Resumo

Neste trabalho, nosso objetivo é estimar o número de ciclos limites do tipo costura em sistemas diferenciais Hamiltonianos lineares por partes planares com duas ou três zonas separadas por retas de modo que os sistemas lineares que definem o por partes têm pontos singulares isolados, ou seja, centros ou selas. Mais precisamente, começaremos determinando o número de ciclos limites de sistemas diferenciais Hamiltonianos lineares por partes contínuos ou descontínuos com duas ou três zonas. Neste caso, mostraremos que se o sistema for descontínuo com três zonas, então ele tem no máximo um ciclo limite, e forneceremos exemplos com um ciclo limite. Em seguida, estimaremos o número de ciclos limites que podem bifurcar de um anel de órbitas periódicas de um sistema diferencial Hamiltoniano linear descontínuo por partes com três zonas, após perturbações polinomiais de grau n, para $n = 1, 2, 3$. Para estes casos, denotando por $H(n)$ o número de ciclos limites que podem bifurcar do anel de órbitas periódicas do sistema, provaremos que se o sistema diferencial linear definido na região entre as duas retas paralelas (chamado de subsistema central) possui um centro na origem e os demais subsistemas possuem centros ou selas, então $H(1) \geq 3$, $H(2) \geq 4$ e $H(3) \geq 7$. Agora, para o caso particular em que o subsistema central possui um centro e os demais subsistemas possuem apenas selas reais, se o centro for real (não necessariamente na origem) ou se estiver sobre a fronteira da região central, então $H(1) \geq 6$, e se for virtual, então $H(1) \geq 4$. Finalmente, se o subsistema central possui uma sela real e os demais subsistemas possuem centros ou selas, então $H(1) \geq 6$. Para isso, estudaremos o número de zeros de suas funções de Melnikov definidas em duas e três zonas. Além disso, fornecemos métodos analíticos detalhados para estudar o número de zeros das funções de Melnikov.

Palavras-chave: Ciclo limite. Sistema diferencial linear por partes. Sistema Hamiltoniano. Função de Melnikov.

Abstract

In this work, our goal is estimate the number of crossing limit cycles in planar piecewise linear Hamiltonian differential systems with two or three zones separated by straight lines such that the linear systems that define the piecewise one have isolated singular points, that is, centers or saddles. More precisely, we will start with the study of the number of limit cycles for continuous or discontinuous piecewise linear Hamiltonian differential systems with two or three zones. In this case, we show that if the system is discontinuous with three zones then it has at most one limit cycle, and we will provide examples with one limit cycle. Next, we will estimate the number of limit cycles that can bifurcate from a periodic annulus in a discontinuous piecewise linear Hamiltonian differential system with three zones, after polynomials functions perturbations of degree n, *for* $n = 1, 2, 3$. *For theses cases, denoting by* $H(n)$ *the number of limit cycles that can bifurcate from this periodic annulus, we prove that if the linear differential system defined in the region between the two parallel lines (called of central subsystem) has a center at the origin and the others subsystems have centers or saddles then* $H(1) \geq 3$, $H(2) \geq 4$ *and* $H(3) \geq 7$. *Now, for the particular case where the central subsystem has a center and the others subsystems have only real saddles, if the central subsystem has a real (not necessarily at the origin) or boundary center then* $H(1) \geq 6$ *and if it has a virtual center then* $H(1) \geq 4$. *Finally, if the central subsystem has a real saddle and the others subsystems have centers or saddles then* $H(1) \geq 6$. *For this, we study the number of zeros of its Melnikov functions defined in two or three zones. Moreover, we provide detailed analytical methods to study the number of zeros from Melnikov functions defined in two or three zones.*

Keywords: Limit cycle. Piecewise linear differential system. Hamiltonian system. Melnikov function.

List of Figures

List of Tables

Summary

Introduction

The qualitative theory of differential systems is important tools for investigate properties inherent to the solutions of the equations that define the problem, without worrying about the possible expressions that such solutions may have. One of the most famous problems of this theory, formulated for the planar case, consist in to determine the number of limit cycles of polynomial differential systems, which was proposed by Hilbert in 1900 as part of Hilbert's 16th problem, (see [25]). Nowadays, this problem has been considered for piecewise differential systems. The study of piecewise differential systems begins with the pioneering work of Andronov [1]. After Filippov [12] establishes the conventions that regulate the transitions of solutions of systems between different regions, piecewise differential systems have piqued the attention of researchers in qualitative theory of differential equations, mainly because many phenomena, for instance in mechanics, electrical circuits, control theory, neurobiology, etc, can be described by models that involve this kind of differential equations (see the books [10, 23, 48] and the papers [7, 13, 45, 47]).

Piecewise linear differential systems are an interesting class of piecewise differential systems and, unlike the smooth case, have a rich dynamic that is far from being fully understood. In addition to numerous applications in various areas of knowledge. In 1990, Lum and Chua [8] conjectured that a continuous piecewise differential systems in the plane with two zones has at most one limit cycle. In 1998 this conjecture was proved by Freire, Ponce, Rodrigo and Torres in [15]. On the other hand, when the piecewise linear differential system is discontinuous, that is, the subsystems do not coincide on the switching curve, is known that the maximum number of limit cycles is at least three. Apparently, for it case, determining the exact number of limit cycles is a hard task. However, important partial results about this problem have been obtained. In summary, the results about the number of limit cycles of discontinuous piecewise linear differential systems with two zones separated by a straight line are given in Table . The symbol "—" indicates that those cases appear repeated in the table and the empty entries on it correspond to cases not studied in the literature, at least as far as we know.

	F_r	F_v	F_b	S_r	S_r^0	N_v	iN_v	C	C_b
F_r		3	2^*	3	2^*	3	3	2^*	2^*
F_v	—	2	2^*	2	1^*		2	1^*	1^*
F_b	—	—	1^*	2^*	1^*	2^*	2^*	1^*	1^*
S_r	—	—	—	2^*	1^*	2	2	1^*	1^*
S_r^0	—	—	—	—	0^*	1^*	1^*	0^*	0^*
N_v	—	—	—	—	—			1^*	1^*
iN_v	—	—	—	—	—	—	2	1^*	1^*
C	—	—	—	—	—	—	—	0^*	0^*
C_b	—	—	—	—	—	—	—	—	0^*

Table 1: Lower bounds (Upper bounds*) of the maximum number of limit cycles of discontinuous piecewise linear differential systems with two zones separated by a straight line. Here F_r, F_v, F_b, S_r, S_r^0, N_v, iN_v, C and C_b denote real focus, virtual focus, boundary focus, real saddle, real saddle with zero trace, virtual node, improper node, center and boundary center, respectively.

We denote the lower bounds in the entrances from Table by the symbols that indicate its position on the table. For example, the lower bound for the case with a real focus F_r and a virtual focus F_v is detonated by F_rF_v, that is, $F_rF_v = 3$. A proof for the lower bound F_rF_v can be found in [32]. A proof for the lower bound F_rS_r can be found in [29]. A proof for the lower bounds F_rN_v and F_riN_v can be found in [17]. A proof for the lower bound F_vF_v can be found in [16]. A proof for the lower bound F_vS_r can be found in [55]. A proof for the lower bound F_viN_v can be found in [56]. A proof for the upper bound S_rS_r can be found in [2]. A proof for the lower bounds S_rN_v and S_riN_v can be found in [33]. A proof for the lower bound iN_viN_v can be found in [27]. The other cases listed in Table can be found in [35]. In the papers [3, 5, 18, 41, 54] we can also find proofs for some lower bounds of Table .

If the curve between two linear zones is not a straight line it is possible to obtain as many cycles as you want. This fact has been conjectured by Braga and Mello in [4] and firstly proved by Novaes and Ponce in [49]. Exact number of limit cycles, for discontinuous piecewise linear systems with two zones separated by a straight line, were obtained in particular cases. Llibre and Teixeira [39] proved that if the linear systems, that define the piecewise one, has no singular point, then it has at most one limit cycle. Medrado and Torregrossa [46] proved that if the straight line has only crossing sewing points and the piecewise linear system has only a monodromic singular point on it, then the system has at most one limit cycle.

Recently, piecewise differential system defined in regions with more than two zones have attracted the attention of researchers, see for instance [11, 26, 36, 53, 60]. Results imposing restrictive hypotheses on the systems, such as symmetry and linearity, have been obtained.

For instance, conditions for nonexistence and existence of one, two or three limit cycles for symmetric continuous piecewise linear differential systems with three zones can be found in [34]. Now, for the nonsymmetric case, examples with two limit cycles surrounding a unique singular point at the origin was found in [38, 42]. When we remove these restrictions, that is, when we consider piecewise discontinuous differential system with three zones, there are few works in the literature. In fact, the works available deal with planar piecewise linear near-Hamiltonian differential systems with three zones, given by

$$\begin{cases} \dot{x} = H_y(x,y) + \epsilon f(x,y), \\ \dot{y} = -H_x(x,y) + \epsilon g(x,y), \end{cases} \tag{1}$$

with

$$H(x,y) = \begin{cases} H^L(x,y) = \dfrac{b_L}{2}y^2 - \dfrac{c_L}{2}x^2 + a_L xy + \alpha_L y - \beta_L x, \ x \leq -1, \\ H^C(x,y) = \dfrac{b_C}{2}y^2 - \dfrac{c_C}{2}x^2 + a_C xy + \alpha_C y - \beta_C x, \ -1 \leq x \leq 1, \\ H^R(x,y) = \dfrac{b_R}{2}y^2 - \dfrac{c_R}{2}x^2 + a_R xy + \alpha_R y - \beta_R x, \ x \geq 1, \end{cases}$$

$$f(x,y) = \begin{cases} f_L(x,y), & x \leq -1, \\ f_C(x,y), & -1 \leq x \leq 1, \\ f_R(x,y), & x \geq 1, \end{cases}$$

$$g(x,y) = \begin{cases} g_L(x,y), & x \leq -1, \\ g_C(x,y), & -1 \leq x \leq 1, \\ g_R(x,y), & x \geq 1, \end{cases}$$

where the functions f_i and g_i are C^∞, for $i = L, C, R$, and $0 \leq \epsilon << 1$. When $\epsilon = 0$ (shortly $(1)|_{\epsilon=0}$) we say that system (1) is a piecewise linear Hamiltonian differential system. We call system (1) of *left subsystem* for $x \leq -1$, *right subsystem* for $x \geq 1$ and *central subsystem* for $-1 \leq x \leq 1$.

In [40], Llibre and Teixeira study the existence of limit cycles for system $(1)|_{\epsilon=0}$ when the subsystems that define the piecewise one have a unique singular point, which is a center. More precisely, in the continuous case, they prove that system $(1)|_{\epsilon=0}$ has no limit cycles. Now, in the discontinuous case, system $(1)|_{\epsilon=0}$ has at most one limit cycle and there are examples with one. Fonseca, Llibre and Mello, in [14], proved that if system $(1)|_{\epsilon=0}$ is discontinuous and without singular points, then the maximum number of limit cycles is also one. Many authors have contributed to estimate the number of limit cycles of system (1) that can bifurcate, after polynomial perturbations, from the periodic annulus of the unperturbed system $(1)|_{\epsilon=0}$. For instance, in [59] the authors showed that at least seven limit cycles can bifurcate from a periodic annulus, after linear perturbations, from system $(1)|_{\epsilon=0}$ with subsystems without singular points and a boundary pseudo–focus. On the other hand, in [61], 5, 7 and 12 limit cycles were obtained by linear, quadratic and cubic polynomials perturbations, respectively. But, in this paper, the periodic annulus of the unperturbed system $(1)|_{\epsilon=0}$ was obtained from a real saddle in the central subsystem and two virtual centers in the others subsystems. For the same type of periodic annulus, in [58], 10 limit cycles were obtained through cubic perturbations.

The search for upper bound for the number of limit cycles that a piecewise linear system with three zones can have is what motivates most of the recently works found in the literature about this topic. However, all cases are interesting in themselves, that is, the search for better quotas for the number of limit cycles can not be used to discourage the study of particular families. So the type of singular points of the subsystems and their positions, that is, whether they are real, virtual or boundary, are important questions in this study and should be considered for all subclasses of piecewise linear systems with three zones. Motivated by the previously mentioned works, in this thesis we contribute along these study lines. Our goal is to estimate the number of crossing limit cycles that can bifurcate from periodic annulus of piecewise linear near-Hamiltonian differential systems with two or three zones. For this purpose, we have the text organized as follows.

In Chapter 1, we introduce some definitions and basic results about piecewise smooth vector fields with two or three zones that will be essential in the development of this work. Moreover, in Section 1.3 we will present the Melnikov function method, that is an important tool to investigate the number of limit cycles that emerge from a periodic annulus of a piecewise Hamiltonian differential system.

In Chapter 2, motivated by the paper [40], we study the existence of crossing limit cycles in continuous and discontinuous planar piecewise linear Hamiltonian differential system with two or three zones separated by straight lines, and parallel if there are more than one, such that the linear systems that define the piecewise one have isolated singular points, that is, centers or saddles. In this case, we show that if the planar piecewise linear Hamiltonian differential system is either continuous or discontinuous with two zones or continuous with three zones, then it has no crossing limit cycles. Now, if the planar piecewise linear Hamiltonian differential system is discontinuous with three zones, that is, if system $(1)|_{\epsilon=0}$ is discontinuous, then it has at most one crossing limit cycle, and there are examples with one limit cycle. More precisely, without taking into account the position of the singular points in the zones, we present examples with the unique crossing limit cycle for all possible combinations of saddles and centers. This chapter is the content of paper I of the list of published and submitted papers derived from this thesis available at the end of this introduction.

In Chapter 3, we study the number of crossing limit cycles that can bifurcated, after linear perturbations, from a periodic annulus formed by periodic orbits of system $(1)|_{\epsilon=0}$ that passes through the three zones. We prove that if the central subsystem has a center at the origin and the others subsystems have centers or saddles, then we have at least three crossing limit cycles visiting the three zones. Our results are obtained by studying the number of zeros of the first order Melnikov function associated to system (1). Moreover, we will present, in Section 1, a normal form for system $(1)|_{\epsilon=0}$ in order to simplify the compute. This normal form will also be used in Chapters 4 and 5. This study resulted in paper II of the list of published and submitted papers derived from this thesis.

In Chapter 4, motivated by the results obtained in Chapter 3, we estimated the lower bounds for the number of crossing limit cycles that can bifurcate from periodic annulus of piecewise system (1), assuming similar hypotheses to system $(1)|_{\epsilon=0}$ from Chapter 3, but with the difference that we perturb system $(1)|_{\epsilon=0}$ by polynomial functions of degree n, for $n = 2, 3$. Denoting by $H(n)$ the number of crossing limit cycles that can bifurcate from this periodic annulus, we prove that $H(2) \geq 4$ and $H(3) \geq 7$. A part of these results are published in the paper III of the list of published and submitted papers derived from this thesis.

In Chapter 5, we study the number of crossing limit cycles that can bifurcated, after linear perturbations, from periodic annulus of system $(1)|_{\epsilon=0}$, assuming that the central subsystem has a center and the other subsystems have only saddles. What differs the results of this chapter from the one of Chapter 3 is that now the center of the central subsystem is not fixed at the origin and therefore, we will have periodic annulus formed by crossing periodic orbits passing through two and three zones. We prove that if the central subsystem from system $(1)|_{\epsilon=0}$ has a real or a boundary center, then the maximum number of limit cycles that bifurcate from the periodic annulus is at least six. Four passing through the three zones and two passing through two zones. Now, if the central subsystem has a virtual center, then we have at least four limit cycles bifurcating from the periodic annulus, three passing through the three zones and one passing through two zones. This study resulted in preprint IV of the list of published and submitted papers derived from this thesis.

Finally, in Chapter 6, we study the number of crossing limit cycles that can bifurcated from periodic annulus of system $(1)|_{\epsilon=0}$, assuming that the central subsystem has a real saddle and the other subsystems have centers or saddles. We prove that the maximum number of limit cycles that can bifurcate from the periodic annulus of this kind of piecewise linear Hamiltonian differential systems, after linear perturbations, is at least five, when the right and left subsystems have saddles, and at least six, when the right subsystem has a saddle or center and the left subsystem has a center (vice-versa). Our study is concentrated in the neighborhood of the double homoclinic loop which separates the periodic annulus of orbits that pass through three zones from the two periodic annulus of orbits that pass through only two zones. Thus, to estimate the zeros of the Melnikov functions we consider its expansions at the point corresponding to this orbit. Moreover, in order to simplifies the computation, we obtain a normal form to system $(1)|_{\epsilon=0}$ when the central subsystem has a real saddle. This study resulted in preprint V of the list of published and submitted papers derived from this thesis.

Chapter

8

Conclusions

In this work, we estimate the number of crossing limit cycles in planar piecewise linear Hamiltonian differential systems with two or three zones separated by straight lines, and parallel if there are more than one, such that the linear systems that define the piecewise one have isolated singular points, i.e. centers or saddles. More precisely, we started with the study of the number of limit cycles for continuous or discontinuous piecewise linear Hamiltonian differential systems with two or three zones. In this case, we showed that if the planar piecewise linear Hamiltonian differential system is either continuous or discontinuous with two zones or continuous with three zones, then it has no limit cycles, but if it is discontinuous with three zones then it has at most one limit cycle, and we provide examples with one limit cycle.

Next, we study the number of crossing limit cycles that can bifurcate from periodic annulis in a discontinuous piecewise linear Hamiltonian differential system with three zones, after polynomials functions perturbations. For theses cases, we prove that if the linear differential system defined in the region between the two parallel lines has a center at the origin and the others subsystems have centers or saddles then we have at least three, four or seven limit cycles bifurcating from a periodic annulus after linear, quadratic or cubic polynomials perturbations, respectively. Now, when the central subsystem has a real or boundary center and the others subsystems have only real saddles then at least six limit cycles can bifurcate simultaneously from two periodic annuli, after linear perturbations. Finally, if the central subsystem has a real saddle and the others subsystems have centers or saddles then at least six limit cycles can bifurcate simultaneously from three periodic annuli, after linear perturbations.

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